University of BRISTOL

Discrete random variables

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Statistical Computing & Empirical Methods (EMATM0061)

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What will we cover today?

- We will introduce the important concept of a random variable and its distribution.
- We will focus on discrete random variables and discuss the probability mass function.
- We will also consider several important examples including Bernoulli and Binomial random variables.
- We will study several important quantities: expectation, variance, covariance, correlation.
- We will generalize our understanding of independence to the random variable setting.

Random variables

We often encounter numerical quantities which are stochastic or random in nature.

Examples

The weight of penguin selected at random from a population on an island;

The number of dots on the top face of dice which has just been thrown;

The number of red balls drawn from a bag.

We model these stochastic numerical quantities as random variables.

Random variables

Suppose we have a probability space $(\Omega, \mathcal{E}, \mathbb{P})$.

A **random variable** is a mapping $X : \Omega \to \mathbb{R}$ such that for every a, $b \in \mathbb{R}$, $\{\omega \in \Omega \ : \ X(\omega) \in [a, b]\} \in \mathcal{E}$ is an event.

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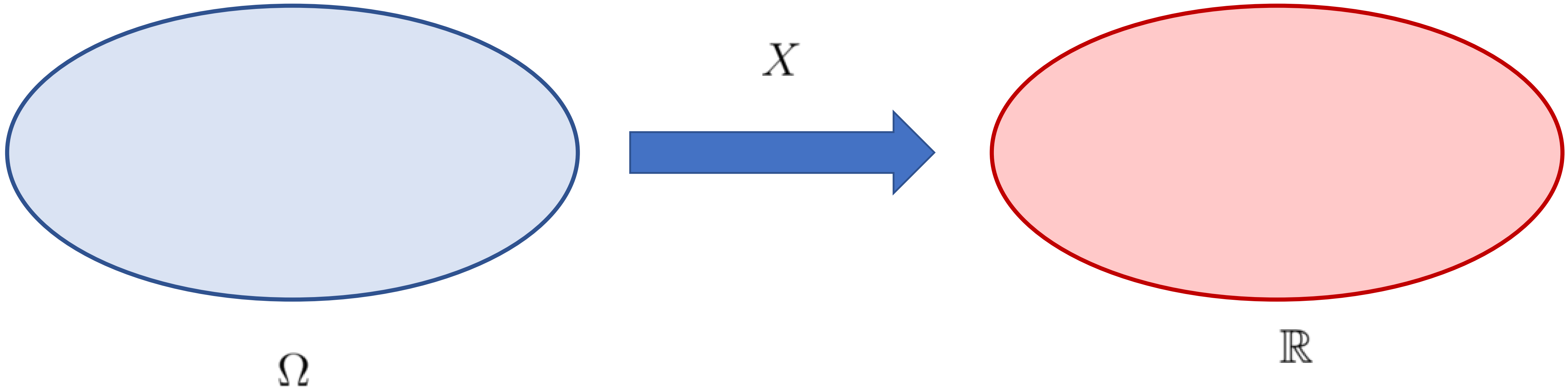

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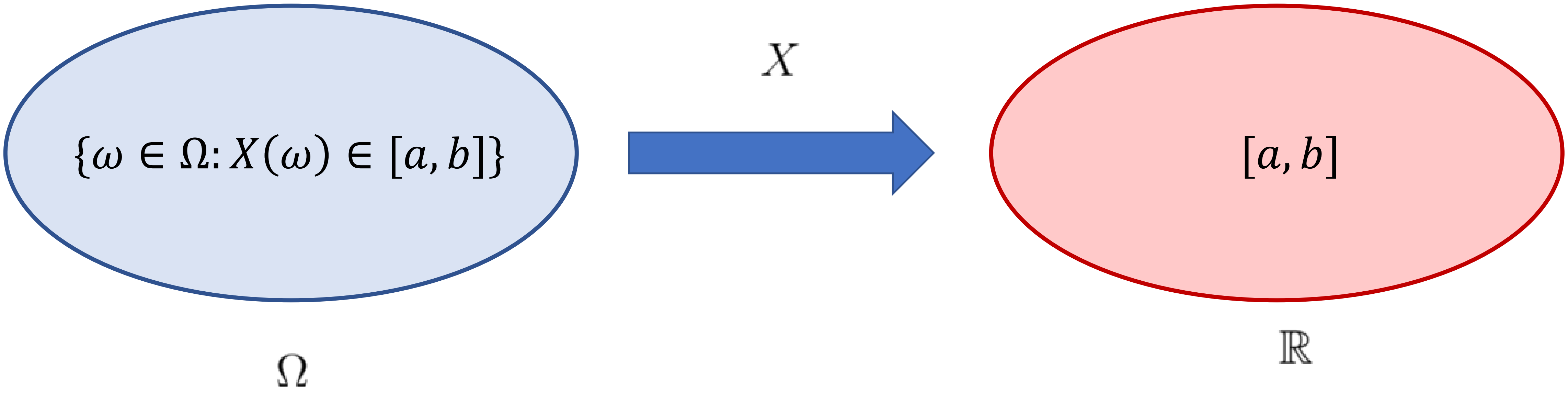

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Example 1

We select 3 at random from a population and record their height and weight.

The sample space is $\Omega = \mathbb{R}^{3\times 2} = \left\{\left((x_1^h, x_1^w), (x_2^h, x_2^w), (x_3^h, x_3^w)\right)\right\}$.

Our random variable is the weight of the third penguin

$$X\left(\left((x_1^h, x_1^w), (x_2^h, x_2^w), (x_3^h, x_3^w)\right)\right) = x_3^w.$$

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Example 2

We roll 5 dice in a row and record each of the results.

The sample space is $\Omega = \{1, \ldots, 6\}^5 = \{(x_1, x_2, x_3, x_4, x_5)\}$.

Our random variable is the result of the final dice roll,

$$X\left((x_1, x_2, x_3, x_4, x_5)\right) = x_5.$$

Random variables

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Example 3

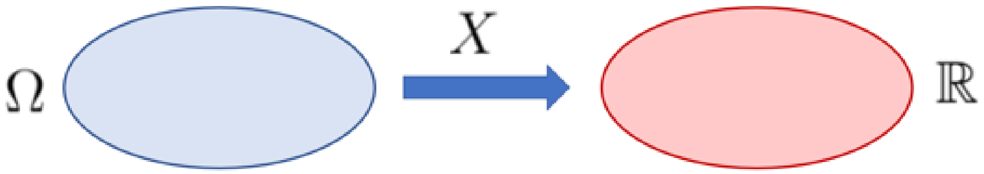
We sample 10 balls with replacement from a bag of 100 balls, 50 of which are red.

The sample space is $\Omega = \{1, \ldots, 100\}^{10} = \{(x_1, \ldots, x_{10})\}$.

Our random variable is the number of red balls sampled

$$X\left((x_1, \ldots, x_{10})\right) = \sum_{i=1}^{10} \mathbb{1}_{\{1,\ldots,50\}}(x_i).$$

Random variables



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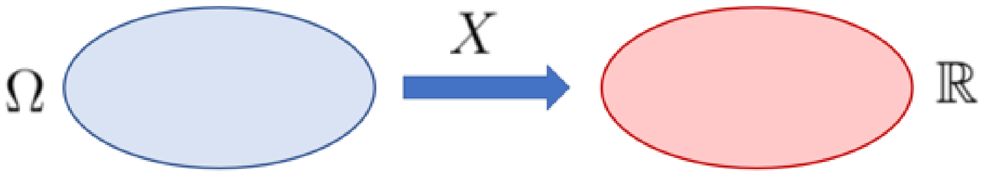

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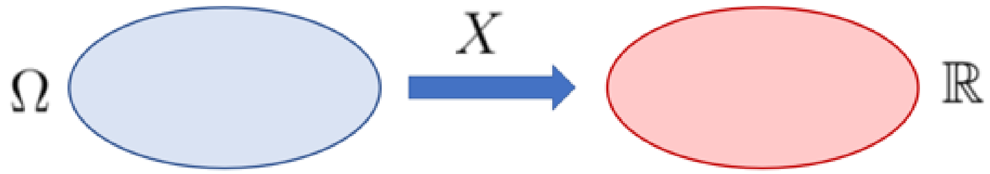

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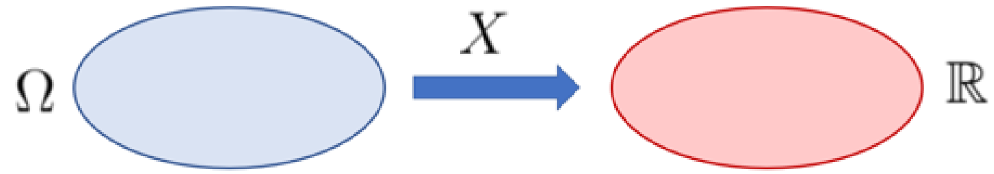

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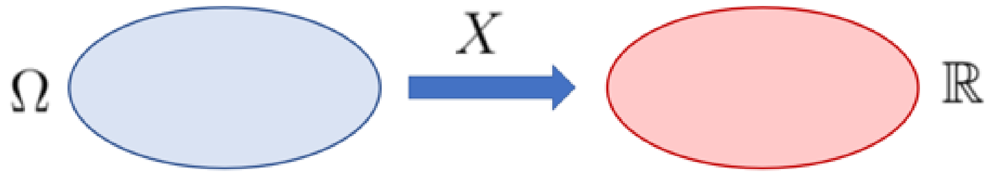
We write $\{X \leq a\}$ for an event $\{\omega \in \Omega \ : \ X(\omega) \leq a\} \in \mathcal{E}$ with $a \in \mathbb{R}$.

In general, we write $\{F(X)\}$ for the event $\{\omega \in \Omega \ : \ F(X(\omega))\} \in \mathcal{E}$, for any predicate "$F$".

Random variables



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Similarly, we write $\mathbb{P}(X \in S)$ for the probability $\mathbb{P}\left(\{\omega \in \Omega \; : \; X(\omega) \in S\right)$.

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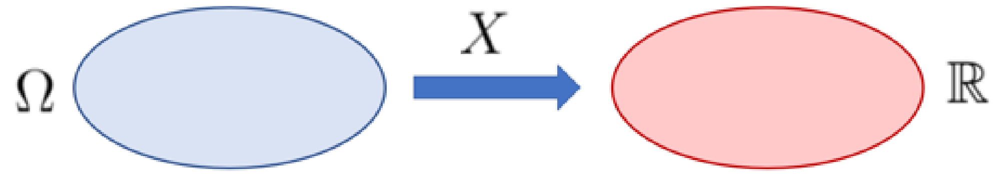

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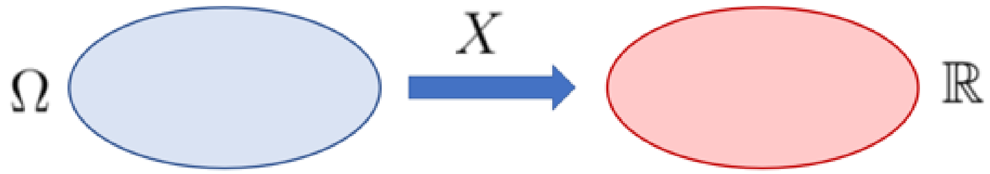
Typically, we ignore the sample space Ω, which may include extraneous information.

Instead, we focus on random variables and interactions between random variables.

Distributions



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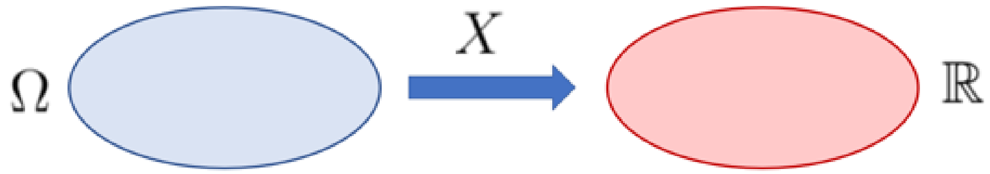

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The **distribution** of a random variable X is the map

$$S \mapsto P_X(S) = \mathbb{P}(X \in S) = \mathbb{P}\left(\{\omega \in \Omega \ : \ X(\omega) \in S\}\right),$$

for all subsets $S \subseteq \mathbb{R}$ such that $\{X \in S\} = \{\omega \in \Omega \ : \ X(\omega) \in S\} \in \mathcal{E}$.

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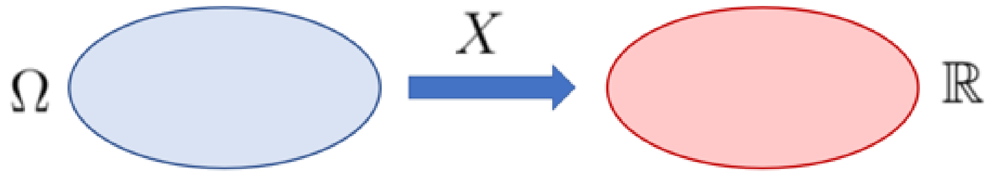

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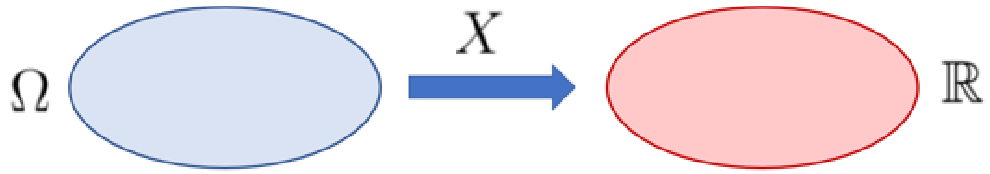
Remark: The distribution P_X of a random variable defines a probability function on (well-behaved) subsets $S \subseteq \mathbb{R}$.

Distributions



We let $\mathfrak{B}(\mathbb{R})$ denote a collection of "well-behaved" (†) subsets $S \subseteq \mathbb{R}$.

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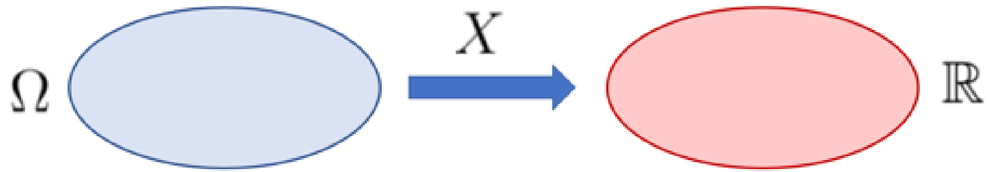

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Theorem . *Suppose we have a probability space* $(\Omega, \mathcal{E}, \mathbb{P})$ *along with a random variable* $X : \Omega \to \mathbb{R}$. *The distribution* P_X *defined by* $P_X(S) = \mathbb{P}(X \in S)$ *for* $S \in \mathfrak{B}(\mathbb{R})$ *satisfies:*

1. *For all* $S \in \mathfrak{B}(\mathbb{R})$ *we have* $P_X(S) = \mathbb{P}(X \in S) \geq 0$;

2. *We have* $P_X(\mathbb{R}) = \mathbb{P}(X \in \mathbb{R}) = 1$;

3. *Given a sequence of disjoint sets* $A_1, A_2, \ldots \in \mathfrak{B}(\mathbb{R})$ *we have* $P_X\left(\bigcup_j A_j\right) = \sum_j P_X(A_j)$.

Hence, $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), P_X)$ *is itself a probability space.*

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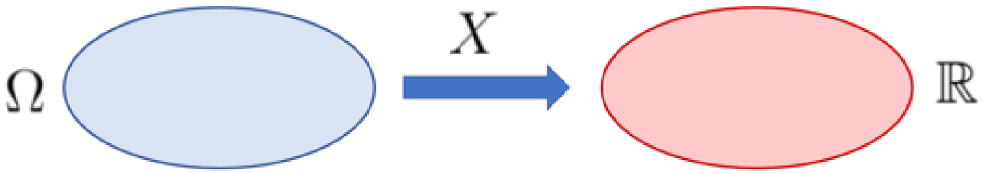

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(†) (Optional technical remark): The collection $\mathfrak{B}(\mathbb{R})$ is the Borel σ-algebra. This the smallest σ-algebra which contains all intervals of the form $[a, b] \subseteq \mathbb{R}$.

Distributions and distribution functions

Given a probability space $(\Omega, \mathcal{E}, \mathbb{P})$, a **random variable** is a mapping $X : \Omega \to \mathbb{R}$ such that for every a, $b \in \mathbb{R}$, $\{\omega \in \Omega \ : \ X(\omega) \in [a, b]\} \in \mathcal{E}$ is an event.

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The **distribution function** of a random variable X is the map $F_X : \mathbb{R} \to [0, 1]$ by

$$F_X(x) := P_X\left\{(-\infty, x]\right\} = \mathbb{P}\left(X \leq x\right) = \mathbb{P}\left(\{\omega \in \Omega \ : \ X(\omega) \leq x\}\right).$$

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The **distribution function** F_X is also referred to as the **probability distribution function** or the **cumulative distribution function**.

Example: Bernoulli distributions

We say that a random variable $X : \Omega \to \mathbb{R}$ is **Bernoulli** if P_X is a Bernoulli distribution:

There exists some $q \in [0, 1]$ such that for any $S \subseteq \mathbb{R}$ we have

$$P_X(S) = \mathbb{P}(X \in S) = \begin{cases} 0 & \text{if } S \cap \{0,1\} = \emptyset \\ 1 - q & \text{if } S \cap \{0,1\} = \{0\} \\ q & \text{if } S \cap \{0,1\} = \{1\} \\ 1 & \text{if } S \cap \{0,1\} = \{0,1\}. \end{cases}$$

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A call option either expires in the money $(X = 1)$ or out of the money $(X = 0)$;

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Example 3

A patient either tests positive $(X = 1)$ or negative $(X = 0)$ for a virus.

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We write $X \sim \mathcal{B}(q)$ for a Bernoulli random variable X with $\mathbb{P}(X = 1) = q$.

Hence, the cumulative distribution function $F_X : \mathbb{R} \to [0,1]$ satisfies,

$$F_X(x) = \mathbb{P}(X \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-q & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

Example: Bernoulli distributions

A Bernoulli random variable $X \sim \mathcal{B}(q)$ has cumulative distribution function $F_X : \mathbb{R} \to [0, 1]$,

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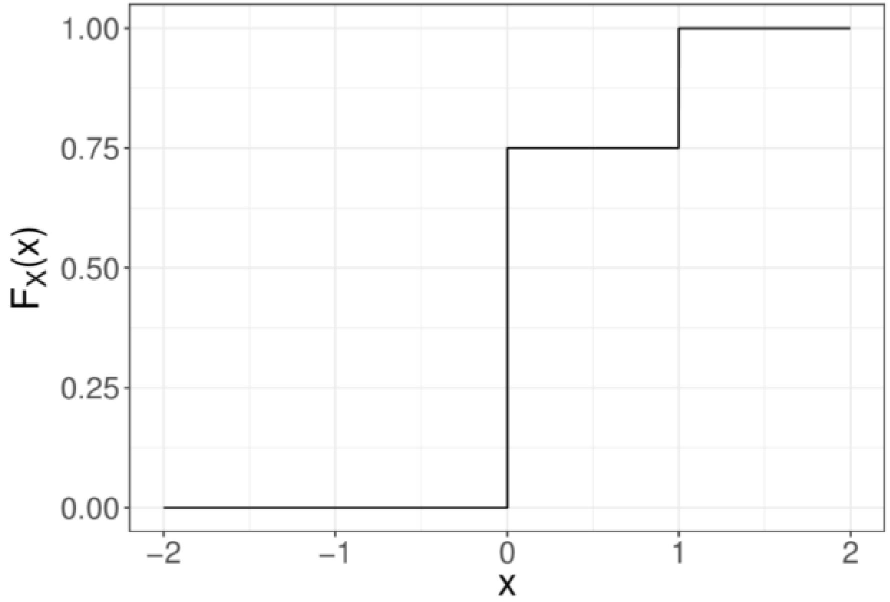

Example: The roll of a dice

Rolling a fair dice corresponds to a random variable $Z : \Omega \to \mathbb{R}$ with

$$P_Z(S) = \mathbb{P}(Z \in S) = \frac{|S \cap \{1, \ldots, 6\}|}{6} = \frac{1}{6} \cdot \sum_{x \in \{1,\ldots,6\}} \mathbb{1}_S(x).$$

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The corresponding cumulative distribution function $F_Z : \mathbb{R} \to [0, 1]$ is,

$$F_Z(x) = \mathbb{P}(Z \leq x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{6} & \text{if } 1 \leq x < 2 \\ \vdots & \\ \frac{5}{6} & \text{if } 5 \leq x < 6 \\ 1 & \text{if } x \geq 6. \end{cases}$$

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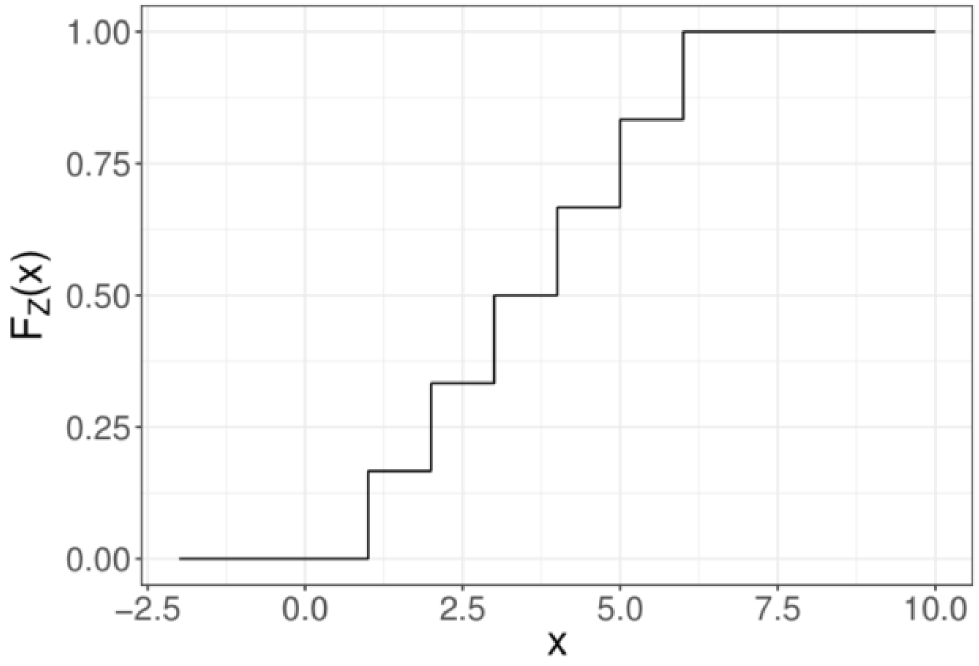

New random variables from old

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Given random variables $X_1, \ldots, X_k : \Omega \to \mathbb{R}$ and a function $f \in \mathfrak{B}(\mathbb{R}^k, \mathbb{R})$ we let $f(X_1, \ldots, X_k)$ denote the random variable $Y : \Omega \to \mathbb{R}$ defined by $Y(\omega) := f\left(X_1(\omega), \ldots, X_k(\omega)\right)$ for $\omega \in \Omega$.

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Example

Let Z_1, Z_2, Z_3 be the outcomes of 3 dice roles. Then $Y = Z_1 + Z_2 + Z_3$ is the total accumulated score. More precisely, we take $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(z_1, z_2, z_3) := z_1 + z_2 + z_3$ so that $Y(\omega) := f\left(Z_1(\omega), Z_2(\omega), Z_3(\omega)\right)$ for all $\omega \in \Omega$.

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(†) (Optional technical remark): The collection $\mathfrak{B}(\mathbb{R}^k, \mathbb{R})$ consists of all Borel measurable functions. These are functions $f : \mathbb{R}^k \to \mathbb{R}$ such that $f^{-1}(A) \in \mathfrak{B}(\mathbb{R}^k)$ whenever $A \in \mathfrak{B}(\mathbb{R})$. Here, $\mathfrak{B}(\mathbb{R}^k)$ is the smallest σ-algebra containing all sets of the form $\prod_{i=1}^k [a_i, b_i]$.

Now take a break!

Discrete random variables

  

We say that the distribution of a random variable $X : \Omega \to \mathbb{R}$ is **supported** on a set $A \subseteq \mathbb{R}$ if $P_X(A) = \mathbb{P}(X \in A) = 1$.

Discrete random variables



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Examples

The distribution of a Bernoulli random variable $X \sim \mathcal{B}(q)$ is supported on $\{0, 1\}$.

The distribution of a random dice roll Z is supported on $\{1, 2, \ldots, 6\}$.

Probability mass functions



Let $X : \Omega \to \mathbb{R}$ be a random variable with distribution supported on a finite or countably infinite set (e.g. a discrete random variable).

The **probability mass function** of the random variable X is the function $p_X : \mathbb{R} \to [0, 1]$ defined by

$$p_X(x) := P_X(\{x\}) = \mathbb{P}\left(X = x\right).$$

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Example

A Bernoulli random variable $X \sim \mathcal{B}(q)$ has probability mass function

$$p_X(x) = \begin{cases} 1 - q & \text{if } x = 0 \\ q & \text{if } x = 1 \\ 0 \text{ otherwise.} \end{cases}$$

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Example

The distribution of a random dice roll Z has probability mass function

$$p_Z(x) = \begin{cases} \frac{1}{6} & \text{if } x \in \{1, 2, \ldots, 6\} \\ 0 & \text{if } x \notin \{1, 2, \ldots, 6\}. \end{cases}$$

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Key Properties

1. For all $x \in \mathbb{R}$, the probability mass function $p_X(x) \geq 0$.
2. The values of the probability mass function sum to unity $\sum_{x \in \mathbb{R}} p_X(x) = 1$.

Expectation of a discrete random variable

Let $X : \Omega \to \mathbb{R}$ be a random variable with distribution supported on a finite or countably infinite set (e.g. a discrete random variable) and let $p_X(x) := \mathbb{P}(X = x)$ be its probability mass function.

The **expectation** $\mathbb{E}(X)$ of the random variable X is defined by $\mathbb{E}(X) := \sum_{x \in \mathbb{R}} x \cdot p_X(x)$.

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Example

Given a $X \sim \mathcal{B}(q)$ be a Bernoulli random variable we have

$$\mathbb{E}(X) := \sum_{x \in \mathbb{R}} x \cdot p_X(x) = (1-q) \cdot 0 + q \cdot 1 + \overbrace{\sum_{x \in \mathbb{R} \setminus \{0,1\}} x \cdot p_X(0)}^{0} = q.$$

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Example

Given a Z be the outcome of a fair dice roll we have,

$$\mathbb{E}(Z) := \sum_{x \in \mathbb{R}} x \cdot p_Z(x) = \frac{1}{6} \cdot (1 + 2 + \ldots + 6) + \overbrace{\sum_{x \in \mathbb{R} \setminus \{0,1\}} x \cdot p_X(0)}^{0} = \frac{7}{2}.$$

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The expectation is often referred to the population average or population mean.

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Expectation is *linear* in the following sense:

Lemma *Given random variables* $X_1, \ldots, X_K : \Omega \to \mathbb{R}$ *and numbers* $\alpha_1, \ldots, \alpha_K \in \mathbb{R}$ *we have*

$$\mathbb{E}\left(\sum_{i=1}^{K} \alpha_i X_i\right) = \sum_{i=1}^{K} \alpha_i \mathbb{E}(X_i).$$

Variance of a discrete random variable

Let $X : \Omega \to \mathbb{R}$ be a random variable with distribution supported on a finite or countably infinite set (e.g. a discrete random variable) and let $p_X(x) := \mathbb{P}(X = x)$ be its probability mass function.

The **variance** $\mathrm{Var}(X)$ of the random variable X is defined by $\mathrm{Var}(X) := \mathbb{E}\left[\{X - \mathbb{E}(X)\}^2\right]$.

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$$\mathrm{Var}(X) := \mathbb{E}\left[\{X - \mathbb{E}(X)\}^2\right] = \sum_{x\in\mathbb{R}} p_X(x) \cdot (x-q)^2 = (1-q) \cdot q^2 + q \cdot (1-q)^2 = q(1-q).$$

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Example

Letting Z be the outcome of a fair dice roll we have,

$$\mathrm{Var}(Z) := \mathbb{E}\left[\{Z - \mathbb{E}\left(Z\right)\}^2\right] = \frac{1}{6}\sum_{x=1}^{6}\left(x - \frac{7}{2}\right)^2 = \frac{35}{12}.$$

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The variance of a random variable is often referred to as the population variance.

The population variance and sample variance are closely connected, as we shall see.

Variance of a discrete random variable


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Given a random variable $X : \Omega \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ we have $\mathrm{Var}\left(\alpha \cdot X\right) = \alpha^2 \cdot \mathrm{Var}(X)$.

This motivates the (population) standard deviation $\sigma(X) := \sqrt{\mathrm{Var}(X)}$.

Given a random variable $X : \Omega \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ we have $\sigma\left(\alpha \cdot X\right) = |\alpha| \cdot \sigma(X)$

Variance of a discrete random variable


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We can view the variance of a random variable as measuring how much it typically fluctuates around its expectation $\mathbb{E}(X)$ upon repeatedly sampling independent copies of X.

What is the variance of a linear combination of random variables $\left(\sum_{i=1}^{K} \alpha_i X_i\right)$?

Independent random variables

Suppose that $X_1, \ldots, X_k : \Omega \to \mathbb{R}$ are random random variables, with distribution functions $F_{X_1}, \ldots, F_{X_k} : \mathbb{R} \to [0,1]$ defined by $F_X(x) := \mathbb{P}(X \leq x)$ for all $x \in \mathbb{R}$.

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We define the joint cumulative distribution function $F_{X_1,\ldots,X_k} : \mathbb{R}^k \to \mathbb{R}$ by

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Equivalently, $X_1, \ldots, X_k$ are independent if for all $x_1, \ldots, x_k \in \mathbb{R}$ the sequence of events $\{X_1 \leq x_1\}, \ldots, \{X_k \leq x_k\}$ are (mutually) independent.

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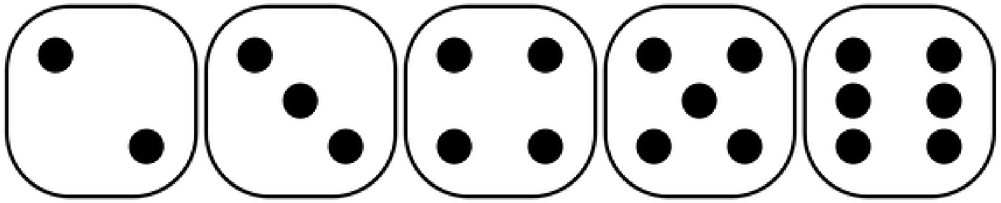
Example

Suppose that I flip k coins and let X_i correspond to the result of the i-th coin flip.

A natural assumption here is that the different coin flips have no interaction with one another.

Hence, we can model $X_1, \ldots, X_k$ as a sequence of independent random variables.

Independent random variables



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Example

Suppose that k patients are treated for a medical condition in a clinical trial.

Let X_i correspond to the recovery time of the i-th patient in the trial.

We may assume that the level of interaction between the patients is very weak.

Hence, we can model $X_1, \ldots, X_k$ as a sequence of independent random variables.

Dependent random variables

Suppose that $X_1, \ldots, X_k : \Omega \to \mathbb{R}$ are random random variables, with distribution functions $F_{X_1}, \ldots, F_{X_k} : \mathbb{R} \to [0,1]$ and joint cumulative distribution function $F_{X_1,\ldots,X_k} : \mathbb{R}^k \to \mathbb{R}$.

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Example

Suppose that we flip k coins and let Z_j be one if the j-th coin was a head and 0 otherwise.

For each $i = 1, \ldots, k$ let $X_i = Z_1 + \ldots Z_i$, the accumulated total.

The sequence $X_1, \ldots, X_k$ is a dependent sequence of random variables.

Dependent random variables

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Example

Let X_i be the AUDUSD exchange rate on the i-th hour of the week.

Exhange rates on consecutive time windows are closely connected, so $X_1, \ldots, X_k$ are dependent.

Dependent random variables

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Example

Let X_i be the average temperature on the i-th day of the year.

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Example

Let X_i be the number of tickets sold in the i-th minute for a music festival.

The total number of tickets sold is limited, so $X_1, \ldots, X_k$ are dependent.

Quantifying dependence through covariation

Suppose that $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ are random random variables.

The **covariance** between X and Y is defined by

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The correlation gives a scale-invariant quantification of the linear relationship between X and Y.

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Key fact:

If X and Y are independent random variables then $\mathrm{Corr}(X, Y) = \mathrm{Cov}(X, Y) = 0$.

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If X and Y are independent random variables then $\mathrm{Corr}(X, Y) = \mathrm{Cov}(X, Y) = 0$.

However, $\mathrm{Cov}(X, Y) = 0$ doesn't necessarily mean that X and Y are independent.

An alternative perspective on independence

Suppose that $X_1, \ldots, X_k : \Omega \to \mathbb{R}$ are random random variables, with distribution functions $F_{X_1}, \ldots, F_{X_k} : \mathbb{R} \to [0,1]$ and joint cumulative distribution function $F_{X_1,\ldots,X_k} : \mathbb{R}^k \to \mathbb{R}$.

We say that $X_1, \ldots, X_k$ are (mutually) **independent** if for all $x_1, \ldots, x_k \in \mathbb{R}$ we have

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In particular, if X and Y are independent random variable then $\mathrm{Cov}(X, Y) = 0$.

The variance of a linear combination of random variables

Let $X : \Omega \to \mathbb{R}$ be a random variable with distribution supported on a finite or countably infinite set (e.g. a discrete random variable) and let $p_X(x) := \mathbb{P}(X = x)$ be its probability mass function.

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In particular, if $X_1, \dots, X_k$ *are independent then* $\mathrm{Var}\left(\sum_{i=1}^K \alpha_i X_i\right) = \sum_{i=1}^K \alpha_i^2 \mathrm{Var}(X_i)$.

Now take a break!

Binomial distributions

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The probability mass function $p_Z : \mathbb{R} \to [0,1]$ is given by

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for $r \in \{0, 1, \ldots, n\}$ and $p_Z(r) = 0$ for $r \notin \{0, 1, \ldots, n\}$.

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Suppose that Z has **Binomial distribution** with parameters n and p.

Hence, $Z = X_1 + \ldots + X_n$ where $X_1, \ldots, X_n$ are independent random variables where each $X_i \sim \mathcal{B}(p)$ has Bernoulli distribution with $\mathbb{E}[X_i] = p$.

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For each sequence $(s_1, \ldots, s_n) \in \{0, 1\}$ with $\sum_{i=1}^n s_i = r$ we have

$$\mathbb{P}\left(X_1 = s_1, \ldots, X_n = s_n\right) = \mathbb{P}\left(X_1 = s_1\right) \cdot \ldots \mathbb{P}\left(X_n = s_n\right) = \left(\prod_{i:s_i=1} p\right) \cdot \left(\prod_{i:s_i=0} (1-p)\right) = p^r \cdot (1-p)^{n-r}.$$

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Hence, we have

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For each $i = 1, \ldots, n$ we have $\mathbb{E}[X_i] = p$ and $\mathrm{Var}(X_i) = p(1-p)$.

Since $X_1, \ldots, X_n$ are independent we have

$$\mathbb{E}\left(Z\right) = \mathbb{E}\left(X_1 + \ldots + X_n\right) = \mathbb{E}(X_1) + \ldots + \mathbb{E}(X_n) = n \cdot p.$$
$$\mathrm{Var}\left(Z\right) = \mathrm{Var}\left(X_1 + \ldots + X_n\right) = \mathrm{Var}(X_1) + \ldots + \mathrm{Var}(X_n) = n \cdot p \cdot (1-p).$$

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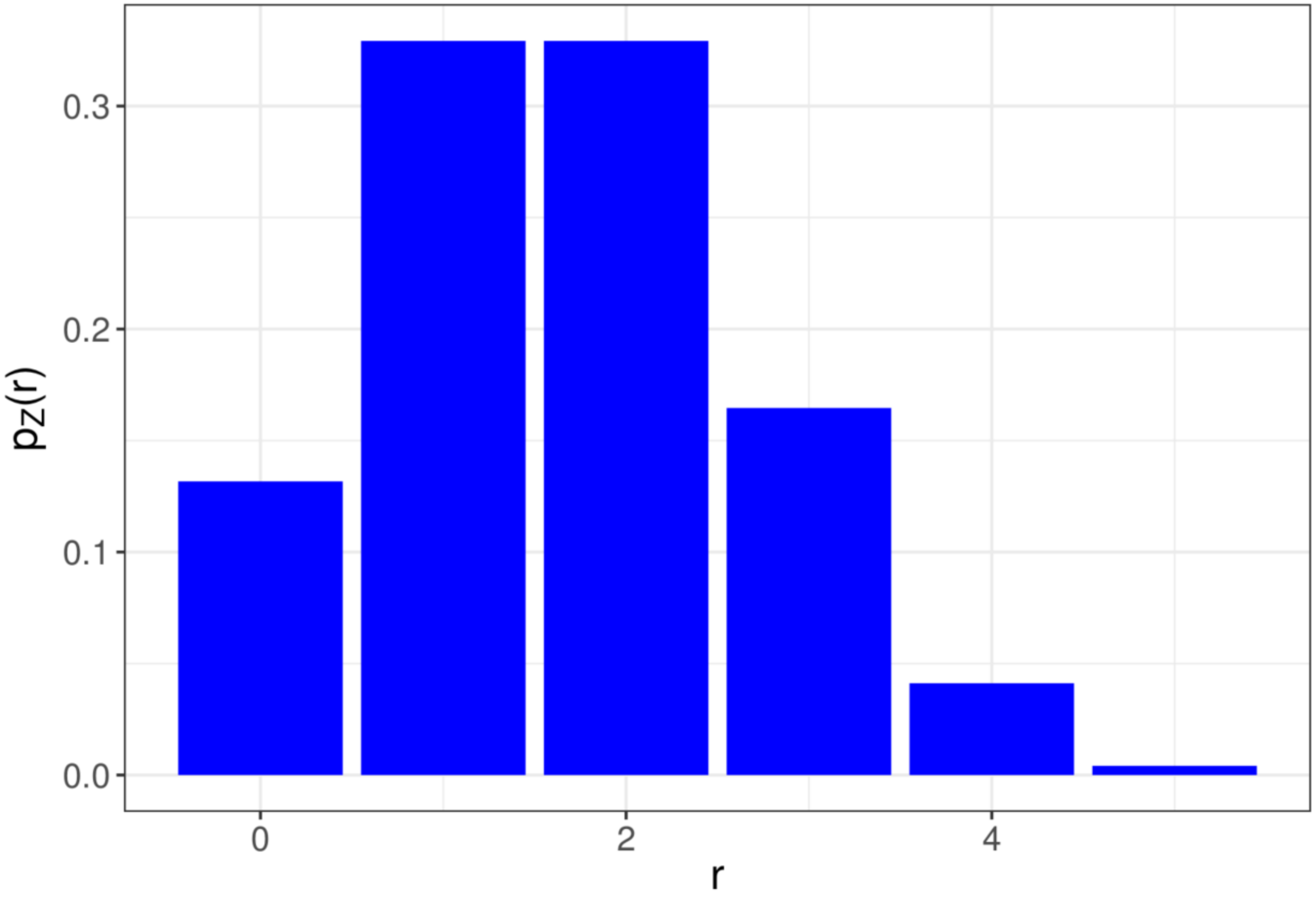

$$n = 5$$

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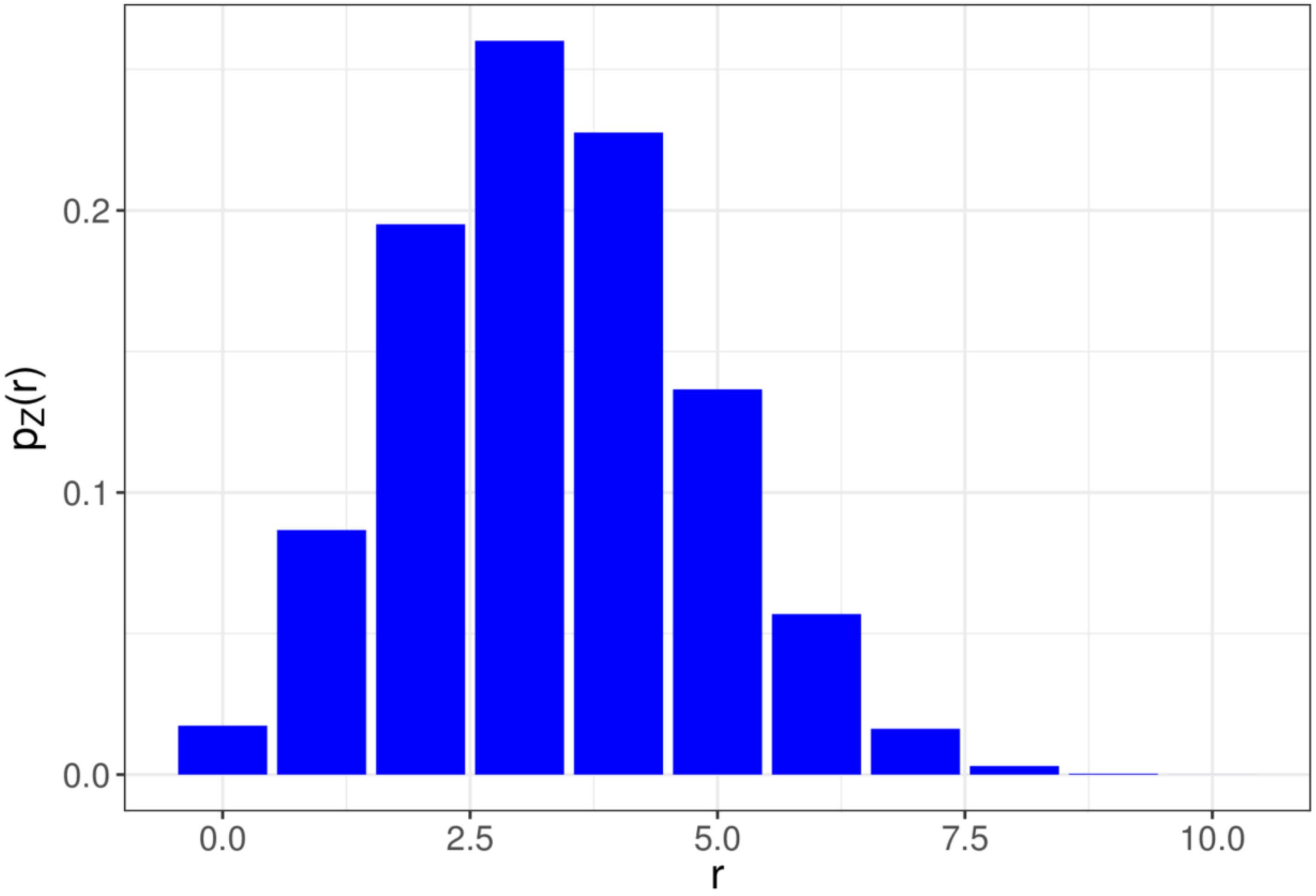

$$n = 10$$

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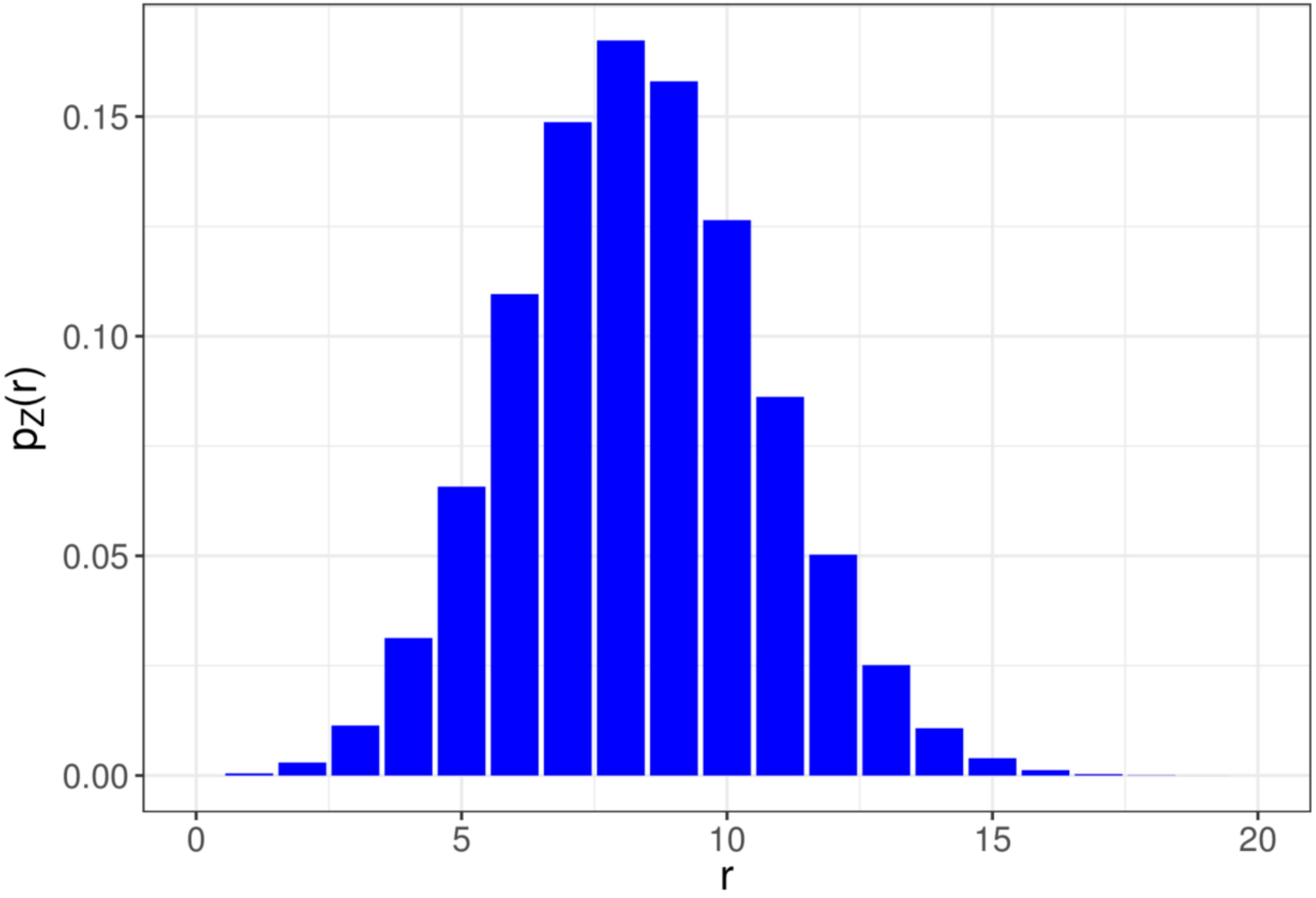

$n = 25$

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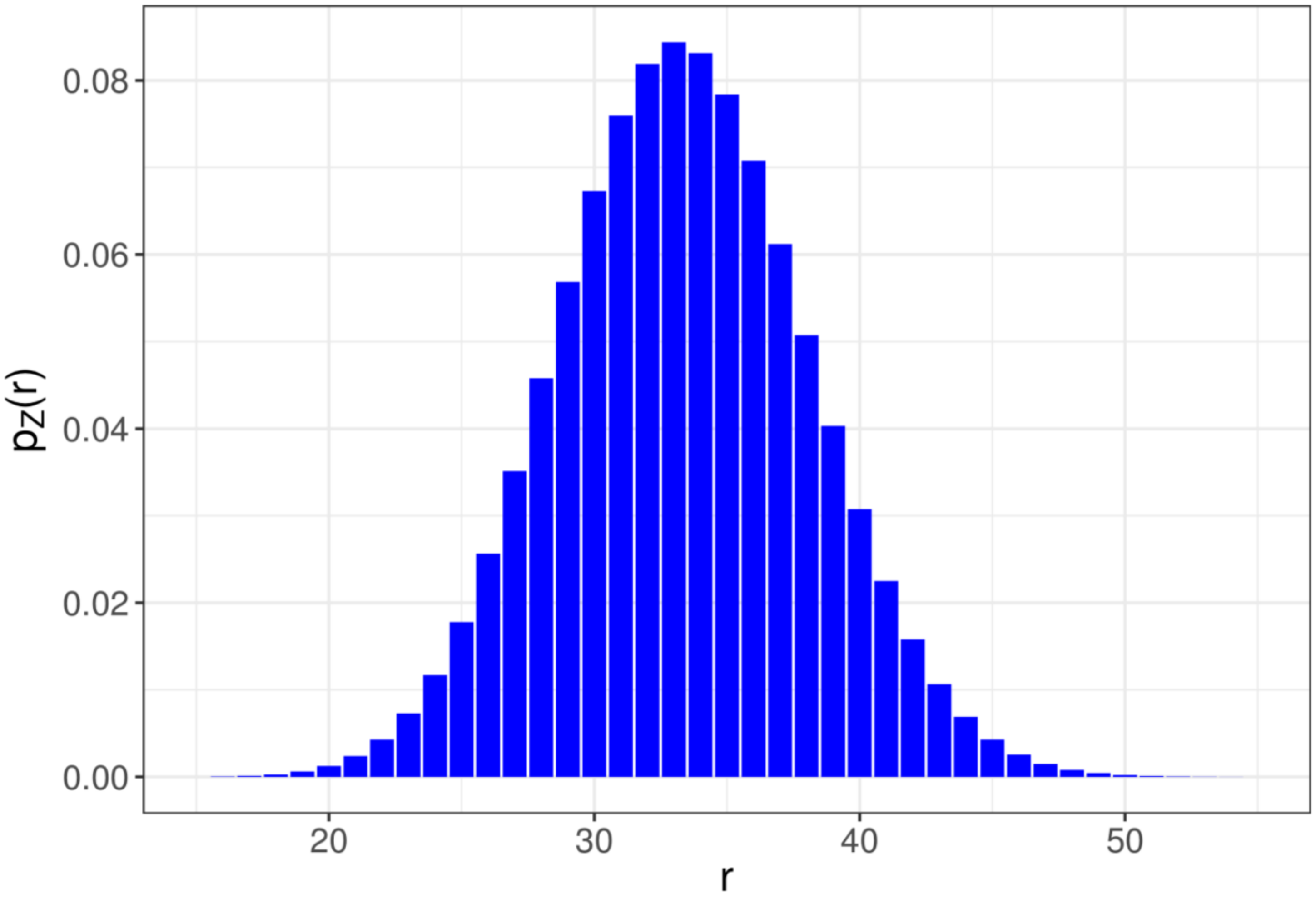

$n = 100$

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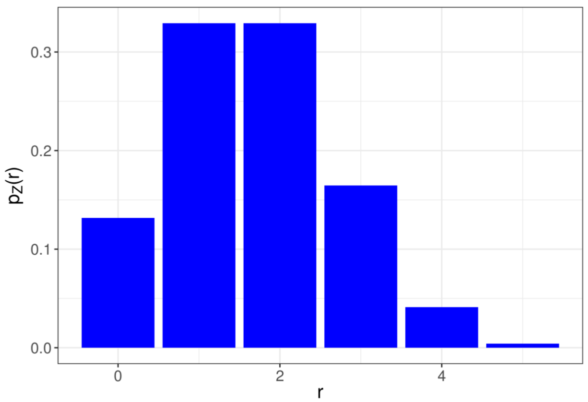
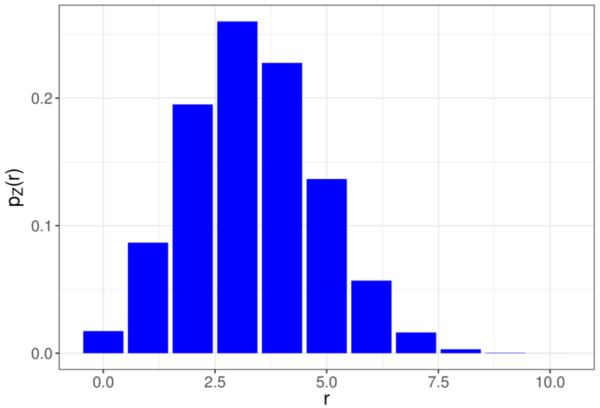


$n = 5$



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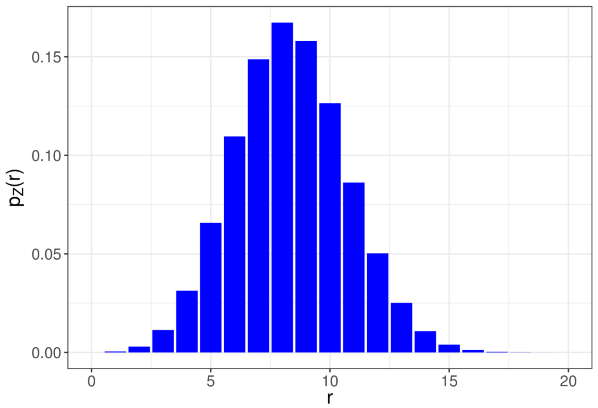

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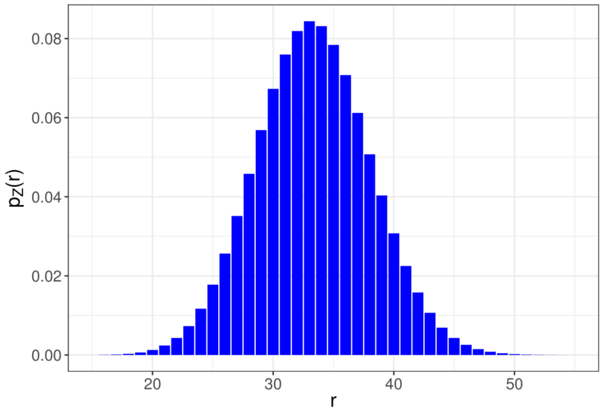

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What have we covered?

- We introduced the important concept of a random variable.
- We saw how random variables can be quantified via its distribution function.
- We introduced the concept of a discrete random variable and discussed the probability mass function.
- We discussed several important examples including Bernoulli and Binomial random variables.
- We also defined the expectation, variance, covariance and correlation of random variables.
- In addition, we generalized our understanding of independence from sequences of events to sequences of random variables.

Thanks for listening!

Henry W J Reeve

henry.reeve@bristol.ac.uk

Include EMATM0061 in the subject of your email.