Spacial Probability Distribution

There are different types of distributions

Discrete Prob. Distribution:

1. Binomial Distribution
2. Negative "
3. Multinomial "
4. Poisson "

Continuous Prob. Distribution:

5. Normal "
6. Exponential "
7. Gamma "

Both + 8. Uniform "

Binomial Distribution

If p and q are the probabilities that an event will happen and fail respectively in any single try, then the probability that the event will happen exactly x times in n trials is -

$$f(x) = P(X=x) = b(x; n, p) = \binom{n}{x} p^x q^{n-x};$$

$$p+q=1$$

$$x = 0, 1, \ldots, n$$

$$(q+p)^n = \binom{n}{0} q^n p^0 + \binom{n}{1} q^{n-1} p^1 + \cdots \cdots$$

$$\cdots \cdots + \binom{n}{n} q^{n-n} p^n$$

$$= b(0; n, p) + b(1; n, p) + b(2; n, p) +$$

$$\cdots + b(n; n, p)$$

$$1 = \sum_{x=0}^{n} b(x; n, p)$$

$$\Rightarrow 1 = \sum f(x)$$

Property:

① $E(X) = np$

② $Var(X) = npq$

* Note:

The special case of binomial distribution with $n=1$ is known as Bernoulli Distribution.

The probability that a patient recovers from a delicate heart operation is $1/10$. If 15 people are known as to have this operation,

Find probability that –

1. At least 3 survived
2. from 4 to 6 survived
3. Exactly 4 survived
4. At most 2 survived

Solⁿ: Let, X is a binomial random variable with probability function,

$$f(x) = P(X=x) = b(x; n, p) = \binom{n}{x} p^x q^{n-x}$$

given that, $n = 15$

$p = 1/10$

$$\therefore f(x) = P(X=x) = \binom{15}{x}\left(\frac{1}{10}\right)^x \left(\frac{9}{10}\right)^{15-x};$$

$$x = 0, 1, 2, \cdots, 15$$

point আকারে লেখা যাবে

① $P(X \geq 3) = 1 - P(X \leq 2) = 1 - 0.8159$

$= 0.1841$

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(2) $P(4 \leq X \leq 6) = P(X \leq 6) - P(X \leq 3)$

$= 0.9997 - 0.9449$

(3) $P(X = 4) = P(X \leq 4) - P(X \leq 3)$

$= 0.9873 - 0.9449$

(4) $P(X \leq 2) = 0.8159$

The screws produced by a certain machine were checked by examining no. of defectives in a sample of 12. The following table shows the distribution of 128 samples according to the no. of defective items they contained.

No. of def in a sample of 12	0	1	2	3	4	5	6	7
No. of samples:	7	6	19	35	30	23	7	1

(1) Fit a binomial distribution and find expected frequencies if the chance of machine being defective is $\frac{1}{2}$.

(2) Find mean & variance of the fitted distribution.

Sol^n: (1) Given that, $p = \frac{1}{2}$ and $q = \frac{1}{2}$

$N = 128$

$\therefore n =$ no. of total item $- 1$

$= 8 - 1 = 7$

$$(q+p)^n = \binom{n}{x} q^{n-x} p^{x} \; ; x = 0, 1, \ldots n$$

$$\Rightarrow \left(\frac{1}{2}+\frac{1}{2}\right)^7 = \binom{7}{0}\left(\frac{1}{2}\right)^7 + \binom{7}{1}\left(\frac{1}{2}\right)^7 + \binom{7}{2}\left(\frac{1}{2}\right)^7 + \cdots + \binom{7}{7}\left(\frac{1}{2}\right)^7$$

multiplying total no. of frequency,

$$(128)\left(\frac{1}{2}+\frac{1}{2}\right)^7 = \left(\frac{1}{2}\right)^7 \cdot 128\,[1+7+21+35+35+21+7+1]$$

$$\Rightarrow \quad = 1+7+21+35+35+21+7+1$$

So, the expected frequencies in binomial distribution is

No. of def. in a sample of 12 :	0	1	2	3	4	5	6	7
No. of samples :	1	7	21	35	35	21	7	1

(2) $E(X) = np = 7 \cdot \frac{1}{2} = \frac{7}{2}$

$Var(X) = npq =$

The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that

(1) Exactly 4 will be defective

(2) None " " "

(3) At most 2 " " "

Solⁿ:

$f(x) = P(X=x) = \binom{12}{x} p^x q^{n-x}$

$n = 12$

$p = \frac{1}{10}$

$$P(X=x) = \binom{12}{x}\left(\frac{1}{10}\right)^{x}\left(\frac{9}{10}\right)^{12-x} \; ; \quad x=0,1,\ldots,12$$

① $P(X=4) = P(X \leq 4) - P(X \leq 3) =$

② $P(X=0) = \binom{12}{0}\left(\frac{9}{10}\right)^{12} =$

③ $P(X \leq 2) =$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$r=3$

$c=4$

Multinomial Distribution

Suppose that, events $A_1, A_2, \ldots, A_K$ are mutually exclusive that can occur with respective probabilities $P_1, P_2, \ldots, P_K$. If $X_1, X_2, \ldots, X_K$ are the random variables respectively giving the no. of times that $A_1, A_2, \ldots, A_K$ occur in a total of n trials. Then,

$$P(X_1 = n_1, X_2 = n_2, \ldots, X_K = n_K) = \cancel{\frac{n!}{n_1!\, n_2! \ldots n_K!}}$$

$$= \frac{n!}{n_1!\, n_2! \ldots n_K!} \, P_1^{n_1} P_2^{n_2} \ldots P_K^{n_K}$$

where, $n_1 + n_2 + \ldots + n_K = n$

$$P_1 + P_2 + \ldots + P_K = 1$$

* No chart

A pair of dice is tossed six times. What is the probability of obtaining a total of 7 or 11 twice?, a ~~max~~ matching pair once and any other combination 3 times?

Soln: Sample space, $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$

$\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots$

$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

Let events, $A_1 = \{\text{total of } 7 \text{ or } 11\}$

$A_2 = \{\text{matching pair}\}$

$A_3 = \{\text{other combinations}\}$

$n = 6$

$P_1 = p(A_1) = \frac{8}{36}$ $\{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3), (5,6), (6,5)\}$

$= \frac{2}{9}$

$P_2 = p(A_2) = \frac{6}{36} = \frac{1}{6}$

$P_3 = p(A_3) = \frac{22}{36} = \frac{11}{18}$

Then, using multinomial distribution,

$$P(X_1 = n_1, X_2 = n_2, \ldots, X_k = n_k) = \frac{n!}{n_1! n_2! \ldots n_k!} P_1^{n_1} P_2^{n_2} \ldots P_k^{n_k}$$

$$\therefore P(X_1 = 2, X_2 = 1, X_3 = 3) = \frac{6!}{2!\,1!\,3!} \left(\frac{2}{9}\right)^2 \left(\frac{1}{6}\right)^1 \left(\frac{11}{18}\right)^3$$

$$= 0.1127$$

Let, 23% of people attending a football match between AKC and MSC live within 5 miles of the stadium, 59% live between 5 and 10 miles from the stadium and 18% live more than 10 miles from the stadium. 20 people are selected randomly from the crowd

attending the match. What is the probability that 7 of them live within 5 miles, 8 live between 5 and 10 miles and 5 live more than 10 miles from stadium.

Sol^n: Let, events $A_1 = \{$ live within 5 miles $\}$

$A_2 = \{$ " " 5–10 " $\}$

$A_3 = \{$ " more than 10 " $\}$

$P_1 = p(A_1) = 0.23$

$P_2 = p(A_2) = 0.59$

$P_3 = p(A_3) = 0.18$

According to multinomial distribution,

$$P(X_1 = 7, X_2 = 8, X_3 = 5) = \frac{20!}{7!\,8!\,5!}(0.23)^7 (0.59)^7 (0.18)^8$$

$$=$$

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Lect. 16

14.05.16

Poisson's Distribution

Let, λ be the mean of successes of a random variable X denotes no. of successes in a given time or region, then the probability distribution function is –

$$f(x;\lambda) = P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \; ; \quad x = 0, 1, 2, \dots \infty$$

$$\lambda > 0$$

$\Downarrow$

$P(x;\lambda)$

① Mean $= E(X) = \lambda$

② $Var(X) = \lambda$

Proof আসবে (term L1)

Theorem

Let, X be a binomial random variable with prob. distribution $b(x;n,p)$. When $n \to \infty$, $p \to 0$ and $\lambda = np$ remains constant, then

$$\lim_{n\to\infty} b(x;n,p) \longrightarrow P(x;\lambda)$$

Proof লাগবে (term 4)

* যদি n বাড়তে থাকে & p কমতে থাকে, তখন binomial থেকে poisson এ convert করব।

In a factory producing cycle tires, one is defective in 500 tires. The tires are supplied in lots of 10. Calculate probability of –

① Non-defective

② 2 defective

in a consignment of 10000 lots.

<u>Solⁿ</u>: Let, X be a poisson random variable.

Then the probability distribution,

$$f(x,\mu) = P(X=x) = \frac{e^{-\mu}\mu^{x}}{x!}\,; \quad x = 0, 1, 2, \ldots \infty, \quad \mu > 0$$

$p = \frac{1}{500} = 0.002$

$n = 10 \times 10,000 = 1,00,000$ tyres

$\lambda = np = 10 \times 0.002 = 0.02$ (for a lot)

(1) Prob of no defective $= \frac{e^{-0.02}(0.02)^0}{0!}$
$P(X=0)$

$= 0.9802$

(2) $P(X=2) = \frac{e^{-0.02}(0.02)^2}{2!} = 0.196 \times 10^{-3}$

In a consignment 10,000 lots,

$P(X=0) = 0.9802 \times 10,000$
$= 9802$ lots

$P(X=2) = 0.196 \times 10^{-3} \times 10,000$
$= 1.96$ lots ≈ 2 lots

The following table gives the no. of days in a 50 day period during which automobile accidents occured in a city. Fit poisson distribution to the data :

let x No. of accidents	0	1	2	3	4
let f No. of days	21	18	7	3	1

Soln : Prob. function of Poisson Distribution

$$f(x, \lambda) = \frac{e^{-\lambda} \lambda^x}{x!} = P(X = x)$$

$$\lambda = \frac{\Sigma fx}{\Sigma f} = 0.9$$

$$N f(x) = \frac{e^{-0.9} (0.9)^x}{x!} \times N$$

$$\therefore \; Nf(0) = \frac{e^{-0.9} (0.9)^0}{0!} \times 50 = 20.33$$

$\therefore N f(1) = \dfrac{e^{-0.9} (0.9)^1}{1!} = 18.3$

$N f(2) = 8.23$

$N f(3) = 2.47$

$N f(4) = 0.56$

So, the data fitting according to Poisson distribution is -

No. of accidents	0	1	2	3	4
No. of days	20.33	18.3	8.23	2.47	0.56

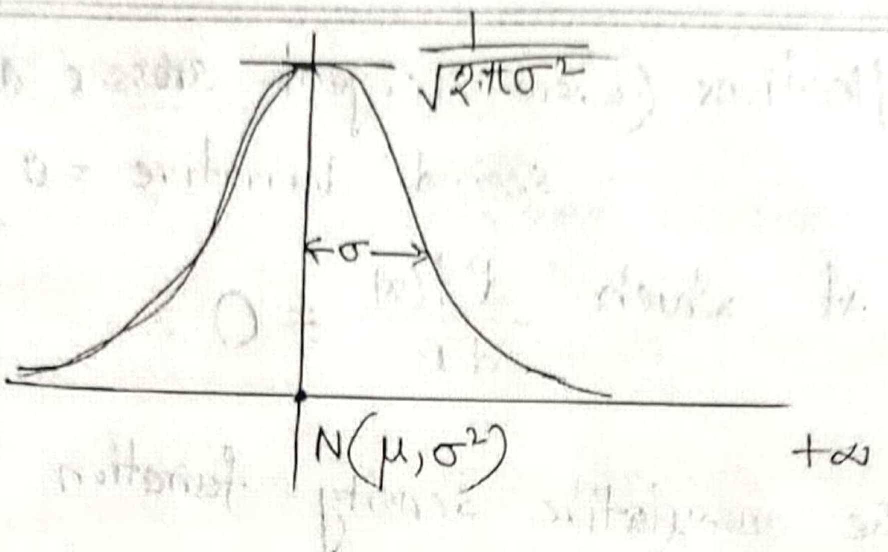
Normal Distribution

This distribution is often referred as Gaussian Distribution

Let, X be the continuous random variable with PDF. $f(x) = N(\mu, \sigma^2) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\left(\frac{x-\mu}{\sigma}\right)^2 \cdot \frac{1}{2}}$

; $-\infty < \mu < \infty$

$0 < \sigma < \infty$



Properties :

① The curve $y = f(x)$ is symmetric about the line $x = \mu$.

② The area under the curve,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

③ Here, $\beta_1 = 0$, $\beta_2 = 3$

④ All odd central moments about mean vanish

⑤ $f(x)$ has max^m value at $x = \mu$ & the value is $\frac{1}{\sqrt{2\pi\sigma^2}}$

⑥ $f(x)$ has point of ~~interaction~~

inflections (where tangent crosses curve, second derivative $= 0$)

at which $\frac{d^2 f(x)}{dx^2} = 0$

The cumulative density function of Z is -

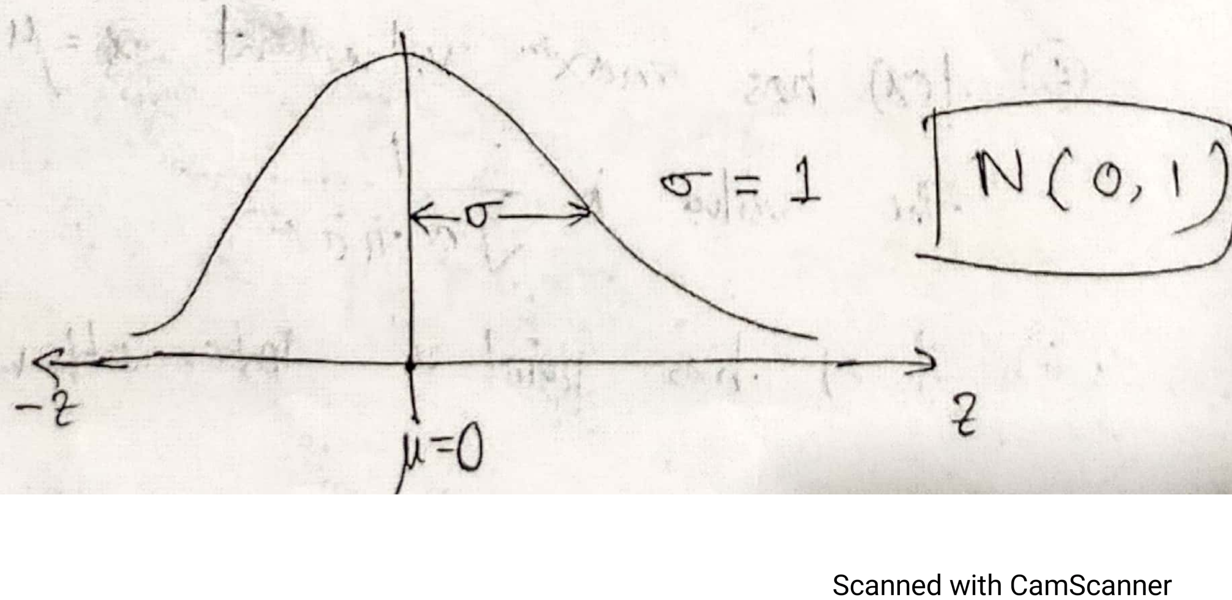
$$\Phi(z) = P(-\infty < Z \leq z)$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

* Note:

① $\Phi(-z) = \Phi(z)$

② $\Phi(z) = 1 - \Phi(-z)$



Standard Normal Distribution

If X is a ~~binomial~~ normal random variable with mean μ and variance σ^2, then $Z = \frac{X-\mu}{\sigma}$ will be standard normal random variable with mean $\mu = 0$ and $\sigma^2 = 1$

Thus, if X is random variable of $N(\mu, \sigma^2)$ then Z is random variable of $N(0, 1)$

The PDF of Z is,

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad ; \quad -\infty < z < \infty$$

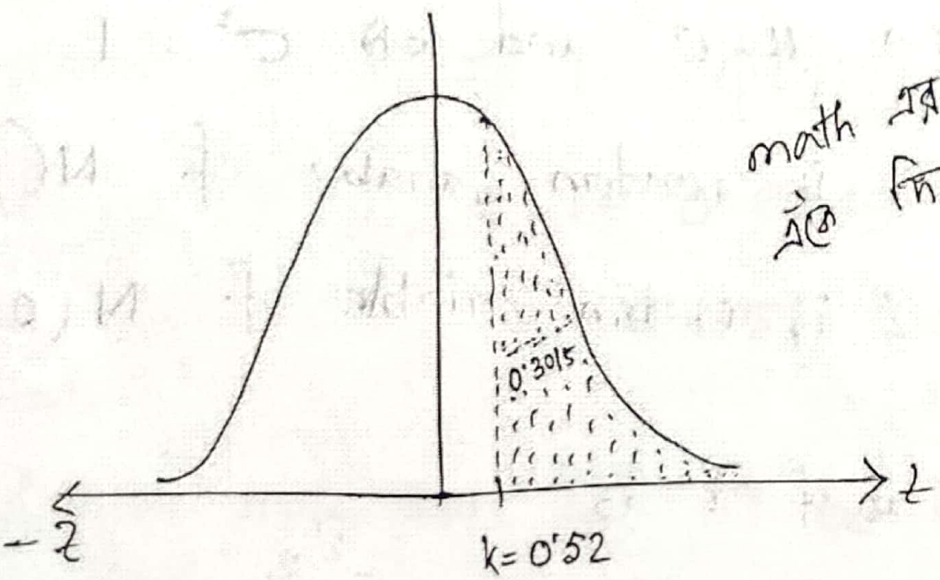
Given the standard normal distribution, find the value of k such that, $P(Z > k) = 0.3015$

Sol^n^:
$$P(Z > k) = 0.3015$$
$$\Rightarrow 1 - P(Z \leq k) = 0.3015$$

$$\Rightarrow P(Z \le k) = 0.6985$$

$$\Rightarrow \Phi(k) = 0.6985$$

$\therefore k = 0.52$ (from chart)



math এর সাথে graph এঁকে দিতে হবে

In a male population of 1000, the mean height is 68.16" (inches) and SD is 3.2". How many male maybe more than 6' ?

<u>Sol^n:</u> Given that, $\mu = 68.16$

$\sigma = 3.2$

and X is a normal random variable

$$P(X > 72'') = 1 - P(X \le 72)$$

$$= 1 - P\left(\frac{X-\mu}{\sigma} \le \frac{72 - 68.16}{3.2}\right)$$

$$= 1 - P(Z \le 1.2)$$

$$= 1 - \Phi(1.2)$$

$$= 1 - 0.8849$$

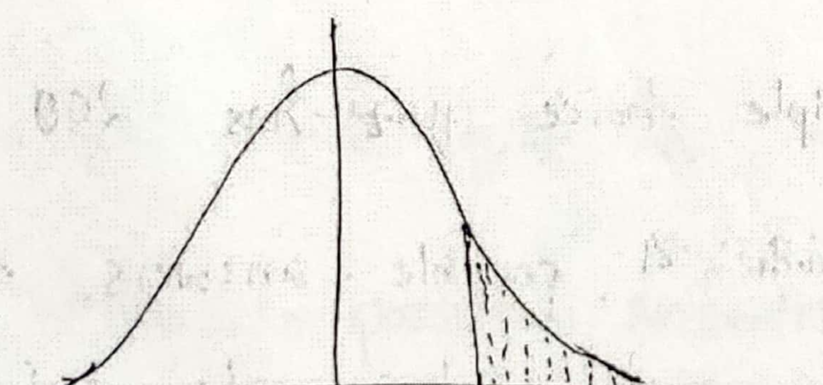
$$= 0.1151$$

Now,

no. of male more than 72" will be (0.1151×1000)

$$= 115.1$$

$$\cong 115 \text{ persons}$$



Normal Approximation to Binomial

Theorem:

If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of

$$z = \frac{X - np}{\sqrt{npq}} \quad \text{as } n \to \infty$$

is the standard Normal Distribution $N(0,1)$

binomial → discrete

normal → continuous

SND → $\text{"}(n \to \infty)$

A multiple choice quiz has 200 questions each with 4 possible answers of which 1 is correct. What is the probability that sheer guesswork yields from 25 to 30 correct answers for 80 of

the 200 problems about which the students have no knowledge?

Sol^n: Given that, $p = \frac{1}{4}$

$$q = \frac{3}{4}$$

$$n = 80.$$

$$\therefore np = 80 \times \frac{1}{4} = 20$$

$$npq = 20 \times \frac{3}{4} = 15$$

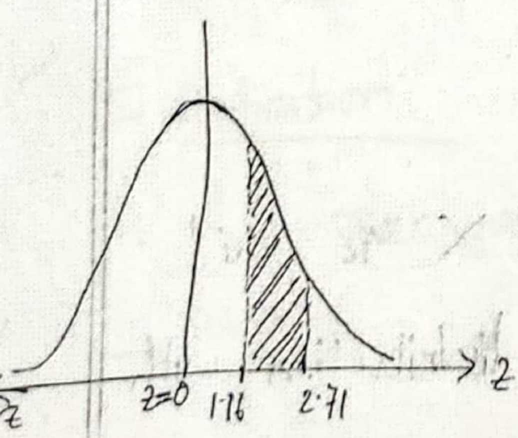
We have to calculate $P(25 \leq X \leq 30)$ where X is a binomial random variable.

$$P(25 \leq X \leq 30) = P\left(\frac{24.5-20}{\sqrt{15}} \leq \frac{X-np}{\sqrt{npq}} \leq \frac{30.5-20}{\sqrt{15}}\right)$$

$$= P(1.16 \leq z \leq 2.71)$$

$$= P(z \leq 2.71) - P(z \leq 1.16)$$

$$= \Phi(2.71) - \Phi(1.16)$$



~~= 0.9986 − 0.877~~

$$= 0.9966 - 0.877 = 0.1196$$

CT : Last Week saturday : 10:00 am

Syllabus : Special Prob. Distribution (all 8)

Exponential Distribution

Application

This distribution has many applications in the field of statistics, particularly in the areas of reliability theory and waiting time or ~~quel~~ queing problems. It can be used as a model for lifetimes of various objects.

Definition

The random variable X is said to have an exponential distribution with parameter λ and the PDF is,