Problem 1 (Problem 6.1)

We check the results directly. We have

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & 1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} \sin(\frac{j\pi}{N+1}) \\ \sin(\frac{2j\pi}{N+1}) \\ \vdots \\ \vdots \\ \sin(\frac{N\pi}{N+1}) \end{bmatrix} = \begin{bmatrix} 2\sin(\frac{j\pi}{N+1}) - \sin(\frac{2j\pi}{N+1}) \\ -\sin(\frac{j\pi}{N+1}) + 2\sin(\frac{2j\pi}{N+1}) - \sin(\frac{3j\pi}{N+1}) \\ \vdots \\ \vdots \\ -\sin(\frac{(N-1)\pi}{N+1}) + 2\sin(\frac{N\pi}{N+1}) \end{bmatrix}$$

For that since

$$\sin(\frac{(k-1)j\pi}{N+1}) + \sin(\frac{(k+1)j\pi}{N+1}) = 2\sin(\frac{kj\pi}{N+1})\cos(\frac{j\pi}{N+1})$$

We have, for $k = 2, ..., N-1$

$$\begin{aligned} -\sin(\frac{(k-1)j\pi}{N+1}) + \sin(\frac{kj\pi}{N+1}) - \sin(\frac{(k+1)j\pi}{N+1}) &= \sin(\frac{kj\pi}{N+1}) - 2\sin(\frac{kj\pi}{N+1})\cos(\frac{j\pi}{N+1}) \\ &= \sin(\frac{kj\pi}{N+1})\left(1 - 2\cos(\frac{j\pi}{N+1})\right) \end{aligned}$$

This proves that the eigenvalue/eigenvector relation holds for $k = 2, ...N-1$. Note that this holds for the last entry since $\sin(\frac{0 \cdot j\pi}{N+1}) = \sin(\frac{(N+1)\cdot j\pi}{N+1}) = 0$

We now prove that the eigenvectors have unit norm. Taking the sum squares and applying trignometric identities, we have

$$\begin{aligned} \sum_{k=1}^{n} \sin\left(\frac{\pi jk}{n+1}\right) &= \sum_{i=1}^{n} \frac{1}{2}\left(1 - \cos(\frac{2\pi jk}{n+1})\right) \\ &= \frac{n}{2} - \frac{1}{2}\sum_{k=1}^{n} \cos(\frac{2\pi jk}{n+1}) \end{aligned}$$

Since

$$\begin{aligned} \cos(\frac{2\pi jk}{n+1}) &= \csc(\frac{\pi}{n+1}) \sin\frac{n\pi}{n+1})\cos(\pi) \\ &= -1 \end{aligned}$$

We have

$$\sum_{k=1}^{n} \sin\left(\frac{\pi jk}{n+1}\right) = \frac{n+1}{2}$$

This means that multiplying the vector by the normalizing constant $\sqrt{\frac{2}{n+1}}$ gives unit vector

Problem 2 (Question 6,4)

Lemma 6.2.1: Proved in class

Lemma 6.2.2:
Note that the right hand side is given by

$$\begin{bmatrix} b_{11}I_m & b_{21}I_m & \cdots & b_{n1}I_m \\ b_{12}I_m & b_{22}I_m & \cdots & b_{n2}I_m \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n}I_m & b_{2n}I_m & \cdots & b_{nn}I_m \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{m1} \\ x_{12} \\ \vdots \end{bmatrix}$$

When multiplied out, the km to $(k+1)m$ entries of the $mn \times 1$ vector is given by

$$b_{1k}\mathbf{x}_1 + b_{2k}\mathbf{x}_2 + ...$$

Where $\mathbf{x}_i$ is a columns of X. Note that the above expression exactly the flattened version of BX. This proves the claim.

Lemma 6.2.3: Proved in class

Lemma 6.3.1:
Note that

$$\begin{aligned}
(A \otimes B)(C \otimes D) &= \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1l}B \\ a_{21}B & a_{22}B & \cdots & a_{2l}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}B & a_{k2}B & \cdots & a_{kl}B \end{bmatrix} \begin{bmatrix} c_{11}D & c_{12}D & \cdots & c_{1m}D \\ c_{21}D & c_{22}D & \cdots & c_{2m}D \\ \vdots & \vdots & \ddots & \vdots \\ c_{l1}D & c_{l2}D & \cdots & c_{lm}D \end{bmatrix} \\
&= \begin{bmatrix} (\sum_{i=1}^l a_{1i}c_{i1})BD & (\sum_{i=1}^l a_{1i}c_{i2})BD & \cdots & (\sum_{i=1}^l a_{1i}c_{im})BD \\ (\sum_{i=1}^l a_{2i}c_{i1})BD & (\sum_{i=1}^l a_{2i}c_{i2})BD & \cdots & (\sum_{i=1}^l a_{2i}c_{im})BD \\ \vdots & \vdots & \ddots & \vdots \\ (\sum_{i=1}^l a_{ki}c_{i1})BD & (\sum_{i=1}^l a_{ki}c_{i2})BD & \cdots & (\sum_{i=1}^l a_{ki}c_{im})BD \end{bmatrix} \\
&= \begin{bmatrix} (AC)_{11}BD & (AC)_{12}BD & \cdots & (AC)_{1m}BD \\ (AC)_{21}BD & (AC)_{22}BD & \cdots & (AC)_{2m}BD \\ \vdots & \vdots & \ddots & \vdots \\ (AC)_{k1}BD & (AC)_{k2}BD & \cdots & (AC)_{km}BD \end{bmatrix} \\
&= (AC) \otimes (BD)
\end{aligned}$$

Lemma 6.3.2:
Apply the result from previous part gives

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = I_m \otimes I_n \implies (A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$$

Lemma 6.3.3:
Note that

$$(A \otimes B)^T = \begin{bmatrix} A_{11}B^T & A_{21}B^T & \cdots & A_{n1}B^T \\ A_{12}B^T & A_{22}B^T & \cdots & A_{n2}B^T \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n}B^T & A_{2n}B^T & \cdots & A_{n2}B^T \end{bmatrix} = A^T \otimes B^T$$

Part 3:
Consider taking vec on both sides, we have

$$\text{vec}(AX - XB) = \text{vec}(AX) - \text{vec}(XB) = (I_n \otimes A)\text{vec}(X) - (B^T \otimes I_m)\text{vec}(X) = \text{vec}(C)$$

This proves the claim.

Part 4:
Note that

$$\text{vec}(AXB) = (I \otimes A)\text{vec}(XB) = (I \otimes A)(B^T \otimes I)\text{vec}(X) = (B^T \otimes A)\text{vec}(X)$$

Problem 3 (Question 6.5)

Part 1:
Note that it suffice verifying

$$(I_m \otimes A + B \otimes I_n)(Y \otimes X) = (Y \otimes X)(I_m \otimes \Lambda_A + \Lambda_B \otimes I_n)$$

The result then follows by expanding the above line. To prove the above equality, consider expanding

$$\begin{aligned}(I_m \otimes A + B \otimes I_n)(Y \otimes X) &= Y \otimes AX + BY \otimes X \\ &= Y \otimes X\Lambda_A + Y\Lambda_B \otimes X \\ &= (Y \otimes X)(I_m \otimes \Lambda_A) + (Y \otimes X)(\Lambda_B \otimes I_n) \\ &= (Y \otimes X)(I_m \otimes \Lambda_A + \Lambda_B \otimes I_n)\end{aligned}$$

Part 2:
From previous exercises, we proved that solving the equation

$$AX + XB^T = C$$

Is equivalent as solving

$$(I_n \otimes A - B^T \otimes I_m)\mathrm{vec}(X) = \mathrm{vec}(C)$$

To solve this linear equation, we need $det(I_n \otimes A - B^T \otimes I_m) \neq 0$. Since the eigenvalues of the matrix is given by $\alpha_i + \beta_j$, it follows that the system is solvable if and only if $\alpha_i + \beta_j \neq 0$. The same holds when we transpose B (since they have the same set of eigenvalues).

Part 3:
It suffice verifying

$$(B \otimes A)(Y \otimes X) = (Y \otimes X)(\Lambda_A \otimes \Lambda_B)$$

Consider expanding the left hand side

$$\begin{aligned}(B \otimes A)(Y \otimes X) &= BY \times AX \\ &= (Y\Lambda_B) \otimes (X\Lambda_A) \\ &= (Y \otimes X)(\Lambda_A \otimes \Lambda_B)\end{aligned}$$

The result then follows by expanding both sides.