Random Variable

Definition :

A random variable (X) is a function that associates a real number with each element in the sample space.

It is of 2 kinds : ① Discrete

② Continuous

① **Discrete**

(There is no limit).

No of defective items, No. of highway facility given by the Govt. per year.

② **Continuous**

Height, weight, Temperature, Distance etc

[Have a limit]

Probability Mass Function :

The ordered pair $(x, f(x))$ is called PMF if

① $f(x) \geq 0$

② $\sum_{x} f(x) = 1$

③ $P(X = x) = f(x)$

Random Variable

Discrete

⊞ Probability Density function :

The function $f(x)$ is PDF if

① $f(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x)\, dx = 1$

③ $P(a < X < b) = \int_a^b f(x)\, dx$

⊞ Joint Probability Mass Function :

$f(x, y)$ is called JPMF if –

① $f(x,y) \geq 0$

② $\sum_x \sum_y f(x,y) = 1$

③ $P(X = a, Y = b) = f(a, b)$

⊞ Joint Probability Density Function :

$f(x,y)$ is called JPDF if –

① $f(x,y) \geq 0$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

③ $P[(X, Y) \in A] = \iint_A f(x,y)\, dA$

Commulative Distribution Function:

$F(x)$ is CDF if

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt \;;\; -\infty < x < \infty$$

$$= \sum_{t \le x} f(t) \;;\; -\infty < x < \infty$$

Find PMF of the no. of tails for tossing a coin twice, mentioned that is biased so that a head is 3 times as likely to occur as a tail.

Soln Sample space, $S = \{HT, TH, TT, HH\}$

Let, X is a random variable which represents no. of tails

$P(H) = 3/4$, $P(T) = 1/4$

X :	0	1	2
$P(X=x) = f(x)$:	$9/16$	$6/16$	$1/16$

$f(0) = P(HH) = P(H)P(H) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$

$f(1) = P(HT) + P(TH) = 2 \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{6}{16}$

$f(2) = P(TT) = \frac{1}{16}$

Here, $\sum f(x) = 1$

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Problem

A coin is tossed three times in which the probability of head is twice as the probability of tail. If number of heads is a random variable, find the probability function of the random variable. Also find (a) $P[x \geq 1]$.

(b) $P[x = 2]$ and (c) $P[x \leq 1]$

Solⁿ Let H and T denote the head and tail of the coin, respectively. The sample space of the experiment is

$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Here $P[H] = 2/3$ and $P[T] = 1/3$.

The probabilities of the different points of the sample space are

$P[HHH] = 2^3/3^3 = 8/27$, $P[HHT] = P[HTH] = P[THH] = 4/27$,

$P[HTT] = P[THT] = P[TTH] = 2/27$, $P[TTT] = 1/27 = \frac{2^0}{27}$.

The sample of the experiment is not simple.* Let X be a random variable defined by the number of heads. The different points of sample space, their probabilities, the different values of the random variable and their respective probabilities are shown in the table:

[* simple → sample point are equal]

Sample pts in Ω: ω_i	Probability of $[\omega_i]$	No. of heads X: x	Probability $P[x]$
HHH	8/27	3	8/27
HHT	4/27	2	
HTH	4/27	2	12/27
THH	4/27	2	
HTT	2/27	1	
THT	2/27	1	6/27
TTH	2/27	1	
TTT	1/27	0	1/27

Hence, the probability function of the random variable X is,

Values of $X: x$	0	1	2	3
$f(x)$	1/27	6/27	12/27	8/27

(i) $P[x \geq 1] = 1 - P[x < 1] = 1 - f(0) = 1 - \frac{1}{27} = \frac{26}{27}$

(ii) $P[x = 2] = f(2) = \frac{12}{27}$.

(iii) $P[x \leq 1] = P[x = 0] + P[x = 1] = f(1) + f(0)$

$= \frac{6}{27} + \frac{1}{27}$

$= \frac{7}{27}$.

(Ans)

(Problem)

Let X be a random variable with probability function defined by $f(-2) = 1/10$, $f(0) = 2/10$, $f(4) = 4/10$ and $f(11) = 3/10$.

Find (i) $P[-2 \le x < 4]$, (ii) $P[x > 0]$ and (iii) $P[x \le 4]$

Soln Here, the random variable X takes the values $-2, 0, 4$ and 11 with respective probabilities $1/10$, $2/10$, $4/10$ and $3/10$.

That is,

Values of $X : x$	-2	0	4	11
$f(x)$	$1/10$	$2/10$	$4/10$	$3/10$

(i) $P[-2 \le x < 4] = P[x = -2] + P[x = 0]$

$= f(-2) + f(0) = 1/10 + 2/10 = 3/10$

(ii) $P[x > 0] = P[x = 4] + P[x = 11] = f(4) + f(11)$

$= 4/10 + 3/10 = 7/10$.

and

(iv) $P[x \le 4] = 1 - P[x = 11] = 1 - f(11) = 1 - 3/10 = 7/10$.

Problem

A random variable X has the following probability function:

Value of X : x	0	1	2	3	4	5	6	7	8
$f(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Determine the values of a.

(ii) Find $P[x<3]$, $P[x \leq 3]$, and $P[0<x<5]$

Sol^n^ Since $f(x)$ is probability function. we have

$\sum f(x) = 1$. That is,

$$f(0)+f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8) = 1$$

or, $a + 3a + 5a + 7a + 9a + 11a + \cancel{13a}\ 13a + 15a + 17a = 1$

or, $a \times 81 = 1 \Rightarrow a = 1/81$

(ii) $P[x<3] = P[x=2] + P[x=1] + P[x=0]$

$= f(2) + f(1) + f(0)$

$= \frac{1}{81} + \frac{3}{81} + \frac{5}{81} = \frac{1}{9}$

$P[x \geq 3] = 1 - P[x<3] = 1 - \frac{1}{9} = \frac{8}{9}$.

$P[0<x<5] = P[x=1] + P[x=2] + P[x=3] + P[x=4]$

$= f(1) + f(2) + f(3) + f(4) = \frac{8}{27}$.

Probability Density Function (PDF):

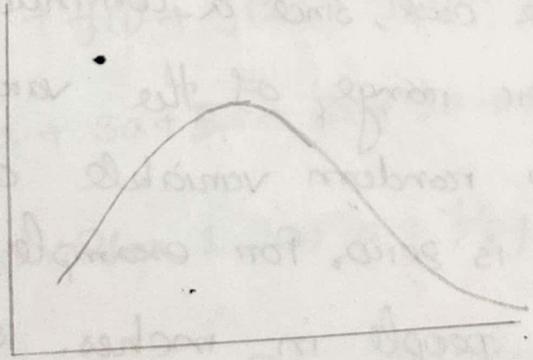
If X is a continous random variable, then the function $f(x)$, defined on the continous sample space Ω with the domain the real line and counterdomain the interval $[0, 1]$ is called probability density function or simply density function.

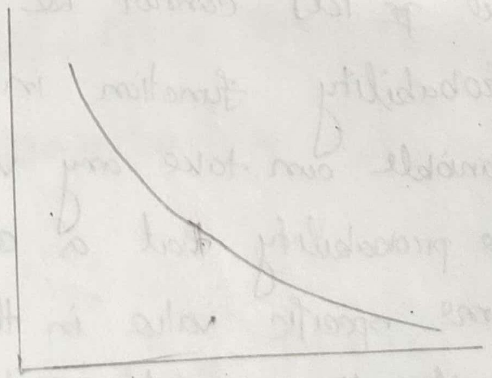
The probability density function ~~cannot be~~ $f(x)$ satisfies the following properties;

(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (iii) $P[a \leq X \leq b] = \int_a^b f(x)\,dx$.

The ~~pr~~ PdF cannot be represented in tabular form like probability function in discrete case, since a continuous random variable can take any values over range of the variable and the probability that a continuous random variable can take some specific value in the range is zero. For example, we consider the height of a adult people in inches. Suppose 50" and 70" are the minimum and maximum heights of these people. Then the height of a particular man can be any values within this range. The probability that the height of a particular man is to be 66" is zero. Because the total no. of sample points of the sample space is nondenumerable infinity. Therefore, in continuous case, the event will be some interval on the range of the variable X. Then probability of the event can be obtained by the method of integration. The density function $f(x)$ of a continuous random variable X can be ~~expressed~~ represented by a mathematical formula. Since X is defined on a continuous sample space, the graph of $f(x)$ will be continuous. That is, the graph of $f(x)$ will be a curve which is known as probability density curve.

The curves given below are the examples of probability density curves.





[5] The pressure measured in pounds per cm² of a certain value is a random variable X whose probability density function is

$$f(x) = \begin{cases} \frac{2}{9}(3x - x^2) & ; \text{ if } 0 < x < 3 \\ 0 & ; \text{ otherwise} \end{cases}$$

Find the probability that the pressure at this value is
(a) is not more than 2 pounds per cm².
(b) greater than 2 pounds per cm².
(c) between 1.5 and 2.5 pounds per cm².
(d) less than 1.5 pounds per cm².

Solⁿ (a) $P[X \le 2] = \frac{2}{9}\int_0^2 (3x - x^2)\, dx$

$$= \frac{2}{9}\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^2 = .7407$$

(b) $P[X > 2] = 1 - P[x \le 2] = 1 - 0.7407$

$$= .2593$$

(c) $P[1.5 \le X \le 2.5] = \int_{1.5}^{2.5} \frac{2}{9}(3x - x^2)\, dx = .4259$

(d) $P[x \le 1.5] = \frac{2}{9}\int_0^{1.5} (3x - x^2)\, dx = .5$

Ans.

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Q) A random variable X has the following functional form:

$$f(x) = kx, \quad 0<x<4$$
$$= 0, \quad \text{elsewhere}$$

(i) Determine k for which $f(x)$ is a density function.

(ii) Find $P(1<X<2)$ and $P(X>2)$

Soln (i) For X to be a density function we must have

$\int_{-\infty}^{\infty} f(x)\,dx = 1$ Thus

$$k\int_0^4 x\,dx = 1 \Rightarrow k\left[\frac{x^2}{2}\right]_0^4 = 1 \Rightarrow k = 1/8.$$

(ii) Again, $P(1<X<2) = \frac{1}{8}\int_1^2 x\,dx = \frac{x^2}{16}\Big|_1^2 = 3/16$,

and $P(X>2) = \frac{1}{8}\int_2^4 x\,dx = \frac{x^2}{16}\Big|_2^4 = 3/4$.

[1] A continuous random variable X has the following density function :

$$f(x) = \frac{2}{27}(1+x) \quad , \quad 2<x<5$$
$$= 0 \qquad \text{elsewhere}$$

(a) Verify that it satisfies the condition $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

(b) Find $P(X<4)$ / Ans: $16/27$

(c) Find $P(3<X<4)$ / Ans: $1/3$.

Joint Probability Distribution : (Discrete & cont.)

* Marginal distribution :

$$g(x) = \sum_y f(x,y)$$
$$h(y) = \sum_x f(x,y)$$
Discrete

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad \text{for } -\infty<x<\infty$$
$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \quad \text{for } -\infty<y<\infty$$
Continuous.

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Find the marginal densities of X and Y from the following joint density function and verify that marginal distributions are also probability distributions.

$$f(x,y) = \frac{1}{8}(6-x-y) \quad , \quad \text{for } 0<x<2,\ 2<y<4$$
$$= 0 \quad , \text{ otherwise.}$$

Also compute $P(X+Y<3)$ and $P(X<\frac{3}{2}, Y<\frac{5}{2})$

Soln By defn, the marginal density of X is,

$$g(x) = \frac{1}{8}\int_2^4 (6-x-y)\,dy = \frac{1}{4}(3-x), \quad 0<x<2.$$

and $h(y) = \frac{1}{8}\int_0^2 (6-x-y)\,dx = \frac{1}{4}(5-y), \quad 2<x<4$

Now, we verify that $g(x)$ and $h(y)$ are probability distributions.

It is clear that in the given range of the variables X and Y, $g(x) \geq 0$ and $h(y) \geq 0$. The other condition to be fulfilled is that

$$\int_{-\infty}^{\infty} g(x)\,dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} h(y)\,dy = 1.$$

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$\int_0^2 g(x)\,dx = \frac{1}{4}\int_0^2 (3-x)\,dx = \frac{1}{4}\left[3x - \frac{x^2}{2}\right]_0^2 = 1$

and $\int_2^4 h(y)\,dy = \int_2^4 (5-y)\,dy = 1$.

Hence the proof.

Now,

$P(X+Y<3) = \frac{1}{8}\int_0^2\int_2^{3-x}(6-x-y)\,dx\,dy$ [Since $x+y<3$]

$= \frac{1}{8}\int_0^2 \left(6y - xy - \frac{y^2}{2}\right)\Big|_2^{3-x} dx$.

$= \frac{1}{8}\int_0^2 \left(\frac{x^2}{2} - 4x + \frac{7}{2}\right)dx$

$= \frac{1}{8}\left[\frac{x^3}{6} - 2x^2 + \frac{7x}{2}\right]_0^2 = \frac{1}{24}$.

and

$P(x<\frac{3}{2}, y<\frac{5}{2}) = \frac{1}{8}\int_0^{3/2}\int_0^{5/2}(6-x-y)\,dy\,dx$

$= \frac{1}{8}\int_0^{3/2}\left[6y - xy - \frac{y^2}{2}\right]_2^{5/2} dx$.

$= \frac{1}{8}\int_0^{3/2}\left(\frac{15}{8} - \frac{x}{2}\right)dx = \frac{9}{32}$. Ans.

Now, $\int_0^2 g(x)\,dx = \frac{1}{4}\int_0^2 (3-x)\,dx = 1$ and

$$\int_2^4 h(y)\,dy = \int_2^4 (5-y)\,dy = \frac{1}{4}\left[5y - \frac{y^2}{2}\right]_2^4 = 1.$$

Hence the proof.

Now,

$$P(X+Y<3) = \frac{1}{8}\int_0^2\int_0^{3-x} (6-x-y)\,dx\,dy \quad [\text{Since } x+y<3]$$

$$= \frac{1}{8}\int_0^2 \left(6y - xy - \frac{y^2}{2}\right)\Big|_0^{3-x} dx$$

$$= \frac{1}{8}\left[\frac{x^3}{6} - 2x^2 + \frac{7x}{2}\right]_0^2 = \frac{1}{24}.$$

and

$$P(x<3/2,\ y<5/2) = \frac{1}{8}\int_0^{3/2}\int_0^{5/2} (6-x-y)\,dy\,dx$$

$$= \frac{1}{8}\int_0^{3/2}\left[6y - xy - \frac{y^2}{2}\right]_0^{5/2} dx.$$

$$= \frac{1}{8}\int_0^{3/2}\left(\frac{15}{8} - \frac{x}{2}\right)dx$$

$$= 9/32.$$ Ans

(2)

Conditional Distribution in Joint Probability Distribution:

$$f(x|y) = \frac{f(x,y)}{\sum_x f(x,y)} = \frac{f(x,y)}{h(y)}, \text{ for } h(y) > 0$$

[Discrete random variable]

And

$$f(y|x) = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dy} = \frac{f(x,y)}{g(x)}, \quad g(x) > 0$$

$$f(x|y) = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dx} = \frac{f(x,y)}{h(y)}, \quad h(y) > 0$$

continuous random variable.

For the following probability distribution show that X and Y are not independent.

$$f(x,y) = \frac{1}{8}(x+y), \quad 0<x<2,\ 0<y<2.$$

Sol^n $g(x) = \frac{1}{8}\int_0^2 (x+y)\,dy = \frac{1}{8}\left[xy + \frac{y^2}{2}\right]_0^2 = \frac{x+1}{4}.$

Similarly, $h(y) = \frac{1}{8}\int_0^2 (x+y)\,dx = \frac{1}{8}\left(xy + \frac{x^2}{2}\right)\Big|_0^2$

$= \frac{y+1}{4}.$

Thus, $g(x) \times h(y) = \frac{(x+1)(y+1)}{16} \neq f(x,y)$

Hence X and Y are not independent.

The non-independence of the random variables can also shown by computing the conditional distributions as below:

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{\frac{1}{8}(x+y)}{\frac{1}{4}(x+1)} = \frac{x+y}{2(x+1)} \neq h(y)$$

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{\frac{1}{8}(x+y)}{\frac{1}{4}(y+1)} = \frac{(x+y)}{2(y+1)} \neq g(x)$$

Since the conditional densities are not equal to the marginal densities, the variables are not independent.

4) The joint probability density of X and Y is given by,

$$f(x,y) = \frac{1}{8}(x+y), \quad \text{for } 0 \le x \le 2,\ 0 \le y \le 2$$
$$= 0 \quad , \text{elsewhere}$$

(a) Verify that it is a joint density function

(b) Find the marginal density function of X.

(c) Find $P(X > Y)$

(d) Find the conditional density function of Y.

Soln It is easy to verify that $f(x,y) \ge 0$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \frac{1}{8}\int_0^2\int_0^2 (x+y)\,dy\,dx = 1$$

Now. $\frac{1}{8}\int_0^2\int_0^2 (x+y)\,dx\,dy = \frac{1}{8}\int_0^2 \left(xy+\frac{y^2}{2}\right)\Big|_0^2 dx$

$$= \frac{1}{8}\int_0^2 (2x+2)\,dx$$

$$= \frac{1}{8}\left[x^2+2x\right]_0^2 = 1.$$

Hence, it is a joint pdf.

(b) The marginal density of X is,

$$g(x) = \int_0^2 \frac{1}{8}(x+y)\,dy = \frac{1}{8}\left[xy+\frac{y^2}{2}\right]_0^2 = \frac{1}{4}(x+1)$$

for $0 \le x \le 2$.

(c) Here $X > Y$ implies that $Y \le X$. Hence

$$P(X>Y) = \frac{1}{8}\int_0^2\int_0^x (x+y)\,dx\,dy$$

$$= \frac{1}{8}\int_0^2 \left(xy+\frac{y^2}{2}\right) dx$$

$$= \frac{1}{8}\int_0^2 \left(x^2+\frac{x^2}{2}\right) dx$$

$$= \frac{1}{2}.$$

(d) The conditional density of ~~X~~ Y is,

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{\frac{1}{8}(x+y)}{\frac{1}{4}(x+1)} = \frac{x+y}{2(x+1)} \text{ for } 0 \le x \le 2,$$

$0 \le y \le 2$.

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Q) Given the following Joint density function of X and

$$f(x,y) = \frac{6}{5}(x+y^2), \quad 0<x<1,\ 0<y<1$$

$$= 0 \qquad \text{elsewhere}$$

Find the following:

(i) $f(x|y)$ (ii) $f(x|0.5)$ and (iii) $P(X<0.5 \mid Y=0.5)$

Soln $f(x|y) = \dfrac{f(x,y)}{h(y)}$

Now, $h(y) = \frac{6}{5}\int_0^1 (x+y^2)\,dx = \frac{6}{5}\left[\frac{x^2}{2} + xy^2\right]_0^1$

$$= \frac{6}{5}\left(\frac{1}{2}+y^2\right), \quad 0<y<1$$

Hence, $f(x|y) = \dfrac{f(x,y)}{h(y)} = \dfrac{\frac{6}{5}(x+y^2)}{\frac{6}{5}(\frac{1}{2}+y^2)}$

$$= \frac{2(x+y^2)}{1+2y^2}, \quad 0<x<1,\ 0<y<1$$

(ii) $f(x|\tfrac{1}{2}) = \dfrac{2(x+\frac{1}{2^2})}{1+2\times\frac{1}{2^2}} = \frac{1}{3}(4x+1), \quad 0<x<1$ [From $f(x|y)$ above]

(iii) $P[X<\tfrac{1}{2} \mid Y=\tfrac{1}{2}] = \frac{1}{3}\int_0^{1/2} (4x+1)\,dx$

$$= \frac{1}{3}\left[2x^2+x\right]_0^{1/2} = \frac{1}{3}.$$

Ans.

For the joint density function

$$f(x,y) = \frac{x(1+3y^2)}{4}, \quad 0<x<2, \quad 0<y<1$$

$$= 0 \qquad \text{elsewhere}$$

Find $g(x)$, $h(y)$, $f(x|y)$ and $P[\frac{1}{4}<X<\frac{1}{2} \mid Y=\frac{1}{2}]$

Soln The marginal density of X is,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{x(1+3y^2)}{4}\,dy$$

$$= \left[\frac{xy}{4} + \frac{xy^3}{4}\right]_{y=0}^{y=1} = x/2.$$

The marginal density of Y is,

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \frac{x(1+3y^2)}{4}\,dx$$

$$= \left(x^2/8 + \frac{3x^2y^2}{8}\right)\Big|_0^2$$

$$= \frac{1+3y^2}{2}, \quad 0<y<1.$$

Again, $f(x|y) = \frac{f(x,y)}{h(y)} = x/2, \quad 0<x<1.$

Thus, $P[\frac{1}{4}<X<\frac{1}{2} \mid Y] = \int_{1/4}^{1/2} f(x|y)\,dx = 3/64.$

Consider the following probability distribution of X and Y.

$$f(x,y) = \frac{x+y}{21}, \quad x = 1,2,3 \text{ and } y = 1,2.$$

(i) Obtain the marginal and conditional distributions of X and Y.

(ii) Find $f(x|1)$ and (iii) $P[X=2 | Y=1]$.

Soln The marginal distributions are.

(i) $$g(x) = \sum_{y=1}^{2} \frac{x+y}{21} = \frac{x+1}{21} + \frac{x+2}{21} = \frac{2x+3}{21} \; ; \; x = 1,2,3$$

$$h(y) = \sum_{x=1}^{3} \frac{x+y}{21} = \frac{y+1}{21} + \frac{y+2}{21} + \frac{y+3}{21} = \frac{3y+6}{21}, \; y=1,2$$

Thus the conditional distributions are.

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{x+y}{3y+6} \text{ and}$$

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{x+y}{2x+3}$$

(ii) $$f(x|1) = \frac{x+1}{9}$$

(iii) $$P[X=2 | y=1] = \frac{2+1}{9} = 3/9 = 1/3$$

Ans.

Commulative distribution or distribution function of continuous random variable:

Like discrete random variable, we are interested in finding the probability of a continuous random variable which is less than or equal to a particular value of the random variable. This probability is called commulative distribution or distribution function of X.

If X is a continuous random variable with density function $f(x)$, then commulative distribution function of X, denoted by $F(X)$, is defined by

$$F(X) = P[X \le x] = \int_{-\infty}^{x} f(x)\, dx$$

The commulative distribution function $F(X)$ of a continuous random variable X is absolutely continuous. If the derivative of $F(x)$ exists, then

$$f(x) = \frac{d}{dx} F(x) = F'(x)$$

Thus, $f(x)\, dx = dF(x)$

The distribution function $F(x)$ possesses the following properties:

(i) $F'(x) = f(x) > 0$

(ii) $F(-\infty) = 0$

(iii) $F(\infty) = 1$

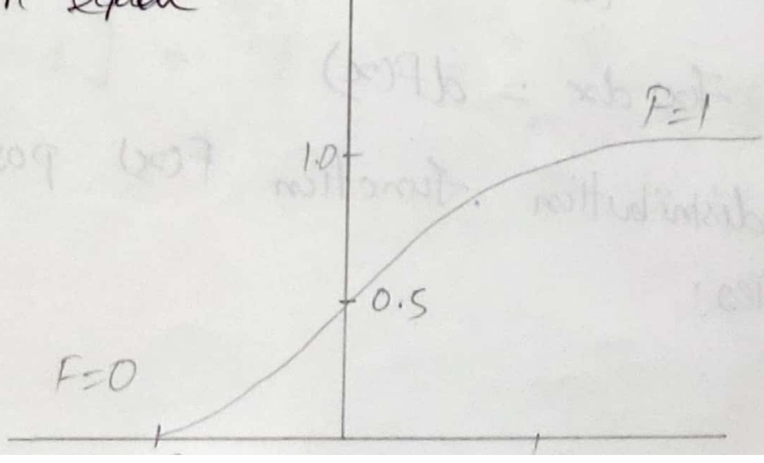
(iv) $f(x)$ is defined at every point in a continuous range and is continuous.

One can easily verify that the distribution function has the following interesting and useful result:

$$P(a < X < b) = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= F(b) - F(a).$$

Commulative distribution curves often have more or less than S shape pattern. A typical curve appears in figure. The horizontal axis shows possible values of the variable T. For any point on the axis t, the height of the curve F(t) is the probability that T is less than or equal to t.



Q) If X has the density function

$$f(x) = \frac{2}{27}(1+x) \quad , \quad 2<x<5$$
$$= 0 \quad , \text{ elsewhere}$$

Find $F(3)$ and $F(4)$ and hence verify that $P(3<X<4) = F(4) - F(3)$.

Soln

$$F(3) = P(X<3) = \frac{2}{27}\int_2^3 (1+x)\,dx = \frac{2}{27}\left[x+\frac{x^2}{2}\right]_2^3$$
$$= \frac{7}{27}.$$

$$F(4) = P(X<4) = \frac{2}{27}\int_2^4 (1+x)\,dx = \frac{2}{27}\left[x+\frac{x^2}{2}\right] = \frac{16}{27}.$$

Hence, $F(4) - F(3) = \frac{16}{27} - \frac{7}{27} = \frac{1}{3}.$

Now,

$$P(3<X<4) = \frac{2}{27}\int_3^4 (1+x)\,dx = \frac{1}{3}.$$

Hence, $P(3<X<4) = F(4) - F(3)$. D.

4) Joint Probability Function.

$$f(x,y) = \begin{cases} 10xy^2 & ; 0<x<y<1 \\ 0 & ; \text{Otherwise} \end{cases}$$

where X is the unit temp. change and Y is the proportion of spectrum sheet that a certain atomic particle produces. Find probability that the spectrum shifts more than half of the total observations, given that temp. is increased to 0.25 unit. Find marginal densities $g(x)$, $h(y)$ and conditional density $f(y|x)$.

Soln

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x}^{1} 10xy^2\,dy = \frac{10x}{3}(1-x^3); \quad 0<x<1.$$

$$h(y) = \int_{0}^{y} 10xy^2\,dx = 5y^4 \; ; \; 0<y<1$$

$$f(y|x) = \frac{10xy^2}{\frac{10x}{3}(1-x^3)} = \frac{3y^2}{1-x^3} \; ; \; 0<x<y<1.$$

$$P(Y > 1/2 \mid X = 0.25) = \int_{1/2}^{1} f(y \mid x = 0.25)\,dy$$

$$= \int_{1/2}^{1} \frac{3y^2}{1-(0.25)^3}\,dy = 8/9$$