Probability Distribution

The Binomial Distribution :-

* Definition : When an experiment has two possible outcomes, success and failure and the experiment is repeated n times independently, and the probability p of success of any given trial remains constant from trial to trial, the experiment is known as binomial experiment.

Let p be the probability that an event will happen in any single Bernoulli trial (called the probability of success). Then $q = 1-p$ is the probability that the event will fail to happen in any single trial (called the probability of failure).

The probability that the event will happen exactly x times in n trials (i.e. successes and $n-x$ failures will occur) is given by the probability function

$$f(x) = P(X=x) = \binom{n}{x} p^x q^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

where the random variable X denotes the number of successes in n trials and $x = 0, 1, \ldots n$.

Problem 01 :- The probability of getting exactly 2 heads in 6 tosses of a fair coin is

$$P(X=2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2}$$

$$= \frac{6!}{2!\,4!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2}$$

$$= \frac{15}{64}$$

$\frac{n!}{x!(n-x)!} p^x q^{n-x}$

The discrete probability function. $f(x) = P(X=x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$ is often called the binomial distribution since for $x = 0, 1, 2, \ldots n$, it corresponds to successive terms in the binomial expansion

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \cdots +$$

$$p^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

The special case of a binomial distribution with $n=1$ is also called the Bernoulli distribution.

Some properties of the Binomial Distribution :-

Some of the important properties of the binomial distribution are listed in Table below.

Mean	$\mu = np$
Variance	$\sigma^2 = npq$
Standard deviation	$\sigma = \sqrt{npq}$
Co-efficient of skewness	$\alpha_3 = \dfrac{q-p}{\sqrt{npq}}$
Co-efficient of kurtosis	$\alpha_4 = 3 + \dfrac{1-6pq}{npq}$
Moment generating function	$M(t) = (q+pe^t)^n$
Characteristic function	$\phi(\omega) = (q+pe^{j\omega})^n$

Problem 02 :- Suppose a fair coin is tossed 100 times, then find the mean number of heads and the standard deviation.

soln:- The expected or mean number of heads is $\mu = (100)\left(\frac{1}{2}\right) = 50$

While the standard deviation is

$$\sigma = \sqrt{(100)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$

$$= 5.$$

Ans.

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Problem 03 :- Find the probability that in tossing a fair coin three times, there will appear

1) 3 heads
2) 2 tails and 1 head
3) At least 1 head
4) not more than 1 tail

Method 1 :

let H denote head and T denote tail,
and suppose that we designate HTH, for example,
to mean head on first toss, tail on second toss,
and then head on third toss.

Since 2 possibilities (head or tail) can occur
on each toss, there are a total of $(2)(2)(2)=8$
possible outcomes, i.e, sample points, in the sample
space. These are

HHH, HHT, HTH, HTT, TTH, THH, THT,
TTT

For a fair coin these are assigned equal probabilities
of $\frac{1}{8}$ each. Therefore,

(a) $P(3 \text{ heads}) = P(HHH) = \frac{1}{8}$

(b) $P(2 \text{ tails and } 1 \text{ head}) = P(HTT \cup TTH \cup THT)$
$= P(HTT) + P(TTH) + P(THT)$
$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

(c) $P(\text{at least 1 head})$

$= P(1, 2, \text{ or } 3 \text{ heads})$

$= P(1 \text{ head}) + P(2 \text{ heads}) + P(3 \text{ heads})$

$= P(HTT \cup THT \cup TTH) + P(HHT \cup HTH \cup THH) + P(HHH)$

$= P(HTT) + P(THT) + P(TTH) + P(HHT) + P(HTH) + P(THH) + P(HHH)$

$= \frac{7}{8}$

Alternatively,

$P(\text{at least 1 head}) = 1 - P(\text{no head})$

$= 1 - P(TTT)$

$= 1 - \frac{1}{8}$

$= \frac{7}{8}$

(d) $P(\text{not more than 1 tail}) = P(\text{0 tails or 1 tail})$

$= P(\text{0 tails}) + P(\text{1 tail})$

$= P(HHH) + P(HHT \cup HTH \cup THH)$

$= P(HHH) + P(HHT) + P(HTH) + P(THH)$

$= \frac{4}{8}$

$= \frac{1}{2}$

~~Method~~ 2 (using formula)

$f(x) = \binom{n}{x} p^x q^{n-x}$

$= \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$

(a) $P(\text{3 heads}) = \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \frac{1}{8}$

(b) $P(\text{2 tails and 1 head}) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1$

$= \frac{3}{8}$

(c) $P(\text{at least 1 head}) = P(\text{1, 2, or 3 heads})$

$= P(\text{1 head}) + P(\text{2 heads}) + P(\text{3 heads})$

$= \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 + \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1$

$+ \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0$

$= \frac{7}{8}$

Alternatively,

$$P(\text{at least 1 head}) = 1 - P(\text{no head})$$

$$= 1 - \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3$$

$$= \frac{7}{8}$$

(d) $P(\text{not more than 1 tail}) = P(\text{0 tails or 1 tail})$

$$= P(\text{0 tails}) + P(\text{1 tail})$$

$$= \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 + \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2}$$

We should mention that if X be the random variable denoting the number of heads in 3 tosses, (c) can be written

$$P(\text{at least 1 head}) = P(X \geq 1)$$

$$= P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{7}{8}$$

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Problem – 04 :- 4.2 Find the probability that in five tosses of a fair die, a 3 will appear

1) twice

2) at most once

3) at least two times.

Soln :- Let the random variable X be the number of times a 3 appears in five tosses of a fair die.

We have

Probability of 3 in a single toss $= p = \frac{1}{6}$

Probability of no 3 in a single toss $= q = 1 - p = \frac{5}{6}$

(a) $P(3 \text{ occurs twice}) = P(X=2) = \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$

$$= \frac{625}{3888}$$

(b) $P(3 \text{ occurs at most once}) = P(X \leq 1) = P(X=0) + P(X=1)$

$$= \binom{5}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 + \binom{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4$$

$$= \frac{3125}{7776} + \frac{3125}{7776} = \frac{3125}{3888}$$

ⓒ P(3 occurs at least 2 times)

$$= P(X \geq 2)$$

$$= P(X=2) + P(X=3) + P(X=4) + P(X=5)$$

$$= \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 + \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 + \binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^1 + \binom{5}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^0$$

$$= \frac{625}{3888} + \frac{125}{3888} + \frac{25}{7776} + \frac{1}{7776}$$

$$= \frac{763}{3888} \quad \text{Ans}$$

4.15

Problem-05 :- Find the probability that in a family of 4 children there will be ⓐ at least 1 boy. ⓑ at least 1 boy and at least 1 girl. Assume that the probability of a male birth is 1/2.

4.3

Soln :-

(a) $p(1 \text{ boy}) = \binom{4}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 = \frac{1}{4}$,

$p(2 \text{ boys}) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{3}{8}$

$p(3 \text{ boys}) = \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 = \frac{1}{4}$.

$p(4 \text{ boys}) = \binom{4}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0 = \frac{1}{16}$

Then $p(\text{at least } 1 \text{ boy}) = p(1 \text{ boy}) + p(2 \text{ boys}) + p(3 \text{ boys}) + p(4 \text{ boys})$

$= \frac{1}{4} + \frac{3}{8} + \frac{1}{4} + \frac{1}{16}$

$= \frac{15}{16}$

Another method:

$p(\text{at least } 1 \text{ boy}) = 1 - p(\text{no boy})$

$= 1 - \left(\frac{1}{2}\right)^4$

$= 1 - \frac{1}{16}$

$= \frac{15}{16}$

(b) P(at least 1 boy and at least 1 girl) =

$1 - P(\text{no boy}) - P(\text{no girl})$

$= 1 - \frac{1}{16} - \frac{1}{16}$

$= \frac{7}{8}$

Let X be a random variable denoting the number of boys in families with 4 children.

$\therefore$ (a) becomes

$P(X \geq 1) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= \frac{15}{16}$.

C.T.

Problem-06 :- Out of 2000 families with 4 children each, how many ~~who~~ would you expect to have

4.4.

(a) at least 1 boy, (b) 2 boys, (c) 1 or 2 girls, (d) no girls ? — H.W.

Page :- 4.16 → Murray R Spiegel.

Problem-07 :- If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen at random.

(a) 1, (b) 0, (c) less than 2, bolts will be defective.

Soln :-

The probability of a defective bolt is $P=0.2$, of a non-defective bolt is $q=1-p=0.8$. Let the random variable X be the number of defective bolts. Then

(a) $P(X=1) = \binom{4}{1}(0.2)^1(0.8)^3 = 0.4096$

(b) $P(X=0) = \binom{4}{0}(0.2)^0(0.8)^4 = 0.4096$

(c) $P(X<2) = P(X=0) = P(X=1)$

$= 0.4096 + 0.4096$

$= 0.8192.$

Problem - 08 :-

4.6 Find the probability of getting a total of 7 at least once in ~~three~~ tosses of a pair of fair dice. H.W.

Soln :- Book - 4.16 (Spiegel)

The Normal Distribution :-

One of the most important examples of a continuous probability distribution is the normal distribution.

Sometimes called the Gaussian distribution.

The density function for this distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty.$$

Where μ and σ are the mean and standard deviation, respectively. The corresponding distribution function is given by

$$F(x) = P(X \leq x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(v-\mu)^2/2\sigma^2}\, dv.$$

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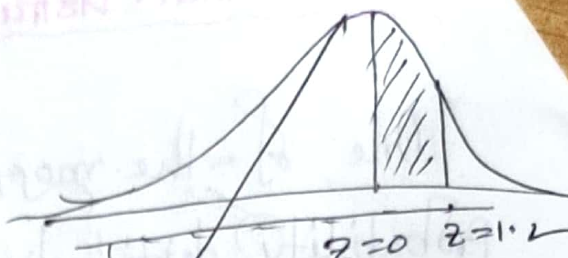
4.12

* Problem :- 01 :- Find the area under the standard normal curve shown in Figure 1 (a) between $z=0$ and $z=1.2$, (b) between $z=-0.68$ and $z=0$, (c) between $z=-0.46$ and $z=2.21$, (d) between $z=0.81$ and $z=1.94$, (e) to the right of $z=-1.28$

Soln:-



Problem:-01: The mean weight of 500 male students at a certain college is 151 lb and the standard deviation is 15 lb.

Assuming that the weights are normally distributed. Find how many students weight

(a) between 120 and 155 lb.

(b) more than 185 lb.

(a) Weights recorded as being between 120 and 155 lb can actually have any value from 119.5 to 155.5 lb. assuming they are recorded to the nearest ~~poo~~ pound (Fig. 1)

119.5 lb in standard units $= (119.5 - 151)/15$

$= -2.10$

155.5 lb in standard units $= (155.5 - 151)/15$

$= 0.30$

Required proportion of students = (area between $z = -2.10$ and $z = 0.30$)

= (area between $z = -2.10$ and $z = 0$)

+ (area between $z = 0$ and $z = 0.30$)

$= 0.4821 + 0.1179$

$= 0.6000$

Then the number of students weighing between 120 and 155 lb is $500(0.6000)$

$= 300$

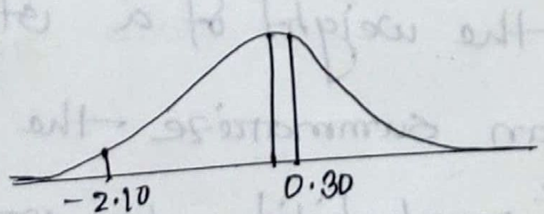


Fig: 1

(b) Students weighing more than 185 lb must weigh at least 185.5 lb (Fig-2)

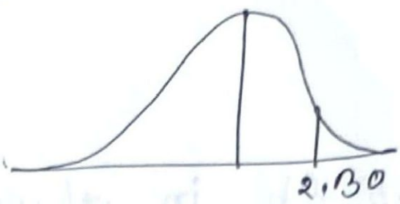


Fig-2

185.5 lb in standard units =

$(185.5 - 151)/15 = 2.30$

Required proportion of students

= (area to right of $z = 2.30$)

= (area to right of $z = 0$)

− (area between $z = 0$ and $z = 2.30$)

$= 0.5 - 0.4893$

$= 0.0107$

Then the number of students weighing more than 185 lb is $500(0.0107) = 5$.

If W denotes the weight of a student chosen at random. We can summarize the above results in terms of probability by writing

$$P(119.5 \le W \le 155.5) = 0.6000$$

$$P(W \geq 185.5) = 0.0107$$

H.W./spot test/Assignment

Find the probability that in successive tosses of a fair die, a 3 will come up for the first time on the fifth toss.

Find the probability that in tossing a fair coin 6 times, there will appear

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4, (f) 5

(g) 6 heads.

Find the probability of getting a total of 11 (a) once (b) twice, in two tosses of a pair of fair dice.

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