review: 1、链式法则推导(差项/商项) 2、反函数求导 3、Taylor 公式 Newton 二项式

第一型曲线积分

定义：$\int_\alpha^\beta f[x(t), y(t)]\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$

$$\sum_{n=1}^{\infty} \rho(x(\eta_i), y(\eta_i)) \int_{t_{i-1}}^{t_i} \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

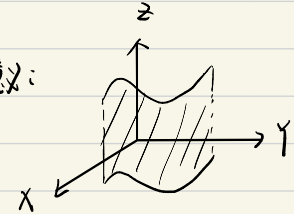
$$= \sum_{n=1}^{\infty} \rho(x(\eta_i), y(\eta_i)) \sqrt{x'(\eta_i)^2 + y'(\eta_i)^2}\,dt$$

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$$

$$\frac{\partial(F,G)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix} \neq 0$$

ensure 局部单调映射、
隐函数映射定理

几何意义：



弧长微元

$$\int_L f(x,y)\,ds = \int_\alpha^\beta f(\varphi(t), \psi(t))\sqrt{\varphi'(t)^2 + \psi'(t)^2}\,dt$$

第二型曲线积分

$F = (P(x,y), Q(x,y))$

$S = (\Delta x, \Delta y)$

$\int_S P(x,y)\,dx + Q(x,y)\,dy$

$$\sum_{i=1}^{n} F_i \cdot S_i = \sum_{i=1}^{n} \|F_i\| \cdot \|S_i\| \cos(F_i, S_i)$$

$$= \sum P(x_i, y_i)\Delta x_i + Q(x_i, y_i)\Delta y_i$$

$$\xrightarrow{\|P\| \to 0} \int_S P(x,y)\,dx + Q(x,y)\,dy$$

换元 $\Updownarrow$

$$\xrightarrow{\begin{cases} x = x(t) \\ y = y(t) \end{cases}} \int_S [P(x(t), y(t))\,x'(t) + Q(x(t), y(t))\,y'(t)]\,dt$$

(proof)

$$\lim_{\|T\| \to 0} \sum_{i=1}^{n} P(x(\xi_i), y(\xi_i))\,\Delta x_i + Q(x(\xi_i), y(\xi_i))\,\Delta y_i$$

$\|T\|$ — 曲线的长度→0时

$t_i \to 0$ 时长度→0

中值 theorem

$$= \lim_{\|T\| \to 0} \sum_{i=1}^{n} P(x(\xi_i), y(\xi_i))\,x'(w_i)\Delta t_i + Q(x_i, y_i)\,y'(w_i')$$

$\Delta x_i = x'(w_i)\,\Delta t_i$ $w_i \in [t_{i-1}, t_i]$.

$$\left| Q(x_i, y_i)\,y'(w_i') - Q(x_i, y_i)\,y'(w_i) \right| < M[y'(w_i) - y'(w_i')] < \varepsilon$$

故可用 w_i 代替 w_i'

$$\int_L P\frac{dx}{ds} + Q\frac{dy}{ds} \cdot ds$$

故：$\int_L P\frac{dx}{ds} + Q\frac{dy}{ds} = \int_L P\,dx + Q\,dy$

第一、二型曲线积分的关系

$\int_L Pdx + Qdy = \int_L (P\frac{dx}{ds} + Q\frac{dy}{ds}) \cdot ds$ $\quad \begin{cases} x = x(s) \\ y = y(s) \end{cases}$ $s_0 \le s \le s_1$

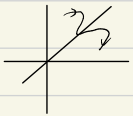
$= \int_L (P\cos\alpha + Q\cos\beta) \cdot ds$ $\quad \frac{dx}{ds} = \cos\alpha \quad \frac{dy}{ds} = \cos\beta \quad ds = \sqrt{d^2x + d^2y}$

prt

③ $f(x,y) = f(y,x)$ $\quad f(x,y,z)$ 三维情况是否可轮换

$\int_L f(x,y)dx \cancel{=} \int_L f(y,x)dy$ 破坏 $\quad \int_L f(x,y)\frac{dx}{ds}ds$

$\int_L f(y,x)\frac{dy}{ds}ds$



计算定积分时，牛莱公式反映了区间上定积分与其端点上原函数之间的联系：$\int_a^b \frac{\partial F(x)}{\partial x}dx = F(b)-F(a)$

格林公式则反映了平面上二重积分与其边界上第二型曲线积分之间的联系：$\iint\limits_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma = \oint_L Q\,dy+P\,dx$

格林公式

定义：若函数 $P(x,y)$，$Q(x,y)$ 在闭区域 D 上有连续的一阶偏导数，（正方向为左手法则）

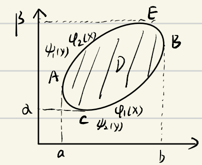
则有 $\iint\limits_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma = \oint_L P\,dx+Q\,dy$

L 为区域 D 的边界曲线，并取正方向

ACB　AEB

$$\begin{cases} \varphi_1(x) \le y \le \varphi_2(x) & (x\in[a,b]) \\ \psi_1(y) \le x \le \psi_2(y) & (y\in[\alpha,\beta]) \end{cases}$$

CAE　CBE



$$\iint\limits_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma = \oint_L P\,dx+Q\,dy$$

proof:

$$\begin{aligned}\iint\limits_D \frac{\partial Q}{\partial x}d\sigma &= \int_\alpha^\beta dy\int_{\psi_1(y)}^{\psi_2(y)}\frac{\partial Q}{\partial x}\,dx \\ &= \int_\alpha^\beta Q(\psi_2(y),y)-Q(\psi_1(y),y)\,dy \\ &= \int_{CBE}Q(\psi_2(y),y)\,dy-\int_{CAE}Q(\psi_1(y),y)\,dy \\ &= \int_{CBE}Q(\psi_2(y),y)\,dy+\int_{EAC}Q(\psi_1(y),y)\,dy \\ &= \oint_L Q\,dy\end{aligned}$$

$$\begin{aligned}\iint\limits_D \frac{\partial P}{\partial y}d\sigma &= \int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)}\frac{\partial P}{\partial y}\,dy \\ &= \int_a^b P(x,\varphi_2(x))-P(x,\varphi_1(x))\,dx \\ &= \int_{AEB}P(x,\varphi_2(x))\,dx-\int_{ACB}P(x,\varphi_1(x))\,dx \\ &= -\int_{BEA}P(x,\varphi_2(x))\,dx-\int_{ACB}P(x,\varphi_1(x))\,dx \\ &= -\oint_L P\,dx\end{aligned}$$

$$\iint\limits_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma = \oint_L Q\,dy+P\,dx$$

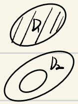
平面区域 D 的面积 S_D 的公式：$S_D = \iint\limits_D d\sigma = \frac{1}{2}\oint_L x\,dy-y\,dx = \begin{cases}\oint_L x\,dy \\ \oint_L y\,dx\end{cases}$

（具体推导与定积分的应用时闭合曲线面积的推导类似）

凑微分法 (Pay attention)

单连通区域：对于平面区域D内任一封闭曲线，皆可不经过D以外的点而连续收缩于属于D的某一点。

复连通区域：



二维连通(面)

一维连通为线

$f(x,y)$在(x,y)这点可微　$\frac{\partial f}{\partial x}\Big|_{(x,y)}$　$\frac{\partial f}{\partial y}\Big|_{(x,y)}$

$f(x+\Delta x, y+\Delta y) - f(x,y) = A\cdot\Delta x + B\cdot\Delta y + o(\rho)$　$\rho = \sqrt{\Delta x^2+\Delta y^2}$

偏导 连续 $\longrightarrow$ 可微 $\rightarrow$ 连续

可微 $\not\longrightarrow$ 偏导∃

可微 $\Longleftrightarrow$ 有切平面，(垂直切平面的法向量) $\rightarrow$ 偏导数

曲线积分与路线的无关性

lagrange.

定理2：设D是单连通闭区域。若函数$P(x,y)$　$Q(x,y)$在D内连续且具有一阶偏导数

TAFE：

1) 沿D内任一按段光滑封闭曲线L，有 $\oint_L P\,dx + Q\,dy = 0$

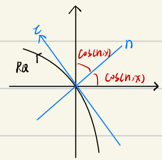
2) 对D中任一按段光滑曲线L，曲线积分 $\oint_L P\,dx+Q\,dy$ 与路线无关，只与始终点有关

3) $P\,dx+Q\,dy$ 是D内某一函数 $u(x,y)$ 的全微分，在D内有 $du = P\,dx+Q\,dy$

4) D内处处成立 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

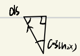
$$\frac{\int_x^{x+\Delta x} P\,dx}{\Delta x} = P\,dx$$

验证格林公式另一形式：$\iint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dxdy = \oint_{\partial D}[P\cos(n,x)+Q\cos(n,y)]\,ds$ （n为D的边界∂D上任一点处的外法线向量）



n：法线

τ：切线



$\uparrow\ ds\cdot\cos(n,x) = dy$

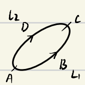
$\leftarrow\ ds\cdot\cos(n,y) = -dx$

$$\oint_L [P\cos(n,x)+Q\cos(n,y)]\,ds = \oint_L (P\cdot dy + (-Qdx)) = \iint_{\partial D} \frac{\partial P}{\partial x} - \frac{-\partial Q}{\partial y}\,d\sigma = \iint_{\partial D} \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\,d\sigma$$

连通的开集 — 开域。

开域及其边界的点 — 闭域

1) → 2)：



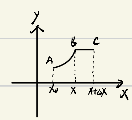
$\oint_L Pdx+Qdy=0$

$\oint_{L_1+L_2} Pdx+Qdy = \oint_{L_1} Pdx+Qdy + \oint_{L_2^-} Pdx+Qdy$

$\oint_{L_1} Pdx+Qdy = \oint_{L_2} Pdx+Qdy$

看偏导数

2) → 3)：



已知曲线积分与路径无关，set (x_0,y_0)为D中固定点，(x,y)为D中任意点

def $u(x,y) = \int_{(x_0,y_0)}^{(x,y)} Pdx+Qdy$ / $u(x,y) = \int_{AB} Pdx+Qdy$

$u(x+\Delta x,y) = \int_{(x_0,y_0)}^{(x+\Delta x,y)} Pdx+Qdy$ $\qquad u(x+\Delta x,y)-u(x,y) = \int_{(x,y)}^{(x+\Delta x,y)} Pdx+Qdy$

令 $\lim\limits_{\Delta x\to 0}\frac{\partial u}{\partial x}(x,y) = \lim\limits_{\Delta x\to 0}\frac{P(\xi,y)\cdot\Delta x}{\Delta x} = P(x,y)$ $\qquad = \int_x^{x+\Delta x} P(x,y)\,dx \xrightarrow{\text{中值定理}} P(\xi,y)\cdot\Delta x$

$\xi\in(x,x+\Delta x)$

同理可得 $\frac{\partial u}{\partial y}(x,y) = Q(x,y)$

?

得 $du = Pdx+Qdy$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}$$

$$= \frac{\int_{AC} Pdx+Qdy - \int_{AB} Pdx+Qdy}{\Delta x}$$

由于D内曲线积分与路径无关，故 $\qquad \xi\in(x,x+\Delta x)$

$$= \frac{\int_{BC} Pdx+Qdy}{\Delta x} = \frac{\int_x^{x+\Delta x} P\,dx}{\Delta x} = P(\xi,y) = P(x,y)$$

$\frac{\partial u}{\partial x} = P(x,y)$

$\frac{\partial u}{\partial y} = Q(x,y)$

3) → 4)：

即 已证明

由3)可知 $\exists\, u(x,y)$ s.t. $\frac{\partial u}{\partial x}=P$ $\quad \frac{\partial u}{\partial y}=Q$

故 $\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial x\partial y}$ $\qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial y\partial x}$

因为 $P(x,y)$ $Q(x,y)$ 具有一阶偏导数，故 $\frac{\partial P}{\partial y}$ $\frac{\partial Q}{\partial x}$ 在D上连续

$\frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial y\partial x} = \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial P}{\partial y}$

连续可使用积分中值定理可PW

4) → 1)：

set L为D内按段光滑曲线，记L所围区域为σ，由于D为单连通区域，σ在D内。

应用 Green 公式 在D内恒有 $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ 得 $\oint_L Pdx+Qdy = \iint_\sigma \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma = 0$

n维空间第一型曲线积分.

$$l\begin{cases} x_1 = x_1(t) \\ \vdots \\ x_n = x_n(t). \end{cases}$$

$y = f(x_1,\cdots,x_n)$　　求 $\int_l y\,ds$

对 l 作分割. $P_0P_1, P_1P_2, \cdots, P_{i-1}P_i, \cdots P_{m-1}P_m$

$x_1: x_1(t_0), x_1(t_1), \cdots, x_1(t_m)$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$x_n: x_n(t_0), x_n(t_1)\cdots, x_n(t_m)$

$$\lim_{\|T\|\to 0}\sum_{i=1}^{m} f(x_1(\xi_i), x_2(\xi_i)\cdots x_n(\xi_i))\sqrt{x_1'(\xi_i)^2+\cdots+x_n'(\xi_i)^2}\,\Delta t_i$$

∞

$$= \int_\alpha^\beta f(x_1(t), x_2(t), \cdots, x_n(t))\sqrt{x_1'(t)^2+\cdots+x_n'(t)^2}\,dt$$

$f(x_0)$ 连续　　　　同胚.

$\Updownarrow$

$f((x_0-\delta),(x_0+\delta)) \subset U(f(x_0),\varepsilon).$

小面积微元.

$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$. 一阶导数.

$\Leftarrow$

$\int_a^b f(x)dx$　$x = x(t)$　$x'(t) \neq 0$. 一阶导数

混合偏导数连续必相等.

3. 本项目成员对于系统的真正完善还需物理以及地理方面的……关知识，并研读国内外相关文献。

数学作业纸

班级：　　　姓名：　　　编号：　　　科目　　　第　　页

1、$|V|=\inf\{\sum_{i=1}^{n}|I_i| \mid V\subset\bigcup_{i=1}^{n}I_i$，$I_i$为长方体$\}$（内/外测度）.

2、$V\subset T$，T是包含V的一个长方体，$F(x,y,z)=\begin{cases} f(x,y,z) & (x,y,z)\in V \\ 0 & (x,y,z)\in T-V \end{cases}$

✓积分区域为 xy 型 — 先一后二 — 几何.

三重积分.　　~~$\forall x(C(x)\to\exists$~~　　~~$\forall y\exists x A(y)\to$~~

1、柱面坐标公式

$\forall x(C(x)\to\exists y(A(y)\wedge\neg B(x,y)))\vdash\forall x(\forall y A(y)\to B(x,y))\to\neg C(x))$

① $\forall x(C(x)\to\exists y(A(y)\wedge\neg B(x,y)))$　前提引入

② $\exists x C(x)\to\forall x\exists y(A(y)\wedge\neg B(x,y))$

$\exists x(\forall y A(y)\to B(x,y)\to\forall x\neg C(x)$.

③ $C(t)\to\exists y(A(y)\wedge\neg B(t,y))$

④ $\neg\exists y(A(y)\wedge\neg B(t,y))\to\neg C(t)$

⑤ ~~$\forall y(A(y)\wedge\neg B(t,$~~

~~$\exists x\ \forall y(\neg A(y)$~~

$\forall y(\neg A(y)\vee B(t,y))\to\neg C(t)$.

$A-(B\cup C)=(A-B)-C$

75−22−7+3

22+53+27

50

50−22−4+3

24+5=27

45　+27

84

42−11+3

45−11

34. 4,6,大

~~75−22−3~~

50−22−4

75−