Conditional Probability

Conditional Probability: Let A and B be two events such that $P(A) > 0$. The probability of an event B when even A is occurred is known as conditional probability. It is denoted by $P(B\backslash A)$. Since A is known to have occurred, it becomes the new sample space replacing the original S.

From this we are led to the definition

$$P(B\backslash A) = \frac{P(A\cap B)}{P(A)} \qquad (1)$$

$$\text{or,} P(A \cap B) = P(B\backslash A)P(A) \qquad (2)$$

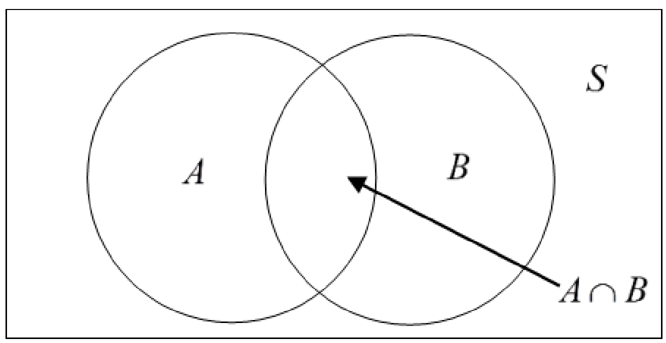

Figure 1 Venn diagram.

Problems Related to Conditional probability:

Problem 1: Compute the probability that a single toss of a die will result in a number less than 4 if (a) no other information is given and (b) it is given that the toss resulted in an odd number.

Solution:

(a) Let B denote the event {less than 4}. Since B is the union of the events 1, 2, or 3 turning up,

$$P(B) = P(1) + P(2) + P(3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

assuming equal probabilities for the sample points.

(b) Letting A be the event {odd number}, we see that $P(A) = \frac{3}{6} = \frac{1}{2}$. Also $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$. Then

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$$

Hence, the added knowledge that the toss results in an odd number raises the probability from 1/2 to 2/3.

Problem 2: Two cards are drawn from a well-shuffled ordinary deck of 52 cards. Find the probability that they are both aces if the first card is (a) replaced, (b) not replaced.

Let A_1 = event "ace on first draw" and A_2 = event "ace on second draw." Then we are looking for $P(A_1 \cap A_2) = P(A_1)\,P(A_2 \mid A_1)$.

(a) Since for the first drawing there are 4 aces in 52 cards, $P(A_1) = 4/52$. Also, if the card is replaced for the second drawing, then $P(A_2 \mid A_1) = 4/52$, since there are also 4 aces out of 52 cards for the second drawing. Then

$$P(A_1 \cap A_2) = P(A_1)P(A_2 \mid A_1) = \left(\frac{4}{52}\right)\left(\frac{4}{52}\right) = \frac{1}{169}$$

(b) As in part (a), $P(A_1) = 4/52$. However, if an ace occurs on the first drawing, there will be only 3 aces left in the remaining 51 cards, so that $P(A_2 \mid A_1) = 3/51$. Then

$$P(A_1 \cap A_2) = P(A_1)P(A_2 \mid A_1) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right) = \frac{1}{221}$$

Problem 3:

One bag contains 4 white balls and 2 black balls; another contains 3 white balls and 5 black balls. If one ball is drawn from each bag, find the probability that (a) both are white, (b) both are black, (c) one is white and one is black.

Solution:

Let W_1 = event "white ball from first bag," W_2 = event "white ball from second bag."

(a) $P(W_1 \cap W_2) = P(W_1)P(W_2 \mid W_1) = P(W_1)P(W_2) = \left(\frac{4}{4+2}\right)\left(\frac{3}{3+5}\right) = \frac{1}{4}$

(b) $P(W_1' \cap W_2') = P(W_1')P(W_2' \mid W_1') = P(W_1')P(W_2') = \left(\frac{2}{4+2}\right)\left(\frac{5}{3+5}\right) = \frac{5}{24}$

(c) The required probability is

$$1 - P(W_1 \cap W_2) - P(W_1' \cap W_2') = 1 - \frac{1}{4} - \frac{5}{24} = \frac{13}{24}$$

Problem 4: Compute the probability that a single toss of a die will result in a number less than 3 if (a) no other information is given and (b) it is given that the toss resulted in an even number. **(HW)**