<u>$\ell 2$ norm</u>

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \|\vec{x}\|_{\ell 2}^2 = x_1^2 + x_2^2 + \cdots x_n^2$$

$$= \vec{x}^H \cdot \vec{x}$$

$$= \langle \vec{x}, \vec{x} \rangle$$

$$D(\vec{x}) = \frac{1}{2} \left\| \vec{y}_k - S_k(\vec{x}) \right\|_{\ell 2}^2$$

$$= \frac{1}{2}\left[(\vec{y}_k - S_k(\vec{x}))^H \cdot (\vec{y}_k - S_k(\vec{x})) \right]$$

$$= \frac{1}{2}\left[(\vec{y}_k^H - S_k(\vec{x})^H) \cdot (\vec{y}_k - S_k(\vec{x})) \right]$$

$$= \frac{1}{2}\left[\vec{y}_k^H \cdot \vec{y}_k - \vec{y}_k^H S_k(\vec{x}) - S_k(\vec{x})^H \cdot \vec{y}_k + S_k(\vec{x})^H \cdot S_k(\vec{x}) \right] \quad \cdots \text{①}$$

$$= \frac{1}{2} \langle \vec{y}_k, \vec{y}_k \rangle - \mathrm{Re}\{ S_k(\vec{x}), \vec{y}_k^H \} + \frac{1}{2} \langle S_k(\vec{x}), S_k(\vec{x}) \rangle$$

Define:

$$\underline{\frac{\partial}{\partial x_j} S_k(\vec{x})} = \begin{bmatrix} \frac{\partial}{\partial x_j}[S_k(\vec{x})]_1 \\ \vdots \\ \frac{\partial}{\partial x_j}[S_k(\vec{x})]_m \end{bmatrix} \qquad \nabla D(\vec{x}) = \left[\frac{\partial}{\partial x_1} D(\vec{x}) \quad \cdots \quad \frac{\partial}{\partial x_n} D(\vec{x}) \right]$$

substitute into each column

Hessian Matrix

$$\frac{\partial}{\partial \vec{x}} S_k(\vec{x}) = \left[\frac{\partial}{\partial x_1} S_k(\vec{x}) \quad \cdots \quad \frac{\partial}{\partial x_n} S_k(\vec{x}) \right] = \begin{bmatrix} \frac{\partial}{\partial x_1}[S_k(\vec{x})]_1 & \cdots & \frac{\partial}{\partial x_n}[S_k(\vec{x})]_1 \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}[S_k(\vec{x})]_m & \cdots & \frac{\partial}{\partial x_n}[S_k(\vec{x})]_m \end{bmatrix}_{m \times n}$$

Starting from ①

$$D(\vec{x}) = \frac{1}{2}\cdot\left[\vec{y}_K^H\cdot\vec{y}_K - \vec{y}_K^H S_K(\vec{x}) - S_K(\vec{x})^H\cdot\vec{y}_K + S_K(\vec{x})^H\cdot S_K(\vec{x})\right]$$

$$= \frac{1}{2}\cdot\left[\vec{y}_K^H\cdot\vec{y}_K - 2\,\mathrm{Re}\{\vec{y}_K^H S_K(\vec{x})\} + S_K(\vec{x})^H\cdot S_K(\vec{x})\right]$$

$$= \frac{1}{2}\vec{y}_K^H\cdot\vec{y}_K - \mathrm{Re}\{\vec{y}_K^H\cdot S_K(\vec{x})\} + \frac{1}{2}S_K(\vec{x})^H\cdot S_K(\vec{x})$$

$$\frac{\partial}{\partial x_j}D(\vec{x}) = \underbrace{\frac{1}{2}\frac{\partial}{\partial x_j}\vec{y}_K^H\cdot\vec{y}_K}_{=0,\text{ since }\vec{y}_K\text{ is not dependent on }\vec{x}} - \frac{\partial}{\partial x_j}\cdot\mathrm{Re}\{\vec{y}_K^H\cdot S_K(\vec{x})\} + \frac{1}{2}\frac{\partial}{\partial x_j}\left[S_K(\vec{x})^H\cdot S_K(\vec{x})\right]$$

$$= -\mathrm{Re}\left\{\frac{\partial}{\partial x_j}\vec{y}_K^H\cdot S_K(\vec{x})\right\} + \frac{1}{2}\underbrace{\frac{\partial}{\partial x_j}\left[S_K(\vec{x})^H\cdot S_K(\vec{x})\right]}_{\text{how do we solve this? lets do derivation for single function value}}$$

Real valued

$$s(x) = a(x) + j\,b(x)$$

$$s^H(x) = s^*(x) = a(x) - j\,b(x)$$

Therefore:

$$s^H(x)\cdot s(x) = a^2(x) + b^2(x)$$

$$\frac{\partial}{\partial x}\left[s^H(x)\cdot s(x)\right] = \frac{\partial}{\partial x}\left[a^2(x) + b^2(x)\right]$$

$$= 2a(x)\frac{\partial}{\partial x}a(x) + 2b(x)\frac{\partial}{\partial x}b(x) \quad \ldots ②$$

Now consider:

$$2\,s^H(x)\cdot\frac{\partial}{\partial x}s(x)$$

$$= 2\left[a(x) - j\,b(x)\right]\cdot\frac{\partial}{\partial x}\left[a(x) + j\,b(x)\right]$$

$$= 2\left[a(x) - j\,b(x)\right]\cdot\left[\frac{\partial}{\partial x}a(x) + j\frac{\partial}{\partial x}b(x)\right]$$

$$= 2\left[a(x)\frac{\partial}{\partial x}a(x) + j\,a(x)\frac{\partial}{\partial x}b(x) - j\,b(x)\frac{\partial}{\partial x}a(x) + b(x)\frac{\partial}{\partial x}b(x)\right]$$

$$= \underbrace{2a(x)\frac{\partial}{\partial x}a(x) + 2b(x)\frac{\partial}{\partial x}b(x)}_{\text{Real component}} + 2j\underbrace{\left[a(x)\frac{\partial}{\partial x}b(x) - b(x)\frac{\partial}{\partial x}a(x)\right]}_{\text{Imaginary component}} \quad ③$$

If we compare ② and ③, we see that:

$$\frac{\partial}{\partial x}\left[s^H(x)\cdot s(x)\right] = \mathrm{Re}\left\{2\ s^H(x)\cdot \frac{\partial}{\partial x}s(x)\right\} \quad \cdots ④$$

We use this property below

$$\frac{\partial}{\partial x_j}D(\vec{x}) = -\mathrm{Re}\left\{\frac{\partial}{\partial x_j}\ \vec{y}_K^H \cdot S_K(\vec{x})\right\} + \frac{1}{2}\underbrace{\frac{\partial}{\partial x_j}\left[S_K(\vec{x})^H\cdot S_K(\vec{x})\right]}$$

$$= -\mathrm{Re}\left\{\frac{\partial}{\partial x_j}\ \vec{y}_K^H \cdot S_K(\vec{x})\right\} + \mathrm{Re}\left\{S_K^H(\vec{x})\cdot\frac{\partial}{\partial x_j}S_K(\vec{x})\right\}$$

constant with respect to x_j

$$= -\mathrm{Re}\left\{\vec{y}_K^H\cdot\frac{\partial}{\partial x_j}S_K(\vec{x})\right\} + \mathrm{Re}\left\{S_K^H(\vec{x})\cdot\frac{\partial}{\partial x_j}S_K(\vec{x})\right\}$$

$$= \mathrm{Re}\left\{\vec{S}_K^H(\vec{x})\cdot\frac{\partial}{\partial x_j}S_K(\vec{x}) - \vec{y}_K^H\cdot\frac{\partial}{\partial x_j}S_K(\vec{x})\right\}$$

$$\frac{\partial}{\partial x_j}D(\vec{x}) = \mathrm{Re}\left\{\left(S_K^H(\vec{x}) - \vec{y}_K^H\right)\cdot\frac{\partial}{\partial x_j}S_K(\vec{x})\right\}$$

$$\nabla D(\vec{x}) = \frac{\partial}{\partial \vec{x}}D(\vec{x})$$

$$= \left[\frac{\partial}{\partial x_1}D(\vec{x}) \quad \frac{\partial}{\partial x_2}D(\vec{x}) \quad \cdots \quad \frac{\partial}{\partial x_n}D(\vec{x})\right]$$

$$= \mathrm{Re}\left\{\left(S_K(\vec{x}) - y_K\right)^H\cdot\frac{\partial}{\partial \vec{x}}S_K(\vec{x})\right\}$$

$$\left[\nabla D(\vec{x})\right]^H = \mathrm{Re}\left\{\left[\frac{\partial}{\partial \vec{x}}S_K(\vec{x})\right]^H\cdot\left(S_K(\vec{x}) - \vec{y}_K\right)\right\}$$

This will be used for gradient descent protocol