Problem 1 (Problem 4.6)

(a)

Let $A = U_A T_A U_A^H, B = U_B T_B U_B^H$ be their respective Schur form, then

$$U_A T_A U_A^H X - X U_B T_B U_B^H = C \implies T_A U_A^H X U_B - U_A^H X U_B T_B = U_A^H C U_B$$

Letting $Y = U_A^H X U_B$, we have

$$T_A Y - Y T_b = C' = U_A^H C U_B$$

Which is the desired transformation.

(b)

The system, in matrix form, looks like

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} - \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & b_{22} & \cdots & b_{2n} \\ & & \ddots & \vdots \\ & & & b_{nn} \end{bmatrix}$$

Note that the nth row of C is given by

$$c_{n,k} = a_{nn} y_{nk} - \sum_{i=1}^{k} b_{ik} y_{ni}$$

Rearranging, we have

$$y_{nk} = \frac{c_{n,k} - \sum_{i=1}^{k-1} b_{ik} y_{ni}}{a_{nn} - b_{kk}}$$

This means that using the last row of C, we can clear the last row of Y iteratively using the formula above. Note that the system is solvable when $a_{nn} \neq b_{kk}$ for all $k = 1, ...n$. This means that Y is solvable when the eigenvectors of A and B are all distinct (since the eigenvectors of a diagonal matrix is the diagonal terms).

By repeating the same argument to the rest of the rows, we see that we can solve Y.

(c)

Once Y has been solved, since $Y = U_A^H X U_B$, it follows that

$$X = U_A Y U_B^H$$

Problem 2 (Question 4.8)

Note that

$$\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$$

Since

$$\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

It follows that the given two matrices are similar. This implies that they have the same characteristic polynomial:

$$\lambda^n \det(\lambda I - AB) = \lambda^m \det(\lambda I - BA)$$

If $\lambda \neq 0$, then $\det(\lambda I - AB) = \det(\lambda I - BA)$. Since AB and BA has the same characteristic polynomial, it follows that they have the same nonzero eigenvalue.

Problem 3 (Question 4.11)

(a)

Consider a scalar multiple of x, y, say $\lambda x, \mu y$, then

$$P = \frac{(\lambda x)(\mu y)^*}{(\mu y)^*(\lambda x)} = \frac{xy^*}{y^*x}$$

This proves that P is uniquely defined.

(b)

$$P^2 = \frac{x(y^*x)y^*}{(y^*x)^2} = \frac{xy^*}{y^*x} = P$$

(c)

$$AP = \frac{Axy^*}{y^*x} = \frac{\lambda xy^*}{y^*x} = \lambda P$$
$$PA = \frac{xy^*A}{y^*x} = \frac{\lambda xy^*}{y^*x} = \lambda P$$

(d)

Recall that the condition number of a simple eigenvalue λ is given by

$$\frac{1}{|xy^*|}$$

Where x, y are the right and left eigenvectors. since

$$||P||_2 = \frac{1}{|xy^*|}||xy^*||_2 = \frac{1}{|xy^*|}||x||_2||y||_2 = \frac{1}{|xy^*|}$$

Since $||x|| = ||y|| = 1$ (we can assumed both to be zero, by part (a)). It follows that $||P||_2$ is the condition number for λ.

Problem 4 (Problem 4.13)

We want to solve

$$(A+E)x = \mu x \implies Ex = -(Ax - \mu x) = -r$$

Clearly, $E = -rx^*$ satisfies the equation (since x is a unit vector). We now check $||E||_F$:

$$||E||_F = \mathrm{Tr}(xr^*rx) = ||r||^2\mathrm{Tr}(xx^*) = ||r||^2$$

This proves the claim.