Dark and Bright Solitons in Left-Handed Nonlinear Transmission Line Metamaterials

Shulabh Gupta and Christophe Caloz
École Polytechnique, Montréal, Québec, H3T 1J4, Canada. Email: shulabh.gupta@polymtl.ca

***Abstract*— The existence of a solitary wave and soliton solutions in a nonlinear (NL) left-handed (LH) transmission line (TL) loaded with nonlinear varactor diodes is demonstrated rigorously for the first time and corresponding closed-form solutions are provided. These solutions are modulated dark and a bright solitons for the cases of heterostructure barrier varactor (HBV) and hyperabrupt junctions varactor (HJV) diodes, respectively. It is shown that NL LH TLs, despite their very different dispersion response, may support solitons identical to those in optical fibers in their normal dispersion regime, in contrast to RH TLs which can support only non-modulated (KdV) and bright solitons. Due to this equivalence, it maybe be anticipated that NL LH TLs will lead to several novel and unique microwave applications, such as UWB pulse shapers and compressors, which will be developed by transposition of concepts from the photonics.**

***Index Terms*— Nonlinear (NL) transmission lines (TLs), metamaterials, solitons, solitary waves, varactor diodes.**

I. Introduction

Recently there has been intense activity in the new field of electromagnetic metamaterials. Specifically, transmission line metamaterials have already lead to a wealth of microwave applications [1]. However, most structures considered to date have been linear. It has been recently suggested that nonlinear (NL) left-handed (LH) TLs may support novel types of solitary waves that could lead to innovative pulse shaping systems [2], but this claim has never been verified so far. This contribution demonstrates for the first time and in a rigorous manner the existence of solitary wave solutions in NL LH TLs and derives closed-form expressions for these solutions for the case of practical nonlinear loads, namely heterostructure barrier varactor (HBV) and hyperabrupt junctions varactor (HJV) diodes.

II. CRLH Transmission line

An ideal LH TL consists of a series capacitance and a shunt inductance, and subsequently exhibits anti-parallel phase and group velocities [3]. However, such a line does not exist in nature because of the presence of parasitic series inductance and shunt capacitance which are responsible for right-handed (RH) contributions. To take into account these effects, Caloz et al. [1] developed the concept of a Composite Right/Left-Handed (CRLH) TL, which acts as a LH TL at low frequencies and RH TL at high frequencies. Depending on the relative values of the LH and RH contributions, this TL can be unbalanced or balanced, i.e. exhibiting with a gapless transition between the LH and RH bands. In the balanced case, which is the condition for intrinsic broadband matching, the propagation constant of the CRLH line splits into uncoupled LH and RH propagation constants, resulting into two uncoupled cascaded TLs, as illustrated in Fig. 1. In this system, the RH TL, being non-dispersive, simply adds a delay term to the overall response, and only the LH TL is responsible for dispersion and subsequent distortion of the signal. Therefore, a balanced CRLH line is equivalent to an *ideal dispersive LH TL* with an extra constant group delay due to the RH contributions. This fact essentially reduces the analysis of a CRLH TL to the analysis of a purely LH TL.

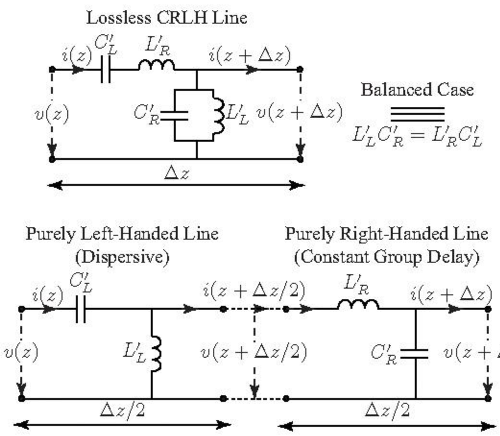

Fig. 1. A balanced CRLH transmission line is equivalent to the series connection of a purely LH (dispersive) line and a purely RH (constant group delay) line [1]. In dispersion analysis, the RH contributions can therefore be ignored and introduced in the end as a simple delay term. A physical CRLH TL is obtained by repeating a lumped CRLH unit cell of subwavelength size.

III. Pulse Propagation along a Linear LH TL

Referring to the LH TL in Fig. 1 (with $\Delta z/2 \to \Delta z$ for notational convenience), the current across the times-unit-length capacitance $C_L' = C_L \cdot \Delta z$ (F·m) and the voltage across the times-unit-length inductance $L_L' = L_L \cdot \Delta z$ (H·m) are $i(z) = \partial/\partial t\{C_L'[v(z) - v(z+\Delta z)]/\Delta z\}$ and $v(z+\Delta z) = \partial/\partial t\{L_L'[i(z) - i(z+\Delta z)]/\Delta z\}$, respectively. In the long wavelength (or metamaterial) regime ($\Delta z/\lambda \to 0$), these expressions reduce to $i(z) = -C_L'\partial^2 v/\partial z \partial t$

and $v(z) = -L'_L \partial^2 i/\partial z \partial t$, respectively. Inserting the first of these expressions into the second one yields the time-domain voltage wave equation

$$\frac{\partial^4 v(z,t)}{\partial z^2 \partial t^2} = \frac{v(z,t)}{L'_L C'_L}. \qquad (1)$$

This equation may be simplified as follows. Firstly, taking the Fourier transform ($\mathcal{F}$) of both sides, multiplying by $d\tilde{v}/dz$, where $\tilde{v}(z,\omega) = \mathcal{F}[v(z,t)] = \int v(z,t)\exp(-j\omega t)dt$, integrating with respect to z, and taking the square root of appropriate sign to ensure causality, results into the spectral-domain version of the voltage wave equation

$$\frac{d\tilde{v}(z,\omega)}{dz} = j\frac{1}{\omega\sqrt{L'C'}}\tilde{v}(z,\omega) = -j\beta_L(\omega)\tilde{v}(z,\omega), \quad (2)$$

where $\beta_L(\omega) = -\omega_L/\omega$ is the propagation constant of a linear LH TL with $\omega_L = 1/\sqrt{(L'_L C'_L)}$ (rad/m·s) [1]. Accounting for the RH contribution $\beta_R(\omega) = \omega/\omega_R$ with $\omega_R = 1/\sqrt{(L'_R C'_R)}$, the propagation constant of a balanced CRLH TL is given by $\beta(\omega) = \beta_R(\omega) + \beta_L(\omega)$. Upon dividing Eq. (2) by $\tilde{v}(z,\omega)$ and integrating the result over z, this equation takes the simpler form $\tilde{v}(z,\omega) = \exp[-j\beta(\omega)z]\tilde{v}(z=0,\omega)$, where $\tilde{v}(z=0,\omega)$ is the spectrum of the input pulse, which may be solved easily for a given input pulse. In order to gain deeper insight into the dispersive pulse propagation phenomenon, we will now simplify Eq. (2) by expanding $\beta(\omega)$ in Taylor series around a modulation frequency ω_0. This expansion reads

$$\beta(\omega) \approx \beta_0 + \beta_1(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2, \qquad (3)$$

where $\beta_0 = (\omega_0/\omega_R - \omega_L/\omega_0)$ is the phase velocity, $\beta_1 = (1/\omega_R + \omega_L/\omega_0^2)$ is the inverse of the nondispersive part of the group velocity [4], and $\beta_2 = -\omega_L/\omega_0^3$ is the group velocity dispersion (GVD) parameter, and where the approximation holds for a narrow band signal ($\Delta\omega \ll \omega_0$), where $\Delta\omega$ is the bandwidth of the pulse. It is interesting to note here that the group velocity of the pulse in a LH TL given by $v_g = 1/\beta_1$ is proportional to the modulation (carrier) frequency. This implies that pulses modulated at different frequencies ω_0 will travel at different group velocities. Fig. 2 shows the evolution of modulated Gaussian pulses along a (linear) LH line. Pulse width broadening accompanied by gradual amplitude decrease, which is an expected consequence of dispersion, is clearly observed in Fig. 2(a). Figs. 2(b) and 2(c) show the evolution of two identical co-propagating Gaussian pulses with different modulation frequencies. In the former case, the two pulses diverge (while broadening) because the earlier pulse has a higher group velocity; it is also apparent here that the pulse with higher modulation frequency experiences less dispersion, due to the fact that $\beta(\omega)$ becomes a more linear function of frequency when frequency increases. In the later case, the earlier pulse has a lower group velocity; consequently the two pulses first converge, then collide and finally diverge (while progressively broadening). Substituting Eq. (3) into Eq. (2) leads

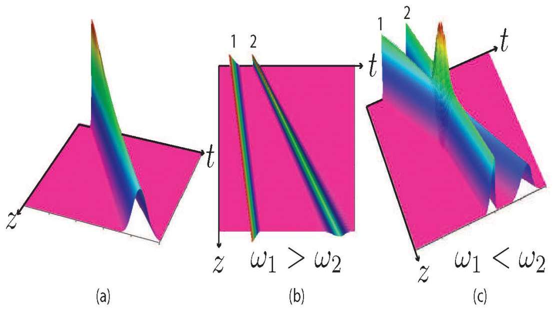

Fig. 2. Propagation of modulated Gaussian pulses (power) along a linear LH TL, i.e. $\mathcal{F}^{-1}[\tilde{v}(z,\omega)] = \mathcal{F}^{-1}\{\exp[-j\beta(\omega)z]\tilde{v}(z=0,\omega)\}$. a) Broadening due to dispersion. b) Dependence of group velocity and dispersion broadening on the modulation frequency, with earlier pulse having higher modulation frequency. (c) Idem, with earlier pulse having lower modulation frequency.

next to the wave equation $d\tilde{v}(z,\omega')/dz = -j(\beta_0 + \beta_1\omega' + 1/2\beta_2\omega'^2)\tilde{v}(z,\omega')$, where $\omega' = \omega - \omega_0$. At this point, we heuristically assume that the system admits a modulated pulse solution, i.e. $v(z,t) = A(z,t)\exp[j(\omega_0 t - \beta_0 z)]$, which corresponds to the spectral-domain expression $\tilde{v}(z,\omega') = \tilde{A}(z,\omega')\exp(-j\beta_0 z)$, where $A(z,t)$ is the slowly-varying envelope of the signal modulated at the frequency ω_0 and traveling with the phase velocity β_0. Finally, applying the operator $\partial/\partial t \leftarrow j\omega'$ with Eq. (3), we obtain the time-domain pulse propagation equation $dA/dz = -\beta_1 dA/dt + j\beta_2/2d^2A/dt^2$ in terms of the envelope A. Using the retarded frame $T = t - \beta_1 z$ and taking into account that $\beta_2 < 0$, simplifies this equation to

$$j\frac{dA}{dz} - \frac{|\beta_2|}{2}\frac{d^2A}{dT^2} = 0, \qquad (4)$$

which governs the broadening of the pulse envelope as it propagates along the line due to the GVD, β_2. It should be noted that this equation for a linear LH TL is identical to the one governing pulse propagation in a linear optical fiber in the anomalous dispersion regime [4].

IV. Pulse Propagation along a NL LH TL

We now investigate whether this GVD can be compensated by the introduction of an appropriate nonlinearity in order to suppress pulse broadening, i.e. whether a NL LH TL may support a solitary wave [4]. For this purpose, we will consider two different types of nonlinear loads. We will show that if incorporating proper nonlinearity, a NL LH TL can support a wide variety of solitary waves and even solitons (solitary waves surviving collisions).

A. Dark Solitons using HBV Diodes

Let us consider that the TL is loaded with nonlinear capacitance of the type $C'_R(v) = C'_R(1 - \alpha|v|^2)$, with $\alpha > 0$. This corresponds to the case of a heterostructure

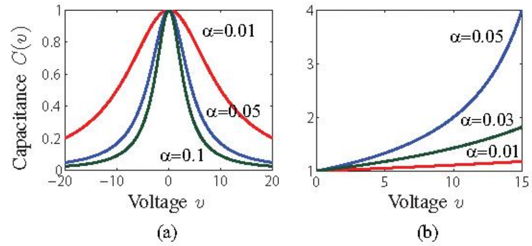

Fig. 3. Qualitative C-V characteristics for two different types of nonlinear capacitance diodes. a) Heterostructure barrier varactor (HBV) diode. b) Hyperabrupt junction varactor (HJV) diode.

barrier varactor (HBV) [5], whose nonlinear characteristic is shown in Fig. 3(a). With the inclusion of this nonlinearity, the angular frequency ω_R defined in Sec. III becomes $\omega_R' = \sqrt{1/[L_R' C_R'(1-\alpha|v|^2)]}$. This expression may be written as $\omega_R' = \omega_R/\sqrt{1-\alpha|v|^2}$, where ω_R corresponds to the linear contribution. As a result, we obtain the following expressions for the new Taylor expansion coefficients of $\beta(\omega)$ assuming weak nonlinearity ($\alpha|v|^2 \ll 1$): $\beta_0' = \omega_0(1-\alpha|v|^2/2)/\omega_R - \omega_L/\omega_0 = \beta_0 - \omega_0/2\omega_R\alpha|v|^2$, $\beta_1' = \beta_1 - \alpha|v|^2/2\omega_R \approx \beta_1$ and $\beta_2' = \beta_2$, where the approximation follow from the fact that the the order of ω_0 in the numerator of the nonlinear term β_0 is higher than as compared to that in β_1' and also that the input pulse is assumed to be narrowband. With this nonlinear perturbation, the propagation constant in Eq. (3) modifies to

$$\beta(\omega) \approx \beta_0 + \beta_1(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 - \frac{\omega_0\alpha}{2\omega_R}|v|^2. \quad (5)$$

Following the same approach as in Sec. III, the envelope propagation equation corresponding to Eq. (4) for this nonlinear case is obtained as

$$j\frac{dA}{dz} - \frac{|\beta_2|}{2}\frac{d^2A}{dT^2} + \gamma|A|^2A = 0 \quad \text{with} \quad \gamma = \frac{\omega_0\alpha}{2\omega_R}, \quad (6)$$

which describes the propagation of a narrow band pulse along a NL LH TL loaded with HBV diodes. This equation is known as *nonlinear Schrödinger* (NLS) equation, which is well-known to admit solitary wave solutions [4]. The nonlinearity γ is responsible for self-phase modulation (SPM), which induces an *intensity* dependent phase across the pulse accompanied by generation of addition spectral components. The solitary wave solution may be derived explicitly using standard NLS transformations [4]. Under the substitutions $U = A/\sqrt{P_0}$, $\xi = |\beta_2|z/T_0^2$ and $\tau = T/T_0$, where P_0 is the peak power and T_0 is the width of the input pulse, Eq. (10) becomes

$$j\frac{dU}{d\xi} - \frac{1}{2}\frac{d^2U}{d\tau^2} + N^2|U|^2U = 0, \quad (7)$$

where $N^2 = |\gamma|P_0T_0^2/|\beta_2|$. By further using the scaling transformation $u = NU$, this equation reduces to the standard-form NLS equation $j\partial u/\partial\xi - (1/2)\partial^2 u/\partial\tau^2 + |u|^2u = 0$. This equation is known to admit the *dark soliton* eigen solution (also existing in optical fibers in the *normal dispersion* regime [4]), whose intensity profile exhibits a dip in a uniform background. The general solution of this equation is given by $u(\xi,\tau) = |u(\xi,\tau)|\exp[j\phi(\xi,\tau)]$ where

$$\phi(\xi,\tau) = \frac{1}{2}\eta^2(3-B^2)\xi + \tan^{-1}\frac{B\tanh(\eta B\tau)}{\sqrt{(1-B^2)}} + \eta\sqrt{(1-B^2)}\tau, |u(\xi,\tau)| = \eta\sqrt{[1-B^2\mathrm{sech}^2(\eta B\tau)]}, \quad (8)$$

where the parameters η and B controls the soliton amplitude and the depth of the dip, respectively. For a given value of η, Eq. (8) describes a family of dark solitons whose width increases inversely to B. Fig. 4 shows a dark soliton for the case $|B| = 1$ (black soliton), $u(\xi,\tau) = \tanh(\tau)\exp(j\xi)$, while in the case of dark solitons with $|B| < 1$ (gray solitons) the dip does reach zero. Thus an input pulse with $|\tanh|$ amplitude, exhibiting an intensity hole at its center, would propagate unchanged in a such a NL LH TL.

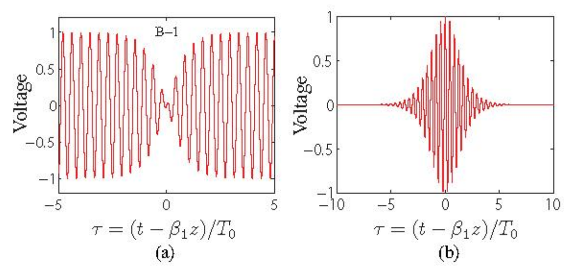

Fig. 4. Modulated solitons supported by a NL LH TL. a) Dark soliton for the case of HBV diodes [Fig. 3(a)]. b) Bright soliton for the case of HJV diodes [Fig. 3(b)].

B. Bright Solitons using HJV Diodes

Let us assume that the NL LH TL is now loaded with a HJV diode, which exhibits a nonlinear capacitance of the type $C_R'(v) = C_R'/(1-\alpha|v|)$, where α is the inverse of the built-in potential [6], whose nonlinear characteristic is shown in Fig. 3(b). With the inclusion of this nonlinearity, the angular frequency ω_R defined in Sec. IV-A becomes $\omega_R' = \sqrt{(1-\alpha|v|)/L_R'C_R'} = \omega_R\sqrt{1-\alpha|v|}$. Assuming weak nonlinearity, this expression may be approximated by $\omega_R' \approx \omega_R/(1-\alpha|v|)^{-1/2} = \omega_R/(1+\alpha|v|/2)$, where ω_R corresponds to the linear contribution. As a result, we obtain the following expressions for the new Taylor expansion coefficients of $\beta(\omega)$: $\beta_0' = \omega_0/\omega_R - \omega_l/\omega_0 + (\omega_0\alpha/2\omega_R)|v| = \beta_0 + (\omega_0\alpha/2\omega_R)|v|$, $\beta_1' = \beta_1 - (\alpha/2\omega_R)|v| \approx \beta_1$ and $\beta_2' = \beta_2$. With this nonlinear perturbation, the propagation constant in Eq. (3) becomes

$$\beta(\omega) \approx \beta_0 + \beta_1(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 + \frac{\omega_0\alpha}{2\omega_R}|v|. \quad (9)$$

Following the same approach as in Sec. III, the envelope propagation equation corresponding to Eq. (4) for this nonlinear case is obtained as

$$j\frac{dA}{dz} - \frac{|\beta_2|}{2}\frac{d^2A}{dt^2} - \gamma|A|A = 0, \text{ where } \gamma = \frac{\omega_0\alpha}{2\omega_R}, \quad (10)$$

which describes the propagation of a narrow band pulse along a NL LH TL loaded with HJV diodes. The nonlinearity γ is again responsible for SPM and induces an *amplitude* (not intensity like for the HBV case!) dependent phase across the pulse as it propagates along the line. Under the substitutions $U = A/P_0$, $\xi = |\beta_2|z/T_0^2$ and $\tau = T/T_0$, where P_0 is the peak power and T_0 is the width of the input pulse, Eq. (10) becomes

$$j\frac{dU}{d\xi} - \frac{1}{2}\frac{d^2U}{d\tau^2} - N|U|U = 0, \quad (11)$$

where $N = \gamma P_0 T_0^2/|\beta_2|$. This equation has the same form as the NLS equation (except for the fact that the nonlinearity term is not squared), and therefore it will support a soliton solution. Further applying the transformation $u = NU$, we get $j\partial u/\partial\xi - (1/2)\partial^2 u/\partial\tau^2 - |u|u = 0$, which is a non-standard equation. Now, assuming that a shape preserving solution of this nonlinear equation exists and is of the form $u(\xi,\tau) = V(\tau)\exp(jk\xi)$, we obtain

$$\frac{1}{2}\frac{d^2V(\tau)}{d\tau^2} = -kV(\tau) - V(\tau)^2. \quad (12)$$

Multiplying both sides by $dV(\tau)/d\tau$ and integrating with respect to $dV(b\tau)$ yields $(1/4)[dV(\tau)/d\tau]^2 = -k[V(\tau)^2/2]-[V(\tau)^3/3]$. Setting the retarded time origin $\tau = 0$ at the peak of the soliton yields $dV(\tau)/d\tau = 0$ and normalizing as $V(\tau) = 1$, we obtain $k = -2/3$. The equation thus becomes $(3/2)d^2V(\tau)/d\tau^2 = 2V(\tau) - 3V(\tau)^2$, which has the closed form solution $V(b\tau) = \text{sech}^2(\tau/\sqrt{3})$. Thus the complete solution reads

$$u(\xi,\tau) = \text{sech}^2(\tau/\sqrt{3})\exp(-j\frac{2\xi}{3}) \quad (13)$$

which is a *bright modulated soliton* solution. This implies that a modulated input pulse with a $|\text{sech}^2|$ amplitude will propagate undistorted arbitrarily long distance along such a NL LT TL.

V. LH TL & Optical Fiber: Prospects

Secs. III and IV-A demonstrated that NL LH TLs admit the same soliton solutions that optical fibers in the *normal* dispersion regime, despite the fact that the two media have opposite handedness.

A. Wave Breaking in LH NL TL using HBV Diodes

Optical fibers operating in the normal dispersion regime and driven in a high nonlinearity regime ($N \gg 1$) are well-known to exhibit the so called "optical wave breaking" phenomenon [4], due to the predominance of SPM over GVD. This phenomenon transforms any input un-chirped (resp. chirped) pulse into a nearly rectangular (resp. triangular) pulse accompanied by a linear chirp across the entire pulse width, which is used for compression in optics. Due to their similarity with optical fibers, HBV-loaded NL LH TLs will produce similar pulse shaping effect which may be exploited in novel types of pulse shaping and compressing devices.

B. Modulation Instability in LH NL TL using HJV Diodes

In the anomalous dispersion regime, an optical fiber exhibits an interesting phenomenon called modulation instability which occurs as a result of inherently unstable continuous wave propagation [4]. This instability leads to the generation of ultrashort optical pulse trains. The same phenomenon is also expected in the case of an HJV loaded NL LH TL where the negative nonlinear parameter γ can be used to induce a modulation instability for a continuous wave solution of Eq. (10). This effect maybe used for the generation of ultrashort electronic pulse generators.

VI. Conclusions

For the first time, the existence of solitary wave and soliton solutions in NL LH TLs has been demonstrated rigorously and corresponding closed-form solutions have been provided. NL LH TLs, despite their very different dispersion response, have been shown to support identical solitons as optical fibers in their normal dispersion regime. In particular, it has been shown that a NL LH TL loaded with HBV diodes supports dark modulatd solitons while a NL LH TL loaded with HJV diodes supports bright modulated solitons, both solutions of the NLS-type equation. This is in contrast to RH TLs which can support only non-modulated (KdV) and bright solitons [7]. Due to the analogy with optical fibers, it maybe be anticipated that several novel microwave applications, such as pulse shapers and compressors, may be developed by transposition of concepts from the photonics.

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