PRECONDITIONERS FOR NONDEFINITE HERMITIAN TOEPLITZ SYSTEMS[*]

RAYMOND H. CHAN[†], DANIEL POTTS[‡], AND GABRIELE STEIDL[§]

Abstract. This paper is concerned with the construction of circulant preconditioners for Toeplitz systems arising from a piecewise continuous generating function with sign changes.

If the generating function is given, we prove that for any $\varepsilon > 0$, only $\mathcal{O}(\log N)$ eigenvalues of our preconditioned Toeplitz systems of size $N \times N$ are not contained in $[-1-\varepsilon, -1+\varepsilon] \cup [1-\varepsilon, 1+\varepsilon]$. The result can be modified for trigonometric preconditioners. We also suggest circulant preconditioners for the case that the generating function is not explicitly known and show that only $\mathcal{O}(\log N)$ absolute values of the eigenvalues of the preconditioned Toeplitz systems are not contained in a positive interval on the real axis.

Using the above results, we conclude that the preconditioned minimal residual method requires only $\mathcal{O}(N \log^2 N)$ arithmetical operations to achieve a solution of prescribed precision if the spectral condition numbers of the Toeplitz systems increase at most polynomial in N. We present various numerical tests.

Key words. nondefinite Toeplitz matrices, circulant matrices, Krylov space methods, preconditioners

AMS subject classifications. 65F10, 65F15, 65T50

PII. S0895479899362521

1. Introduction. Let $\mathcal{L}_{2\pi}$ be the space of 2π-periodic Lebesgue integrable real-valued functions and let $C_{2\pi}$ be the subspace of 2π-periodic real-valued continuous functions with norm

$$\|f\|_\infty := \max_{t \in [-\pi, \pi]} |f(t)| (f \in C_{2\pi}).$$

The Fourier coefficients of $f \in \mathcal{L}_{2\pi}$ are given by

$$a_k = a_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \mathrm{e}^{-\mathrm{i}kt} \,\mathrm{d}t (k \in \mathbb{Z}),$$

and the sequence $\{\boldsymbol{A}_N(f)\}_{N=1}^\infty$ of (N, N)-Toeplitz matrices generated by f is defined by

$$\boldsymbol{A}_N = \boldsymbol{A}_N(f) := (a_{j-k}(f))_{j,k=0}^{N-1}.$$

Since $f \in \mathcal{L}_{2\pi}$ is real-valued, the matrices $\boldsymbol{A}_N(f)$ are Hermitian.

We are interested in the iterative solution of Toeplitz systems

$$(1.1) \qquad \boldsymbol{A}_N(f)\,\boldsymbol{x} = \boldsymbol{b},$$

[†]Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong (rchan@math.cuhk.edu.hk).

[‡]Medizinische Universität Lübeck, Institut für Mathematik, Wallstraße 40, D-23560 Lübeck, Germany (potts@math.mu-luebeck.de).

[§]Universität Mannheim, Fakultät für Mathematik und Informatik, D-68131 Mannheim, Germany (steidl@math.uni-mannheim.de).

where the generating function $f \in \mathcal{L}_{2\pi}$. To be more precise, we are looking for good preconditioning strategies so that Krylov space methods applied to the preconditioned system converge in a small number of iteration steps. Note that by the Toeplitz structure of $\boldsymbol{A}_N$ each iteration step requires only $\mathcal{O}(N \log N)$ arithmetical operations by using fast Fourier transforms.

Preconditioning techniques for Toeplitz systems have been well studied in the past 10 years. However, most of the papers in this area are concerned with the case where the generating function f is either positive or nonnegative; see, for instance, [5, 3, 20, 7, 17, 10] and the references therein. In this paper, we consider f that has sign changes. The method we propose here will also work for generating functions that are positive or nonnegative.

Up to now iterative methods for Toeplitz systems with generating functions having different signs were only considered in [20, 19, 22, 24] and in connection with non-Hermitian systems in [8, 6]. In [8], we have constructed circulant preconditioners for non-Hermitian Toeplitz matrices with known generating function of the form

$$f = p\,h,$$

where p is an arbitrary trigonometric polynomial and h is a function from the Wiener class with $|h| > 0$. We proved that the preconditioned matrices have *singular values* properly clustered at 1. Then, if the *spectral condition number* of $\boldsymbol{A}_N(f)$ fulfills $\kappa_2(\boldsymbol{A}_N(f)) = \mathcal{O}(N^\alpha)$, the *conjugate gradient (CG) method* applied to the normal equation requires only $\mathcal{O}(\log N)$ iteration steps to produce a solution of fixed precision. However, in general, nothing can be said about the eigenvalues of the preconditioned matrix.

In this paper, we consider real-valued functions $f \in \mathcal{L}_{2\pi}$ of the form

$$(1.2) \qquad\qquad\qquad\qquad f = p_s h,$$

where

$$(1.3) \qquad\qquad p_s(t) := \prod_{j=1}^{\mu} (2 - 2\cos(t - t_j))^{s_j}, \ \ s := \sum_{j=1}^{\mu} s_j$$

is a trigonometric polynomial with a finite number of zeros $t_j \in [-\pi, \pi)$ $(j = 1, \dots, \mu)$ of even order $2s_j$ and where $h \in \mathcal{L}_{2\pi}$ is a piecewise continuous function with simple discontinuities at ξ_j $(j = 1, \dots, \nu)$, i.e., there exist $h(\xi_j \pm 0)$ and $h(\xi_j + 0) - h(\xi_j - 0) = \alpha_j \neq 0$. For simplicity let $h(\xi_j) = (h(\xi_j - 0) + h(\xi_j + 0))/2$. Further, we assume that

$$(1.4) \qquad\qquad \overline{\{|h(t)| : t \in [-\pi, \pi); \ |h(t)| > 0\}} \subseteq [h_-, h_+],$$

where $0 < h_- \leq h_+ < \infty$. In particular, we are interested in the Heaviside function h.

A similar setting was also considered in [20]. Serra Capizzano suggested the application of band-Toeplitz preconditioners $\boldsymbol{A}_N(p_s)$ in combination with CG applied to the normal equation. He proved, beyond a more general result which cannot directly be used for preconditioning, that at most $o(N)$ eigenvalues of the preconditioned matrix $\boldsymbol{A}_N(p_s)^{-1}\boldsymbol{A}_N(f)$ have absolute values not contained in a positive interval on the real axis.

The same author suggested in [19] a preconditioning method based on the Sherman–Morrison–Woodbery formula and some kind of normal equation for generating functions with zeros of odd order.

A result with $o(N)$ outliers was also obtained in [23], where the application of preconditioned GMRES was examined.

In the following, we construct circulant preconditioners for the *minimal residual method* (MINRES). Note that preconditioned MINRES avoids the transformation of the original system to the normal equation but requires Hermitian positive definite preconditioners. Then, the preconditioned matrices are again Hermitian, so that the absolute values of their eigenvalues coincide with their singular values. If the generating function is given, we prove that for any $\varepsilon > 0$, only $\mathcal{O}(\log N)$ singular values of the preconditioned matrices are not contained in $[1 - \varepsilon, 1 + \varepsilon]$. We also construct circulant preconditioners for the case that the generating function of the Toeplitz matrices is not explicitly known. For this, we use positive reproducing kernels with special properties previously applied by the authors in [17, 10] and show that $\mathcal{O}(\log N)$ singular values of the preconditioned matrices are not contained in a positive interval on the real axis. Then, if in addition $\kappa_2(\boldsymbol{A}_N(f)) = \mathcal{O}(N^\alpha)$, preconditioned MINRES converges in at most $\mathcal{O}(\log N)$ iteration steps. In summary, the proposed algorithm requires only $\mathcal{O}(N \log^2 N)$ arithmetical operations.

Note that the theoretical verification of the above assumption on the condition number of $\boldsymbol{A}_N(f)$ is not straightforward. See [4] for examples of banded indefinite Toeplitz matrices with exponentially (or even faster) growing condition numbers.

This paper is organized as follows. In section 2, we introduce circulant preconditioners for (1.1) under the assumption that the generating function of the sequence of Toeplitz matrices is known and prove clustering results for the eigenvalues of the preconditioned matrices. Section 3 deals with the construction of preconditioners if the generating function of the Toeplitz matrices is not explicitly known. In section 4, we modify the results of section 2 with respect to trigonometric preconditioners. The convergence of MINRES applied to our preconditioned Toeplitz systems is considered in section 5. Finally, we present numerical results in section 6.

2. Circulant preconditioners involving generating functions. First we introduce some basic notation. By $\boldsymbol{R}_N(M)$ we denote arbitrary (N, N)-matrices of rank *at most* M. Let $\boldsymbol{M}_N(g)$ be the circulant (N, N)-matrix

$$\boldsymbol{M}_N(g) := \boldsymbol{F}_N \, \mathrm{diag} \left(g\left(\frac{2\pi l}{N} \right) \right)_{l=0}^{N-1} \boldsymbol{F}_N^*,$$

where $\boldsymbol{F}_N$ denotes the Nth *Fourier matrix*

$$\boldsymbol{F}_N := \frac{1}{\sqrt{N}} \left(\mathrm{e}^{-2\pi \mathrm{i} jk/N} \right)_{j,k=0}^{N-1}$$

and where $\boldsymbol{F}^*$ is the transposed complex conjugate matrix of $\boldsymbol{F}$. For a trigonometric polynomial $q(t) := \sum_{k=-n_1}^{n_2} q_k \mathrm{e}^{\mathrm{i}kt}$, the matrices $\boldsymbol{A}_N(q)$ and $\boldsymbol{M}_N(q)$ are related by

$$(2.1) \qquad \boldsymbol{A}_N(q) = \boldsymbol{M}_N(q) + \boldsymbol{R}_N(n_1 + n_2)$$

(see [15]). For a function g with a finite number of zeros we define the set $I_N(g)$ by

$$I_N(g) := \left\{ l = 0, \dots, N - 1 : g\left(\frac{2\pi l}{N} \right) \neq 0 \right\}$$

and the points $x_{N,l}(g)$ $(l = 0, \dots, N - 1)$ by

$$x_{N,l}(g) := \begin{cases} \frac{2l\pi}{N} & \text{if} \quad l \in I_N(g), \\ \frac{2\tilde{l}\pi}{N} & \text{otherwise}, \end{cases}$$

where $\tilde{l} \in \{0, \dots, N-1\}$ is the next higher index to l so that $\tilde{l} \in I_N(g)$. For N large enough we can simply choose $\tilde{l} = l+1 \mod N$. By $\boldsymbol{M}_{N,g}(f)$ we denote the circulant matrix

$$(2.2) \qquad \boldsymbol{M}_{N,g}(f) := \boldsymbol{F}_N \operatorname{diag}\left(f(x_{N,l}(g))\right)_{l=0}^{N-1} \boldsymbol{F}_N^*.$$

If g has m zeros, then we have by construction that

$$(2.3) \qquad \boldsymbol{M}_N(f) = \boldsymbol{M}_{N,g}(f) + \boldsymbol{R}_N(m).$$

Now assume that the sequence $\{\boldsymbol{A}_N(f)\}_{N=1}^{\infty}$ of nonsingular Toeplitz matrices is generated by a known piecewise continuous function $f \in \mathcal{L}_{2\pi}$ of the form (1.2)–(1.4). Then we suggest the Hermitian positive definite circulant matrix $\boldsymbol{M}_{N,f}(|f|)$ as preconditioner for MINRES.

We examine the distribution of the eigenvalues of $\boldsymbol{M}_{N,f}(|f|)^{-\frac{1}{2}}\boldsymbol{A}_N(f)\boldsymbol{M}_{N,f}(|f|)^{-\frac{1}{2}}$.

The following theorem is Lemma 10 of [26] written with respect to our notation.

THEOREM 2.1. *Let $h \in \mathcal{L}_{2\pi}$ be a piecewise continuous function having only simple discontinuities at $\xi_j \in [-\pi, \pi)$ $(j = 1, \dots, \nu)$. By $\mathcal{F}_N$ we denote the* Fejér kernel

$$(2.4) \qquad \mathcal{F}_N(t) := \sum_{k=-(N-1)}^{N-1} \left(1 - \left|\frac{k}{N}\right|\right) e^{ikt} = 1 + 2\sum_{k=1}^{N-1}\left(1 - \frac{k}{N}\right)\cos kt$$

$$(2.5) \qquad = \begin{cases} \frac{1}{N}\left(\sin\left(\frac{Nt}{2}\right)/\sin\left(\frac{t}{2}\right)\right)^2, & t \neq 0, \\ \frac{1}{N}, & t = 0, \end{cases}$$

and by $\mathcal{F}_N * h$ we denote the cyclic convolution *of $\mathcal{F}_N$ and h. Then, for any $\varepsilon > 0$, there exist constants $0 < c_1 \le c_2 < \infty$ independent of N so that the number $\nu(\varepsilon; \boldsymbol{A}_N)$ of eigenvalues of $\boldsymbol{A}_N(h) - \boldsymbol{M}_N(\mathcal{F}_N * h)$ with absolute value exceeding ε can be estimated by*

$$c_1 \log (N) \le \nu(\varepsilon; \boldsymbol{A}_N) \le c_2 \log (N).$$

In other words, we have by Theorem 2.1 that

$$(2.6) \qquad \boldsymbol{A}_N(h) = \boldsymbol{M}_N(\mathcal{F}_N * h) + \boldsymbol{V}_N + \boldsymbol{U}_N,$$

where $\boldsymbol{V}_N$ is a matrix of spectral norm $\le \varepsilon$ and where

$$c_1 \log N \le \operatorname{rank}(\boldsymbol{U}_N) \le c_2 \log N.$$

Using Theorem 2.1, we can prove the following lemma.

LEMMA 2.2. *Let $f = p_s h \in \mathcal{L}_{2\pi}$ be given by (1.2)–(1.4). Then, for any $\varepsilon > 0$ and sufficiently large N, the number of singular values of $\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{A}_N(h)\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}$ which are not contained in the interval $[1-\varepsilon, 1+\varepsilon]$ is $\mathcal{O}(\log N)$.*

Proof. By (2.6) and since the eigenvalues of $\boldsymbol{M}_{N,f}(|h|)$ are restricted from below by h_-, it remains to show that for any $\varepsilon > 0$ and sufficiently large N, except for $\mathcal{O}(\log N)$ eigenvalues, all eigenvalues of $\boldsymbol{M}_{N,f}(|h|)^{-1}\boldsymbol{M}_N(\mathcal{F}_N * h)$ have absolute values in $[1-\varepsilon, 1+\varepsilon]$. Indeed we will prove that there are only $\mathcal{O}(1)$ outliers.

For this we mainly follow the lines of proof of Gibb's phenomenon. Without loss of generality we assume that $h \in \mathcal{L}_{2\pi}$ has only one jump at $\xi_1 = 0$ of height α_1.

First we examine $\mathcal{F}_N * g$, where g is given by

$$g(x) := \begin{cases} \frac{1}{2}(\pi - x), & x \in (0, \pi), \\ \frac{1}{2}(-x - \pi), & x \in (-\pi, 0), \\ 0, & x = 0. \end{cases}$$

By (2.4) and since g has Fourier series

$$g(x) \sim \sum_{k=1}^{\infty} \frac{1}{k} \sin kx,$$

we obtain

$$\int_0^x \mathcal{F}_N(t)\, dt = x + 2 \sum_{k=1}^{N-1} \left(\frac{1}{k} - \frac{1}{N} \right) \sin kx = x + 2 \left(\mathcal{F}_N * g \right)(x),$$

and further by (2.5)

$$(\mathcal{F}_N * g)(x) = \frac{1}{2N} \int_0^x \left(\frac{\sin \frac{Nt}{2}}{\sin \frac{t}{2}} \right)^2 dt - \frac{x}{2}$$

$$= \frac{1}{2N} \int_0^x \left(\frac{\sin \frac{Nt}{2}}{\frac{t}{2}} \right)^2 dt + \frac{1}{2N} \int_0^x \left(\frac{1}{(\sin \frac{t}{2})^2} - \frac{1}{(\frac{t}{2})^2} \right) \left(\sin \frac{Nt}{2} \right)^2 dt - \frac{x}{2}$$

$$= \int_0^{\frac{Nx}{2}} \left(\frac{\sin t}{t} \right)^2 dt + \mathcal{O}(N^{-1}) - \frac{x}{2},$$

and by partial integration and definition of g

$$(\mathcal{F}_N * g)(x) - g(x) = \frac{-(\sin \frac{Nx}{2})^2}{\frac{Nx}{2}} + \mathrm{si}\,(Nx) - \frac{\pi}{2} + \mathcal{O}(N^{-1}) \quad (x \in (0, \pi)),$$

where $\mathrm{si}\,(y) := \int_0^y \frac{\sin t}{t}\, dt$. We are interested in the behavior of

$$(\mathcal{F}_N * g) \left(\frac{2\pi l}{N} \right) - g \left(\frac{2\pi l}{N} \right) = \mathrm{si}\,(2\pi l) - \frac{\pi}{2} + \mathcal{O}(N^{-1}) \quad \left(l = 0, \ldots, \left\lceil \frac{N}{2} \right\rceil - 1 \right).$$

Here $\lceil x \rceil$ denotes the smallest integer $\geq x$. It is well known that $\lim_{x \to \infty} \mathrm{si}\,(x) = \frac{\pi}{2}$. Thus, if $l = l(N) \to \infty$ for $N \to \infty$, then, for any $\varepsilon > 0$, there exists $N_0 = N_0(\varepsilon)$ so that

$$(2.7) \qquad \left| (\mathcal{F}_N * g) \left(\frac{2\pi l}{N} \right) - g \left(\frac{2\pi l}{N} \right) \right| < \frac{\pi h_-}{2\alpha_1} \varepsilon \quad \text{for all } N \geq N_0.$$

The same holds if we approach 0 from the left, i.e., if we consider $2\pi l/N$ for $l = \left\lceil \frac{N}{2} \right\rceil, \ldots, N - 1$.

Next we have by definition of g and h that

$$\tilde{h}(x) := h(x) - \frac{\alpha_1}{\pi} g(x)$$

is a continuous function. Since $\mathcal{F}_N$ is a reproducing kernel, for any $\varepsilon > 0$, there exists $\tilde{N}_0 = \tilde{N}_0(\varepsilon)$ so that for all $l \in \{0, \dots, N-1\}$

$$(2.8) \qquad \left| (\mathcal{F}_N * \tilde{h}) \left(\frac{2\pi l}{N} \right) - \tilde{h} \left(\frac{2\pi l}{N} \right) \right| < \frac{\varepsilon}{2} h_- \quad \text{for all } N \geq \tilde{N}_0.$$

Assume that $l = l(N) \to \infty$ for $N \to \infty$ ($l \in \{0, \dots, \lceil \frac{N}{2} \rceil - 1\}$). Then we obtain by (2.7) and (2.8) that for any $\varepsilon > 0$ there exists $N(\varepsilon) = \max(N_0, \tilde{N}_0)$ so that

$$\left| (\mathcal{F}_N * h) \left(\frac{2\pi l}{N} \right) - h \left(\frac{2\pi l}{N} \right) \right| \leq \left| (\mathcal{F}_N * \tilde{h}) \left(\frac{2\pi l}{N} \right) - \tilde{h} \left(\frac{2\pi l}{N} \right) \right|$$
$$+ \frac{\alpha_1}{\pi} \left| (\mathcal{F}_N * g) \left(\frac{2\pi l}{N} \right) - g \left(\frac{2\pi l}{N} \right) \right|,$$

$$\left| (\mathcal{F}_N * h) \left(\frac{2\pi l}{N} \right) - h \left(\frac{2\pi l}{N} \right) \right| \leq \varepsilon \, h_- \qquad \text{for all } N \geq N(\varepsilon),$$

and consequently, since $\left| h \left(\frac{2\pi l}{N} \right) \right| \geq h_- \quad (l \in I_N(f))$,

$$(2.9) \qquad 1 - \varepsilon \leq \frac{\left| (\mathcal{F}_N * h) \left(\frac{2\pi l}{N} \right) \right|}{\left| h \left(\frac{2\pi l}{N} \right) \right|} \leq 1 + \varepsilon \qquad (l \in I_N(f)).$$

Let $m \leq \mu + \nu$ denote the number of zeros of f which are equal to one of the points $2\pi l/N$ ($l = 0, \dots, N-1$). Then the set

$$\left\{ \frac{\left| (\mathcal{F}_N * h) \left(\frac{2\pi l}{N} \right) \right|}{\left| h \left(\frac{2\pi l}{N} \right) \right|} : l \in I_N(f) \right\}$$

contains at least $N - m$ absolute values of eigenvalues of $\boldsymbol{M}_{N,f}(|h|)^{-1} \boldsymbol{M}_N(\mathcal{F}_N * h)$ and we conclude by (2.9) that except for $\mathcal{O}(1)$ eigenvalues and sufficiently large N, all eigenvalues of $\boldsymbol{M}_{N,f}(|h|)^{-1} \boldsymbol{M}_N(\mathcal{F}_N * h)$ have absolute values contained in $[1 - \varepsilon, 1 + \varepsilon]$. This completes the proof.   □

Remark 2.3.   In a similar way as above we can prove that for any $\varepsilon > 0$ and N sufficiently large, the number of eigenvalues of $\boldsymbol{A}_N(h)$ with absolute values not in the interval $[h_- - \varepsilon, h_+]$ is $\mathcal{O}(\log N)$.

Note that the property that at most $o(N)$ eigenvalues of $\boldsymbol{A}_N(h)$ have absolute values not contained in $[h_- - \varepsilon, h_+]$ follows simply from the fact that the singular values of $\boldsymbol{A}_N(h)$ are distributed as $|h|$ [14, 23].   □

THEOREM 2.4.   *Let* $f = p_s h \in \mathcal{L}_{2\pi}$ *be given by* (1.2)–(1.4). *Then, for any* $\varepsilon > 0$ *and sufficiently large* N, *except for* $\mathcal{O}(\log N)$ *singular values, all singular values of*

$$\boldsymbol{M}_{N,f}(|f|)^{-\frac{1}{2}} \boldsymbol{A}_N(f) \boldsymbol{M}_{N,f}(|f|)^{-\frac{1}{2}}$$

are contained in $[1 - \varepsilon, 1 + \varepsilon]$.

Proof. The polynomial p_s in (1.3) can be rewritten as

$$p_s = p\bar{p},$$

where

$$p(t) := \prod_{j=1}^{\mu} (1 - e^{-it_j} e^{it})^{s_j}, \quad \sum_{j=1}^{\mu} s_j = s,$$

and $\bar{p}(t)$ is the complex conjugate of $p(t)$. By straightforward computation it is easy to check that

$$
\begin{aligned}
\boldsymbol{A}_N(f) = \boldsymbol{A}_N(p_s\, h) \;&=\; \boldsymbol{A}_N(p\, h\, \bar{p}) \\
&= \boldsymbol{A}_N(p\, h)\, \boldsymbol{A}_N(\bar{p}) \;+\; \boldsymbol{R}_N^c(s) \\
&= \boldsymbol{A}_N(p)\, \boldsymbol{A}_N(h)\, \boldsymbol{A}_N(\bar{p}) \;+\; \boldsymbol{R}_N^r(s)\, \boldsymbol{A}_N(\bar{p}) \;+\; \boldsymbol{R}_N^c(s) \\
&= \boldsymbol{A}_N(p)\, \boldsymbol{A}_N(h)\, \boldsymbol{A}_N(\bar{p}) \;+\; \boldsymbol{R}_N(2s),
\end{aligned}
$$

(2.10)

where only the first s columns (rows) of $\boldsymbol{R}_N^{c\,(r)}(s)$ are nonzero columns (rows).

Since $|f| = p\,\bar{p}\,|h|$ the eigenvalues of $\boldsymbol{M}_{N,f}(|f|)^{-1}\boldsymbol{A}_N(f)$ coincide with the eigenvalues of

$$
(2.11) \qquad \boldsymbol{B}_N(f) := \boldsymbol{M}_{N,f}(|h|)^{-1/2}\, \boldsymbol{M}_{N,f}(p)^{-1}\, \boldsymbol{A}_N(f)\boldsymbol{M}_{N,f}(\bar{p})^{-1}\, \boldsymbol{M}_{N,f}(|h|)^{-1/2}.
$$

Now we obtain by (2.10), (2.1), and (2.3) that

$$
\begin{aligned}
\boldsymbol{B}_N(f) = \;& \boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{M}_{N,f}(p)^{-1}\boldsymbol{A}_N(p)\boldsymbol{A}_N(h)\boldsymbol{A}_N(\bar{p})\boldsymbol{M}_{N,f}(\bar{p})^{-1}\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}} \\
& +\boldsymbol{R}_N(2s) \\
= \;& \boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{M}_{N,f}(p)^{-1}(\boldsymbol{M}_{N,f}(p) + \boldsymbol{R}_N(s+m))\boldsymbol{A}_N(h) \\
& \cdot\, (\boldsymbol{M}_{N,f}(\bar{p}) + \boldsymbol{R}_N(s+m))\boldsymbol{M}_{N,f}(\bar{p})^{-1}\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}} + \boldsymbol{R}_N(2s) \\
(2.12)\qquad = \;& \boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{A}_N(h)\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}} \;+\; \boldsymbol{R}_N(4s + 2m).
\end{aligned}
$$

By Lemma 2.2, for any $\varepsilon > 0$ and N sufficiently large, except for $\mathcal{O}(\log N)$ singular values, all singular values of $\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{A}_N(h)\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}$ are contained in $[1 - \varepsilon, 1 + \varepsilon]$. Now the assertion follows by (2.12) and Weyl's interlacing theorem [13, p. 184]. $\quad\square$

3. Circulant preconditioners involving positive kernels. In many applications we know only the entries $a_k(f)$ of the Toeplitz matrices $\boldsymbol{A}_N(f)$ and not the generating function itself. In this case, we use even positive reproducing kernels $K_N \in C_{2\pi}$. These are trigonometric polynomials of the form

$$
K_N(t) := c_{N,0} + 2\sum_{k=1}^{N-1} c_{N,k} \cos kt, \quad c_{N,k} = a_k(K_N) \in \mathbb{R}
$$

satisfying $K_N \geq 0$,

$$
(3.1) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\,\mathrm{d}t = 1,
$$

and the *reproducing property*

$$
\lim_{N\to\infty} \|f - K_N * f\|_\infty = 0 \quad \text{for all } f \in C_{2\pi}.
$$

Since

$$
(K_N * f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_N(x - t)\,\mathrm{d}t = \sum_{k=-(N-1)}^{N-1} a_k(f)\, c_{N,k}\, \mathrm{e}^{\mathrm{i}kx},
$$

the cyclic convolution of K_N and f is determined by the first N Fourier coefficients of f. As a preconditioner which can be constructed from the entries of $\boldsymbol{A}_N(f)$ without explicit knowledge of f we suggest the circulant matrix $\boldsymbol{M}_{N,K_N*f}(|K_N*f|)$.

In order to obtain a suitable distribution of the eigenvalues of the preconditioned matrices, we need kernels with a special property which is related to the order

$$\sigma := \max_{j=1,\dots,\mu} s_j$$

of the zeros of p_s.

The *generalized Jackson kernels* $\mathcal{J}_{m,N}$ of degree $\leq N-1$ are defined by

$$(3.2) \qquad K_{m,N}(t) = \mathcal{J}_{m,N}(t) := \lambda_{m,N} \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^{2m} \quad (m \in \mathbb{N}),$$

where $n := \lfloor \frac{N-1}{m} \rfloor + 1$ and where $\lambda_{m,N}$ is determined by (3.1). Here $\lfloor t \rfloor$ denotes the largest integer $\leq t$. In particular, we have that

$$\lambda_{m,N} \sim N^{1-2m},$$

i.e., there exist positive constants c_1, c_2 so that $c_1 N^{1-2m} \leq \lambda_{m,N} \leq c_2 N^{1-2m}$. See [11, pp. 203–204]. A possibility for the construction of the Fourier coefficients of $\mathcal{J}_{m,N}$ is prescribed in [10].

The *B-spline kernels* $\mathcal{B}_{m,N}$ of degree $\leq N-1$ are defined by

$$(3.3) \qquad K_{m,N}(t) = \mathcal{B}_{m,N}(t) := \frac{N}{m} \frac{1}{M_{2m}(0)} \sum_{r \in Z} \left(\mathrm{sinc}\left(\frac{N}{m} \left(\frac{t + 2\pi r}{2} \right) \right) \right)^{2m},$$

where M_m denotes the *centered cardinal B-spline* of order m and

$$\mathrm{sinc}\, t := \begin{cases} \frac{\sin t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

See [17, 9]. Since

$$\mathcal{B}_{m,N}(t) := 1 + \frac{2}{M_{2m}(0)} \sum_{k=1}^{N-1} M_{2m}\left(\frac{mk}{N} \right) \cos kt,$$

the Fourier coefficients of $\mathcal{B}_{m,N}$ are given by values of centered cardinal B-splines. Note that $\mathcal{J}_{1,N} = \mathcal{B}_{1,N}$ is just the Fejér kernel $\mathcal{F}_N$.

The above kernels have the following important property.

THEOREM 3.1. *Let* $f = p_s h \in \mathcal{L}_{2\pi}$ *be given by* (1.2)–(1.4). *Assume that for all* t_j $(j \in \{1,\dots,\mu\})$ *with* $t_j = \xi_k$ *for some* $k \in \{1,\dots,\nu\}$ *and* $\mathrm{sgn}\, h(\xi_k+0) \neq \mathrm{sgn}\, h(\xi_k-0)$ *there exists a neighborhood* $[t_j - \varepsilon_j, t_j + \varepsilon_j]$ $(\varepsilon_j > 0)$ *of* t_j *so that* f *is a monotone function in this neighborhood and moreover* $f(t_j - t) = -f(t_j + t)$ $(0 \leq t \leq \varepsilon_j)$. *Let* $K_N = K_{m,N}$ *be given by* (3.2) *or* (3.3), *where*

$$m \geq \sigma + 1.$$

Then there exist $0 < \alpha \leq \beta < \infty$ *so that for* $N \to \infty$, *except for* $\mathcal{O}(1)$ *points, all points of the set* $\{2\pi l/N : l \in I_N(f)\}$ *fulfill*

$$(3.4) \qquad \frac{1}{\beta} \leq \frac{|(K_N * f)(\frac{2\pi l}{N})|}{|f(\frac{2\pi l}{N})|} \leq \frac{1}{\alpha}.$$

Proof. (1) First we consider the upper bound. Since p_s and K_N are nonnegative, we obtain

$$|(K_N * f)(x)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(t)| \, p_s(t) \, K_N(x - t) \, dt$$

$$\leq h_+ \frac{1}{2\pi} \int_{-\pi}^{\pi} p_s(t) K_N(x - t) \, dt \;=\; h_+ \, (K_N * p_s)(x).$$

In [17, 10], we proved that $m \geq \sigma + 1$ implies that for all $x \in I_N(p_s) \supseteq I_N(f)$, there exists a constant $0 < c < \infty$ so that

$$\frac{(K_N * p_s)(x)}{p_s(x)} \leq c.$$

Thus, since $|h(x)| \geq h_-$ for $(x \in I_N(f))$, we obtain

$$\frac{|(K_N * f)(x)|}{|f(x)|} \leq \frac{h_+}{h_-} \frac{(K_N * p_s)(x)}{p_s(x)} \leq \frac{h_+}{h_-} c \qquad (x \in I_N(f)).$$

(2) Next we deal with the lower bound.

(2.1) Let $x \in I_N(f)$ be not in the neighborhood of t_j $(j = 1, \ldots, \mu)$, i.e., there exist $b_j > 0$ independent of N so that $|x - t_j| \geq b_j > 0$ $(j = 1, \ldots, \mu)$. Then $|f(x)| \geq c > 0$ for all $x \in I_N(f)$. Further, since K_N is a reproducing kernel and by using the same arguments as in the proof of Lemma 2.2 if x is in the neighborhood of some ξ_k $(k = 1, \ldots, \nu)$, we obtain that, for any $\varepsilon > 0$ there exists $N(\varepsilon)$, so that except for at most a constant number of points, all considered points $x \in I_N(f)$ satisfy

$$|(K_N * f)(x) - f(x)| \leq c\varepsilon \qquad (N \geq N(\varepsilon)),$$

and thus

$$\frac{|(K_N * f)(x)|}{|f(x)|} \geq 1 - \frac{c\varepsilon}{|f(x)|} \geq 1 - \varepsilon.$$

(2.2) It remains to consider the points $x = x(N) \in I_N(f)$ with $\lim_{N\to\infty} x(N) = t_j$ $(j = 1, \ldots, \mu)$.

For simplicity we assume that

$$p_s(t) = (2 - 2\cos t)^s = (2 \sin(t/2))^{2s},$$

i.e., p_s has only a zero of order $2s$ at $t_1 = 0$. Let $x = x(N) \in I_N(f)$ with

$$\lim_{N\to\infty} x(N) = 0.$$

For any fixed $0 < b < \pi$ we obtain

$$(K_N * f)(x) = \frac{1}{2\pi} \left(\int_{-b}^{b} f(t) K_N(x - t) \, dt + \int_{-\pi}^{-b} f(t) K_N(x - t) \, dt \right.$$

$$\left. + \int_{b}^{\pi} f(t) K_N(x - t) \, dt \right)$$

$$= \frac{1}{2\pi} \left(\int_{-b}^{b} f(t) K_N(x - t) \, dt + \int_{b+x}^{\pi+x} f(x - t) K_N(t) \, dt \right.$$

$$\left. + \int_{b-x}^{\pi-x} f(x + t) K_N(t) \, dt \right),$$

and since f is bounded

$$(K_N * f)(x) - \frac{1}{2\pi} \int_{-b}^{b} f(t) K_N(x - t) \, dt \ \sim \ \left(\int_{b+x}^{\pi+x} + \int_{b-x}^{\pi-x} \right) K_N(t) \, dt.$$

By definition of K_N we see that for any fixed $0 < \tilde{b} \leq \pi$

$$(3.5) \qquad \int_{\tilde{b}}^{\pi} K_N(t) \, dt \ \leq \ \text{const } N^{-2m+1},$$

so that we get for small x (e.g., $x < b/2$)

$$(3.6) \qquad (K_N * f)(x) = \frac{1}{2\pi} \int_{-b}^{b} f(t) K_N(x - t) \, dt \ + \ \mathcal{O}(N^{-2m+1}).$$

(2.2.1) Assume that h has no jump at $t_1 = 0$ with sign change. Then there exists $\varepsilon > 0$ so that $h(t) \geq h_-$ or $h(t) \leq -h_-$ for $t \in [-\varepsilon, \varepsilon]$. We restrict our attention to the case $h \geq h_-$. Since $0 < h_- \, p_s(t) \leq f(t) \leq h_+ \, p_s(t)$ $(t \in [-\varepsilon, \varepsilon])$ and p_s is monotone increasing on $(0, \pi)$, we obtain for $x(N) \in (0, \varepsilon) \cap I_N(f)$ and N sufficiently large that

$$\int_{-\varepsilon}^{\varepsilon} \frac{f(t)}{f(x(N))} K_N(t - x(N)) \, dt \geq \int_{x(N)}^{\varepsilon} \frac{f(t)}{f(x(N))} K_N(t - x(N)) \, dt$$

$$\geq \frac{h_-}{h_+} \int_{x(N)}^{\varepsilon} \frac{p_s(t)}{p_s(x(N))} K_N(t - x(N)) \, dt$$

$$(3.7) \qquad \geq \frac{h_-}{h_+} \int_{0}^{\varepsilon - x(N)} \frac{p_s(t)}{p_s(x(N))} K_N(t) \, dt \ \geq \ c$$

with a positive constant c independent of N. On the other hand, we have by definition of p_s and by assumption $s \leq m - 1$ that $f(x(N)) \geq h_- \, \tilde{c} N^{-2s} \geq h_- \, \tilde{c} N^{-2m+2}$. Then we obtain by (3.6) with $b = \varepsilon$ and (3.7) that for N large enough

$$\frac{(K_N * f)(x(N))}{f(x(N))} \geq \text{const}$$

with a positive constant const independent of N.

The proof for $x(N) \in (-\varepsilon, 0) \cap I_N(f)$ follows the same lines.

(2.2.2) Finally, we assume that h has a jump at $t_1 = 0$ with $\text{sgn } h(0 + 0) \neq \text{sgn } h(0 - 0)$. Without loss of generality let $h(0 + 0) > 0$. Then, by assumption on f, there exists $\varepsilon_1 > 0$ so that $h(t) = -h(-t)$ for $t \in [0, \varepsilon_1]$. Thus,

$$(3.8) \qquad \int_{-\varepsilon_1}^{\varepsilon_1} f(t) K_N(x - t) \, dt = \int_{0}^{\varepsilon_1} f(t) (K_N(t - x) - K_N(t + x)) \, dt.$$

We consider points of the form

$$y = y_k(N) := \frac{2\pi m}{N \, \gamma} k \qquad (k \in \mathbb{N})$$

with $\lim_{N \to \infty} y_k(N) = 0$, where $\gamma := mn/N$ in case of Jackson kernels and $\gamma := 1$ in case of B-spline kernels. Then we have for $t \in [0, \varepsilon_1]$ that

$$(3.9)$$

$$\mathcal{J}_{m,N}(t - y) - \mathcal{J}_{m,N}(t + y) = \lambda_{m,N} \left(\left(\frac{\sin(nt/2)}{\sin((t - y)/2)} \right)^{2m} - \left(\frac{\sin(nt/2)}{\sin((t + y)/2)} \right)^{2m} \right),$$

and consequently, for sufficiently small ε_1 and y, since sin is odd and monotone increasing on $(0, \pi/2)$ we have that

$$\mathcal{J}_{m,N}(t-y) - \mathcal{J}_{m,N}(t+y) > 0 \quad \text{for all } t \in (0, \varepsilon_1).$$

Further, by definition of the B-spline kernels

$$\mathcal{B}_{m,N}(t-y) - \mathcal{B}_{m,N}(t+y) = \mathcal{B}_{m,N}^0(t-y) - \mathcal{B}_{m,N}^0(t+y) + \mathcal{O}(N^{-2m+1}),$$

where $\mathcal{B}_{m,N}^0(t) := \frac{N}{m} \frac{1}{M_{2m}(0)} \left(\text{sinc}\left(\frac{N}{m}\frac{t}{2}\right)\right)^{2m}$, and similarly as in (3.9) we see that

$$\mathcal{B}_{m,N}^0(t-y) - \mathcal{B}_{m,N}^0(t+y) > 0 \quad \text{for all } t \in (0, \varepsilon_1).$$

By assumption h does not change the sign in $(0, \varepsilon_1)$. Then we obtain by (3.8), monotonicity of p_s in $(0, \pi)$ and $m \geq s + 1$ that

$$(3.10) \quad \int_{-\varepsilon_1}^{\varepsilon_1} \frac{f(t)}{f(y)} K_N(y-t) \, \mathrm{d}t \geq \frac{h_-}{h_+} \int_y^{\varepsilon_1} K_N^0(t-y) - K_N^0(t+y) \, \mathrm{d}t + \mathcal{O}(N^{-1}),$$

where $K_N^0 \in \{\mathcal{J}_{m,N}, \mathcal{B}_{m,N}^0\}$. Set $w = w(N) := \frac{2\pi m}{N\gamma}$. Then $y_k = y_k(N) = w\,k$ and there exist $r = r(N) \in \mathbb{N}$ $(r > k)$ so that $\varepsilon_1 = w\,r + \tilde{\varepsilon}_1$, where $0 \leq \tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(N) < w$. Now it follows that

$$\int_{y_k}^{wr} K_N^0(t-y_k) - K_N^0(t+y_k) \, \mathrm{d}t = \sum_{l=0}^{r-k-1} \int_{y_k+wl}^{y_k+w(l+1)} K_N^0(t-y_k) - K_N^0(t+y_k) \, \mathrm{d}t$$

$$= \sum_{l=0}^{2k-1} \int_{wl}^{w(l+1)} K_N^0(t) \, \mathrm{d}t - \sum_{l=r-k}^{r+k-1} \int_{wl}^{w(l+1)} K_N^0(t) \, \mathrm{d}t$$

$$\geq \int_0^w K_N^0(t) \, \mathrm{d}t - \int_{\varepsilon_1+y_k-w}^{\varepsilon_1+y_k} K_N^0(t) \, \mathrm{d}t,$$

and further by (3.5) and since $\lim_{N\to\infty} y_k = 0$,

$$\int_{y_k}^{\varepsilon_1} K_N^0(t-y_k) - K_N^0(t+y_k) \, \mathrm{d}t \geq \int_0^w K_N^0(t) \, \mathrm{d}t + \mathcal{O}(N^{-2m+1}).$$

Straightforward computation yields

$$\int_0^{2\pi m/(N\gamma)} K_N^0(t) \, \mathrm{d}t \geq \text{const} \int_0^\pi \left(\frac{\sin u}{u}\right)^{2m} \mathrm{d}u \geq \text{const}.$$

Hence we get for N large enough that

$$\int_{y_k}^{\varepsilon_1} K_N^0(t-y_k) - K_N^0(t+y_k) \, \mathrm{d}t \geq \text{const}$$

and by (3.10) that

$$(3.11) \qquad \int_{-\varepsilon_1}^{\varepsilon_1} \frac{f(t)}{f(y_k)} K_N(y_k - t) \, \mathrm{d}t \geq \text{const}$$

with positive constants const independent of N.

Now we consider $x(N) \in I_N(f)$ with $y_k(N) \leq x(N) < y_{k+1}(N)$.
Let $z(N) := x(N) - y_k(N) > 0$. Then

$$\int_{-\varepsilon_1}^{\varepsilon_1} f(t) \, K_N(t - x(N)) \, dt = \int_{-\varepsilon_1 - z(N)}^{\varepsilon_1 - z(N)} f(t + z(N)) \, K_N(t - y_k(N)) \, dt$$

$$= \int_{-\varepsilon_1}^{\varepsilon_1 - z(N)} f(t + z(N)) \, K_N(t - y_k(N)) \, dt$$

$$+ \int_{-\varepsilon_1 - z(N)}^{-\varepsilon_1} f(t + z(N)) \, K_N(t - y_k(N)) \, dt,$$

and since f is by assumption monotone increasing on $[-\varepsilon_1, \varepsilon_1]$

$$\int_{-\varepsilon_1}^{\varepsilon_1} f(t) \, K_N(t - x(N)) \, dt \geq \int_{-\varepsilon_1}^{\varepsilon_1 - z(N)} f(t) \, K_N(t - y_k(N)) \, dt$$

$$+ \int_{-\varepsilon_1}^{-\varepsilon_1 + z(N)} f(t) \, K_N(t - x(N)) \, dt$$

$$= \int_{-\varepsilon_1}^{\varepsilon_1} f(t) \, K_N(t - y_k(N)) \, dt$$

$$+ \int_{-\varepsilon_1}^{-\varepsilon_1 + z(N)} f(t) \, K_N(t - x(N)) \, dt$$

$$- \int_{\varepsilon_1 - z(N)}^{\varepsilon_1} f(t) \, K_N(t - y_k(N)) \, dt,$$

and by (3.5) and since f is bounded

$$(3.12) \quad \int_{-\varepsilon_1}^{\varepsilon_1} f(t) \, K_N(t - x(N)) \, dt \; \geq \; \int_{-\varepsilon_1}^{\varepsilon_1} f(t) \, K_N(t - y_k(N)) \, dt \; + \; \mathcal{O}(N^{-2m+1}).$$

By assumption $x(N) = \zeta \, y_k(N) \; (0 < \zeta < 2)$. Thus

$$\frac{\int_{-\varepsilon_1}^{\varepsilon_1} f(t) \, K_N(t - x(N)) \, dt}{f(x(N))} \; \geq \; \mathrm{const} \, \frac{\int_{-\varepsilon_1}^{\varepsilon_1} f(t) \, K_N(t - y_k(N)) \, dt}{f(y_k(N))},$$

and since $f(y_k(N) \geq \mathrm{const} \, N^{-2s}$ and $m \geq s + 1$ we obtain by (3.12), (3.11) that for N large enough

$$\int_{-\varepsilon_1}^{\varepsilon_1} f(t) \, K_N(t - x(N)) \, dt \, / \, f(x(N)) \; \geq \; \mathrm{const}$$

with a nonnegative constant const independent of N. Finally, we use (3.6) with $b = \varepsilon_1$ and again $m \geq s + 1$ to finish the proof. $\square$

To show our main result we also need the following lemma.

LEMMA 3.2. *Let $\boldsymbol{A} \in \mathbb{C}^{N,N}$ be a Hermitian positive definite matrix having $N - n_1$ eigenvalues in $[a_-, a_+]$, where $0 < a_- \leq a_+ < \infty$. Let $\boldsymbol{B} \in \mathbb{C}^{N,N}$ be a Hermitian matrix with $N - n_2$ singular values in $[b_-, b_+]$, where $0 < b_- \leq b_+ < \infty$. Then at least $N - 4n_1 - n_2$ eigenvalues of $\boldsymbol{A} \, \boldsymbol{B}$ are contained in $[-a_+ b_+, -a_- b_-] \cup [a_- b_-, a_+ b_+]$.*

Proof. (1) Assume first that $n_1 = 0$, i.e., $\boldsymbol{A}$ has only eigenvalues in $[a_-, a_+]$. Let $\lambda_j(\boldsymbol{B})$ denote the jth eigenvalue of the matrix $\boldsymbol{B}$. We consider the eigenvalues of

$\boldsymbol{B} - t\boldsymbol{A}^{-1}$ with respect to $t \in \boldsymbol{R}$. By Weyl's interlacing theorem (see [13, p. 184]) we obtain for $t \geq 0$ that

$$(3.13) \qquad \lambda_j(\boldsymbol{B}) - \frac{t}{a_-} \leq \lambda_j\left(\boldsymbol{B} - t\boldsymbol{A}^{-1}\right) \leq \lambda_j(\boldsymbol{B}) - \frac{t}{a_+}$$

and for $t < 0$ that

$$(3.14) \qquad \lambda_j(\boldsymbol{B}) - \frac{t}{a_+} \leq \lambda_j\left(\boldsymbol{B} - t\boldsymbol{A}^{-1}\right) \leq \lambda_j(\boldsymbol{B}) - \frac{t}{a_-}.$$

Let $\lambda_j(\boldsymbol{B}) \in [-b_+, -b_-]$. Then we obtain by (3.13) and (3.14) that $\lambda_j\left(\boldsymbol{B} - t\boldsymbol{A}^{-1}\right) < 0$ for all $t > -a_- b_-$. On the other hand, we see by (3.13) and (3.14) that $\lambda_j\left(\boldsymbol{B} - t\boldsymbol{A}^{-1}\right) > 0$ for all $t < -a_+ b_+$. Thus, since $\lambda_j\left(\boldsymbol{B} - t\boldsymbol{A}^{-1}\right) = \lambda_j(t)$ is a continuous function in $t \in \boldsymbol{R}$, there exists $t_j \in [-a_+ b_+, -a_- b_-]$ such that $\lambda_j\left(\boldsymbol{B} - t_j \boldsymbol{A}^{-1}\right) = 0$. This implies that $t_j \in [-a_+ b_+, -a_- b_-]$ is an eigenvalue of $\boldsymbol{AB}$. Consequently, every $\lambda_j(\boldsymbol{B}) \in [-b_+, -b_-]$ corresponds to an eigenvalue $t_j \in [-a_+ b_+, -a_- b_-]$ of $\boldsymbol{AB}$. (Eigenvalues are called with multiplicities.)

The examination of $\lambda_j(\boldsymbol{B}) \in [a_- b_-, a_+ b_+]$ follows the same lines.

In summary, $N - n_2$ eigenvalues of $\boldsymbol{AB}$ are contained in $[-a_+ b_+, -a_- b_-] \cup [a_- b_-, a_+ b_+]$.

(2) Let n_1 eigenvalues of $\boldsymbol{A}$ be outside $[a_-, a_+]$. Then, since $\boldsymbol{A}$ is positive definite, the matrix can be split as

$$(3.15) \qquad \boldsymbol{A}^{1/2} = \tilde{\boldsymbol{A}}^{1/2} + \boldsymbol{R}(n_1),$$

where $\tilde{\boldsymbol{A}}^{1/2}$ is Hermitian with all eigenvalues in $[a_-^{1/2}, a_+^{1/2}]$ and $\boldsymbol{R}(n_1)$ is a Hermitian matrix of rank n_1. The eigenvalues of $\boldsymbol{AB}$ coincide with the eigenvalues of $\boldsymbol{A}^{1/2}\boldsymbol{B}\boldsymbol{A}^{1/2}$. Hence it remains to show that at most $4n_1 + n_2$ singular values of $\boldsymbol{A}^{1/2}\boldsymbol{B}\boldsymbol{A}^{1/2}$ are not contained in $[a_- b_-, a_+ b_+]$. By (3.15) we have

$$\boldsymbol{A}^{1/2}\boldsymbol{B}\boldsymbol{A}^{1/2} = \tilde{\boldsymbol{A}}^{1/2}\boldsymbol{B}\tilde{\boldsymbol{A}}^{1/2} + \boldsymbol{R}(2n_1),$$
$$(3.16) \qquad \left(\boldsymbol{A}^{1/2}\boldsymbol{B}\boldsymbol{A}^{1/2}\right)^2 = \left(\tilde{\boldsymbol{A}}^{1/2}\boldsymbol{B}\tilde{\boldsymbol{A}}^{1/2}\right)^2 + \boldsymbol{R}(4n_1).$$

By (1) all but n_2 singular values of $\tilde{\boldsymbol{A}}^{1/2}\boldsymbol{B}\tilde{\boldsymbol{A}}^{1/2}$ are contained in $[a_- b_-, a_+ b_+]$. Then (3.16) and Weyl's interlacing theorem yield the assertion. $\square$

THEOREM 3.3. *Let $f = p_s h \in \mathcal{L}_{2\pi}$ be given by (1.2)–(1.4). Assume that for all t_j $(j \in \{1, \dots, \mu\})$ with $t_j = \xi_k$ for some $k \in \{1, \dots, \nu\}$ and $\mathrm{sgn}\, h(\xi_k + 0) \neq \mathrm{sgn}\, h(\xi_k - 0)$ there exists a neighborhood $[t_j - \varepsilon_j, t_j + \varepsilon_j]$ $(\varepsilon_j > 0)$ of t_j so that f is a monotone function in this neighborhood and moreover $f(t_j - t) = -f(t_j + t)$ $(0 \leq t \leq \varepsilon_j)$. Let $K_N = K_{m,N}$ be given by (3.2) or (3.3), where*

$$m \geq \sigma + 1.$$

By α, β we denote the constants from Theorem 3.1.

Then, for any $\varepsilon > 0$ and sufficiently large N, except for $\mathcal{O}(\log N)$ singular values, all singular values of $\boldsymbol{M}_N(|K_N * f|)^{-\frac{1}{2}}\boldsymbol{A}_N(f)\boldsymbol{M}_N(|K_N * f|)^{-\frac{1}{2}}$ are contained in $[\alpha - \varepsilon, \beta + \varepsilon]$.

Proof. Let $\boldsymbol{B}_N(f)$ be defined by (2.11). Then we obtain by (2.12) that

$$(3.17) \quad \boldsymbol{M}_{N,K_N*f}(|K_N*f|)^{-\frac{1}{2}} \boldsymbol{A}_N(f) \boldsymbol{M}_{N,K_N*f}(|K_N*f|)^{-\frac{1}{2}}$$

$$= \boldsymbol{M}_{N,K_N*f}(|K_N*f|)^{-\frac{1}{2}} \boldsymbol{M}_{N,f}(p) \boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}} \boldsymbol{B}_N(f)$$

$$\cdot \boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}} \boldsymbol{M}_{N,f}(\bar{p}) \boldsymbol{M}_{N,K_N*f}(|K_N*f|)^{-\frac{1}{2}}$$

$$= \boldsymbol{M}_{N,K_N*f}(|K_N*f|)^{-\frac{1}{2}} \boldsymbol{M}_{N,f}(p) \boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}}$$

$$\cdot \boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}} \boldsymbol{A}_N(h) \boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}$$

$$(3.18) \qquad \cdot \boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}} \boldsymbol{M}_{N,f}(\bar{p}) \boldsymbol{M}_{N,K_N*f}(|K_N*f|)^{-\frac{1}{2}} + \boldsymbol{R}(4s+2m).$$

The distribution of the eigenvalues of $\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}} \boldsymbol{A}_N(h) \boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}$ is known by Lemma 2.2. It remains to examine the eigenvalues of the Hermitian positive definite matrix

$$\boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}} \boldsymbol{M}_{N,f}(\bar{p}) \boldsymbol{M}_{N,K_N*f}(|K_N*f|)^{-1} \boldsymbol{M}_{N,f}(p) \boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}}.$$

These eigenvalues coincide with the reciprocal eigenvalues of $\boldsymbol{M}_{N,f}(|f|)^{-1} \boldsymbol{M}_{N,K_N*f}$ $(|K_N*f|)$. By definition of $\boldsymbol{M}_{N,g}$ and since K_N is a reproducing kernel, except for $\mathcal{O}(1)$ eigenvalues, all eigenvalues of $\boldsymbol{M}_{N,f}(|f|)^{-1} \boldsymbol{M}_{N,K_N*f}(|K_N*f|)$ are given by $|(K_N*f)(2\pi l/N)|/|f(2\pi l/N)|$ $(l \in I_N(f))$. Thus, by Theorem 3.1, for $N \to \infty$ only $\mathcal{O}(1)$ eigenvalues of $\boldsymbol{M}_{N,f}(|f|) \boldsymbol{M}_{N,K_N*f}(|K_N*f|)^{-1}$ are not contained in $[\alpha, \beta]$. Consequently, by (3.18), Lemma 2.2, Lemma 3.2, and Weyl's interlacing theorem at most $\mathcal{O}(\log N)$ singular values of $\boldsymbol{M}_{N,K_N*f}(|K_N*f|)^{-\frac{1}{2}} \boldsymbol{A}_N(f) \boldsymbol{M}_{N,K_N*f}(|K_N*f|)^{-\frac{1}{2}}$ are not contained in $[\alpha - \varepsilon, \beta + \varepsilon]$. □

4. Trigonometric preconditioners. In addition to section 2, we suppose that the Toeplitz matrices $\boldsymbol{A}_N \in \mathbb{R}^{N,N}$ are symmetric, i.e., the generating function $f \in \mathcal{L}_{2\pi}$ is even. This suggests the application of so-called trigonometric preconditioners. Note that in the symmetric case the multiplication of a vector with $\boldsymbol{A}_N$ can be realized using *fast trigonometric transforms* instead of fast Fourier transforms (see [15]). In this way complex arithmetic can be completely avoided in the iterative solution of (1.1). This is one of the reasons to look for preconditioners which can be diagonalized by trigonometric matrices corresponding to fast trigonometric transforms instead of the Fourier matrix $\boldsymbol{F}_N$.

In practice, four discrete sine transforms (DST I–IV) and four discrete cosine transforms (DCT I–IV) were used (see [25]). Any of these eight trigonometric transforms can be realized with $\mathcal{O}(N \log N)$ arithmetical operations. Likewise, we can define preconditioners with respect to any of these transforms.

In this paper, we restrict our attention to the so-called discrete cosine transform of type II (DCT-II) and discrete sine transform of type II (DST-II), which are determined by the following transform matrices:

$$\text{DCT–II}: \quad \boldsymbol{C}_N^{II} := \left(\frac{2}{N}\right)^{1/2} \left(\epsilon_j^N \cos \frac{j(2k+1)\pi}{2N}\right)_{j,k=0}^{N-1} \in \mathbb{R}^{N,N},$$

$$\text{DST–II}: \quad \boldsymbol{S}_N^{II} := \left(\frac{2}{N}\right)^{1/2} \left(\epsilon_{j+1}^N \sin \frac{(j+1)(2k+1)\pi}{2N}\right)_{j,k=0}^{N-1} \in \mathbb{R}^{N,N},$$

where $\epsilon_k^N := 2^{-1/2} (k = 0, N)$ and $\epsilon_k^N := 1 (k = 1, \dots, N-1)$. We propose the

preconditioners

$$\text{DCT} - \text{II}: \quad \boldsymbol{M}_{N,f}(|f|, \boldsymbol{C}_N^{II}) \quad := \quad (\boldsymbol{C}_N^{II})' \operatorname{diag}(|f(\tilde{x}_{N,l})|)_{l=0}^{N-1} \ \boldsymbol{C}_N^{II},$$

$$\text{DST} - \text{II}: \quad \boldsymbol{M}_{N,f}(|f|, \boldsymbol{S}_N^{II}) \quad := \quad (\boldsymbol{S}_N^{II})' \operatorname{diag}(|f(\tilde{x}_{N,l})|)_{l=1}^{N} \ \boldsymbol{S}_N^{II},$$

where

$$\tilde{x}_{N,l} := \begin{cases} \frac{l\pi}{N} & \text{if} \quad f\left(\frac{l\pi}{N}\right) \neq 0, \\ \frac{\tilde{l}\pi}{N} & \text{otherwise} \end{cases}$$

and where $\tilde{l} \in \{0, \dots, N-1\}$ is the next higher index to l such that $|f(\tilde{x}_{N,l})| > 0$. (See [16].)

Then we can prove in a completely similar way as in section 2 that for any $\varepsilon > 0$ and sufficiently large N except for $\mathcal{O}(\log N)$ singular values, all singular values of

$$\boldsymbol{M}_{N,f}(|f|, \boldsymbol{O})^{-\frac{1}{2}} \boldsymbol{A}_N(f) \boldsymbol{M}_{N,f}(|f|, \boldsymbol{O})^{-\frac{1}{2}} \quad (\boldsymbol{O} \in \{\boldsymbol{S}_N^{II}, \boldsymbol{C}_N^{II}\})$$

are contained in $[1 - \varepsilon, 1 + \varepsilon]$.

5. Convergence of preconditioned MINRES. In order to prescribe the convergence behavior of preconditioned MINRES with our preconditioners of the previous sections, we have to estimate the smaller outliers for increasing N.

LEMMA 5.1. *Let $f \in \mathcal{L}_{2\pi}$ be defined by (1.2)–(1.4). Assume that $\kappa_2(\boldsymbol{A}_N(f)) = \mathcal{O}(N^\alpha)$ $(\alpha > 0)$. Then the smallest absolute values of the eigenvalues of*

$$\boldsymbol{M}_{N,f}(|f|)^{-1} \boldsymbol{A}_N(f)$$

and

$$\boldsymbol{M}_{N,K_N*f}(|K_N * f|)^{-1} \boldsymbol{A}_N(f)$$

behave for $N \to \infty$ as $\mathcal{O}(N^{-\alpha})$.

Proof. Since

$$\|\boldsymbol{A}_N(f)^{-1} \boldsymbol{M}_{N,f}(|f|)\|_2 \leq \frac{\|\boldsymbol{M}_{N,f}(|f|)\|_2}{\|\boldsymbol{A}_N(f)\|_2} \kappa_2(\boldsymbol{A}_N(f)),$$

$$\|\boldsymbol{A}_N(f)^{-1} \boldsymbol{M}_{N,K_N*f}(|K_N * f|)\|_2 \leq \frac{\|\boldsymbol{M}_{N,K_N*f}(|K_N * f|)\|_2}{\|\boldsymbol{A}_N(f)\|_2} \kappa_2(\boldsymbol{A}_N(f)),$$

and both $\|\boldsymbol{M}_{N,f}(|f|)\|_2$ and $\|\boldsymbol{M}_{N,K_N*f}(|K_N * f|)\|_2$ are restricted from above, it remains to show that there exists a constant $c > 0$ independent of N so that

$$\|\boldsymbol{A}_N(f)\|_2 > c.$$

The above inequality follows immediately from the fact that the singular values of $\boldsymbol{A}_N(f)$ are distributed as $|f|$ (see [14, 23]). $\square$

We want to combine our knowledge of the distribution of the eigenvalues of our preconditioned matrices with results concerning the convergence of MINRES.

THEOREM 5.2. *Let* $\boldsymbol{A} \in \mathbb{C}^{N,N}$ *be a Hermitian matrix with* p *and* q *isolated large and small singular values, respectively:*

$$0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_q < a \leq \sigma_{q+1} \leq \cdots \sigma_{N-p} \leq b$$
$$< \sigma_{N-p+1} \leq \sigma_{N-p+2} \leq \cdots \leq \sigma_N \quad (0 < a \leq b < \infty).$$

Let $\nu(k) := 0$ *if* $k - p - q \equiv 0 \bmod 2$ *and* $\nu(k) := 1$ *otherwise. Then MINRES requires for the solution of* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$$k \leq 2 \left(\ln \frac{2}{\tau} + \sum_{k=1}^{q} \ln \left(1 + \frac{b}{\sigma_k} \right) + p \ln 2 \right) \Big/ \left(\ln \frac{1 + (\frac{a}{b})}{1 - (\frac{a}{b})} \right) + p + q + \nu(k)$$

iteration steps to achieve precision τ, *i.e.,* $\frac{||\boldsymbol{r}^{(k)}||_2}{||\boldsymbol{r}^{(0)}||_2} \leq \tau$, *where* $\boldsymbol{r}^{(k)} := \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^{(k)}$ *and* $x^{(k)}$ *is the kth iterate.*

The theorem can be proved by using the same technique as in [1, pp. 569–573]. Namely, based on the known estimate

$$\frac{||\boldsymbol{r}^{(k)}||_2}{||\boldsymbol{r}^{(0)}||_2} \leq \min_{p_k \in \Pi_k^0} \max_{\lambda_j} |p_k(\lambda_j)|,$$

where Π_k^0 denotes the space of polynomials of degree $\leq k$ with $p_k(0) = 1$ and λ_j are the eigenvalues of $\boldsymbol{A}$, we choose p_k as the product of the linear polynomials passing through the $p + q$ outliers and the modified Chebyshev polynomials

$$T_{\lfloor (k-p-q)/2 \rfloor} \left(1 + 2 \frac{a^2 - x^2}{b^2 - a^2} \right) \Big/ T_{\lfloor (k-p-q)/2 \rfloor} \left(1 + 2 \frac{a^2}{b^2 - a^2} \right).$$

The above summand $p \ln 2$ can be further reduced if we use polynomials of higher degree for the larger outliers.

Note that a similar estimate can be given for the CG method applied to the normal equation $\boldsymbol{A}^* \boldsymbol{A} \boldsymbol{x} = \boldsymbol{A}^* \boldsymbol{b}$. Here we need

$$k \leq \left(\ln \frac{2}{\tau} + \sum_{k=1}^{q} \ln \left(\frac{b}{\sigma_k^2} \right) \right) \Big/ \left(\ln \frac{1 + (\frac{a}{b})}{1 - (\frac{a}{b})} \right) + p + q$$

iteration steps to achieve precision $\frac{||\boldsymbol{e}^{(k)}||_{\boldsymbol{A}}}{||\boldsymbol{e}^{(0)}||_{\boldsymbol{A}}} \leq \tau$, where $\boldsymbol{e}^{(k)} := \boldsymbol{x}_* - \boldsymbol{x}^{(k)}$. Note that the latter method requires two matrix-vector multiplications in each iteration step.

By Theorem 2.4, Theorem 3.3, and Lemma 5.1 our preconditioned MINRES with both preconditioners $\boldsymbol{M}_{N,f}(|f|)$ and $\boldsymbol{M}_{N,K_N * f}(|K_N * f|)$ produces a solution of (1.1) of prescribed precision in $\mathcal{O}(\log N)$ iteration steps and with $\mathcal{O}(N \log^2 N)$ arithmetical operations. The same holds for preconditioned CG applied to the normal equation.

6. Numerical results. In this section, we test our circulant and trigonometric preconditioners in connection with different iterative methods on a SGI O2 work station. As transform length we use $N = 2^n$, as right-hand side $\boldsymbol{b}$ of (1.1) we use the vector consisting of N entries "1," and as start vector we use the zero vector.

We begin with a comparison of MINRES applied to

(6.1) $$\boldsymbol{M}_{N,f}(|f|, \boldsymbol{O})^{-1} \boldsymbol{A}_N(f) \boldsymbol{x} = \boldsymbol{M}_{N,f}(|f|, \boldsymbol{O})^{-1} \boldsymbol{b},$$

TABLE 1
$f(t) = h_1(t)\, t^2 \quad h_1(t) = (t^2 + 1)\, sgn\,(t) \quad (t \in [-\pi, \pi)).$

Method	$\boldsymbol{M}_{N,f}$	4	5	6	7	8	9	10
MINRES	$\boldsymbol{I}_N$	23	71	277	*	*	*	*
MINRES	$\boldsymbol{M}_{N,f}(|f|, \boldsymbol{F}_N)$	15	17	17	19	21	23	23
MINRES	$\boldsymbol{M}_{N,\mathcal{F}_N * f}(|\mathcal{F}_N * f|, \boldsymbol{F}_N)$	19	31	35	41	43	47	51
MINRES	$\boldsymbol{M}_{N,\mathcal{B}_{2,N} * f}(|\mathcal{B}_{2,N} * f|, \boldsymbol{F}_N)$	19	23	23	25	25	27	29
CGNE	$\boldsymbol{I}_N$	11	37	164	*	*	*	*
CGNE	$\boldsymbol{M}_{N,f}(|f|, \boldsymbol{F}_N)$	8	8	9	9	9	10	10

where $\boldsymbol{O} \in \{\boldsymbol{F}_N, \boldsymbol{C}_N^{II}, \boldsymbol{S}_N^{II}\}$ and CGNE (Craig's method) (cf. [18, p. 239]) applied to

(6.2)
$$(\boldsymbol{M}_{N,f}(|f|, \boldsymbol{O})^{-\frac{1}{2}}\, \boldsymbol{A}_N(f)\, \boldsymbol{M}_{N,f}(|f|, \boldsymbol{O})^{-\frac{1}{2}})\, (\boldsymbol{M}_{N,f}(|f|, \boldsymbol{O})^{\frac{1}{2}}\, \boldsymbol{x}) = \boldsymbol{M}_{N,f}(|f|, \boldsymbol{O})^{-\frac{1}{2}}\, \boldsymbol{b}.$$

For both algorithms we have used MATLAB implementations of Fischer (see also [12]). In particular, his implementation of preconditioned MINRES avoids the splitting (6.2).

In order to make the following computations with MINRES and CGNE comparable, we have stopped both computations if

$$||\boldsymbol{b} - \boldsymbol{A}_N \boldsymbol{x}^{(k)}||_2 \,/\, ||\boldsymbol{b}||_2 < 10^{-7}.$$

Example 1. We begin with Hermitian Toeplitz matrices $\boldsymbol{A}_N(f)$ arising from the generating function

$$f_1(t) = h_1(t)\, t^2 \quad \text{with } h_1(t) = (t^2 + 1)\, \text{sgn}\,(t) \quad (t \in [-\pi, \pi)).$$

Table 1 presents the number of iterations for circulant preconditioners. The first row of the table contains the exponent n of the transform length $N = 2^n$. According to Theorem 2.4 and Theorem 5.2, the preconditioners $\boldsymbol{M}_N(|f|, \boldsymbol{F}_N)$ lead to very good results. As expected, the preconditioners $\boldsymbol{M}_{N,K_N * f}(|K_N * f|, \boldsymbol{F}_N)$ with the Fejér kernels $K_N = \mathcal{F}_N$ are not suitable for (1.1) (cf. also [17]), while the preconditioners with $K_N = \mathcal{B}_{2,N}$ do their job.

Further, CGNE needs half the number of iterations but twice the number of matrix-vector multiplications per iteration as MINRES needs. See also section 5.

Example 2. Next, we consider the symmetric Toeplitz matrices $\boldsymbol{A}_N(f)$ arising from the generating function

$$f_2(t) = h_2(t)\, (\cos(t + 2) + 1)\, (\cos(t - 2) + 1)$$

with

$$h_2(t) = \text{sgn}(t - \pi + 2)\, \text{sgn}(t + \pi - 2).$$

Table 2 presents the number of iterations for trigonometric preconditioners. The results are similar to those of Example 1, except that CGNE requires nearly the same number of iterations as MINRES.

TABLE 2
$f_2(t) = h_2(t)\,(\cos(t+2)+1)\,(\cos(t-2)+1)$ $(t \in [-\pi, \pi))$.

Method	M_N	4	5	6	7	8	9	10
MINRES	I_N	9	17	45	142	401	*	*
MINRES	$M_{N,f}(|f|, C_N^{II})$	8	9	10	11	14	13	16
MINRES	$M_{N,f}(|f|, S_N^{II})$	9	10	11	12	14	13	16
MINRES	$M_{N,\mathcal{F}_N*f}(|\mathcal{F}_N*f|, C_N^{II})$	10	15	20	26	30	39	53
MINRES	$M_{N,\mathcal{F}_N*f}(|\mathcal{F}_N*f|, S_N^{II})$	10	15	19	25	30	39	53
MINRES	$M_{N,\mathcal{B}_{2,N}*f}(|\mathcal{B}_{2,N}*f|, C_N^{II})$	9	15	17	16	20	18	18
MINRES	$M_{N,\mathcal{B}_{2,N}*f}(|\mathcal{B}_{2,N}*f|, S_N^{II})$	9	14	16	18	19	18	18
CGNE	I_N	10	29	99	413	*	*	*
CGNE	$M_{N,f}(|f|, C_N^{II})$	7	9	11	11	17	16	17
CGNE	$M_{N,f}(|f|, S_N^{II})$	7	7	10	10	12	14	15

TABLE 3
$f_3(t) = ((\frac{t}{\pi})^2 - 1)^2 - 0.9$ $(t \in [-\pi, \pi))$.

Method	M_N	4	5	6	7	8	9	10
MINRES	I_N	9	17	33	66	133	*	*
MINRES	$M_{N,f}(|f|, C_N^{II})$	6	7	7	8	7	7	7
MINRES	$M_{N,f}(|f|, S_N^{II})$	7	8	8	7	9	8	8
MINRES	$M_{N,\mathcal{F}_N*f}(|\mathcal{F}_N*f|, C_N^{II})$	8	11	15	17	16	17	17
MINRES	$M_{N,\mathcal{F}_N*f}(|\mathcal{F}_N*f|, S_N^{II})$	8	11	15	16	15	15	15
MINRES	$M_{N,\mathcal{B}_{2,N}*f}(|\mathcal{B}_{2,N}*f|, C_N^{II})$	8	10	10	11	9	7	7
MINRES	$M_{N,\mathcal{B}_{2,N}*f}(|\mathcal{B}_{2,N}*f|, S_N^{II})$	8	10	10	10	9	9	8
CGNE	I_N	8	22	65	164	378	*	*
CGNE	$M_{N,f}(|f|, C_N^{II})$	5	6	6	8	6	5	6
CGNE	$M_{N,f}(|f|, S_N^{II})$	6	6	6	6	7	7	7

Example 3. Finally, we consider an example from [21, Table 4.9]:

$$f_3(t) = \left(\left(\frac{t}{\pi}\right)^2 - 1\right)^2 - 0.9 \quad \left(t \in [-\pi, \pi)\right).$$

This generating function doesn't fit into our setting (1.2). However, our preconditioning technique leads to very good practical results; see Table 3. For $N = 512$ the preconditioned MINRES with our preconditioner $M_{N,\mathcal{B}_{2,N}*f}(|\mathcal{B}_{2,N}*f|, C_N^{II})$ requires only 7 iterations. For comparison, PCG with the banded Toeplitz preconditioner of *bandwidth* 37 suggested in [21] requires 2 iterations. The theoretical justification of these numerical results is part of further research.

Acknowledgments. We wish to thank B. Fischer for the MATLAB implementations of MINRES and CGNE. Many thanks to the referees for their valuable com-

ments, in particular for pointing out various references to us.

REFERENCES

[1] O. Axelsson, *Iterative Solution Methods*, Cambridge University Press, Cambridge, UK, 1996.

[2] R. Barrett, M. W. Berry, T. F. Chan, J. Demmel, J. Donato, J. Dongarra, V. Eijkhout, R. Pozo, C. Romine, and H. van der Vorst, *Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods,* 2nd Edition, SIAM, Philadelphia, PA, 1994.

[3] F. Di Benedetto, G. Fiorentino, and S. Serra, *C. G. preconditioning for Toeplitz matrices*, Comput. Math. Appl., 25 (1993), pp. 35–45.

[4] A. Böttcher and S. M. Grudsky, *Toeplitz band matrices with exponentially growing condition numbers*, Electron. J. Linear Algebra, 5 (1999), pp. 104–125.

[5] R. Chan, *Toeplitz preconditioners for Toeplitz systems with nonnegative generating functions*, IMA J. Numer. Anal., 11 (1991), pp. 333–345.

[6] R. H. Chan and W.-K. Ching, *Toeplitz–circulant preconditioners for Toeplitz systems and their applications to queueing networks with batch arrivals*, SIAM J. Sci. Comput., 17 (1996), pp. 762–772.

[7] R. H. Chan and M. K. Ng, *Conjugate gradient methods for Toeplitz systems*, SIAM Rev., 38 (1996), pp. 427–482.

[8] R. H. Chan, D. Potts, and G. Steidl, *Preconditioners for Non-Hermitian Toeplitz Systems*, Numer. Linear Algebra Appl., to appear.

[9] R. H. Chan, T. Tso, and H. Sun, *Circulant preconditioners from B-splines*, in Proceedings of the SPIE Symposium, Advanced Signal Processing: Algorithms, Architectures, and Implementations VII, Vol. 3162, San Diego, CA, 1997, F. Luk, ed., Society of Photo-optical Instrumentation Engineers, 1997, pp. 338–347.

[10] R. H. Chan, M. Yip, and M. Ng, *The Best Circulant Preconditioners for Hermitian Toeplitz Matrices*, SIAM J. Numer. Anal., 38 (2000), pp. 876–896.

[11] R. DeVore and G. Lorentz, *Constructive Approximation*, Springer–Verlag, Berlin, Germany, 1993.

[12] B. Fischer, *Polynomial Based Iteration Methods for Symmetric Linear Systems*, Wiley–Teubner, New York, 1996.

[13] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1985.

[14] S. V. Parter, *On the distribution of singular values of Toeplitz matrices*, Linear Algebra Appl., 80 (1986), pp. 115–130.

[15] D. Potts and G. Steidl, *Optimal trigonometric preconditioners for nonsymmetric Toeplitz systems*, Linear Algebra Appl., 281 (1998), pp. 265–292.

[16] D. Potts and G. Steidl, *Preconditioners for ill–conditioned Toeplitz matrices*, BIT, 39 (1999), pp. 513–533.

[17] D. Potts and G. Steidl, *Preconditioners for Ill–Conditioned Toeplitz Matrices Constructed from Positive Kernels*, SIAM J. Sci. Comput., to appear.

[18] Y. Saad, *Iterative Methods for Sparse Linear Systems*, PWS Publishers, Boston, MA, 1996.

[19] S. Serra, *New PCG based algorithms for the solution of Hermitian Toeplitz systems*, Calcolo, 32 (1995), pp. 153–176.

[20] S. Serra, *Preconditioning strategies for Hermitian Toeplitz systems with nondefinite generating functions*, SIAM J. Matrix Anal. Appl., 17 (1996), pp. 1007–1019.

[21] S. Serra Capizzano, *Toeplitz preconditioners constructed from linear approximation processes*, SIAM J. Matrix Anal. Appl., 20 (1998), pp. 446–465.

[22] S. Serra, *How to choose the best iterative strategy for symmetric Toeplitz systems*, SIAM J. Numer. Anal., 36 (1999), pp. 1078–1103.

[23] E. E. Tyrtyshnikov, *A unifying approach to some old and new theorems on distribution and clustering*, Linear Algebra Appl., 232 (1996), pp. 1–43.

[24] E. E. Tyrtyshnikov, A. Yeremin, and N. Zamarashkin, *Clusters – preconditioners – convergence*, Linear Algebra Appl., 263 (1997), pp. 25–48.

[25] Z. Wang, *Fast algorithms for the discrete W transform and for the discrete Fourier transform*, IEEE Trans. Acoust. Speech Signal Process, 32 (1984), pp. 803–816.

[26] M.-C. Yeung and R. H. Chan, *Circulant preconditioners for Toeplitz matrices with piecewise continuous generating functions*, Math. Comp., 61 (1993), pp. 701–718.