University of BRISTOL

Foundations of statistical estimation: Consistency, bias and variance

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Statistical Computing & Empirical Methods (EMATM0061)

MSc in Data Science, Teaching block 1, 2021.

What will we cover today?

- We will view sample statistics as estimators of parameters of interest.
- We will discuss the concept of statistical consistency.
- We will also introduce the ideas of statistical bias and the bias-variance decomposition.
- We will also consider the concept of a minimum variance unbiased-estimator.

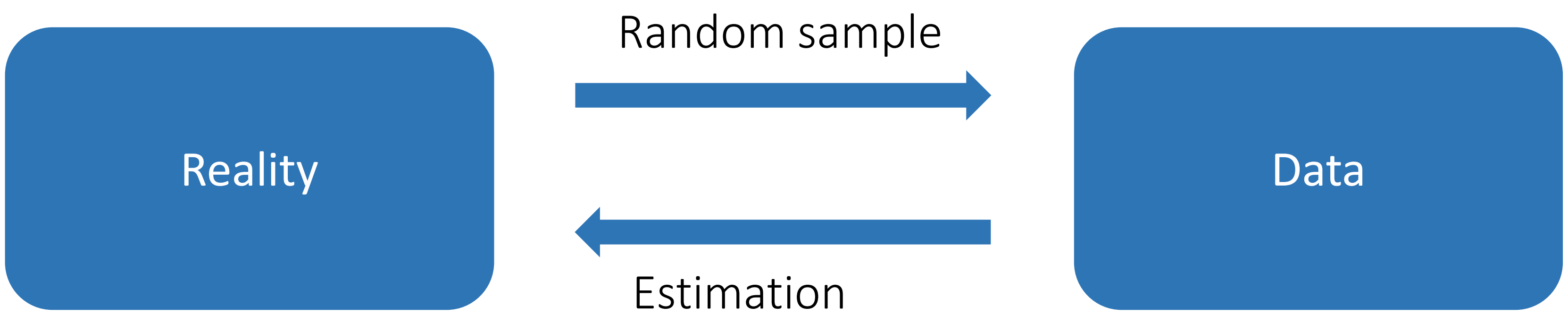
Samples and populations

We attempt to understand populations of penguins by looking at random samples.

Population

Sample

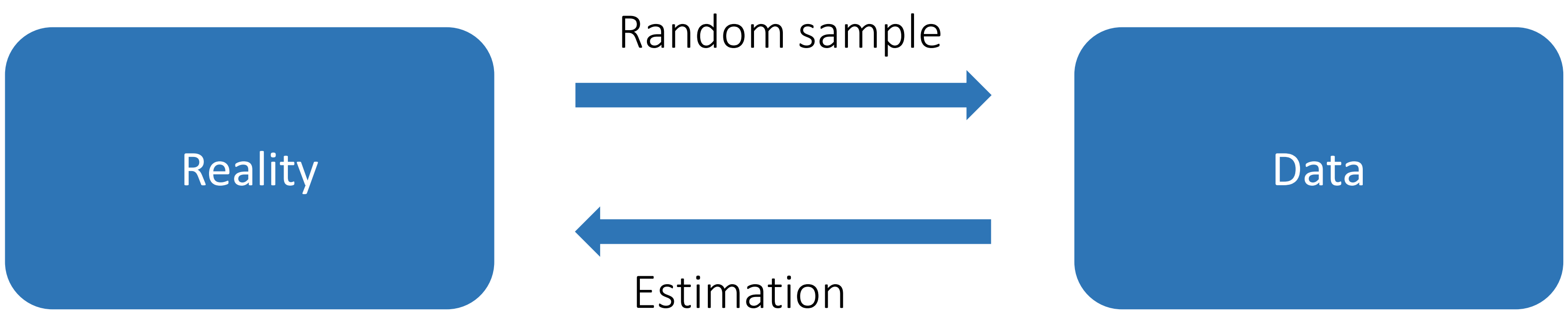
Statistical estimation



Population statistic

Sample statistic

Statistical estimation



Population statistic

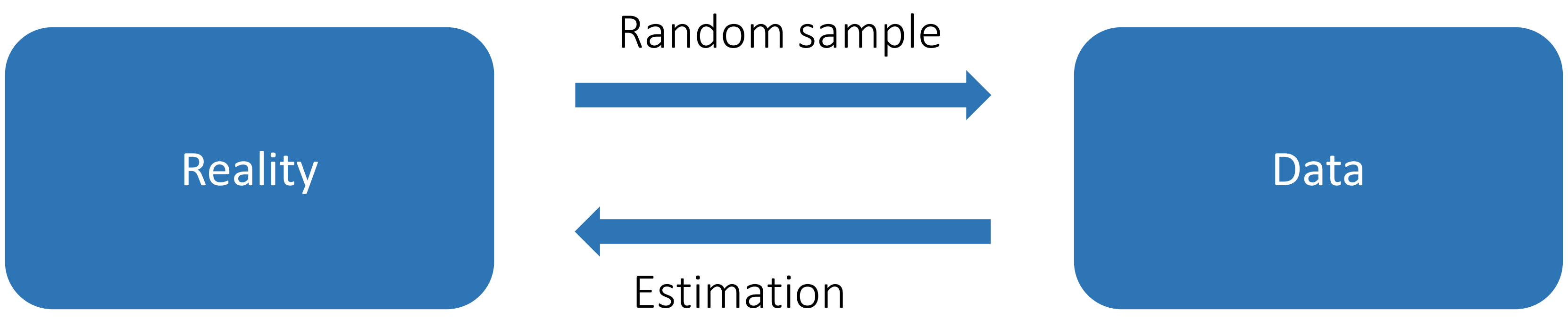
Sample statistic

Mean

$$\mu = \mathbb{E}[X_i]$$

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Statistical estimation



	Population statistic	Sample statistic
Mean	$\mu = \mathbb{E}[X_i]$	$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$
Variance	$\sigma^2 = \mathbb{E}\left[(X_i - \mu)^2\right]$	$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$

Probabilistic modelling

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Examples

1. Suppose we have a sequence $(X_i)_{i=1}^n$ in $\{0,1\}^n$ corresponding to pass or fail for a driving test.

   We can model $(X_i)_{i=1}^n$ as a sequence of independent and identically distributed Bernoulli RVs

$$X_1, \cdots, X_n \sim \mathcal{B}(q) \qquad\qquad \theta = q$$

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   $$X_1, \cdots, X_n \sim \mathcal{B}(q) \qquad\qquad \theta = q$$

2. Suppose we have a sequence $(X_i)_{i=1}^n$ in $\mathbb{R}^n$ corresponding to the height of a penguin.

   We can model $(X_i)_{i=1}^n$ as a sequence of independent and identically distributed Gaussian RVs

   $$X_1, \cdots, X_n \sim \mathcal{N}(\mu, \sigma^2) \qquad\qquad \theta = (\theta_1, \theta_2) = (\mu, \sigma^2)$$

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Functions of your sample $\hat{\theta} = g(X_1, \cdots, X_n)$ that don't depend on θ.

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1. Suppose $X_1, \cdots, X_n \sim \mathcal{B}(q)$ are i.i.d. with a single parameter $\theta = q$

   We estimate q with the sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

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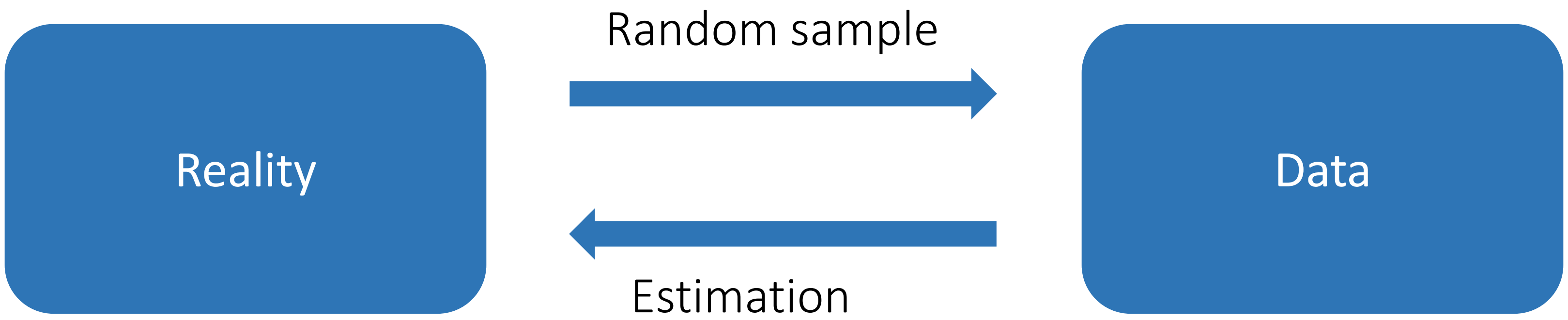
2. Suppose $X_1, \cdots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ are i.i.d. with parameters $\theta = (\theta_1, \theta_2) = (\mu, \sigma^2)$

   We estimate μ with the sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

   Similarly, we estimate σ^2 with the sample variance $s^2 = \frac{1}{n-1}\sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$

Statistical estimation



Population

Sample

$$\theta$$

$$\hat{\theta} = g(X_1, \cdots, X_n)$$

Sample statistics are computed from a random sample so are themselves random variables.

The goal is to find sample statistics which closely approximate corresponding population statistics.

Examples of statistical estimation

Example 1

Suppose $X_1, \ldots, X_n \sim \mathcal{B}(q)$ are i.i.d. with a single parameter $\theta = q$.

We estimate estimate $q = \mathbb{E}(X_i)$ with the sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

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  mutate(sample_mean=map_dbl(.x=simulation,.f=mean))
  # compute the sample means
```

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```
ggplot()+labs(x="Mean",y="Density")+theme_bw()+
  geom_density(data=simulation_df,
                aes(x=sample_mean,color="Sample",
                    linetype="Sample"))+
  # kernel density plot of sample means
  geom_vline(aes(xintercept=q,color="Population",
               linetype="Population"))+
  # vertical line displaying population mean
  scale_color_manual(name = "Legend",
                    values=c("Sample"="red", "Population"="blue"))+
  scale_linetype_manual(name="Legend",
                       values=c("Sample"="solid", "Population"="dashed"))
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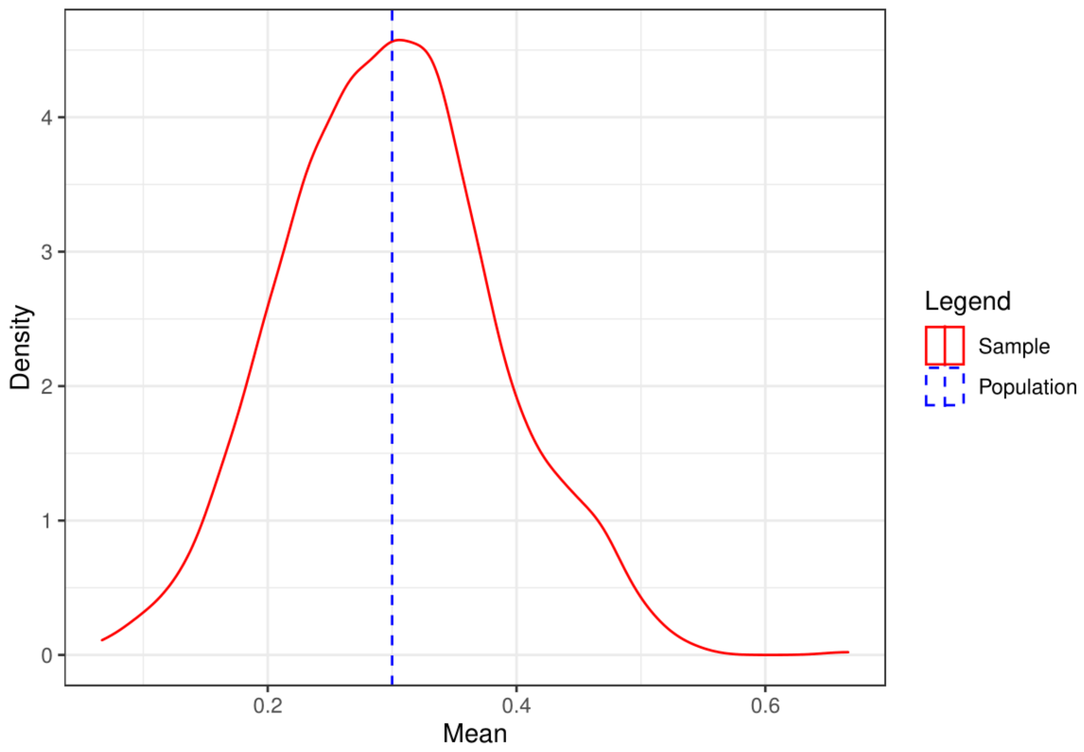

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num_trials_per_sample_size<-10
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q<-0.3


sim_by_n_df<-crossing(trial=seq(num_trials_per_sample_size),
                      sample_size=seq(to=sqrt(max_sample_size),by=0.1)**2)%>%
  # create data frame of all pairs of sample_size and trial
```

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  # simulate sequences of Bernoulli random variables
  mutate(sample_mean=map_dbl(.x=simulation,.f=mean))%>%
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```

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```
ggplot()+labs(x="Sample size",y="Mean")+theme_bw()+
  geom_point(data=sim_by_n_df,
                aes(x=sample_size,y=sample_mean,color="Sample",
                    linetype="Sample"),size=0.1)+
  # scatter plot of sample means
  geom_hline(aes(yintercept=q,color="Population",
                linetype="Population"),size=1)+
  # horizontal line displaying population mean
  scale_color_manual(name = "Legend",
                    values=c("Sample"="blue", "Population"="red"))+
  scale_linetype_manual(name="Legend",
                    values=c("Sample"="dashed", "Population"="solid"))+
  scale_x_sqrt()
```

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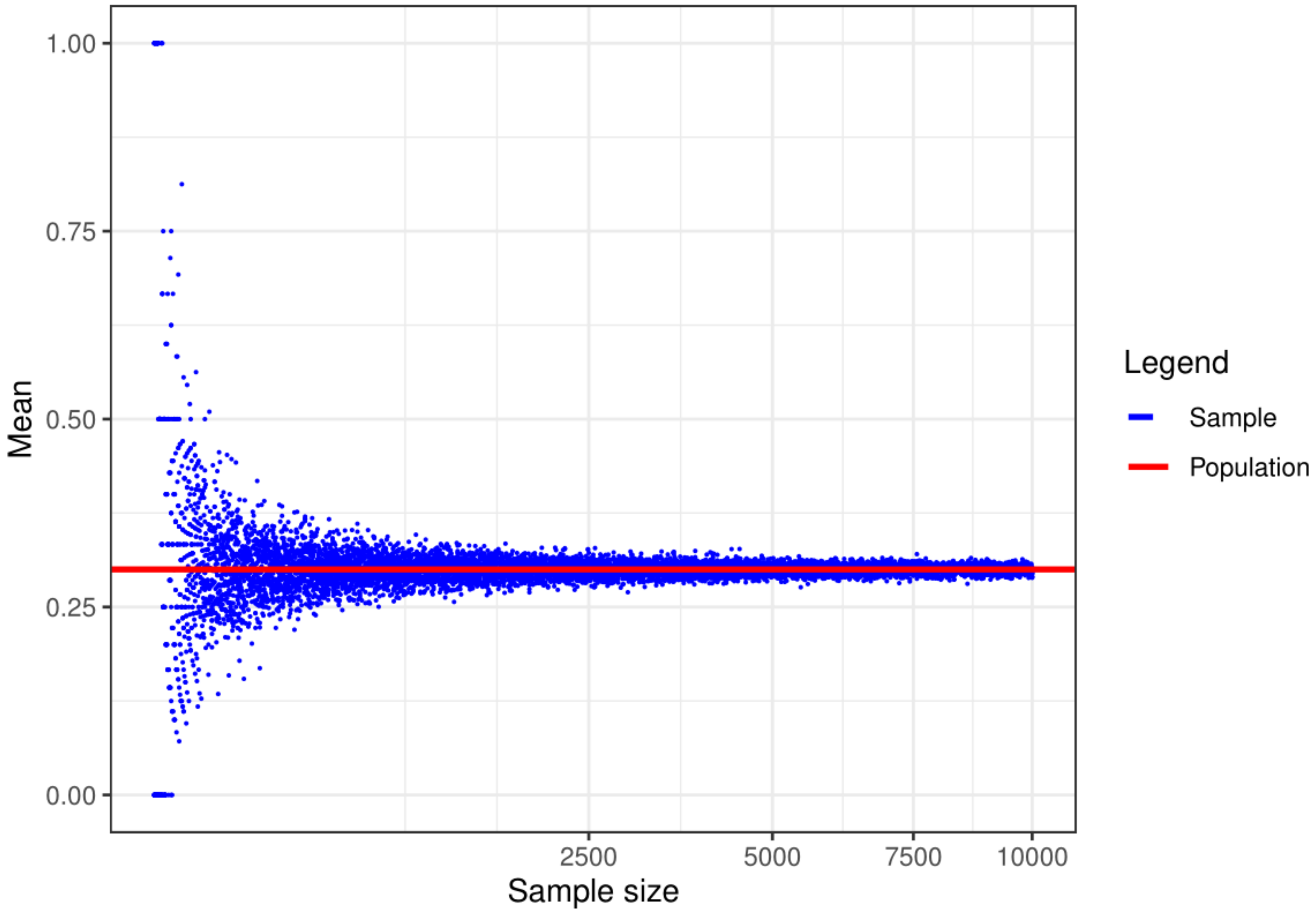

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Example 2

Suppose $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ are i.i.d. with parameters $\theta = (\mu, \sigma^2)$.

We estimate estimate $\sigma^2 = \mathrm{Var}(X_i)$ with the sample variance $\frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$.

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```r
set.seed(0)
num_trials<-1000
sample_size<-30
mu<-1
sigma_sqr<-3

simulation_df<-data.frame(trial=seq(num_trials))%>%
  mutate(simulation=map(.x=trial,
                        .f=~rnorm(sample_size,mean=mu,sd=sqrt(sigma_sqr))))%>%
  # simulate sequences of Gaussian random variables
  mutate(sample_var=map_dbl(.x=simulation,.f=var))
  # compute the sample variances
```

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```
ggplot()+labs(x="Variance",y="Density")+theme_bw()+
  geom_density(data=simulation_df,
                aes(x=sample_var,color="Sample",
                    linetype="Sample"))+
  # kernel density plot of sample variances
  geom_vline(aes(xintercept=sigma_sqr,color="Population",
                 linetype="Population"))+
  # vertical line displaying population mean
  scale_color_manual(name = "Legend",
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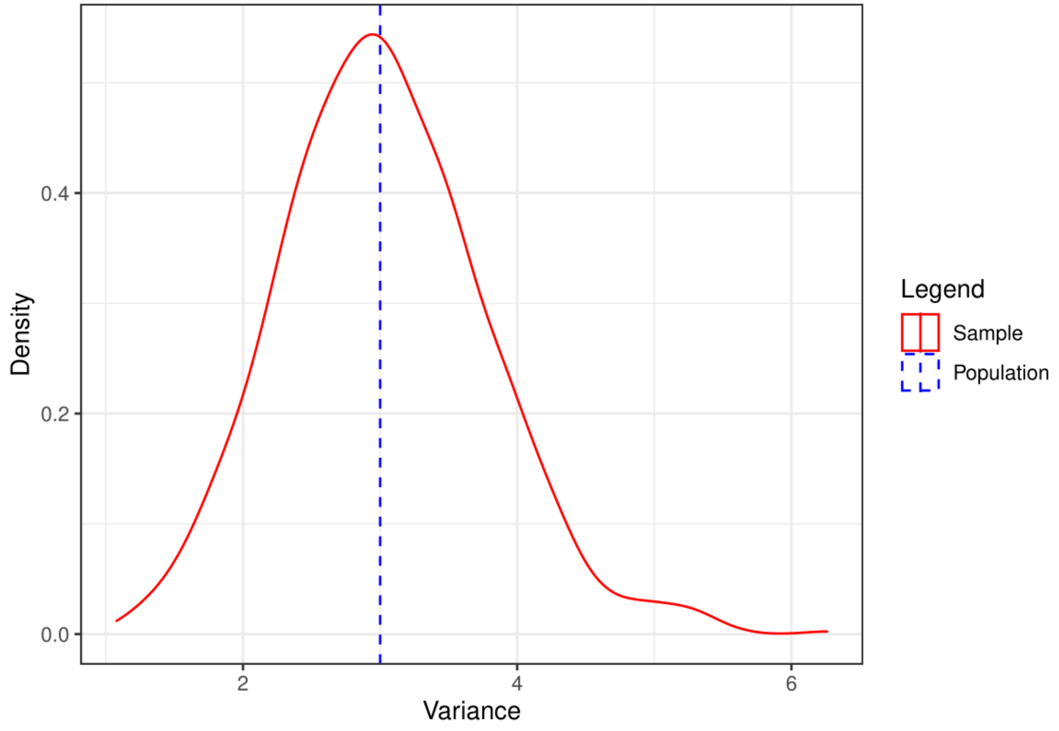

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max_sample_size<-10000
mu<-1
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sim_by_n_df<-crossing(trial=seq(num_trials_per_sample_size),
                      sample_size=seq(to=sqrt(max_sample_size),by=0.1)**2)%>%
  # create data frame of all pairs of sample_size and trial
  mutate(simulation=pmap(.l=list(trial,sample_size),
                         .f=~rnorm(.y,mean=mu,sd=sqrt(sigma_sqr))))%>%
  # simulate sequences of Gaussian random variables
  mutate(sample_var=map_dbl(.x=simulation,.f=var))
  # compute the sample variances
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ggplot()+labs(x="Sample size",y="Variance")+theme_bw()+
  geom_point(data=sim_by_n_df,
                  aes(x=sample_size,y=sample_var,color="Sample",
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  # scatter plot of sample variances
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Examples of statistical estimation

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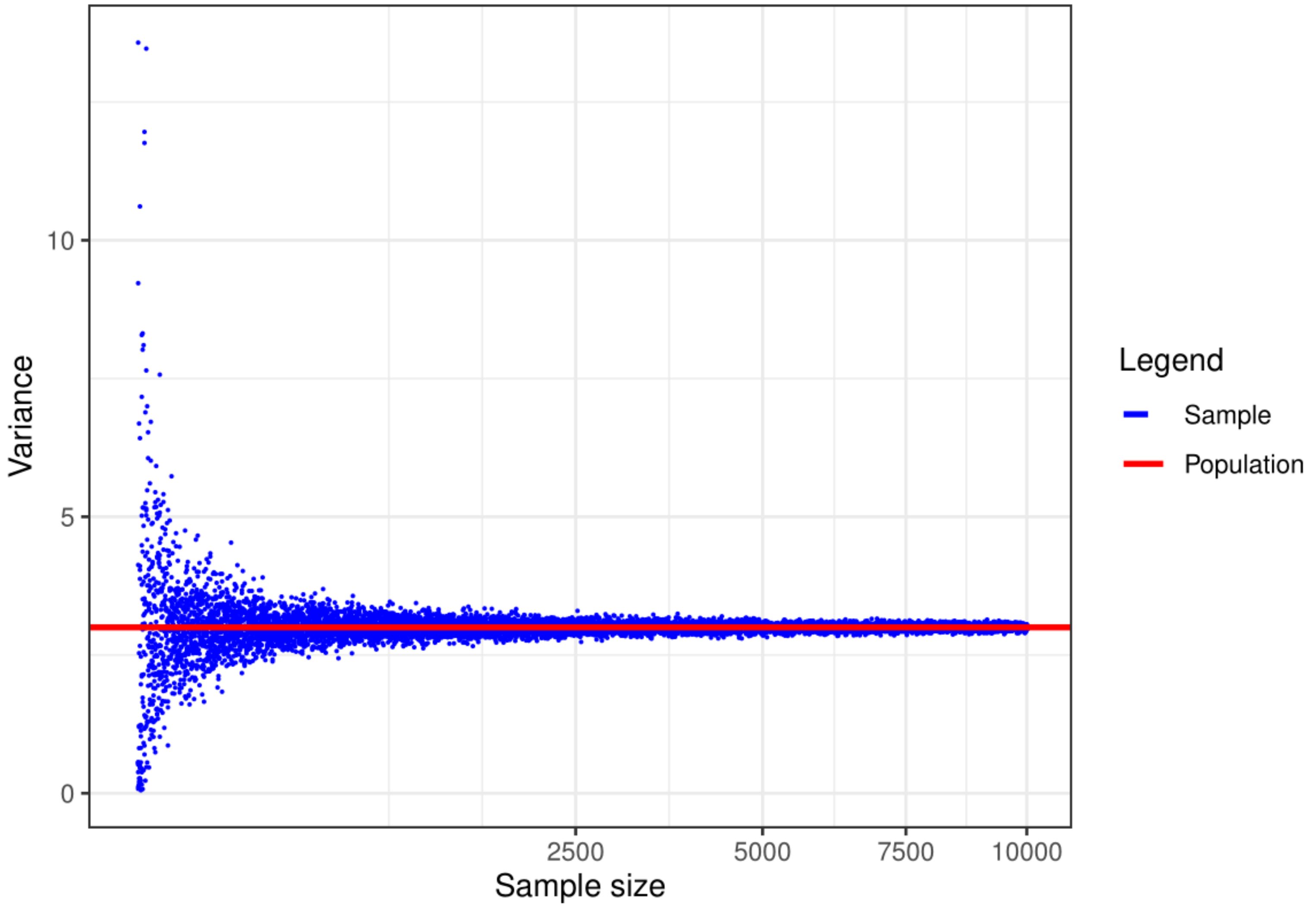

The Cauchy distribution

A random variable X has a Cauchy distribution with location parameter θ if its density is

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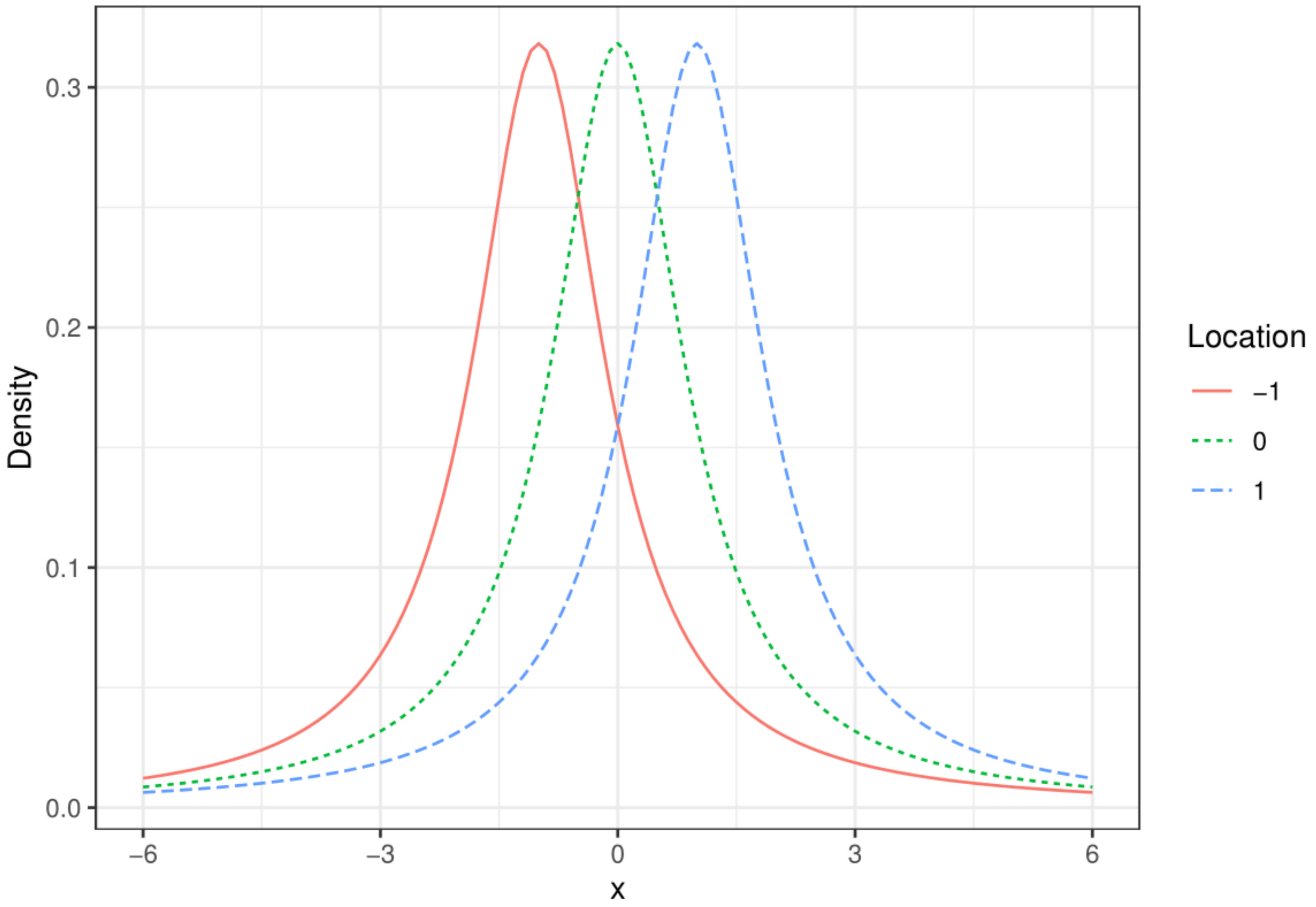

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  # create data frame of all pairs of sample_size and trial
  mutate(simulation=pmap(.l=list(trial,sample_size),.f=~rcauchy(.y,location=theta)))%>%
  # simulate sequences of Cauchy random variables
  mutate(sample_mean=map_dbl(.x=simulation,.f=mean))
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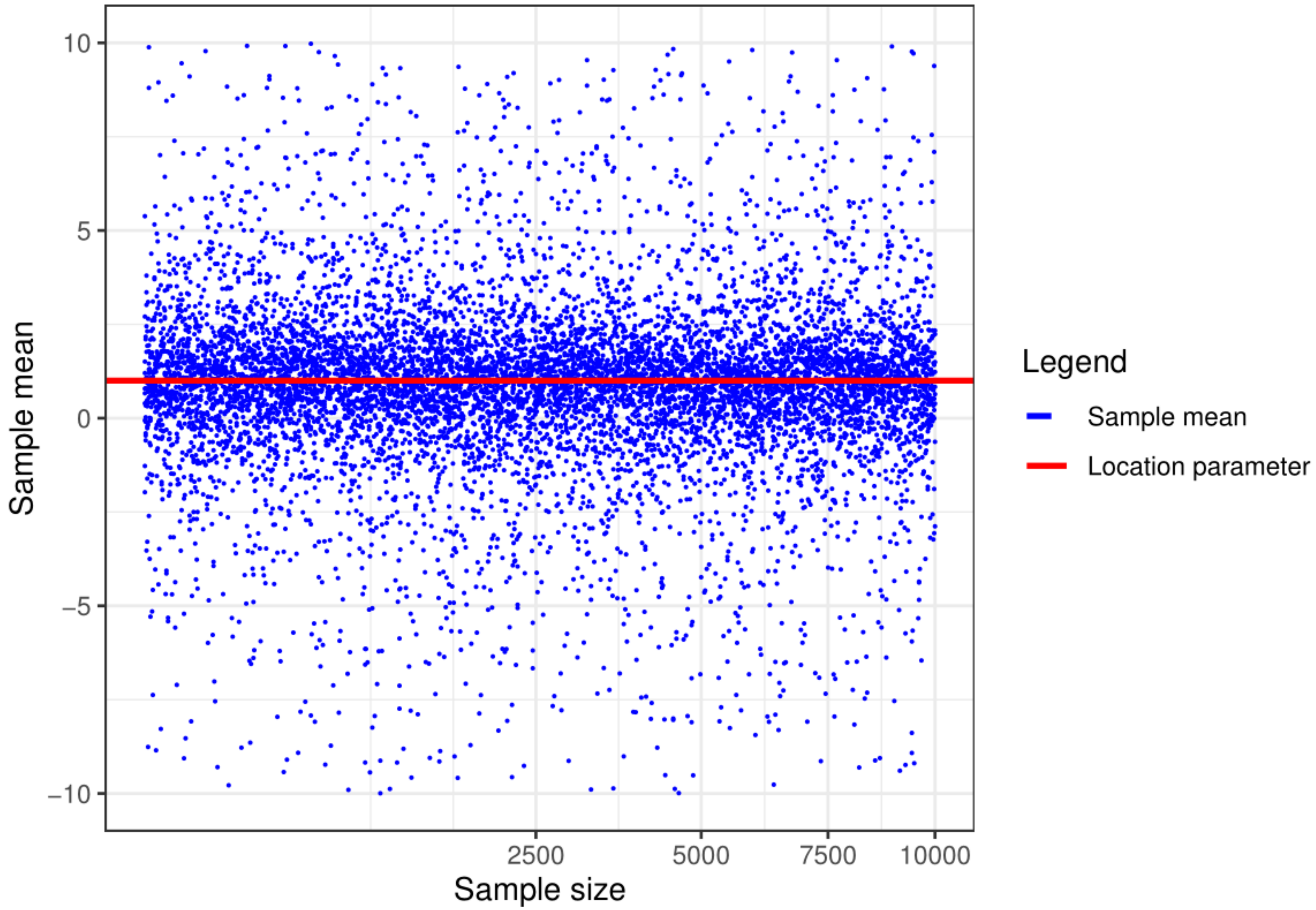

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A Cauchy random variable has cumulative distribution function

$$F_\theta(x) = \mathbb{P}\left(X \leq x\right) = \int_{-\infty}^{x} f_\theta(z) = \frac{1}{\pi}\arctan\left(x-\theta\right) + \frac{1}{2}.$$

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The population median of the Cauchy random variable is

$$F_\theta^{-1}\left(\frac{1}{2}\right) = \inf\left\{x \in \mathbb{R} \;:\; F_\theta(x) \geq \frac{1}{2}\right\} = \theta.$$

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Suppose we have i.i.d. data $X_1, \ldots, X_n$ with Cauchy density f_θ.

A natural estimator for θ is the sample median $\hat{\theta} = \text{Median}(X_1, \ldots, X_n)$.

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set.seed(0)
num_trials_per_sample_size<-10
max_sample_size<-10000
theta<-1


sim_by_n_df<-crossing(trial=seq(num_trials_per_sample_size),
                      sample_size=seq(to=sqrt(max_sample_size),by=0.1)**2)%>%
  # create data frame of all pairs of sample_size and trial
  mutate(simulation=pmap(.l=list(trial,sample_size),.f=~rcauchy(.y,location=theta)))%>%
  # simulate sequences of Cauchy random variables
  mutate(sample_median=map_dbl(.x=simulation,.f=median))
  # compute the sample median
```

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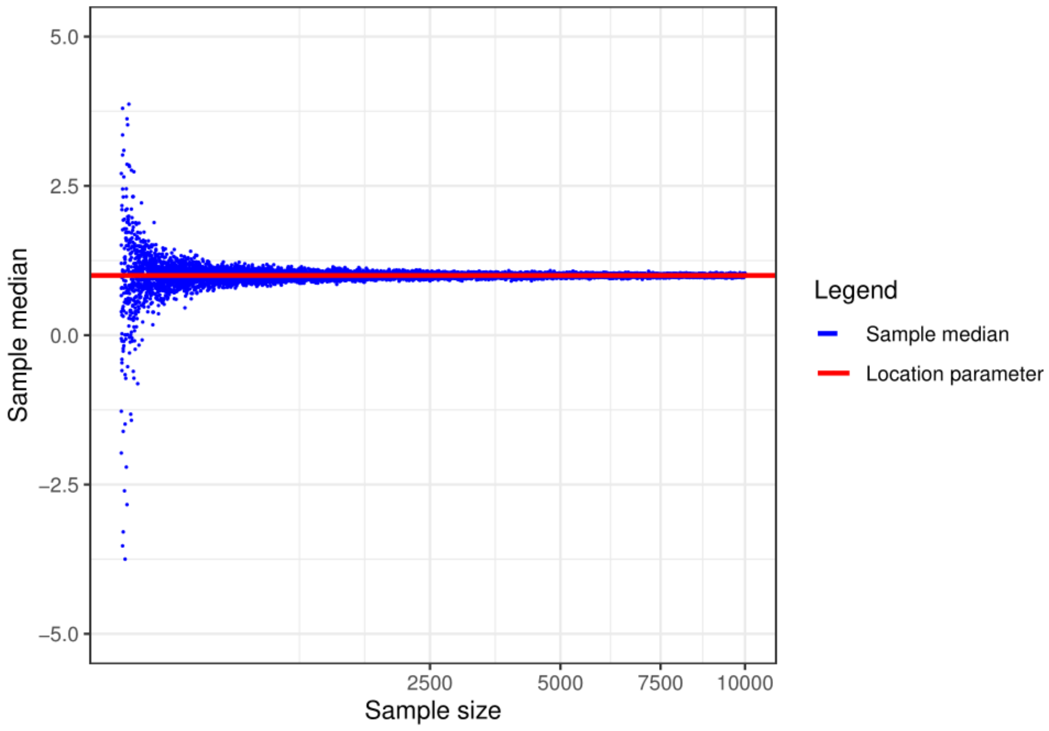
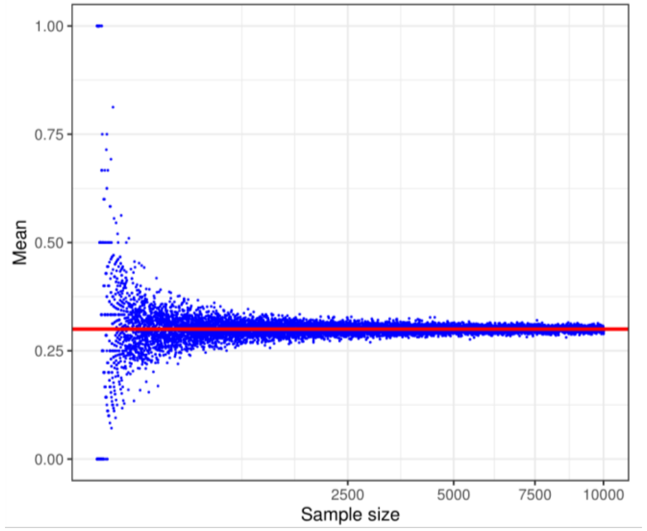


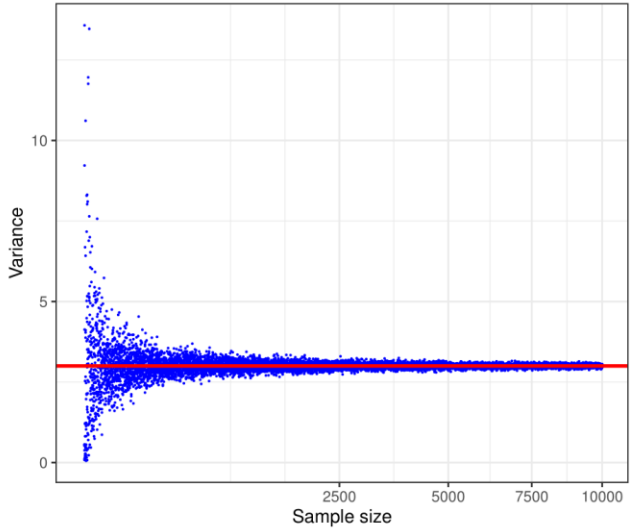
Consistency

We are interested in statistical estimators $\hat{\theta}$ which tend towards θ as $n \to \infty$.

Example 1



Example 2



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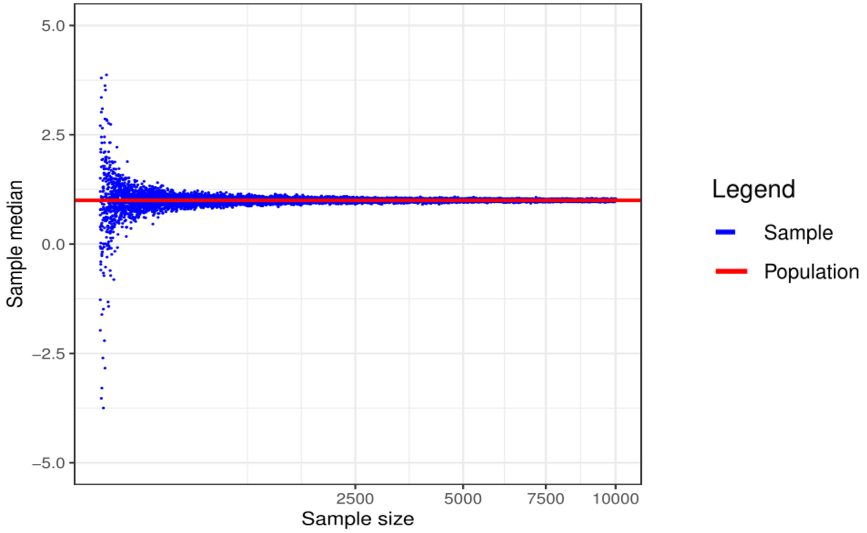

We refer to such $\hat{\theta}$ as **consistent** estimators of θ.

Consistent estimators

A sample statistic $\hat{\theta} = g(X_1, \cdots, X_n)$ of a population parameter θ is **consistent** if

$$\hat{\theta} = g(X_1, \cdots, X_n) \quad \to \quad \theta \quad \text{as} \quad n \to \infty$$

More precisely, $\hat{\theta} = g(X_1, \cdots, X_n)$ is a consistent estimator of θ if for all $\epsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left[|g(X_1, \cdots, X_n) - \theta| > \epsilon\right] = 0.$$

Consistency and the law of large numbers

A sample statistic $\hat{\theta}_n$ is a **consistent** estimator of θ if for all $\epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\left(|\hat{\theta}_n - \theta| \geq \epsilon\right) = 0$.

Theorem (Bernoulli, circa. 1700). *Let $X : \Omega \to \mathbb{R}$ be a random variable with a well-defined expectation $\mu = \mathbb{E}(X)$. Let $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ be a sequence of independent copies of X. Then for all $\epsilon > 0$,*

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Example 1

Suppose that $X_1, \ldots, X_n$ are i.i.d. with expectation $\mu = \mathbb{E}(X_i) \in \mathbb{R}$.

Then the sample mean $\overline{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$ is a consistent estimator of μ.

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Example 2

Suppose that $X_1, \ldots, X_n$ are i.i.d. with expectation $\mu = \mathbb{E}(X_i) \in \mathbb{R}$ and variance $\sigma^2 = \mathrm{Var}(X_i) \in \mathbb{R}$.

Then the sample variance $\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ is a consistent estimator of σ^2.

Consistency of sample statistics

A sample statistic $\hat{\theta}_n$ is a **consistent** estimator of θ if for all $\epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\left(|\hat{\theta}_n - \theta| \geq \epsilon\right) = 0$.

Examples

Suppose that $X_1, \ldots, X_n$ is a well-behaved sequence of independent and identically distributed data.

1. The sample mean $\overline{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$ is a consistent estimator of μ.
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4. Suppose that $X_1, \ldots, X_n$ are continuous random variables with continuous density f.

   Given $q \in (0,1)$, let $x_q \in \mathbb{R}$ be the population q-quantile and suppose $f(x_q) > 0$.

   Then the sample q-quantile is a consistent estimator or the population q-quantile x_q.

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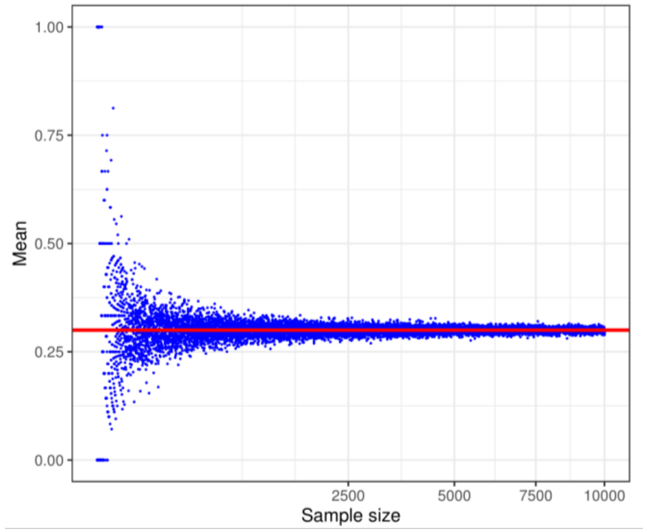
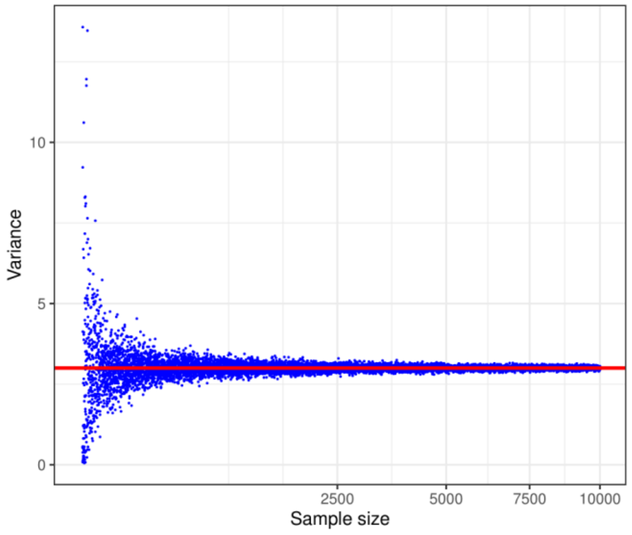


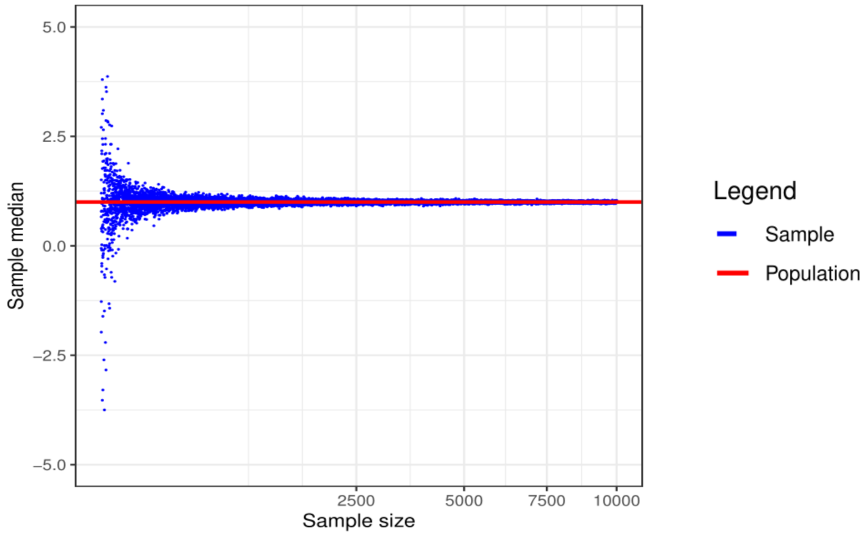
Sample mean

Example 2



Sample variance

Example 3



Sample median

Statistical bias

The **bias** of an estimator $\hat{\theta} = g(X_1, \cdots, X_n)$ of a population parameter θ is

$$\mathsf{Bias}(\hat{\theta}) := \mathbb{E}(\hat{\theta}) - \theta.$$

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Given an independent and identically distributed sample $X_1, \cdots, X_n$ we have

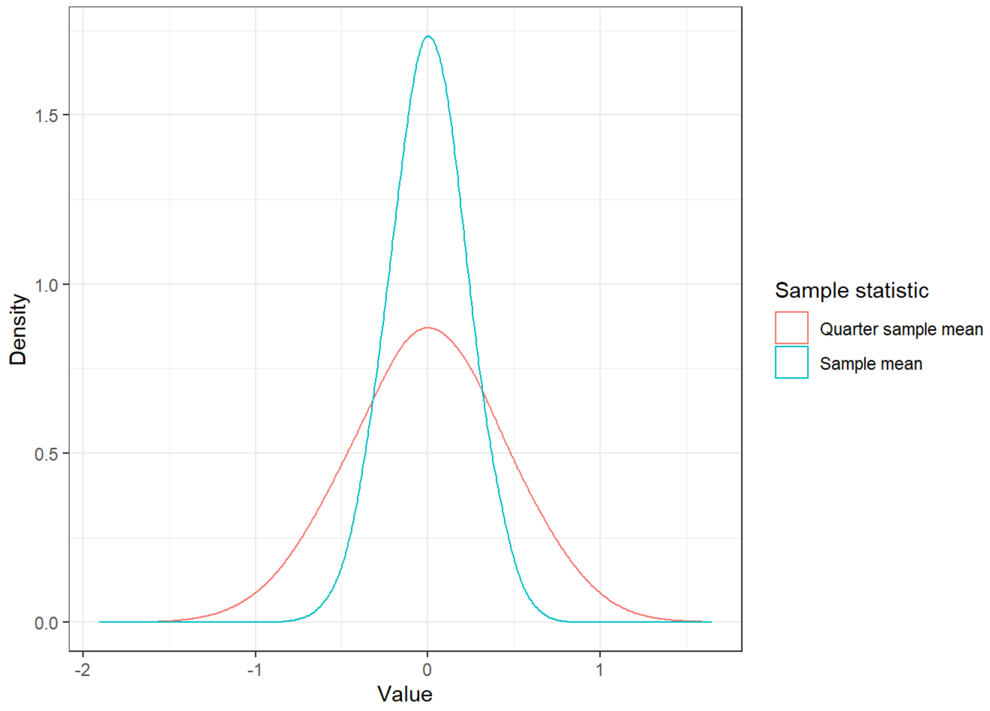
$$\mathrm{Bias}(\overline{X}) = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] - \mu = 0.$$

$$\mathrm{Bias}(s^2) = \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2\right] - \sigma^2 = 0.$$

The variance of an estimator

The sample mean is an unbiased estimator for the population mean….

…. but so is the sample mean computed from just the first quarter of the data!



There are many unbiased estimators… however, many such estimators are very high variance!

Bias and variance of an estimator

Suppose that $\hat{\theta}$ is an estimator of a population parameter θ .

The **bias** of the estimator is defined by

$$\mathsf{Bias}(\hat{\theta}) := \mathbb{E}(\hat{\theta}) - \theta.$$

The **variance** of the estimator is defined by

$$\mathsf{Var}(\hat{\theta}) := \mathbb{E}\left[\left\{\hat{\theta} - \mathbb{E}(\hat{\theta})\right\}^2\right]$$

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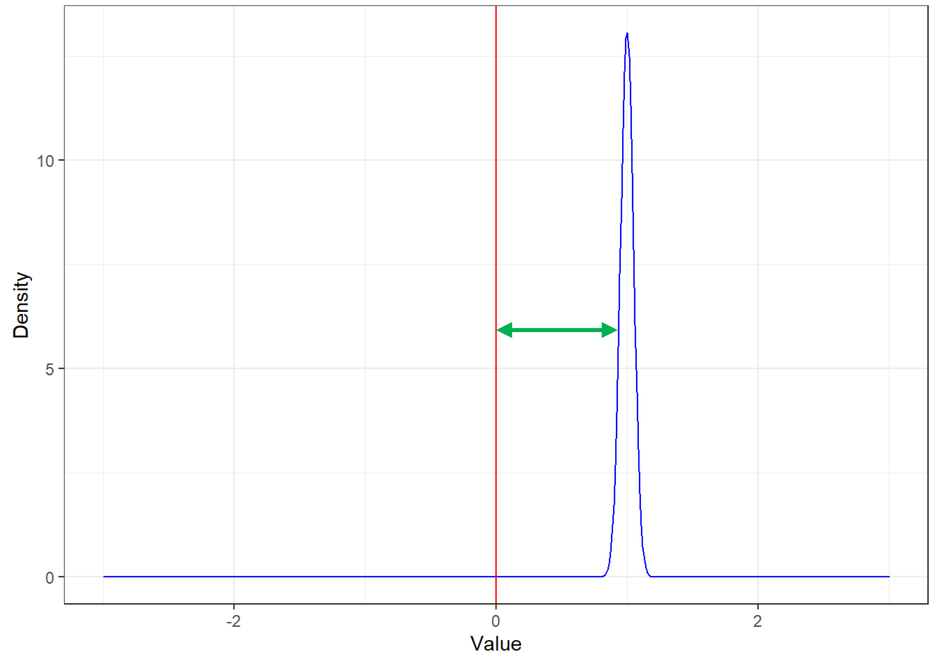

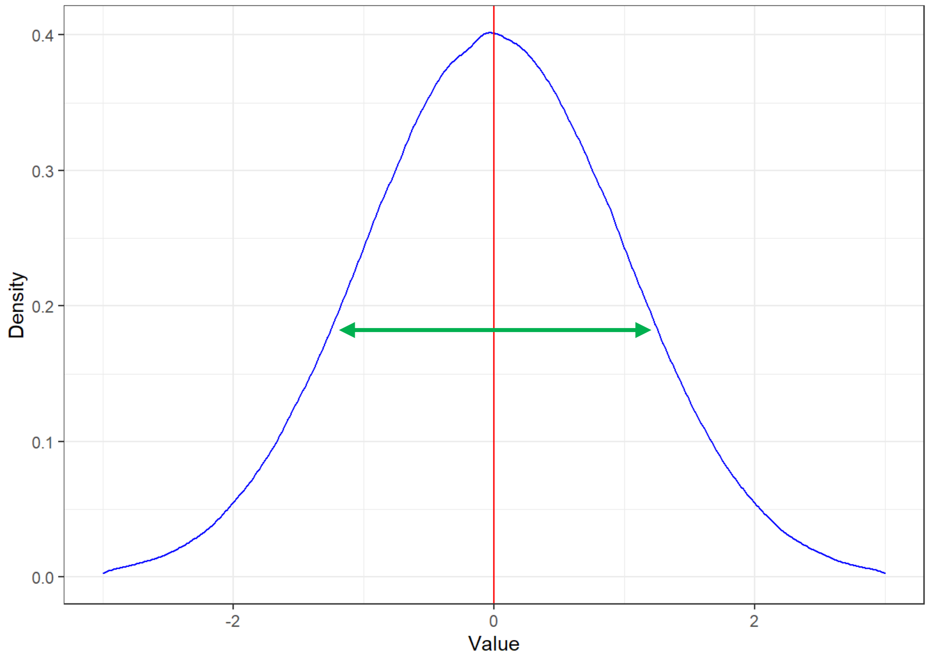

Mean squared error

Suppose that $g(X_1, \cdots, X_n)$ is an estimator of a population parameter θ

The **bias** of the estimator is $\mathsf{Bias}(\hat{\theta}) := \mathbb{E}(\hat{\theta}) - \theta.$

The **variance** of the estimator is $\mathsf{Var}(\hat{\theta}) := \mathbb{E}\{\hat{\theta} - \mathbb{E}(\hat{\theta})\}^2$

The **mean square error** of an estimator is $\mathsf{MSE}(\hat{\theta}) := \mathbb{E}\{(\hat{\theta} - \theta)^2\}.$

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Mean squared error

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Proof.

$$\begin{aligned}
\mathsf{MSE}(\hat{\theta}) &:= \mathbb{E}\left[\left(\hat{\theta}-\theta\right)^2\right] \\
&= \mathbb{E}\left[\left\{\left(\hat{\theta}-\mathbb{E}[\hat{\theta}]\right)+\left(\mathbb{E}[\hat{\theta}]-\theta\right)\right\}^2\right] \\
&= \mathbb{E}\left[\left(\hat{\theta}-\mathbb{E}[\hat{\theta}]\right)^2+\left(\mathbb{E}[\hat{\theta}]-\theta\right)^2+2\left(\hat{\theta}-\mathbb{E}[\hat{\theta}]\right)\left(\mathbb{E}[\hat{\theta}]-\theta\right)\right] \\
&= \mathbb{E}\left[\left(\hat{\theta}-\mathbb{E}[\hat{\theta}]\right)^2\right]+\left(\mathbb{E}[\hat{\theta}]-\theta\right)^2 \\
&= \mathsf{Var}(\hat{\theta})+\mathsf{Bias}(\hat{\theta})^2.
\end{aligned}$$

$\square$

Minimum variance unbiased estimator

Suppose that $\hat{\theta}$ is a statistical estimator of a population parameter θ.

We say that $\hat{\theta}$ is **unbiased** if $\mathbb{E}(\hat{\theta}) = \theta$.

We say that $\hat{\theta}$ is a **minimum variance unbiased estimator** (MVUE) if

1. $\hat{\theta}$ is unbiased i.e. $\mathbb{E}(\hat{\theta}) = \theta$;
2. $\hat{\theta}$ is minimum variance i.e. we have $\mathrm{Var}(\hat{\theta}) \leq \mathrm{Var}(\tilde{\theta})$, for all $\tilde{\theta}$ with $\mathbb{E}(\tilde{\theta}) = \theta$.

Remark: A MVUE also has minimal mean-square error over all unbiased estimators.

This is a consequence of the bias-variance decomposition $\mathsf{MSE}(\hat{\theta}) = \mathsf{Bias}(\hat{\theta})^2 + \mathsf{Var}(\hat{\theta})$.

Minimum variance unbiased estimator

A minimum variance unbiased estimator (MVUE) has minimum variance over all unbiased estimators.

<u>Example 1</u>

Suppose $X_1, \cdots, X_n \sim \mathcal{B}(q)$ are independent and identically distributed. Then

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$ is a MVUE of q

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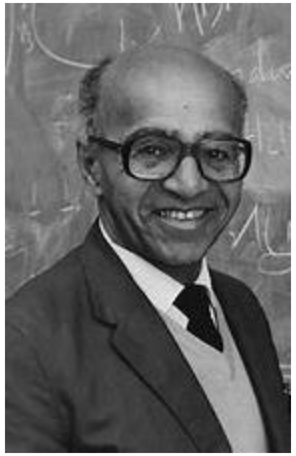
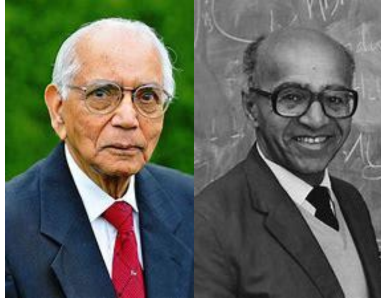


This is a consequence of the Rao-Blackwell theorem.

Minimum variance unbiased estimator



Example 2

Suppose $X_1, \cdots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ are independent and identically distributed. Then

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{is a MVUE of} \quad \mu = \mathbb{E}[X_i]$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \quad \text{is a MVUE of} \quad \sigma^2 = \mathbb{E}\left[(X_i - \mu)^2\right]$$

This is also a consequence of the Rao-Blackwell theorem.

What have we covered?

- We considered sample statistics as estimators of parameters of interest.
- We introduced the concept of statistical consistency.
- We also considered the idea of statistical bias and the bias-variance decomposition.
- We will discussed the concept of a minimum variance unbiased-estimator.
- In the next lecture we will consider a general-purpose approach to deriving statistical estimators.

University of BRISTOL

Thanks for listening!

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Include EMATM0061 in the subject of your email.

Statistical Computing & Empirical Methods (EMATM0061)

MSc in Data Science, Teaching block 1, 2021.