# Ecole Nationale Supérieure d'Informatique pour l'Industrie et l'Entreprise

# ENSIIE


# An Overview of Portfolio Insurances : CPPI and OBPI strategies


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# Abstract

We introduce a comprehensive framework for evaluating alternative portfolio insurance strategies within realistic scenarios. Our study contributes to existing literature by providing a broader perspective on the comparison between option-based and constant proportion portfolio insurance strategies (OBPI and CPPI). We determine the optimal payoffs for OBPI and CPPI by maximizing expected utilities, considering different levels of risk aversion and investment time frames, and employing a general, two-parameter HARA utility function. We examine two scenarios : one where predefined payoffs are acquired at fair prices and another where replication is employed, as commonly seen in the implementation of portfolio insurance strategies. The dynamics of asset prices are modeled using either a geometric Brownian process or a time-changed geometric Brownian process. Our findings consistently demonstrate the superiority of CPPI over OBPI across all scenarios. Furthermore, we investigate the impact of discrete replication and discontinuous price processes through simulation and compare the results to the theoretically optimal CPPI payoff obtained when the underlying process follows a geometric Brownian motion.

# Introduction

Portfolio insurance strategies aim to mitigate downside risk while allowing for participation in upward-moving markets. Two commonly used methods for achieving this are Option Based Portfolio Insurance (OBPI) and Constant Proportion Portfolio Insurance (CPPI).
OBPI involves investing in a risky asset while simultaneously purchasing a put option on that asset. This put option acts as a form of insurance, ensuring that the portfolio value remains above the strike price of the put, regardless of market conditions. On the other hand, CPPI dynamically allocates assets based on a predetermined floor and multiple. The floor represents the minimum acceptable portfolio value, and the multiple determines the allocation to the risky asset, with any excess funds being invested in a risk-free asset.
Comparisons between OBPI and CPPI shed light on their performance under various market conditions. This paper systematically examines the dynamics of both methods, providing insights into their effectiveness. Section 2 presents numerical results for both strategies under different models and utilizes real-world data to further analyze their performance.

# Chapitre 1

# Portfolio Insurances

Portfolio insurances are capital guarantee derivative securities that embed a dynamic trading strategy in order to make a contribution to the performance of a certain underlying assets. Two different types of portfolio insurances are considered here. First, the constant proportion portfolio insurance [1] and, second, the option-based portfolio insurance. Both invest partially in a risk-free way and combine this with a risky asset.

## 1.1 Constant Proportion Portfolio Insurance (CPPI)

The family of constant proportion portfolio insurance consists of investments for which the amount necessary for guaranteeing a repayment of a fixed amount N at maturity T is invested in a risk-free way, typically a bond, B, and only the exceeding amount will be invested in one or more risky assets, $S_i$. This way an investor can limit its downside risks and maintain some upside potential. This type of portfolio insurance has first been introduced by Black and Jones[2] and Perold [3].
The product manager will take larger risks when the market is performing well. But if the market is going down he will reduce the risk rapidly. The following factors play a key role in the risk strategies an investor will take :
• Price : The current value of the CPPI. The value at time t  [0, T] will be denoted as Vt.
• Floor : The reference level to which the CPPI is compared. This level will guarantee the possibility of repaying the fixed amount N at maturity T, hence it could be seen as the present value of N at maturity. Typically this is a zero-coupon bond and its price at time t will be denoted as Bt.
• Cushion : The cushion is defined as the difference between the price and the floor,

Cushion = Price Floor.
• Cushion% = Cushion/Price.
• Multiplier : The multiplier is a fixed value which represents the amount of leverage an investor is willing to take.
• Investment level : is the percentage invested in the risky asset portfolio ; this also known as the exposure and is for each step fixed at :
e = Multiplier × Cushion%.
• "gap" risk : is the probability that the CPPI value will fall under the Floor, see e.g [4].
The level of risk an investor will take is equal to the investment level as long as the value of the CPPI exceeds the floor. For any time t the future investment decision will be made according to the following rule :
• if Vt Floor = $B_t$ , we will invest the complete portfolio in a into the zero-coupon bond,
• if Vt > Floor, we will invest an amount equal to e in the risky asset portfolio.

## 1.2 Option Based Portfolio Insurance (OBPI)

Besides the CPPI this strategy of insuring a pay-off of a portfolio is also popular. The OBPI, introduced by Leland and Rubinstein [5], consists essentially in buying simultaneously a risky asset S (usually a financial index such as the S&P) and a put option written on it. Investing this way, independently of the value of S at maturity date T, the OBPI portfolio value will always be greater than the strike K of the put. Hence a pay-off value of K can be guaranteed.
It might seem that the goal of the OBPI method is to guarantee a fixed amount only at the terminal date but in fact it can be shown that the OBPI method allows one to get a portfolio insurance at any time. Note that the OBPI has just one parameter, the strike K of the put while the CPPI method is based on the choice of two parameters : the initial floor F0 and the multiplier m. The strike K will therefore play the same role as F0e rT in the CPPI model. In this paper we will not discuss the OBPI further as the OBPI will only be used as a tool to discuss the CPPI performance[5].

The OBPI method consists basically of purchasing $q$ shares of the asset $S$ and $q$ shares of European put options on $S$ with maturity $T$ and exercise price $K$ [6].

Thus, the portfolio value $V^{OBPI}$ is given at the terminal date by :

$$V_T^{OBPI} = q\left(S_T + (K - S_T)^+\right)$$

which is also : $V_T^{OBPI} = q\left(K + (S_T - K)^+\right)$, due to the Put/Call parity. This relation shows that the insured amount at maturity is the exercise price times the number of shares, $qK$.

The value $V_t^{OBPI}$ of this portfolio at any time $t$ in the period $[0,T]$ is :

$$V_t^{OBPI}=q\left(S_t+P\left(t,S_t,K\right)\right)=q\left(K.e^{-r(T-t)}+C\left(t,S_t,K\right)\right)$$

where $P\left(t,S_t,K\right)$ and $C\left(t,S_t,K\right)$ are the no-arbitrage values calculated under a given risk-neutral probability $Q$ (if coefficient functions $\mu,a$ and $b$ are constant, $P\left(t,S_t,K\right)$ and $C\left(t,S_t,K\right)$ are the usual Black-Scholes values of the European Put and Call).

Note that, for all dates $t$ before $T$, the portfolio value is always above the deterministic level $qKe^{-r(T-t)}$.

Usually, an investor is willing to recover a percentage $p$ of her initial investment $V_0$ (with $p\leq e^{rT}$ ). Then, her portfolio manager has to choose the two adequate parameters, $q$ and $K$.

First, since the insured amount is equal to $qK$, it is required that $K$ satisfies the relation :

$$pV_0=pq\left(K.e^{-rT}+C\left(0,S_0,K\right)\right)=qK$$

which implies that :

$$\frac{C\left(0,S_0,K\right)}{K}=\frac{1-pe^{-rT}}{p}$$

Therefore, the strike $K$ is a function $K(p)$ of the percentage $p$, which is increasing.

Second, the number of shares $q$ is given by :

$$q=\frac{V_0}{S_0+P\left(0,S_0,K(p)\right)}$$

Thus, for any initial investment value $V_0$, the number of shares $q$ is a decreasing function of the percentage $p$.

Note that the portfolio value has the homogeneity property with respect to $q$. In order to simplify the notations, in what follows we normalize $q$ to 1 without loss of generality.

# Chapitre 2

# Numerical study

## 2.1 CPPI strategy

In this analysis, we focus on implementing and evaluating the Constant Proportion Portfolio Insurance (CPPI) strategy to understand its behavior in various market conditions. Our primary objective is to examine the strategy's effectiveness through key performance metrics : the number of violations (instances where the strategy fails to guarantee the predetermined amount), the mean and median of the portfolio's wealth at maturity, and the portfolio's expected shortfall. By scrutinizing these metrics, we aim to gain insights into the resilience and reliability of the CPPI strategy in ensuring the minimum guaranteed return to investors, even in volatile market scenarios.

### 2.1.1 Methodology and Analytical Framework

#### Bond Price Simulation

We simulate bond prices over time considering a fixed interest rate and the bond's maturity value. This method projects future bond prices by reversing the discounting process to understand how bond values respond to interest rate changes.

#### Merton's Jump-Diffusion Model for Asset Prices

Asset price paths are simulated using Merton's jump-diffusion model, which accounts for both continuous market risks (via geometric Brownian motion) and discrete jump risks. The model uses parameters such jump frequency (lam), mean jump size (me), and jump size volatility (v). When jump parameters are zero, the model reduces to geometric Brownian motion, simulating a continuous market risk without jumps.

### Floor Types in CPPI Strategy Simulations

Within this framework, we introduce the concept of "floor" as the minimum amount of wealth that the portfolio should not fall below, operationalized through two distinct floor types :

1. **Classical CPPI Floor :** This floor is tied to the bond price, which is updated over time. It ensures that if the portfolio value hits this threshold, all wealth is reallocated to the non-risky asset to guarantee the minimum return. The start value of the portfolio, or the amount of money invested by the client, is denoted as N, which also represents the guaranteed amount the client expects to receive at least at maturity. if the account value hit the floor , the entire portfolio is shifted toward the non-risky asset , so the floor and the account value will grow at the same rate allowing for the strategy to guarantee the amount N at the maturity.
2. **Maximum Drawdown Floor :**This innovative floor prevents the portfolio's value from dropping below a specified percentage of the highest wealth achieved, thus preserving a portion of the peak wealth even if the value of the risky asset declines. This approach allows the portfolio the possibility to exceed its previous maximum wealth, offering a more dynamic response to market changes compared to the classical strategy. Under this strategy, when the portfolio value decreases and even hit the floor level,the floor itself stays constant so despite that the entire wealth is shifted to the non-risky asset the account value will growth at the risk-free interest rate until the cushion becomes positive again, thereby permitting reinvestment in the risky asset. This mechanism aims to avoid the pitfall of the classical strategy, where hitting the floor could stifle future growth potential.

### Failure Scenarios in CPPI Strategy Simulations

The simulation addresses failure scenarios, where the final portfolio value falls short of the guaranteed amount. Such failures or violations can stem from two main issues : infrequent portfolio adjustments or unfavorable market conditions that prevent the portfolio's floor value from achieving the guaranteed amount by maturity.

1. **First Strategy (Classical CPPI Floor)** : In theory, there should be no failure with the classical CPPI strategy because, upon reaching the floor, all assets are shifted to the non-risky investment, ensuring the guaranteed amount N N by the end of the period. However, this assumes continuous portfolio adjustments as market prices evolve. In practice, adjustments are made periodically (e.g., weekly or daily) due to the impracticalities of constant monitoring and the costs associated with frequent transactions. We model

this with an 'adjustment frequency' variable. For instance, if the adjustment frequency is set to every two time steps, the portfolio is only rebalanced at these intervals. The risk here is that the portfolio value could dip below the floor between adjustments. However, if adjustments are made frequently enough, the expected shortfall should be minimal.

2. **Second Strategy (Maximum Drawdown Floor) :** This strategy allows for recovery even if the portfolio's value drops below the floor, provided the risky asset's value subsequently increases. Initially, the floor value might be lower than that required for the classical strategy, posing a risk if the risky asset does not appreciate in value, and the floor does not reach the guaranteed amount N by maturity. The flexibility of this strategy lies in its ability to recover from downturns, offering a chance for the portfolio to regain and potentially exceed its peak value, unlike the classical strategy where hitting the floor could limit growth opportunities.

These scenarios highlight the importance of the adjustment frequency in managing the CPPI strategy's effectiveness. While the classical floor provides a straightforward safeguard by reallocating to non-risky assets upon reaching the floor, the maximum drawdown floor introduces a dynamic element that preserves growth potential even after significant market drops. Both strategies, however, depend on the careful timing of portfolio adjustments to mitigate the risk of failure and ensure the investor's minimum guaranteed amount is protected under various market conditions.

## 2.2 CPPI strategy simulation

Initial Portfolio Value (N) : We assume the start value of the portfolio, or the amount of money invested by the client, is denoted as N. This amount represents not only the initial investment but also the guaranteed amount that Guaranteed Amount at Maturity : The guaranteed amount the client expects to receive at least at the maturity of the investment is also N. This assumption is critical for calculating the floor values in our CPPI strategies and assessing the performance against the guaranteed amount.

### 2.2.1 comparing the standard GBM with the jump model

We will use the classic CPI floor for the sake of this comparison We will study the effect of suddent jumps on the results of the stratery for that we will begin by simulating the cppi on the geometric brownian motion we used these parametres for that :

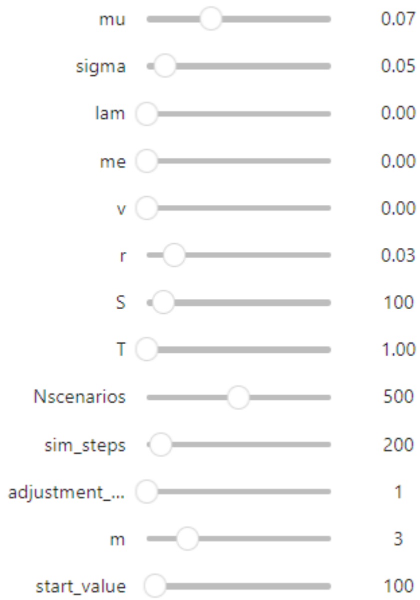



FIGURE 2.1 – CPPI parameters

we obtain this result :

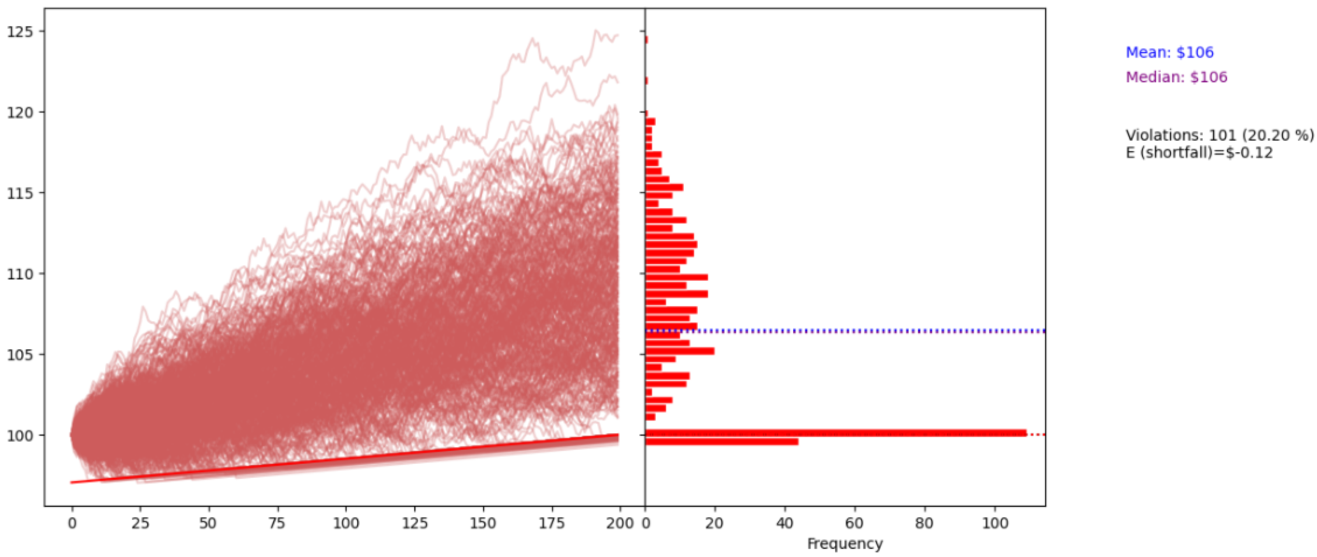



FIGURE 2.2 – CPPI simulation with GBM model

now , we will add jumps by setting v=0.04 , lam = 0.25, me =0.0 we obtain this result :

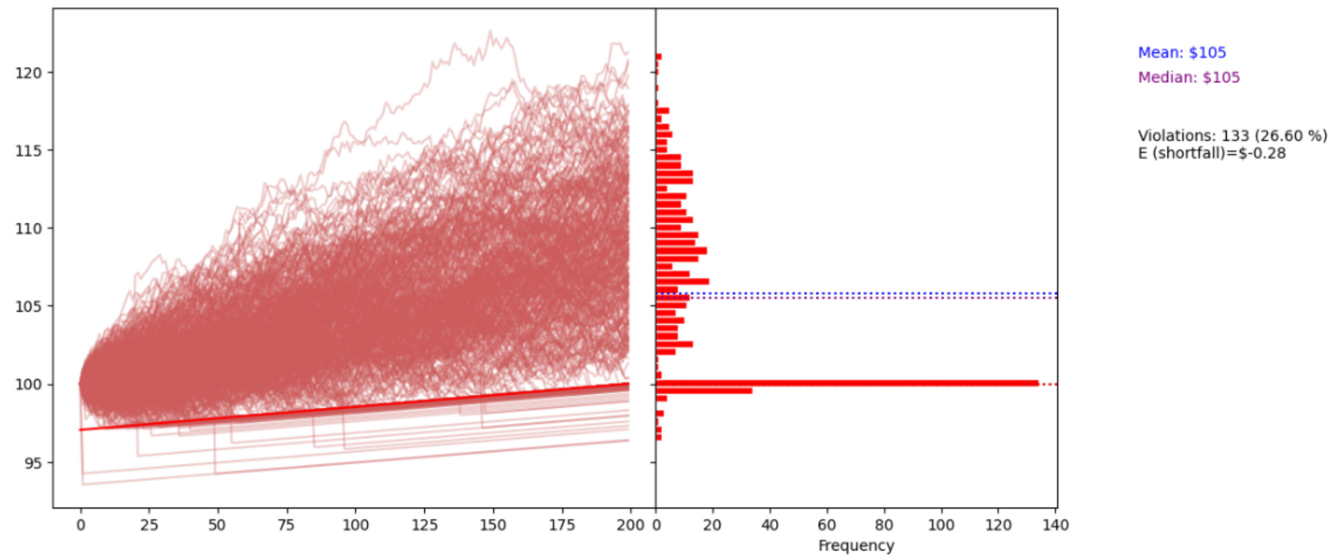



FIGURE 2.3 – CPPI simulation with jump model

As expected , the jump model present a higher level of violations , we observe also an increase in the expected shortfall due to extreme jumps. we will continue using the Merton jump diffusion model in this part as it resembles more the behaviour of the risky asset.

### 2.2.2 comparing with the Maximum Drawdown Floor

Using the same parameters for the Merton jump model and setting the floor to be 0.9 of the maximum wealth ever reached we get the following simulation :

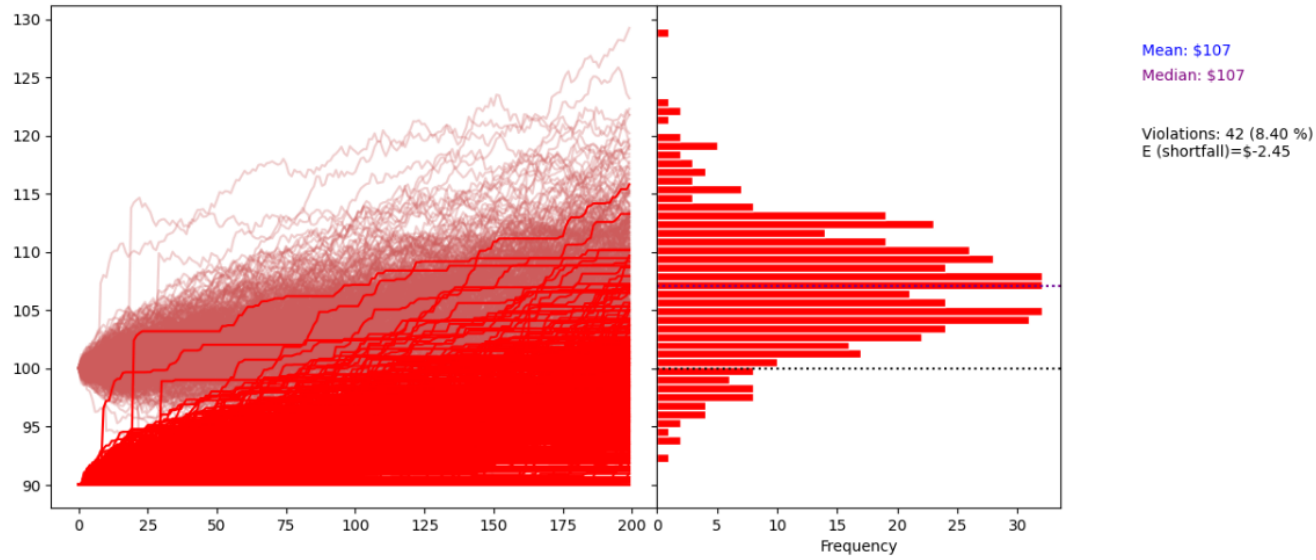



FIGURE 2.4 – CPPI simulation with max drowdown floor (jump model)

We got more expected return as the mean and median increased for 105 dollars to 107 dollars , We also observe less number of violations as the floor is a non-decreasing function. the only problem that the expected shortfall has gone down a lot from 0.28 dollars to 2.45 dollars. the reason is that the floor as 0.9 of the wealth is initially less than the classic floor at 97.05 dollars , that means that the floor can in some scenarios never reach the value N at maturity unlike the classic floor. one way to deal with that is to increase the floor or combining the two floors so that the wealth should never go below both of them.

### 2.2.3 Comparing the adjustment frequency

finally we will see the effect of the adjustment frequency , we will test it with the **GBM model** with the **classical floor** . In the first simulation we set the adjustment frequency to 1 , means that we will adjust the portfolio every time step. we will change now this parameter to 20. we get this result :

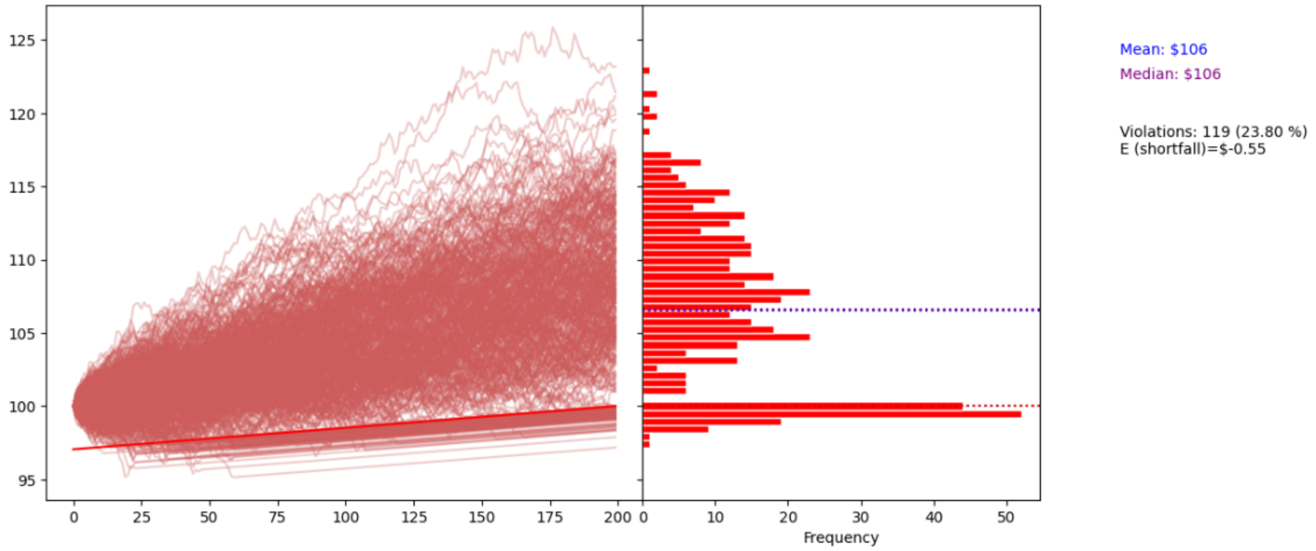



FIGURE 2.5 – CPPI simulation when adjusting every 20 step (GBM ,classical floor

we can clearly see that the number of violations has increased from 101 to 113 as well as the expected shortfall from 0.12 to 0.55.

## 2.3 OBPI strategy

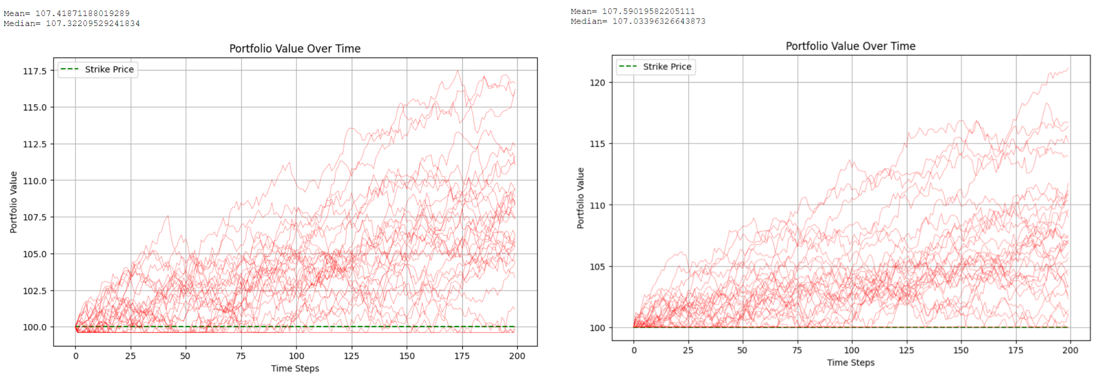
In this section, we focus on implementing and evaluating the Option-Based Portfolio Insurance (OBPI) strategy to understand its behavior. Our primary objective is to examine the strategy's effectiveness through the mean and median of the portfolio's wealth at maturity.

The portfolio value over time fluctuates due to changes in the underlying asset's price, influenced by market dynamics and the specific parameters of the OBPI strategy. Initially, investors secure their investment by allocating it across the chosen assets or securities. As time progresses, the portfolio value reacts to market movements, which may lead to fluctuations in value.
The OBPI strategy often incorporates options contracts, such as put options, to hedge against potential losses. These options provide investors with the right, but not the obligation, to sell the underlying asset at a predetermined price (the strike price) within a specified timeframe. By strategically employing put options, investors can protect their portfolio value from significant downturns in the market.

explain more this paragraph Additionally, the OBPI strategy often incorporates options contracts, such as put options, to hedge against potential losses. By strategically employing put options, investors can protect their portfolio value from significant downturns in the market.

At maturity, if the price of the asset falls below the strike price, the investor has the option to purchase the asset at the predetermined strike price. In this scenario, the investor effectively hedges against losses resulting from the decline in the asset's price. Conversely, if the asset's price exceeds the strike price, the investor acquires the asset at its prevailing market price, incurring only the cost of the put option. Thus, the OBPI strategy enables investors to minimize their losses by leveraging put options to protect against adverse market movements.

Indeed, the portfolio value and potential losses within this strategy are intricately tied to market parameters. When one of these parameters is varied, it significantly impacts the portfolio value. This is primarily because the price of options, which play a crucial role in the strategy, is closely linked to these parameters. As a result, any fluctuations in market parameters directly influence the valuation of options, thereby exerting a profound effect on the overall performance of the portfolio and the extent of potential losses.
For example, if we adjust the value of the risk-free interest rate (r) from 0.05 to 0.1, it can have substantial ramifications on the portfolio value and potential losses within the strategy. This change in the interest rate directly influences the pricing of options, given that interest rates are a critical component in option pricing models like the Black-Scholes model. Consequently, a higher interest rate would lead to adjustments in option prices, potentially altering the risk profile and performance of the portfolio. In particular, an increase in the interest rate might result in higher option premiums.




((a)) portfolio values under the OBPI strategy for r=0.05

((b)) portfolio values under the OBPI strategy for r=0.2

To compare the results between the CPPI strategy and the OBPI strategy, I plotted the portfolio values of the two strategies for various scenarios, utilizing identical parameters as in the previous section. Upon examination, it becomes evident that both the mean and median values for the OBPI strategy surpass those of the CPPI strategy. This observation underscores the enhanced efficiency of the OBPI strategy under the Merton's model framework in this context.

```
Mean= 106.49353190876593
Median= 105.97165828095548
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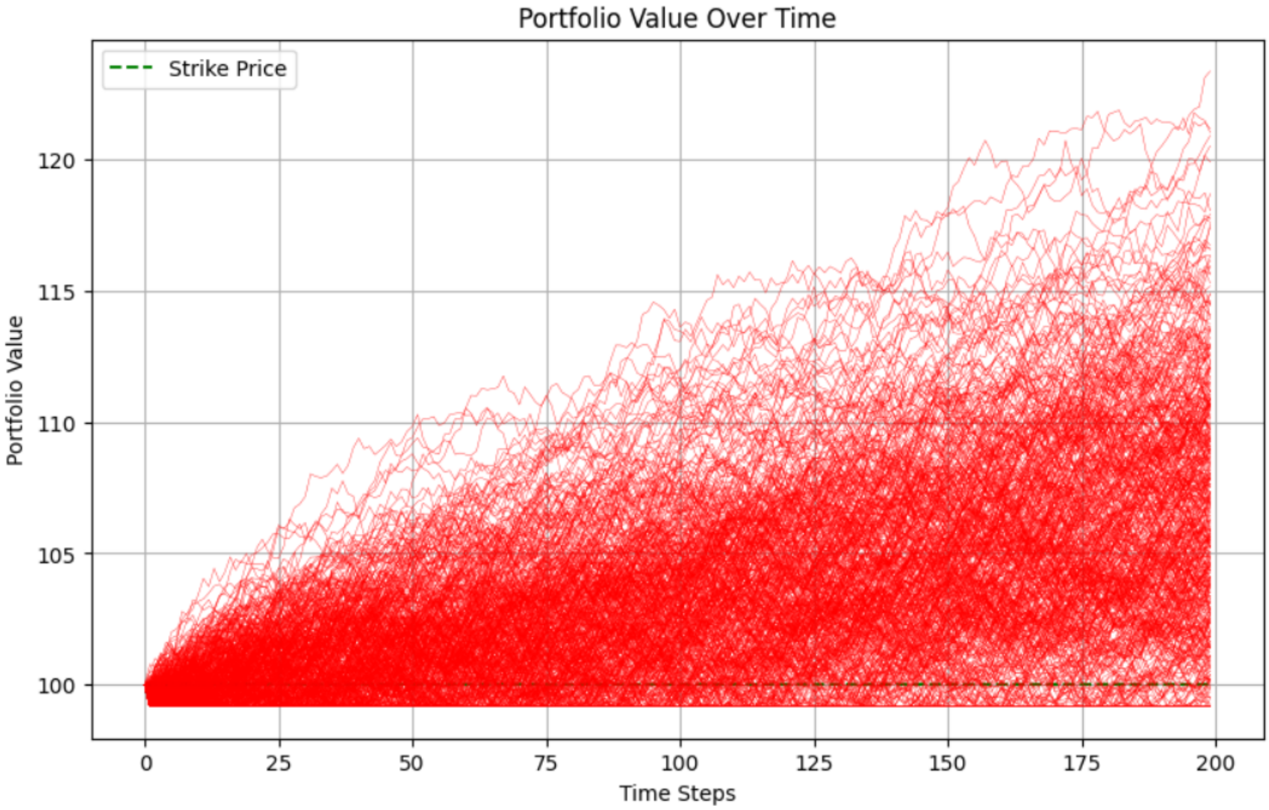



FIGURE 2.7 – OBPI simulation with jump model

1. **Implementation of the BDT Model in Continuous State Within Discrete Time** :

The paper HOGAN-WEINTRAUB SINGULARITY AND EXPLOSIVE BEHAVIOR IN THE BLACK-DERMAN-TOY MODEL studies the model of black derman toy in the continious state variable and try to study its explosive behaviour in certain conditions with respect to the tree model to show that the behavious of the contiious state variable doesn't align with the result of the simple tree model and represent an explosive behavious making the continous state variable model give result that doesn't correspond to the price maket ,, we will tackle in this section this behaviour , its resons and in what condition this behavious appears.

We consider here the BDT model in a discrete time setting but with a continuous state variable we then consider the continuous time limit of this model.

we will first set the model by giving expressions to the libor rates Li , the zero coupon bond price that we are trying to calculate its expression using the model. The BDT model is traditionally set in discrete time, yet it incorporates a continuous state variable. This model presumes that the Libor rate, $L_i$, for the interval $(t_i, t_{i+1})$, adheres to a geometric Brownian motion across discrete time points, described by the formula :

$$L_i = r_i e^{\sigma_i x_i - \frac{1}{2}\sigma_i^2 t_i}, \quad i = 0, 1, \cdots, n-1$$

we will assume that $\sigma_i$ and $r_i$ values are known and we don't need to calibrate them using the initial yield curve through zero coupon bonds and the market caplet volatilities on the Libor rates $L_i$.

For simulating the model, it's crucial to calculate the zero coupon bonds $P_{i,j}(x_i)$ as functions of the stochastic variable $x_i$ at time $t_i$. These bonds are defined via risk-neutral expectations as follows :

$$P_{i,j}(x_i) = \frac{1}{1 + L_i \tau} \mathbb{E}_Q \left[ \frac{1}{\prod_{k=i+1}^{j-1} (1 + L_k \tau)} \middle| \mathcal{F}_{t_i} \right].$$

in order to be able to calculate the expectation in the zero bond expression, we will try to convert the expectation to a recursive integral calculation for that we wil get the recursive expression in the classic tree implementation we will then generalise the expression for the continious state variable. the paper models the tree points with index i representing the time evolution and k that creates the different scenarios in the tree. the node $x_k^{(i)}$ is repserented as :

$$t_i : \quad x_k^{(i)} = (-i + 2k)\sqrt{\tau}, \quad k = 0, 1, \cdots, i.$$

the general expression of The zero coupon bonds on the tree $P_{i,j}(x)$ defined on the points $x = x_k^{(i)}$ is :

$$P_{i,j}(x) = \frac{1}{1 + L_i(x)\tau} \left\{ \frac{1}{2} P_{i+1,j}(x + \sqrt{\tau}) + \frac{1}{2} P_{i+1,j}(x - \sqrt{\tau}) \right\}$$

we can extend this calculation to the continious state implementation of the BDT model by taking an itegral as the sum of all the points in the time step.

$$P_{i,j}(x_i) = \frac{1}{1 + L_i(x_i)\tau} \int_{-\infty}^{\infty} \frac{dx_{i+1}}{\sqrt{2\pi\tau}} e^{-\frac{1}{2\tau}(x_i - x_{i+1})^2} P_{i+1,j}(x_{i+1})$$

with the initial condition This recursion relation can be simplified by introducing the functions $\pi_{i,j}(x_i)$ defined as

$$P_{i,j}(x_i) = \frac{1}{1 + L_i(x_i)\tau}\pi_{i,j}(x_i)$$

They satisfy the backwards recursion

$$\pi_{i,j}(x_i) = \int_{-\infty}^{\infty} \frac{dx_{i+1}}{\sqrt{2\pi\tau}} e^{-\frac{1}{2\tau}(x_i - x_{i+1})^2} \frac{\pi_{i+1,j}(x_{i+1})}{1 + L_{i+1}(x_{i+1})\tau}$$

with initial condition $\pi_{j-1,j}(x_{j-1}) = 1$.

the paper has proven that the functions $\pi_{i,j}(x_i)$ have the asymptotic behavior

$$\pi_{i,j}(x_i) \sim Ce^{-(j-i-1)\sigma x_i}, \quad x \to +\infty$$

with $C$ a constant. So, we can rewrite $\log P_{i,j}(x_i)$ as :

$$-\log P_{i,j}(x_i) = \log(1 + L_i\tau) + (j - i - 1)\left(\log(L_i/r_i) + \frac{1}{2}\sigma^2 t_i\right) - \log C$$

When volatility is low, the dominance shifts towards the first term, particularly if the number of simulation steps spanned by the bond $P_{i,j}$, represented by $j - i$, is relatively small.

Conversely, with an increase in volatility, the importance of the second term grows, notably when $j-i$ represents a larger span (e.g., $j-i = 10$). This introduces a logarithmic dependence on $L_i$, thereby modifying the observed relationship at lower volatility thus creating an inconsistency with results obtained from the tree implementation

black derman toy with continious time limit and continious state variable :

We consider in this section the approach to the continuous time limit of the BDT model with continuous state variable. This corresponds to taking the time step very small $\tau \to 0$, while preserving the log-normal distributional specification of the one-step Libor rates $L_i$. We will assume for simplicity uniform volatility $\sigma_i = \sigma$.

Consider a short rate model with short rate specification of the form (2.4) $r_t = \bar{r}(t)e^{\sigma W_t - \frac{1}{2}\sigma^2}$. The process for $r_t$ follows from (2.5) by assuming a constant volatility $\sigma(t) = \sigma$

$$\frac{dr_t}{r_t} = \sigma dW_t + \left[\partial_t \log \bar{r}(t)\right] dt$$

Denoting the Eurodollar future price at time $t$ with $V_t$, and assuming discrete settlement at times $t_i$,

$$V_t = 100 \left(1 - \mathbb{E}_Q \left[L(T, T+\delta) \mid \mathcal{F}_t\right] \delta\right)$$

In order to observe the Hogan-Weintraub singularity in a simulation of the BDT model, one must consider the Libor rate over several periods $(t_i, t_i + n_\delta \tau)$, and take $\tau$ to zero while keeping $n_\delta \tau = t_j - t_i$ and the volatility $\sigma$ finite. In this limit one expects that the following expectation should diverge

$$\lim_{\tau \to 0, n_\delta \tau = \text{ finite}} \mathbb{E}_Q \left[L\left(t_i, t_i + n_\delta \tau\right)\right] \to \infty$$

where

$$L\left(t_i, t_i + n_\delta \tau\right) = \frac{1}{n_\delta \tau} \left(P_{i,j}^{-1} - 1\right)$$

$Theappearanceofthisdivergencecanbeseenexplicitlyfromthesolutionofthe BDT modelpresentedin$

$\mathrm{P}_{i,j}^{-1}\left(x_i\right)$

$$\mathbb{E}_Q\left[P_{i,j}^{-1}\left(x_i\right)\right] = \int_{-\infty}^{\infty} \frac{dx_i}{\sqrt{2\pi t_i}} e^{-\frac{1}{2t_i} x_i^2} \left(1 + L_i(x)\tau\right) \frac{1}{\pi_{i,j}\left(x_i\right)}$$

with $\pi_{i,j}\left(x_i\right)$ defined in Eq. (3.17).

The appearance of this divergence can be seen explicitly from the solution of the BDT model presented in Sec. 3. The starting point is the following expression for the expectation of $P_{i,j}^{-1}\left(x_i\right)$

$$\mathbb{E}_Q\left[P_{i,j}^{-1}\left(x_i\right)\right] = \int_{-\infty}^{\infty} \frac{dx_i}{\sqrt{2\pi t_i}} e^{-\frac{1}{2t_i} x_i^2} \left(1 + L_i(x)\tau\right) \frac{1}{\pi_{i,j}\left(x_i\right)}$$

the paper elaborates an uppper bound for the expectation by application the recursion in equation (3.8)

$$\mathbb{E}\left[P_{i,j}^{-1}(x_i)\right] \geq \exp\left(\sum_{k=i+1}^{j-1}\log(r_k\tau) - \frac{1}{6}\sigma^2\tau n_\delta\left(n_\delta^2-1\right) + \frac{1}{2}(n_\delta-1)(n_\delta-2)\sigma^2 t_i\right)$$

Term Analysis we will examines a specific scenario where the number of time steps between two points in time, denoted as $n{=}ji$, is greater than one.

-The first term in the exponent, described as diverging negatively like $-n_\delta \log n_\delta$, suggests that as the number of steps ($n_\delta$) increases, this term decreases significantly.

-The sum of the last two terms in the exponent increases as $\left(\frac{1}{2}\sigma^2 t_i - \frac{1}{6}\sigma^2(t_j - t_i)\right) n_\delta^2$, suggesting a quadratic increase as the number of steps increases. This behavior is contingent upon the condition $t_i > \frac{1}{3}(t_j - t_i)$, meaning the start time ($t_i$) must be significantly before the midpoint between $t_i$ and $t_j$. -The divergence is attributed to the rapid decrease of the function $\pi_{i,j}(x_i)$ as $x_i$ approaches infinity with an increasing number of time steps.

the paper afterward introduced the concept of the convexity adjustment factor $\kappa(T, \delta, n_\delta)$ defined as

$$\mathbb{E}[L(T, T+\delta)] = \kappa(T, \delta, n_\delta)\, L^{\mathrm{fwd}}(T, T+\delta)$$

In his analysis, the author illustrated the convexity adjustment $\kappa$ for a term $T = 5$ using simulations with a time step $\tau = 0.25$, as a function of volatility $\sigma$ within both a continuous state framework and a tree (lattice) model. For volatilities up to roughly $\sigma \approx 30\%$, the findings from both models were in close agreement.

However, as volatility increased beyond this level, discrepancies emerged, with the continuous state model demonstrating a significant increase at a critical volatility, $\sigma_{\mathrm{cr}} \approx 47\%$.

To further explore the cause of this anomaly, the author examined the integrand of expectation equation (4.6) for $t_i = 5$ and $t_j = 7.5$, across various volatilities near the critical point $\sigma_{\mathrm{cr}} \approx 47\%$. For volatilities below $\sigma_{\mathrm{cr}}$, the integrand was predominantly peaked around the origin with a width of approximately $\sqrt{t_i} \approx 2.24$. This region is adequately covered by the tree model, which spans $x$ values from $-i\sqrt{\tau}$ to $i\sqrt{\tau}$, equivalent to a range of $(-10, +10)$ at the final time step $t_{20} = 20\tau = 5$. This accounts for the good agreement between the models at lower volatilities. However, at volatilities above $\sigma_{\mathrm{cr}}$, a second peak forms at a substantially higher $x$ value, around 24, and grows significantly as volatility exceeds the critical point. Since this peak falls outside the range covered by the tree model, it is not captured in its simulations, explaining the divergence in results between the models at high volatilities.

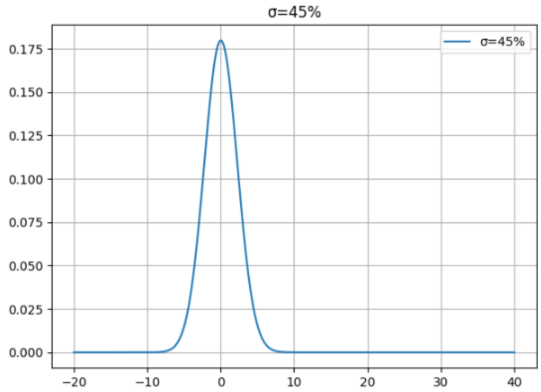



FIGURE 2.8 – Enter Caption

we have tried to simulate the integrand wilth the volatility 45we got the following result that aligns with the paper result.

we know that the tree covers the region $x = (-i\sqrt{\tau}, i\sqrt{\tau}) = \left(-\frac{t_i}{\sqrt{\tau}}, \frac{t_i}{\sqrt{\tau}}\right)$ at the specific time slice $t_i$, which translates to $(-20\sqrt{\tau}, +20\sqrt{\tau}) = (-10, +10)$ at the terminal time $t_{20} = 20\tau = 5$. This explains the accurate correspondence with the continuous state outcome for low volatility levels. However, as the volatility nears a pivotal point, the integrand forms a second peak at a considerably high value of $x \approx 24$, which swiftly escalates in magnitude as the volatility exceeds the critical threshold. Past this critical volatility $\sigma_{\text{cr}}$, the second peak predominantly influences the integrand. Since this peak is beyond the range of $x$ values encompassed by the tree model, it remains unobserved in this simulation approach, leading to the divergence observed between the two simulation methods under extremely high volatility conditions.

# Chapitre 3

# Results

## 3.1 Application of CPPI on historical data

In this section, we explore the application of the Constant Proportion Portfolio Insurance (CPPI) strategy to historical stock data for three prominent companies : Ford Motor Company (F), Apple Inc. (AAPL), and General Electric Company (GE). The period of analysis spans from January 1, 2016, to January 1, 2021.

### 3.1.1 Data description

The selection of Ford Motor Company (F), Apple Inc. (AAPL), and General Electric Company (GE) for testing the Constant Proportion Portfolio Insurance (CPPI) strategy was deliberate and aimed at capturing a diverse range of market dynamics. This graph shows the historical performance of $1,000 investments made in each of these stocks.

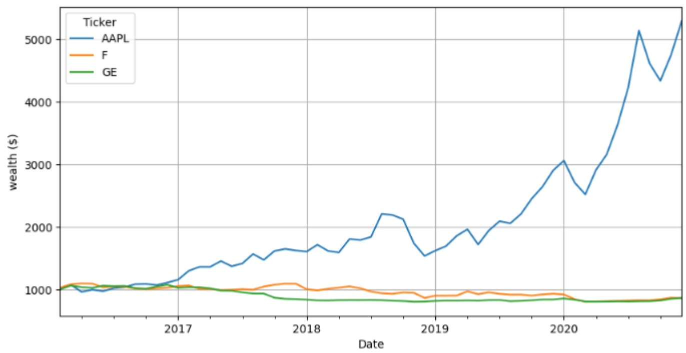
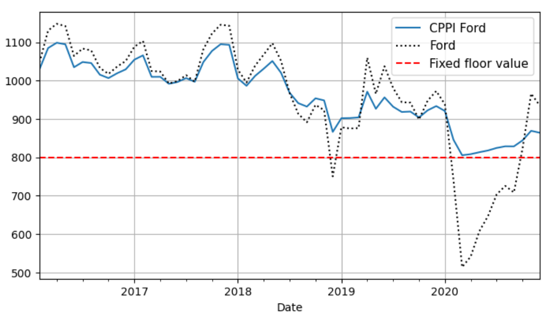



FIGURE 3.1 – Accumulated Wealth For Each Stock Portfolio

Ford, representing the automotive industry, is known for its sensitivity to economic cycles, making it a suitable candidate to assess the CPPI strategy's effectiveness during periods of industry cyclicality. Apple, a leading technology company, was included due to its global market impact and historical growth trends, providing insights into how the CPPI strategy performs in the context of a technology-driven sector. General Electric, operating across various industries, offers a mix of market exposures and economic sensitivities, allowing for a comprehensive evaluation of the CPPI strategy's adaptability to diverse economic conditions. By choosing these three companies, we aimed to analyze the strategy's performance across different industries and under various market scenarios, providing a holistic understanding of its potential benefits and limitations.

### 3.1.2 Implementation Of CPPI

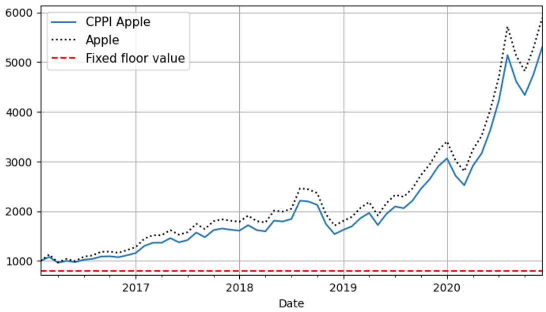
In this section, we implemented the CPPI strategy for each stock portfolio. Each portfolio began with an initial account value of 1000, a floor set at 0.8, and a multiplier of 3. Additionally, for the safe asset component, we established an artificial set of assets ensuring a guaranteed annual return of 3% with an initial allocation of 0.6 in risky assets.
The subsequent figures depict the accumulated wealth comparison between the CPPI portfolio and the portfolio consisting solely of risky assets.



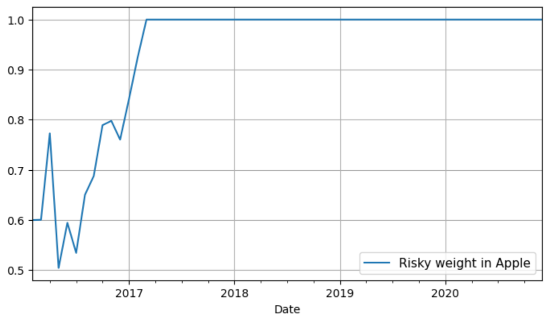

((a)) Comparison of Risky Asset and CPPI Portfolio Performance for Ford


((b)) Risky Asset Allocation in CPPI Portfolio

For the case of Ford, the CPPI portfolio performed similarly to the risky asset portfolio. However, in 2020, employing the CPPI strategy enabled us to shield a portion of our portfolio from the abrupt stock price decline during the COVID-19 pandemic which would have made our portfolio's value as low as 510.

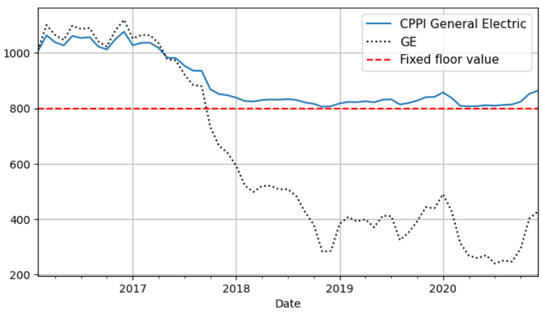



((a)) Comparison of Risky Asset and CPPI Portfolio Performance for Apple




((b)) Risky Asset Allocation in CPPI Portfolio

For the case of Apple, a consistently high-performing stock, it's evident that the CPPI portfolio's value would be lower than the risky asset portfolio. This is because initially, we didn't invest 100% in risky assets.




((a)) Comparison of Risky Asset and CPPI Portfolio Performance for GE


((b)) Risky Asset Allocation in CPPI Portfolio

The portfolio value of GE experienced a significant decline, plummeting to as low as \$230. However, through the implementation of CPPI, we successfully safeguarded a portion of our wealth, ensuring that the portfolio value didn't dip below our floor value of \$800.

These examples illustrate how by incorporating Constant Proportion Portfolio Insurance (CPPI) into investment strategies, investors can fortify their portfolios against significant losses during market downturns, ensuring a level of downside protection. At the same time, CPPI facilitates participation in the potential gains of risky assets, striking a balance between risk mitigation and return potential. The dynamic adjustments inherent in CPPI further enhance its effectiveness by allowing the strategy to adapt and allocate resources based on the evolving market conditions. In essence, CPPI serves as a versatile risk management tool, providing investors with a comprehensive approach to navigate various market scenarios.

# Conclusion

As the market for structured credit products continues to expand, the demand for protection mechanisms in structured credit transactions remains high, driving a continuous evolution in this field. Recently developed products such as CPPI and OBPI aim to provide this protection.
CPPI, which emerged around a decade ago, offers a predetermined principal payment at maturity. It operates on a constant proportion rule, dynamically rebalancing investments in risky and safe assets at each step to optimize profits.
On the other hand, OBPI, introduced recently, is designed to provide protection against market downturns. It utilizes put options to safeguard invested capital, allowing investors to limit losses in case of market declines while retaining upside potential.
The objective of this document is to provide an in-depth understanding of the dynamics and risks associated with CPPI and OBPI. We begin by explaining their functioning and investment decision processes step by step.
For CPPI, the risk of the total portfolio falling below the floor value exists, resulting in losses, while for OBPI, losses occur during triggered protection events. Quantifying these risks is crucial, and existing research explores various approaches for both strategies.

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