Column Space & Null Space of a Matrix

- Column space of a matrix A denoted as Col (A) is the span of columns of A.
- Null space of a matrix is the set- {x | Ax=0}.
- Note both are subspaces.
- Find column space and null space of following matrix

$$\begin{bmatrix} 0 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = Ax$

$Col(A) = \{ b \mid Ax = b \text{ for some } x \}$

Basis of a subspace

$A \sim R$ $Nul(A) = Nul(R)$ $Ax = 0$ $EAx = 0$ $Rx = 0$

- A basis for a subspace **H** of $\mathbf{R}^n$ is a linearly independent set which spans **H**.
- A basis of $\mathbf{R}^n$ is $\{e_1, e_2, \ldots, e_n\}$.
- Another basis of $\mathbf{R}^n$ is $\{e_1+e_2, e_2, \ldots, e_n\}$.
- The pivot columns of a matrix **A** form a basis for the column space of **A**.
- Note: pivot columns of REF of A do not form basis for col(A).

$H = \mathbb{R}^3$

$\{e_1, e_2, e_1 + e_3\}$

$x_1 e_1 + x_2 e_2 + x_3(e_1 + e_3) = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x - z)e_1 + ye_2 + z(e_3 + e_1)$$

$H = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \middle| x, y \in \mathbb{R} \right\}$ $\mathcal{B} = \{e_1, e_2\}$ $[v]_{\mathcal{B}} = \begin{bmatrix} x \\ y \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$H \subseteq \mathbb{R}^n$

① Find $0 \neq v_1 \in H$

$v_2 \in H \setminus \operatorname{Span}\{v_1\}$

Dimension & Rank

- Each vector in a subspace **H** can be written in only one way as a linear combination of the basis vectors.

Suppose the set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a ordered basis for a subspace H. For each $\mathbf{x}$ in H, the **coordinates of x relative to the basis** $\mathcal{B}$ are the weights $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$, and the vector in $\mathbb{R}^p$

$$\cancel{x = c_1' b_1' + \cdots + c_p' b_p}$$

$$\sum (c_i - c_i') b_i = 0$$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

$\mathcal{B} = \{v_1, v_2\}$

$v = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = (3x-2y)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (y-x)\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$[v]_{\mathcal{B}} = \begin{bmatrix} 3x-2y \\ y-x \end{bmatrix}$

$d = y - x$

$c + 2d = x$

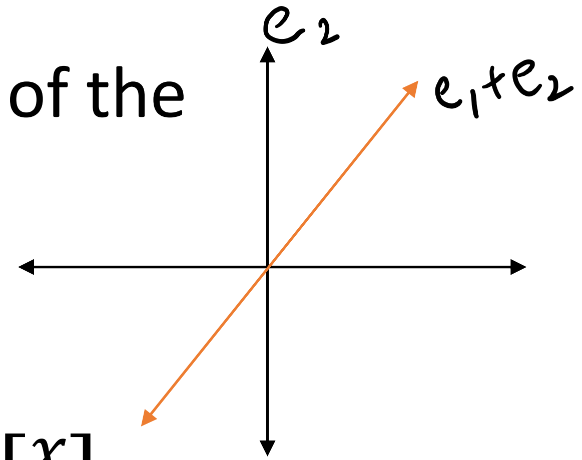
$c + 3d = y$

is called the **coordinate vector of x (relative to** $\mathcal{B}$**)** or the $\mathcal{B}$**-coordinate vector of x**.[1]

$\begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}$

- $V=R^2$. $B=\{e_1+e_2, e_2\}$ be a basis. Then coordinate vector of the *vector* $\begin{bmatrix}1\\1\end{bmatrix}$ is $\begin{bmatrix}1\\0\end{bmatrix}$.

$\left(\begin{bmatrix}1\\0\end{bmatrix}\right)_\beta = \begin{bmatrix}1\\-1\end{bmatrix}$



$e_1 = 1(e_1+e_2) - 1(e_2)$

- For $V=\left\{\begin{bmatrix}x\\0\\0\end{bmatrix} \middle| \, x \in R\right\}$ and $B=\{e_1\}$, coordinate vector of any $\begin{bmatrix}x\\0\\0\end{bmatrix} \in V$ is

$\beta = \{2e_1\}$

[x]

$\left(\begin{bmatrix}x\\0\\0\end{bmatrix}\right)_\beta = [x/2]$

Dimension & Rank

$\beta_1 = \{v_1 \ldots v_k\}$ $\quad w_i = \sum a_{ij} v_j$

$\beta_2 = \{w_1 \ldots w_\ell\}$ $\quad v_j = \sum b_{ji} w_i$

The **dimension** of a nonzero subspace H, denoted by dim H, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.[2]

The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A.

Nullity of A = dim(Nul(A))

$Im(T_A \circ T_B) \subseteq Im(T_A)$, $\quad T_{AB}$

Theorem: rank(A)=rank(A^T) for any matrix A.

Theorem: rank(AB) ≤ rank(A) for any matrix A.

≤ rank(B)

$\mathbb{R}^n \xrightarrow{T_B} \mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^n$

$\{v_1, v_2\}$ $\quad \{w_1\}$

$v_1 = c_1 w_1$

$v_2 = c_2 w_1$

$v_2 - \frac{c_2}{c_1} v_1 = 0$

$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix} \to \text{rank}(A) = 2$

Nullity = 1

$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{-1} & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \text{Null}(A) = \left\{ \begin{bmatrix} 3\alpha \\ -3\alpha \\ \alpha \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\}$

$-3z - 2y = -3\alpha + 6\alpha$

$A_{m \times n} : \mathbb{R}^n \to \mathbb{R}^m$

The Rank Theorem / Rank - Nullity Theorem

If a matrix A has n columns, then rank A + dim Nul A = n.

Suppose $\dim(H) = p$

Let $\beta = \{v_1 \ldots v_p\}$ is L.I. $(\Leftrightarrow)$ $\text{Span}\{v_1, \ldots v_p\} = H$

The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

Ex. Suppose $\dim(H) = p$ & $\text{span}\{v_1, \ldots v_p\} = H$ Then $\{v_1, \ldots v_p\}$ is Lin. Indep.

Invertible Matrix Theorem

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix. $A = [v_1 \cdots v_n]$

m. The columns of A form a basis of $\mathbb{R}^n$. $\dim(\text{Span}\{v_1, \cdots v_n\}) = n$

$\cap$

$\mathbb{R}^n$

n. $\text{Col}\, A = \mathbb{R}^n$

o. $\dim \text{Col}\, A = n$

p. $\text{rank}\, A = n$

q. $\text{Nul}\, A = \{\mathbf{0}\}$

r. $\dim \text{Nul}\, A = 0$

Vector Space

$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 7 & 10 \\ 0 & 0 \end{bmatrix}$

$\text{rank}(AB) = \text{rank}(A)$

$BA = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$A_{n\times n}$ $B_{n\times n}$ $(AB)_{n\times n}$

$T_A : \mathbb{R}^n \to \mathbb{R}^n$: T_B T_{AB}

$x \quad \mathbb{R}^n \xrightarrow{T_B} \mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^n$

T_{AB}

$\boxed{Im(T_{AB}) \subseteq Im(T_A)}$

v

x s.t. $T_{AB}(x) = v$

$\underline{V \subseteq \mathbb{R}^n}$ is called subspace if

① for $v_1, v_2 \in V$, $v_1 + v_2 \in V$

② for any $c \in \mathbb{R}$. $c \cdot v_1 \in V$

$V = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \middle| \, a, b, \ldots, i \in \mathbb{R} \right\}$

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a+j & b+k & c+l \\ d+m & e+n & f+o \\ g+p & h+q & i+r \end{bmatrix}$$

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. Commutativity
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$. → Associativity.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.

$$\alpha A = \begin{bmatrix} \alpha a & \alpha b & \alpha c \\ \alpha d & \alpha e & \alpha f \\ \alpha g & \alpha h & \alpha i \end{bmatrix}$$

7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.

Let $A \neq \emptyset$

$V = \{ f : A \to \mathbb{R} \mid f \text{ is a function} \}$ is a vector space.

$f_0 : A \to \mathbb{R}$

$f_0(a) = 0$

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$$

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \operatorname{Span}\{v_1\} = \left\{ \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \middle| c \in \mathbb{R} \right\}$$

$$v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \operatorname{Span}\{v_1, v_2\} = \left\{ \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} \middle| c, d \in \mathbb{R} \right\}$$

$$v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \operatorname{Span}\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} c & d \\ e & c \end{bmatrix} \middle| c, d, e \in \mathbb{R} \right\}$$

$$v_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \operatorname{Span}\{v_1, v_2, v_3, v_4\} = V$$

$$W = \left\{ \sum_{i=0}^{n} a_i x^i \middle| \begin{array}{l} n \in \mathbb{N} \cup \{0\} \\ a_i \in \mathbb{R} \end{array} \right\}$$

$v_1 = 1$

$v_2 = x$

$\vdots$

$v_n = x^n$