Problem 1

(a)

Consider the PDE

$$-a\frac{\partial^2 v}{\partial x^2} - b\frac{\partial^2 v}{\partial y^2} = f(x,y)$$

Discretizing, we have

$$(-av_{i+1,j} + 2av_{i,j} - av_{i-1,j}) + (-bv_{i,j+1} + 2bv_{i,j} - bv_{i,j-1}) = h^2 f_{i,j}$$

Let T_N be the tridiagonal matrix

$$T_N = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

And define $T_a = aT_N, T_b = bT_N$, then the discretized equations can be written as

$$T_a \cdot V + V \cdot T_b = h^2 F$$

Using Kronecker product notation, we have

$$\underbrace{(I_N \otimes T_a + T_b \otimes I_N)}_{T(a,b)} \text{Vec}(V) = h^2 \text{Vec}(F)$$

(b)

Note that both T_a and T_b has the same set of eigenvectors as T_N. The eigenvalues of T_a are precisely scalar multiples (by factor of a, b respectively) of the eigenvalues of T_a, T_b.

In particular, let $T_N = Z\Lambda Z^T$ be the eigendecomposition of T_N, then

$$T(a,b) = I_N \otimes T_a + T_b \otimes I_N = (Z \otimes Z) \cdot (I \otimes a\Lambda + b\Lambda \otimes I) \cdot (Z \otimes Z)^T$$

This can be verified by multiplying the RHS. This implies the eigenvectors of $T(a,b)$ is $Z \otimes Z$ and the eigenvalues are $I \otimes a\Lambda + b\Lambda \otimes I$.
Since Z is orthogonal and $I \otimes a\Lambda + b\Lambda \otimes I$ is diagonal where each entry is positive, it follows that $T(a,b)$ is symmetric and positive definite.
Finally, assuming the eigenvalues are sorted in ascending order

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$$

the condition number is given by

$$\kappa(T(a,b)) = \frac{(a+b)\lambda_n}{(a+b)\lambda_1} = \frac{\lambda_n}{\lambda_1}$$

Which is the same as T_N.

(c)

Consider performing Jacobi's method. We split the matrix $T(a,b)$ into

$$T = M - (M-K) = 2(a+b)I - (2(a+b)I - T)$$

We now compute the spectral radius of

$$R = M^{-1}K = I - \frac{1}{2(a+b)}T$$

In part (b), we've shown that the eigenvalues of T takes in the form $a\lambda_i + b\lambda_j$, therefore

$$\rho(R) = 1 - \min_{i,j} \frac{1}{2(a+b)}(a\lambda_i + b\lambda_j) = 1 - \frac{\lambda_1}{2} = \cos(\frac{\pi}{N+1})$$

Since $\rho(R) < 1$, it follows that Jacobi's method converges. However, as $n \to \infty$, we see that $\rho(R) \to 1$, meaning that the rate of convergence becomes slower.

Problem 2

(a)

By the same argument as problem 1, and by the results for 3d Poisson PDE shown in class, it can be seen that using Kronecker product notation, we have

$$(I_N \otimes I_N \otimes T_a + I_N \otimes T_b \otimes I_N + T_c \otimes I_N \otimes I_N)\text{Vec}(V) = h^2\text{Vec}(F)$$

(b)

Like before, let $T_N = Z\Lambda Z^T$ be the eigendecomposition of T_N, then

$$\begin{aligned} T(a,b,c) &= (I_N \otimes I_N \otimes T_a + I_N \otimes T_b \otimes I_N + T_c \otimes I_N \otimes I_N) \\ &= (Z \otimes Z) \cdot (I \otimes I \otimes a\Lambda + I \otimes b\Lambda \otimes I + c\Lambda \otimes I \otimes I) \cdot (Z \otimes Z)^T \end{aligned}$$

Which again, can be verified by expanding the RHS. Since the middle term is diagonal and th eigenvalue are positive, it follows that the matrix is symmetric and positive definite.

(c)

Performing Jacobi'ss method and splitting gives us

$$T = M - (M - K) = 2(a+b+c)I - (2(a+b+c)I - T)$$

The spectral radius of

$$R = M^{-1}K = I - \frac{1}{2(a+b+c)}T$$

Since the eigenvalues of T takes in the form $a\lambda_i + b\lambda_j + c\lambda_k$, it follows that

$$\rho(R) = 1 - \min_{i,j,k} \frac{1}{2(a+b+c)}(a\lambda_i + b\lambda_j + c\lambda_k) = 1 - \frac{\lambda_1}{2} = \cos\left(\frac{\pi}{n+1}\right) < 1$$

This proves that Jacobi's method converges but the rate of convergence becomes slower as n becomes larger.

Problem 3 (Question 6.10)

(b)

Let T_N be the Poisson matrix for 1D Poisson equation defined in class, consider the matrix

$$\hat{T} = T_N - 2I = \begin{bmatrix} & -1 & & \\ -1 & & \ddots & \\ & \ddots & & -1 \\ & & -1 & \end{bmatrix}$$

Then the eigenvalues of $\hat{T}$ are $2\cos(\frac{j\pi}{n+1})$ with eigenvectors

$$z_j = \begin{bmatrix} \sin(\frac{j\pi}{n+1}) & \sin(\frac{2j\pi}{n+1}) & \cdots & \sin(\frac{nj\pi}{n+1}) \end{bmatrix}$$

We can then use the fact that the desired matrix is of the form $\alpha_I - \theta\hat{T}$ to conclude that the eigenvalues are $\alpha - 2\theta\cos(\frac{j\pi}{n+1})$ with eigenvectors

$$z_j = \begin{bmatrix} \sin(\frac{j\pi}{n+1}) & \sin(\frac{2j\pi}{n+1}) & \cdots & \sin(\frac{nj\pi}{n+1}) \end{bmatrix}$$

(c)

Let $\hat{T}$ be defined above, then

$$T = I \otimes A - \hat{T} \otimes H$$

It follows that

$$\begin{aligned}(I \otimes Q)T(I \otimes Q)^T &= (I \otimes Q)(I \otimes A)(I \otimes Q)^T - (I \otimes Q)(\hat{T} \otimes H)(I \otimes Q)^T \\ &= I \otimes QAQ^T - \hat{T} \otimes QHQ^T \\ &= I \otimes \Lambda_A - \hat{T} \otimes \Lambda_H\end{aligned}$$

Where Λ_A, Λ_H denote the eigenvalues for A and H respectively. From the previous problem, we see that the eigenvalues of $-\hat{T} \otimes \Lambda_H$ is $\lambda_{ij} = 2\theta_i \cos(\frac{j\pi}{n+1})$ and the eigenvectors are

$$z_j = \begin{bmatrix} \sin(\frac{j\pi}{n+1}) & \sin(\frac{2j\pi}{n+1}) & \cdots & \sin(\frac{nj\pi}{n+1}) \end{bmatrix} \otimes e_j$$

Let Z be the matrix of eigenvectors then we have

$$(I \otimes \Lambda_A - \hat{T} \otimes \Lambda_H)(Z \otimes I) = (Z \otimes I)\mathrm{diag}(\alpha_1\lambda_{11}, ..., \alpha_n\lambda_{nn})$$

It follows that the eigenvectors are $(I \otimes Q)(Z \otimes I)$

(d)

Let

$$T = S^{-1}\Lambda S$$

be the eigendecomposition of T, solving $Tx = b$ is equivalent as solving

$$(S^{-1}\Lambda S)x = b$$

Which implies that dense LU factorization takes $O(n^6)$ time, and band LU takes $O(n^3)$ time.