Problem 1 (Problem 5.4 Part I)

Note that

$$H = \begin{bmatrix} I_{n-1} & 0 \end{bmatrix} \begin{bmatrix} H & b \\ b^T & u \end{bmatrix} \begin{bmatrix} I_{n-1} \\ 0 \end{bmatrix} = P^T A P$$

By Courant Fisher minimax theorem

$$\begin{aligned} \theta_i &= \min_{S^{n-i}} \max_{x \in S^{n-i}} \frac{x^T H x}{x^T x} \\ &= \min_{S^{n-1}} \max_{x \in S^{n-i}} \frac{(Px)^T A (Px)}{(Px)^T (Px)} \\ &\geq \min_{S^{n-1}} \max_{y = Px \in S^{n-i}} \frac{y^T A y}{y^T y} \\ &= \alpha_{i+1} \end{aligned}$$

The third line follows since if $x_1, ...x_k$ is linearly independent, then $Px_1, Px_2, ...Px_k$ must also be linearly independent (this can be justified by noting that P is injective). Likewise, we have

$$\begin{aligned} \theta_i &= \max_{R^i} \min_{x \in R^i} \frac{x^T H x}{x^T x} \\ &= \max_{R^i} \min_{x \in R^i} \frac{(Px)^T A (Px)}{(Px)^T (Px)} \\ &\leq \max_{R^i} \min_{y \in R^i} \frac{y^T A y}{y^T y} \\ &= \alpha_i \end{aligned}$$

This proves that the eigenvalues interleaves.

Problem 2 (Question 5.5)

By Courant Fisher minimax theorem

$$\begin{aligned}
\lambda_i &= \min_{S^{n-i+1}} \max_{x \in S^{n-i+1}} \frac{x^T(A+H)x}{x^Tx} \\
&\leq \min_{S^{n-i+1}} \left(\max_{x \in S^{n-i+1}} \frac{x^TAx}{x^Tx} + \max_{x \in S^{n-i+1}} \frac{x^THx}{x^Tx} \right) \\
&\leq \min_{S^{n-i+1}} \left(\max_{x \in S^{n-i+1}} \frac{x^TAx}{x^Tx} + \theta_1 \right) \\
&= \alpha_i + \theta_1
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lambda_i &= \max_{S^i} \min_{x \in S^i} \frac{x^T(A+H)x}{x^Tx} \\
&\geq \max_{S^i} \left(\min_{x \in S^i} \frac{x^TAx}{x^Tx} + \min_{x \in S^i} \frac{x^THx}{x^Tx} \right) \\
&\geq \max_{S^i} \left(\min_{x \in S^i} \frac{x^TAx}{x^Tx} + \theta_n \right) \\
&= \alpha_i + \theta_n
\end{aligned}$$

This proves that

$$\alpha_i + \theta_n \leq \lambda_i \leq \alpha_i + \theta_1$$

If H is in addition symmetric positive definite, then $\theta_n > 0$ and consequently

$$\alpha_i < \alpha_i + \theta_n \leq \lambda_i$$

Problem 3 (Question 5.14)

Note that

$$\begin{bmatrix} 1 & 0 \\ x & I_n \end{bmatrix} \begin{bmatrix} 1 & -y^T \\ 0 & I_n + xy^T \end{bmatrix} = \begin{bmatrix} 1 + x^T y & -y^T \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & I_n \end{bmatrix}$$

Using block determinant formula and the fact the the determinant of an upper triangular matrix is the product of diagonals, we see that

$$\det(I_n)\det(I_n + xy^T) = \det((1 + x^T y)I_n)\det(I_n) \implies \det(I_n + xy^T) = 1 + x^T y$$

Problem 4 (Problem 5.18)

We construct the polar decomposition explicitly. Let $A = U\Sigma V^T$ be the SVD of A, then consider

$$A = (UV^T)(V\Sigma V^T)$$

Clearly, UV^T is orthogonal and $V\Sigma V^T$ is positive semidefinite.

We now prove uniqueness. Suppose $A = QP$ is nonsingular, then $A^T A$ is also nonsingular, meaning that

$$(A^T A)^{\frac{1}{2}} = P$$

Is also nonsingular. Since A and P are both nonsingular, it follows that P^{-1} exists and

$$Q = AP^{-1}$$

Exists uniquely. This proves uniqueness.