Joint Distribution

Marginal probability function

Let X and Y be two discrete random variables with joint probability function $f(x,y)$, then the marginal probability function of X, denoted by $g(x)$ is defined by

$$g(x) = P[X=x] = \sum_{j=1}^{n} f(x, y_j), \text{ for all } x.$$

The marginal probability function of Y, denoted by $h(y)$, is similarly defined.

That is,

$$h(y) = P[Y=y] = \sum_{i=1}^{m} f(x_i, y), \text{ for all } y.$$

Marginal probability functions of X and Y are identical to the probability function of X and Y.

Independence of two discrete random variables

Two random variables are independent if the values of either random variable do not affect in any way the values of the other random variable.

Let X and Y be two discrete random variables with marginal probability functions $g(x)$ and $h(y)$, respectively.

Let $f(x,y)$ be the joint probability function of X and Y, then X and Y are said to be independent if and only if

$$f(x,y) = g(x) \cdot h(y)$$

for all values of X and Y.

[Note: This means that the joint probability function of X and Y is equal to the product of their marginal probability functions.

— These types of random variables are known as independent random variables.]

Sometimes independence of events are defined in terms of distribution functions.

Ex: Let X and Y be two discrete random variables with the following joint probability function:

Values of $Y : y$ / Values of $X : x$	2	3	4	$g(x)$
1	.12	.16	.12	.40
2	.18	.24	.18	.60
$h(y)$	.30	.40	.30	1

Are X and Y are independent?

Soln: Here $g(x)$ and $h(y)$ are the marginal probability functions of X and Y, respectively.

So, we need to verify $f(x,y) = g(x)\,h(y)$.

where $f(x,y)$ is the joint probability function of X and Y.

For example:

$f(X=2, Y=4) = .18$

$g(X=2) = .6$ and $h(Y=4) = .3$.

3)

There-fore,

$$f(X=2, Y=4) = g(X=2)\, h(Y=4) = .6 \times .3 = .18.$$

Hence $f(X=2, Y=4) = g(X=2) \cdot h(Y=4)$.

which shows that X and Y are independent random variables.

Ex: Consider the following joint distribution of X and Y

Values of $Y : y$ / Values of $X : x$	-4	2	7	$g(x)$
1	$1/8$	$1/4$	$1/8$	$1/2$
5	$1/4$	$1/8$	$1/8$	$1/2$
$h(y)$	$3/8$	$3/8$	$1/4$	1

Are X and Y independent?

Soln: The marginal probability function of X is

$$g(x) = \sum_{y} f(x,y) \;;\; \text{for } x = 1, 5$$

When $X = 1$, $g(X=1) = \sum_{y=-4}^{7} f(1, y) = \cancel{f(1,-4)}$

$= f(1,-4) + f(1,2) + f(1,7)$

$= 1/8 + 1/4 + 1/8$

$= 1/2$

When $X = 5$, $g(X=5) = \sum_{y=-4}^{7} f(5,y)$

$= f(5,-4) + f(5,2) + f(5,7)$

$= 1/4 + 1/8 + 1/8$

$= 1/2$

Similarly, the marginal probability function of Y is

$h(y) = \sum_{x} f(x,y)$; for $y = -4, 2$ and 7

when $Y = -4$, $h(Y=-4) = \sum_{x} f(x,-4)$

$= f(1,-4) + f(5,-4)$

$= 1/8 + 1/4$

$= 3/8$

When $Y = 2$, $h(Y=2) = \sum_x f(x, 2)$

$= f(1,2) + f(5,2)$

$= 1/4 + 1/8$

$= 3/8$

When $Y = 7$, $h(Y=7) = \sum_x f(x, 7)$

$= f(1,7) + f(5,7)$

$= 1/8 + 1/8$

$= 1/4$

The random variables X and Y will be independent if

$$f(1,-4) = g(1)\, h(-4)$$

Here $f(1,-4) = 1/8$ and

$g(1) \cdot h(-4) = 1/2 \cdot 3/8$

$= \frac{3}{16}$

Thus $f(1,-4) \neq g(1) \cdot h(-4)$

Therefore, X and Y are not independent

Marginal probability density functions

If X and Y are two continuous random variables with joint density function $f(x,y)$, then the marginal density function of X and Y, denoted as $g(x)$ and $h(y)$, are defined by

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy, \quad \text{for } -\infty < x < \infty \text{ and}$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx, \quad \text{for } -\infty < y < \infty$$

Conditional probability density functions.

Let X and Y be two continous random variable with joint probability density function $f(x,y)$ and if $g(x)$ and $h(y)$ be their marginal probability density functions, then the conditional probability density function of X given $Y=y$, denoted by $f(x|y)$, is defined by

$$f(x|y) = \frac{f(x,y)}{h(y)}, \quad -\infty < x < \infty$$

Similarly, the conditional probability density function of Y for given $X=x$, denoted by $f(y|x)$, is defined by

$$f(y|x) = \frac{f(x,y)}{g(x)}, \quad -\infty < x < \infty.$$

Ex: Suppose X and Y have a continuous joint probability density function as follows

$$f(x,y) = c(x+y^2) \; ; \; 0 \le x \le 1 \text{ and } 0 \le y \le 1$$

1. Find the value of c
2. Determine the conditional probability density function of X for any given

Value of Y and find

$$P\left[X < \tfrac{1}{2} \mid Y = \tfrac{1}{2}\right].$$

Soln: (1) The value of c ~~and~~ can be found from the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1,$$ since $f(x,y)$ is a joint probability density function.

Therefore, $$\int_0^1\int_0^1 c(x+y^2)\,dx\,dy = 1$$

or $$c\left[\frac{x^2}{2}\right]_0^1 + c\left[\frac{y^3}{3}\right]_0^1 = 1$$

or $$c\ \frac{5}{6} = 1$$

$$\therefore\quad c = \frac{6}{5}$$

(2) Now, by definition, the conditional probability density function of X for given Y = y is

$$f(x|y) = \frac{f(x,y)}{h(y)}, \quad 0 < x \leq 1$$

Here, $h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

$$= \frac{6}{5} \int_0^1 [x + y^2]\, dx$$

$$= \frac{6}{5} \left[\frac{x^2}{2} + y^2 x \right]_0^1$$

$$= \frac{6}{5} \left[\frac{1}{2} + y^2 \right]$$

Therefore, $f(x|y) = \dfrac{6/5\, [x + y^2]}{6/5\, [1/2 + y^2]}$

$$= \frac{2(x + y^2)}{1 + 2y^2} \quad 0 \leq x \leq 1$$

Now,

$$P\left[X < 1/2 \,\middle|\, Y = \frac{1}{2}\right] = \int_0^{1/2} f(x|y = 1/2)$$

$$= \int_0^{1/2} \frac{2(x + 1/4)}{(1 + 2 \cdot 1/4)}\, dx$$

$$= \frac{2}{3} \int_0^{1/2} \left(2x + \frac{1}{2}\right) dx = \frac{2}{3} \left[x^2 + \frac{1}{2}x\right]_0^{1/2}$$

$$= \frac{2}{3} \left[\frac{1}{4} + \frac{1}{4}\right]$$

$$= \frac{1}{3}.$$

A joint probability density function of two random variables X and Y is given by

$$f(x,y) = 12xy(1-y) \quad ; \quad 0 < x < 1, \; 0 < y < 1$$

Are X and Y independent ?

Sol^n: The random variables X and Y will be independent if

$$f(x,y) = g(x)\, h(y).$$

Now, $g(x) = 12 \int_0^1 xy(1-y)\, dx = 2x \; ; \; 0 < x < 1$

$$h(y) = 12 \int_0^1 xy(1-y)\, dx = 12 \left[y(1-y)\frac{x^2}{2}\right]_0^1$$

$= 6y(1-y) \quad 0<y<1$

Hence, $f(x,y) = 12xy(1-y)$ and

$$g(x)\,h(y) = 12xy(1-y)$$
$$= f(x,y)$$

Therefore X and Y are independent.