Problem 1 (Problem 6.14)

To prove the relationship. We proceed by induction. For the base case, we have

$$\hat{z} = \hat{A}\hat{p}_1 = M^{-\frac{1}{2}}AM^{-1}b = M^{-\frac{1}{2}}z$$

Consequently,

$$\hat{v}_1 = \frac{\hat{r}_0^T\hat{r}_0}{\hat{p}_1^T\hat{z}} = \frac{b^TM^{-1}b}{b^TM^{-1}z} = v_1$$

$$\hat{x}_1 = \hat{x}_0 + \hat{v}_1\hat{p}_1 = v_1M^{\frac{1}{2}}p_1 = M^{\frac{1}{2}}x_1$$

$$\hat{r}_1 = \hat{r}_0 - \hat{v}_1\hat{z} = M^{-\frac{1}{2}}r_0 - v_1M^{-\frac{1}{2}}z = M^{-\frac{1}{2}}r_1$$

$$\hat{\mu}_2 = \frac{\hat{r}_1^T\hat{r}_1}{\hat{r}_0^T\hat{r}_0} = \mu_2$$

$$\hat{p}_2 = \hat{r}_1 + \hat{\mu}_2\hat{p}_1 = M^{-\frac{1}{2}}(r_1 + \mu_2p_1) = M^{\frac{1}{2}}p_2$$

Assume the claim holds for $k-1$, then at step k, we have

$$\hat{z} = \hat{A}\hat{p}_k = M^{-\frac{1}{2}}Ap_k = M^{-\frac{1}{2}}z$$

$$\hat{v}_k = \frac{\hat{r}_{k-1}^T\hat{r}_{k-1}}{\hat{p}_k^T\hat{z}} = \frac{r_{k-1}^TM^{-1}r_{k-1}}{\hat{p}_k^T\hat{z}} = v_k$$

$$\hat{x}_k = \hat{x}_{k-1} + \hat{v}_k\hat{p}_k = M^{\frac{1}{2}}x_{k-1} + v_kM^{\frac{1}{2}}p_k = M^{\frac{1}{2}}x_k$$

$$\hat{r}_k = \hat{r}_{k-1} - \hat{v}_k\hat{z} = M^{-\frac{1}{2}}r_{k-1} - v_kM^{-\frac{1}{2}}z = M^{-\frac{1}{2}}r_k$$

$$\hat{\mu}_{k+1} = \frac{\hat{r}_k^T\hat{r}_k}{\hat{r}_{k-1}^T\hat{r}_{k-1}} = \frac{r_k^TM^{-1}r_k}{r_{k-1}^TM^{-1}r_{k-1}} = \mu_{k+1}$$

$$\hat{p}_{k+1} = \hat{r}_k + \hat{\mu}_{k+1}\hat{p}_k = M^{-\frac{1}{2}}(r_k + \mu_{k+1}p_k) = M^{\frac{1}{2}}p_{k+1}$$

By induction, the relationship is true. Therefore, we see that x_k converges to the solution of $\hat{A}\hat{x} = \hat{b}$