Problem 1 (Problem 4.1)

Consider

$$A = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$$

Where A_{11}, A_{22} are blocks of size $m \times m, n \times n$. We claim that

$$\det A = \det A_{11} \det A_{22}$$

Note that given any permutation σ that sends one of the indices $\{1, 2...n\}$ to $\{n+1, ...n+m\}$, σ must also send one of the indices $\{n+1, ...n+m\}$ to $\{1, 2...n\}$. However, this means that the sum

$$a_{1,\sigma(1)}a_{2,\sigma(2)}...a_{n+m,\sigma(n+m)} = 0$$

Since one of the entries lies in O. This means that we only need to consider permutations that maps $\{1, 2...n\}$ and $\{n+1, ...n+m\}$ to itself. This suggests that factor σ into two permutations σ_1, σ_2 that permutes $\{1, 2...n\}$ and $\{n+1, ...n+m\}$ respectively (such factorization, or map, from σ to σ_1, σ_2 is a bijection). Consequently,

$$\begin{aligned}
\det A &= \sum_{\sigma} a_{1,\sigma(1)}a_{2,\sigma(2)}...a_{n+m,\sigma(n+m)} \\
&= \sum_{\sigma_1} a_{1,\sigma_1(1)}a_{2,\sigma_1(2)}...a_{n,\sigma_1(n)} \sum_{\sigma_2} a_{n+1,\sigma_2(n+1)}a_{n+2,\sigma_2(n+2)}...a_{n+m,\sigma_2(n+m)} \\
&= \det A_{11} \det A_{22}
\end{aligned}$$

To solve the original problem, note that

$$\det A = \det A_{11} \det \begin{bmatrix} A_{22} & \cdots & A_{2b} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{bb} \end{bmatrix}$$

And note the the large matrix is again in the same form. By repeatedly applying the same results, we have the conclusion

$$\det A = \det A_{11} \det A_{22} ... \det A_{bb}$$

Problem 2 (Question 4.2)

Let A be normal, then

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{nn} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \overline{a}_{11} & 0 & \cdots & 0 \\ \overline{a}_{12} & \overline{a}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{1n} & \overline{a}_{2n} & \cdots & \overline{a}_{nn} \end{bmatrix} = \begin{bmatrix} \overline{a}_{11} & 0 & \cdots & 0 \\ \overline{a}_{12} & \overline{a}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{1n} & \overline{a}_{2n} & \cdots & \overline{a}_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{nn} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Comparing the $(1,1)$ entry on both sides, we see that

$$\sum_{i=1}^{n} |a_{1i}|^2 = |a_{11}|^2$$

This suggests that $a_{1i} = 0$ for all $i \neq 1$. This reduces the matrix A down to

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Note that the $(2,2)$ submatrix is again normal and upper triangular. By applying the same argument to the remaining submatrix, we have $a_{ij} = 0$ whenever $i \neq j$, or A is diagonal.

Now with this result, suppose A is normal, then there exists utinary matrix U such that $U^*AU = T$, where T is triangular. Note that T is again normal since

$$TT^* = U^*AUU^*A^*U = U^*AA^*U = U^*A^*AU = U^*A^*UU^*AU = T^*T$$

Since T is upper triangular and normal, by the previous result, T is diagonal. Since

$$AU = UT$$

And T is diagonal, it follows that U is a matrix of orthogonal eigenvectors. The other direction can be proven by following the proof in the opposite direction.

Problem 3 (Question 4.3)

Since

$$\overline{y}^T Ax = (\overline{y}^T A)x = \mu\overline{y}^T x$$

And

$$\overline{y}^T Ax = \overline{y}^T(Ax) = \lambda\overline{y}^T x$$

It follows that

$$(\mu - \lambda)\overline{y}^T x = 0 \implies \overline{y}^T x$$

Since μ, λ are distinct. This suggests that x, y are orthogonal.

Problem 4 (Problem 4.4)

(1)
Note that

$$A^n = QTQ^*QTQ^*...QTQ^* = QT^nQ^*$$

(The case where $n < 0$ follows by taking inverse). Hence,

$$f(A) = \sum_{i=-\infty}^{\infty} a_i A^i = \sum_{i=-\infty}^{\infty} a_i Q T^i Q^* = Q\bigg(\sum_{i=-\infty}^{\infty} a_i T^i\bigg)Q^* = Qf(T)Q^*$$

(2)
Note the it suffuce proving $(T^i)_{nn} = (T_{nn})^i$. We proceed by induction. The base case is trivially true. Suppose the claim is true for $i = k$, then for $i = k+1$, we have

$$\begin{aligned}
(T^{k+1})_{nn} &= (T^k T)_{nn} \\
&= \sum_j (T^k)_{nj}(T)_{jn} \\
&= (T^k)_{nn}(T)_{nn} \\
&= (T_{nn})^k (T)_{nn} \\
&= (T_{nn})^{k+1}
\end{aligned}$$

The third line follows because both T^k anf T are upper triangular, so $(T^k)_{nj} = 0$ when $j < n$ and $(T)_{jn} = 0$ when $j > n$, so the only term remain is the term where $j = n$. With this result, we have

$$\begin{aligned}
(f(T))_{nn} &= (\sum_{i=-\infty}^{\infty} a_i T^i)_{nn} \\
&= \sum_{i=-\infty}^{\infty} a_i (T^i)_{nn} \\
&= \sum_{i=-\infty}^{\infty} a_i (T_{nn})^i \\
&= f(T_{nn})
\end{aligned}$$

(3)

$$Tf(T) = T\sum_{i=-\infty}^{\infty} a_i T^i = \sum_{i=-\infty}^{\infty} a_i T T^i = \sum_{i=-\infty}^{\infty} a_i T^i T = f(T)T$$

(4)
Note that is suffice proving the result for T^n. We proceed by induction. The base case is trivial. Suppose the claim holds for T^k, in other words, $T^k(i, i+j)$ can be determined

by the $(j-1)$ previous superdiagonals, then

$$T^{k+1}(i,i+j) = \sum_{m=i}^{i+j} T^k(i,m)T(m,i+j)$$

Since both $T^k(i,m)$ and $T(m,i+j)$ can be determined using $(j-1)$ previous superdiagonals, the claim follows by induction.
Since the claim holds for T^n, the claim holds for $f(T)$ as well.

Problem 5 (Problem 4.5)

Let $A = U^*TU$ be the Schur form. Note that $f(A)$ and $f(T)$ has the same eigenvalue since

$$f(A) = U^* f(T) U$$

From (2), since

$$f(T)_{ii} = f(T_{ii}) = f(\lambda_i)$$

It follows that $f(\lambda_i)$ is an eigenvalue of $f(T)$ for all i. Since $f(A)$ and $f(T)$ has the same eigenvalue, the result follows