Mathematics Analysis and Approaches

Internal Assessment

Modeling the Growth of Rabbits

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1.INTRODUCTION

As I grew up, one of my memories was visiting my grandfather's farm. There are different kinds of animals on the farm but rabbits fascinated me the most. They were everywhere and I was wondering how they seemed to multiply so quickly, almost like magic. As I spent more time around them, I became curious about how their number grew. I began to notice a strange yet beautiful pattern. If you start with just two rabbits and imagine how many there might be in a month, then two months, then three. The population doesn't increase randomly, there's a sequence that starts small but grows rapidly in a way that feels almost alive. But because I was still young I had a lot of confusion and I didn't understand well what was going on. Later, as I grew up, I started to study about different sequences that reminded me of my Grandfather's farm. So, I decided to figure out the sequence that was behind the growth of rabbits.So, starting from the growth of rabbits, I'll explain the Fibonacci sequence and get to know its proof and others get used to it.

2.AIM

During Maths HL class, we were taught different sequences such as Arithmetic and Geometric sequences. But I need to understand how the growth of rabbits is modeled by the Fibonacci sequence . That is defined as:

$F_n = F_{n-1} + F_{n-2}$ where $n > 1$ and $F_0 = 0$, $F_1 = 1$

This shown:

F_n is the number of rabbits in a certain time.

F_{n-1} is the number of rabbits in the first previous time

F_{n-2} is the number of rabbits in the second previous time

F_0 is the number of rabbits at the initial time

F_1 is the number of rabbits after a certain increase in the first time

So, the aim of this IA is to understand how the growth of rabbits can be modeled by the Fibonacci sequence.

3.Fibonacci Sequence

The Fibonacci sequence is a series of numbers where each number is the sum of the two preceding ones, usually starting with 0 and 1. The sequence begins: 0,1,1,2,3,5,8,13,21, and so forth. This mathematical concept was introduced by An Italian Mathematician, Leonardo of Pisa (Fibonacci) in his book “Liber Abaci” published in 1202.

It can be defined recursively as follows:

$F_n = F_{n-1} + F_{n-2}$ where $n > 1$ and $F_0 = 0$, $F_1 = 1$

Where; F_n is the term to find

F_{n-1} is the previous first term to what you want to find.

F_{n-2} is the previous second term to what you want to find.

F_0 is the initial value.

F_1 Is the first value.

Let us find the solution to this sequence as it is a recurrence relation.

The approach to solving such recurrence is to assume that:

$F_n = r^n$ also $F_{n-1} = r^{n-1}$, $F_{n-2} = r^{n-2}$, $F_{n-3} = r^{n-3}$,... where r is a constant and $r \neq 0$ which is to be determined.

Substituting this into the recurrence relation gives:

$F_n = F_{n-1} + F_{n-2} \Rightarrow r^n = r^{n-1} + r^{n-2}$

Dividing both sides by r^{n-2} we get:

$\frac{r^n}{r^{n-2}} = \frac{r^{n-1}}{r^{n-2}} + \frac{r^{n-2}}{r^{n-2}} \Leftrightarrow r^2 = r + 1$

By rearranging this we get the characteristic equation:

$r^2 - r - 1 = 0$

Solving this using the quadratic formula:

$r = \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}$

Simplifying:

$r = \frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2}$

Therefore, the two roots are:

$$r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$$

To remind that the general solution to the recurrence relation is a linear combination of the two independent solutions r_1^n and r_2^n:

$$F_n = Ar_1^n + Br_2^n$$

Where A and B are constants to be determined by the initial conditions.

So, now we use the initial conditions $F_0 = 0$ and $F_1 = 1$ to find A and B

For n=0, $F_o = 0 = Ar_1^0 + Br_2^0 = A + B$ .....Equation 1

For n=1, $F_1 = 1 = Ar_1 + Br_2$ …..Equation 2

Then let us solve for A and B

From Equation 1, we have: $B =- A$

Substituting this into Equation 2:

$$Ar_1 - Ar_2 = 1$$

Factoring out A:

$$A(r_1 - r_2) = 1$$

Now, using the values of r_1 and r_2 :

$$r_1 - r_2 = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \sqrt{5}$$

So:

$$A\sqrt{5} = 1 \Leftrightarrow A = c$$

Therefore, from $B =- A$:

$$B =- \frac{1}{\sqrt{5}}$$

Substituting the values of A and B into the general solution:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

This formula provides the n-th Fibonacci number directly without needing to compute all the previous Fibonacci numbers.

So far we’ve seen that a fibonacci term can be written in that way using recurrence relation; then let’s use induction to prove that it works for all terms.

For n=1 thus

$$F_1 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}}$$

$$= \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}}$$

$$= \frac{\frac{1+\sqrt{5}-1+\sqrt{5}}{2}}{\sqrt{5}}$$

$$= \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}}$$

$$= \frac{\sqrt{5}}{\sqrt{5}}$$

$$= 1$$

Suppose that

$$F_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} \text{ to mean that } F_{k-1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

And then $F_{k+1} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$

Also $F_{k+1} = F_k + F_{k-1}$

$$F_k + F_{k-1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$= \frac{\left(\left(\frac{1+\sqrt{5}}{2}\right)+1\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\left(\frac{1-\sqrt{5}}{2}\right)+1\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$= \frac{\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^2\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$$

$$= F_{k+1}$$

Hence, Proven !!!!

4.Growth of Rabbits

Let us try to model the growth of Rabbits using the Fibonacci sequence.

In order to find it let us assume:

1.A single newly born pair of rabbits (one male, one female) are put in field;

2.Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits;

3.Rabbits in this field will not die and mating pair always produces one new pair (one male, one female) every month from the second month on.

Let us abbreviate these rules by using “W” for a pair of adult rabbits and “w’ for a pair of baby rabbits. And then let us try to find the pattern.

In the first month, we’re told that there is one pair of rabbits.

w

In the second month, the baby rabbits have grown into adults. One pair still.

W

In the third month, the first pair produced a pair of baby rabbits. Two pairs total.

W w

In the fourth month, the adult pair has produced another baby pair, and the previous baby pair has grown up. Three pairs.

W w W

In the fifth month, both adult pairs have generated baby pairs, and the previous baby pair has grown up. Five pairs.

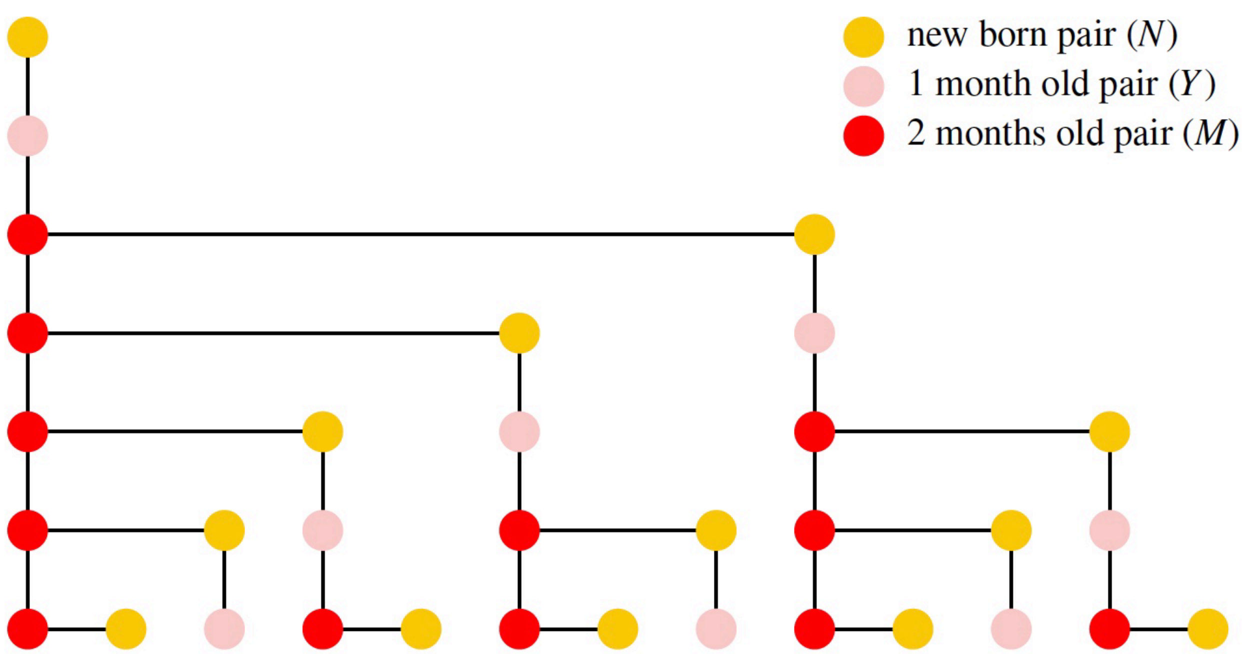
W w   W W w

In the sixth month, all three adult pairs generate pairs and both baby pairs grow up, and we've eight pairs.

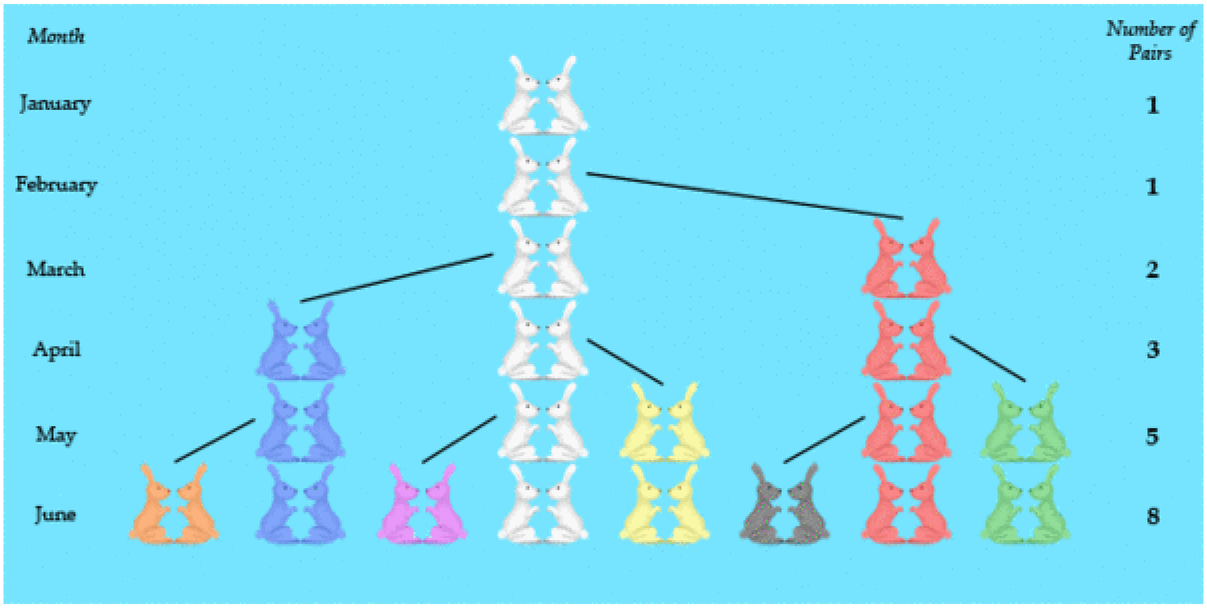
W w W W w W w W

And so on.

By trying to represent it graphically in order to understand it well as :



Where each row represents a month and the graph or follow chart goes from the top to the bottom. So, let us try to count them again using our picture in order to get it well:



In the picture, the black arrows show after mating the new born pair.:

Observation: - Each baby pair comes from an adult pair in the month before it.

- Also each adult pair comes from the baby pair in the month before it.

- The sum of total rabbits in a certain month is equal to the sum of rabbits previous 2 months.

Let us return back again in our following chart and see where those observations come from. Let us align them as following:

w

W

W w

W w W

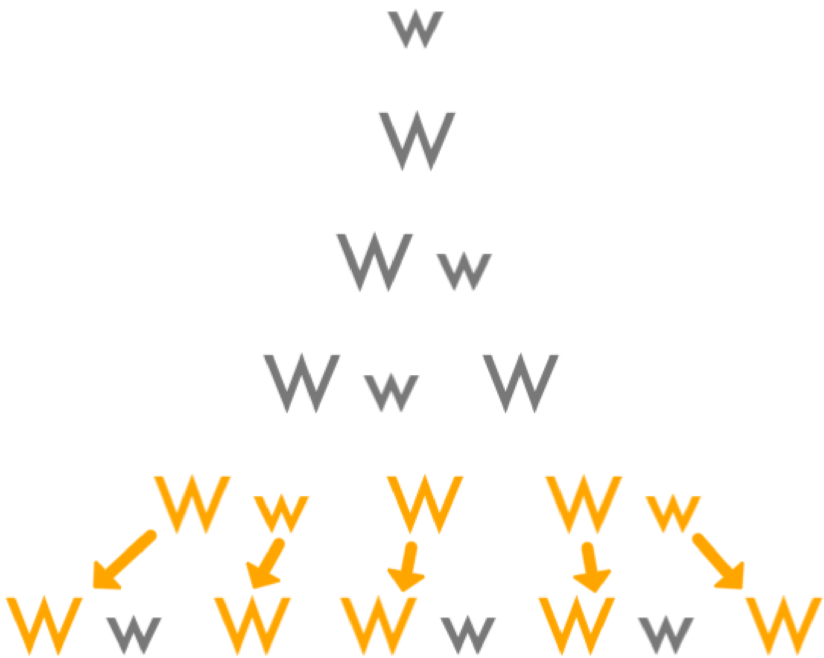
W w W W w

W w W W w W w W

Each row represents rabbits in each month from the Top as January to Bottom as June.

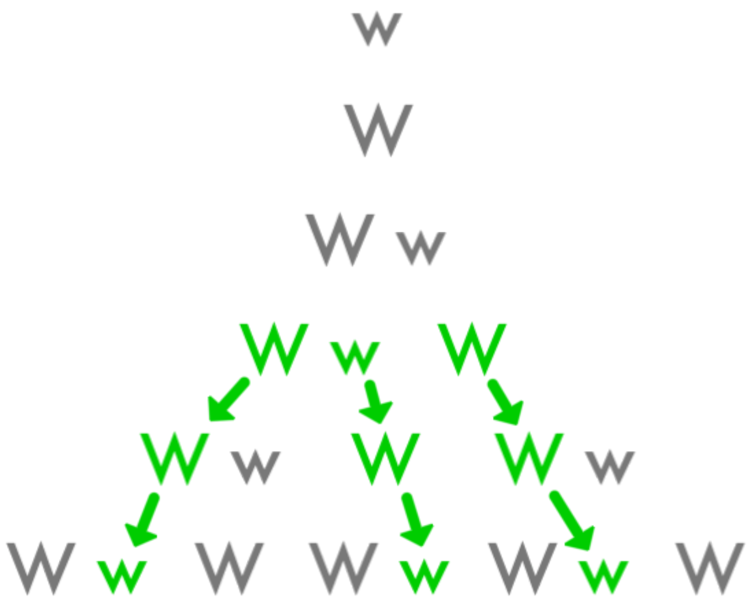
Let us investigate deeply. We’ll need to use the fact that each generation consists of some number of pairs of adult rabbits and some number of pairs of baby rabbits.

Let’s start with the adult pairs. We know that they’re adults, so by definition they must have had time to grow up. Specifically, we know that they must have been around for at least one month. For example, all the adult rabbits in June (Sixth row) had to have existed in May (Fifth row). As :



But when we draw it out like this, we see something else as well: Every rabbit pair in May (Fifth row) is an adult by June (Sixth row). Either it's an adult pair that's going to still be an adult pair or it's a baby pair that's going to grow up to be an adult pair. Therefore, we realize something important: This one-to-one correspondence means that we can count the number of rabbits in June by counting the total number of rabbit pairs in May.

It tells us that each baby pair comes from an adult pair in the generation before it. And we also know that every adult pair in that generation before is going to produce a baby pair in the next generation. So we can count the number of baby pairs in any generation by the number of adult pairs in the previous generation. As shown below:



Okay, so we count the number of adult rabbit pairs by counting the total numbers of rabbit pairs one generation before, and we count the number of baby rabbit pairs by counting the total number of pairs two generations before. So to find the total number of rabbit pairs we add those two categories. And we end up adding two consecutive generations to get the next one. Which gives us a sequence known as the Fibonacci sequence where : $F_n = F_{n-1} + F_{n-2}$ where F_n is the new term, F_{n-1} is the previous term and F_{n-2} is the previous 2 terms.

So we can then summarize our calculations for the number of rabbits in the whole year in the table as :

Month	J	F	M	A	M	J	J	A	S	O	N	D
New-Born	1	0	1	1	2	3	5	8	13	21	34	55
Adult	0	1	1	2	3	5	8	13	21	34	55	89
Total	1	1	2	3	5	8	13	21	34	55	89	144

5.The Golden Ratio

So two positive numbers x and y where x > y > 0 are said to be in the golden ratio if the ratio between the larger number and the smaller number is the same as the ratio between their sm and the larger number, that is ; $\frac{x}{y} = \frac{x+y}{x}$

So we take $\frac{x}{y} = \Phi$ to be the golden ratio (Φ is the capital Greek letter Phi), which turns our relation to: $\Phi = 1 + \frac{1}{\Phi} \Leftrightarrow \Phi^2 - \Phi - 1 = 0$

By solving that quadratic equation we get that $\Phi = \frac{\sqrt{5}+1}{2} = 1.618033988749895$

Also the negative of the negative root of the quadratic equation is what we call the golden ratio conjugate φ ,(the small Greek letter phi)and is equal to

$\varphi = \frac{\sqrt{5}-1}{2} = 0.6180339887$

The relation between Golden ratio and golden ratio conjugate is given by:

$\varphi = \Phi - 1$ or using $\varphi = \frac{1}{\Phi}$

6.Relation between Fibonacci numbers and the Golden ratio

Let us consider again the recursion relation for the Fibonacci number:

$$F_{n+1} = F_n + F_{n-1}$$

Dividing by F_n on both sides yields :

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$$ …….. Call it equation 1

Let us assume that the ratio of two consecutive Fibonacci numbers approaches a limit as n approaches infinity and call that limit a.

Define $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = a$

Also $\lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{1}{a}$ for we're getting a limit of a smaller number divided by a larger number. Therefore, when we find a limit of whole equation 1 on both sides we'll get;

$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n} \Rightarrow a = 1 + \frac{1}{a}$ ……. Call it equation 2

So, equation 2 is the same identity as satisfied by the golden ratio. To mean that $a = \Phi$. Therefore, if the limits exists, the ratio of two consecutive Fibonacci numbers must approach the golden ratio for large n, that is,

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \Phi$$

Let us show the value of first 10 ratio of consecutive Fibonacci numbers in the table below:

n	F_{n+1}/F_n	Value	F_{n+1}/F_n - Φ
1	1/1	1.0000	-0.6180
2	2/1	2.0000	0.3820
3	3/2	1.5000	-0.1180
4	5/3	1.6667	0.0486
5	8/5	1.6000	-0.0180
6	13/8	1.6250	0.0070
7	21/13	1.6154	-0.0026
8	34/21	1.6190	0.0010
9	55/34	1.6176	-0.0004
10	89/55	1.6182	0.0001

From above table we can see that we're getting convergence slowly by slowly to the golden ration where it is going positive-negative where F_{n+1}/F_n - Φ is getting closer to 0.

7.Binet's Formula

Binet's formula was invented and developed by a French Mathematician, Jacques Philippe Marie Binet in 1843.It is a mathematical equation that provides a direct way to calculate any Fibonacci number, F_n directly based on its position in the sequence, using the golden ratio.

So, as we've seen it earlier, let us write recursion term of Fibonacci sequence as follow;

$$F_{n+1} = F_n + F_{n-1} \Leftrightarrow F_{n+1} = c_1\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + c_2\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

And then remember that: $\frac{1+\sqrt{5}}{2} = \Phi$ and $\frac{1-\sqrt{5}}{2} =- \varphi$

So our recurrence relation turns into:

$$F_{n+1} = c_1\Phi^{n+1} + c_2(-\varphi)^{n+1}$$

For n=0, $F_o = 0 = c_1 + c_2$

For n=1, $F_1 = 1 = c_1\Phi + c_2(-\varphi)$

Let us find those constants using that : $F_0 = 0$ and $F_1 = 1$

By solving those set of two equations we get that :

$\Rightarrow c_1 = \frac{1}{\sqrt{5}}$ and $c_2 =- \frac{1}{\sqrt{5}}$

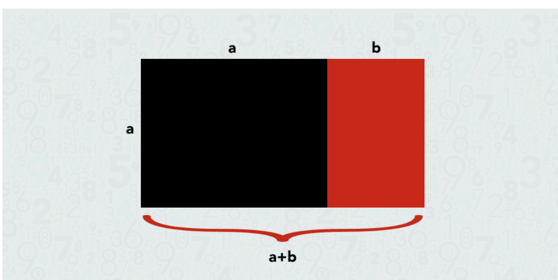
So then the Fibonacci formula came us: $F_{n+1} = \frac{1}{\sqrt{5}}(\Phi^{n+1}) - \frac{1}{\sqrt{5}}(-\varphi)^{n+1}$

$$\Leftrightarrow F_{n+1} = \frac{\Phi^{n+1}-(-\varphi)}{\sqrt{5}}$$

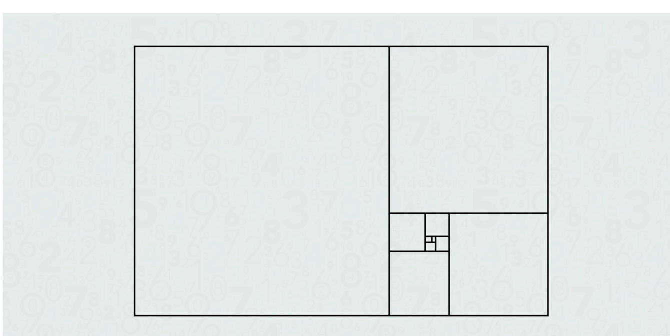
Which can be used to calculate fibonacci terms directly when you know their location.

8.Golden Rectangle

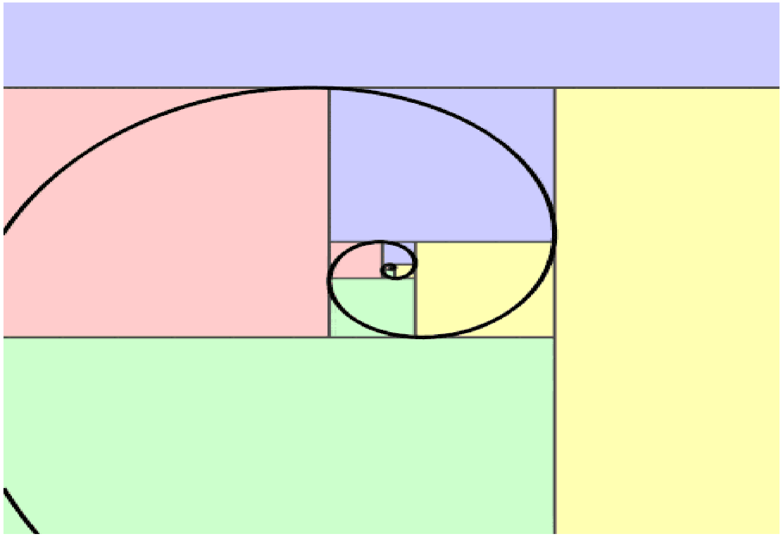
The concept of the Golden Rectangle is attributed to the Ancient Greeks, with the first known mathematical description appearing in Euclid's element around 300 BCE. It is a rectangle whose side lengths are in the golden ratio. It is also a rectangle that can be cut up into a square and a rectangle similar to the original one. All golden rectangles are similar and the ratio length/width is equal to the golden ratio which is also equal to $\frac{1+\sqrt{5}}{2}$. An example of a Golden rectangle is shown below:



You can continue creating those rectangles by placing a square inside that rectangle and the sides of this square are the same length as the shortest side of the rectangle. This process can continue forever, making a sequence of rectangles, each following the golden ratio. As:



As the sequence goes on, we compose the famous golden spiral-an image often used to represent the golden ratio:



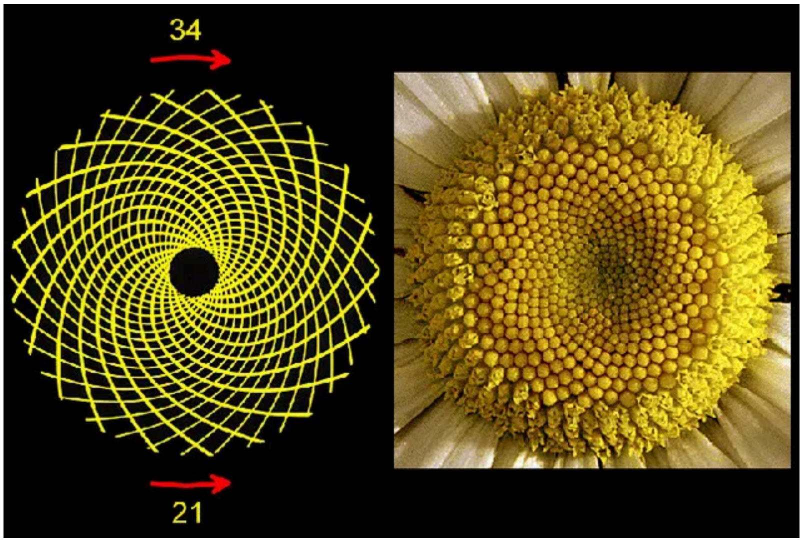
9.Fibonacci numbers in nature

Flower Petals: The number of petals on many flowers often follows a Fibonacci number. For example:Lilies have 3 petals.Buttercups have 5 petals.Chicory has 8, the delphinium 13,Daisies can have 34, 55, or even 89 petals….

Here is Buttercup flower petals:

Pinecones and Pineapples: The arrangement of scales on pinecones and pineapples follows Fibonacci spirals. If you count the spirals in one direction, you will find a Fibonacci number, and if you count in the opposite direction, you'll find the next Fibonacci number. Here are images of Pinecones:

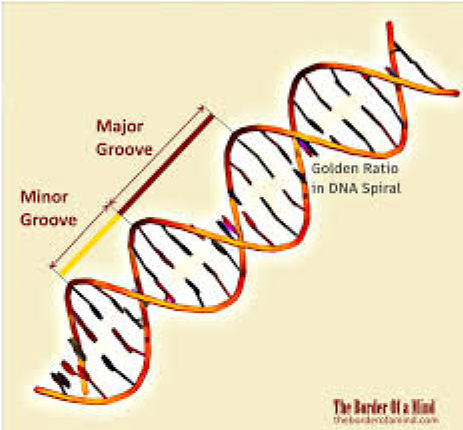
Sunflowers: The seeds of sunflowers are arranged in a spiral pattern that follows Fibonacci numbers. There are usually two sets of spirals running in opposite directions, and if you count them, they often correspond to Fibonacci numbers like 21 and 34 or 34 and 55. Here there's a picture of sunflower seeds:

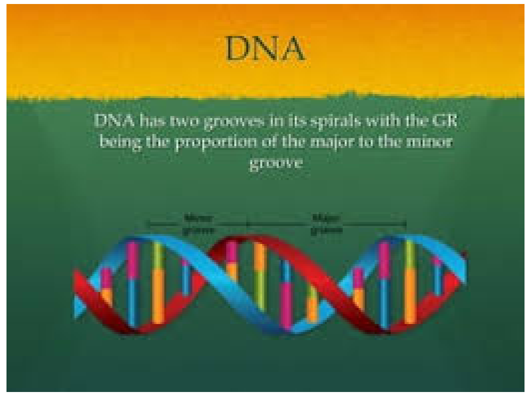


10.Golden Ratio in Nature

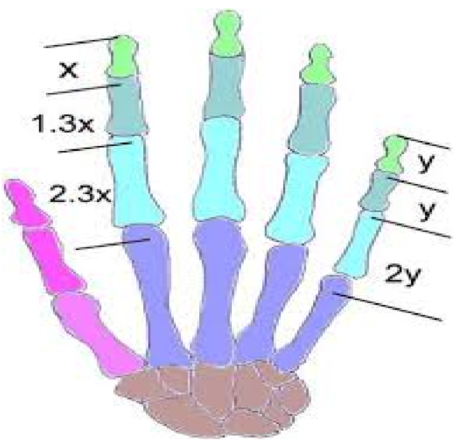
Spiral Patterns in Nature: The golden ratio is observed in the spiral shapes of galaxies, hurricanes, and shells (such as the nautilus shell). These spirals grow outward in a logarithmic pattern that mirrors the golden ratio. Here there is image of Nautilus Shell:

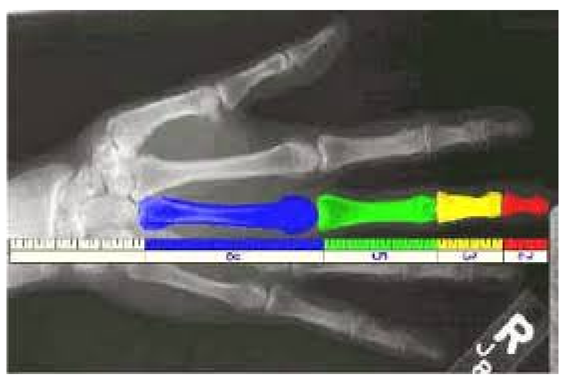
DNA: The structure of DNA itself reflects the golden ratio. The length of each turn of the double helix is 34 angstroms, and its width is 21 angstroms, a ratio of 34:21, which is close to the golden ratio.





Animal Body Proportions: Many animals display proportions that approximate the golden ratio in the relative lengths of different body parts. For example, the ratio between the lengths of the bones in the human hand (finger bones, for example) often shows a relationship to the golden ratio.





11.Critical Reflection

In my research on modeling rabbit population growth using the Fibonacci sequence, I demonstrated how the sequence effectively can be used to represent idealized rabbit growth, showing how each generation depends on the previous two. A strength of the research was proving that the ratio of successive Fibonacci numbers converges to the Golden Ratio, linking mathematical patterns to both natural growth and design. However, a key weakness lies in the model's oversimplification of real-world rabbit populations, as it ignores critical factors like resource limitations, predation, and environmental challenges. This limitation highlights the need for future research to incorporate more realistic variables for accurate population modeling. Despite this, the study successfully illustrates the power of the Fibonacci sequence in describing growth patterns and its relationship to the Golden Ratio

12. Conclusion

In my research, I looked at how the Fibonacci sequence can explain the growth of rabbits over time. The Fibonacci sequence is a pattern where each number is the sum of the two before it, and it can show how rabbit populations grow if they reproduce without limits. I found that as these numbers get bigger, the ratio between them approaches something called the Golden Ratio, which is a special number that appears in many places in nature and design. While this model works for understanding simple growth, it doesn’t account for real-life problems like running out of food or other challenges that would affect rabbit populations. Overall, my research shows how this mathematical pattern can be used to describe growth, but also explains why it's important to think about real-world limits.

13. References

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4.Golden Ratio in real nature

5. Golden Ratio and Fibonacci Numbers