线性变换的像与核

$\mathscr{A}(\varepsilon_1 \cdots \varepsilon_n) = (\varepsilon_1 \cdots \varepsilon_m) A_{m\times n}$

def: set $\mathscr{A} \in Hom(V, W)$，则

$Im\,\mathscr{A} = \mathscr{A}(V) = \{\mathscr{A}(\alpha) \mid \alpha \in V\}$ 是 W 的子空间，称为 $\mathscr{A}$ 的像

$ker\,\mathscr{A} = \mathscr{A}^{-1}(0) = \{\alpha \mid \mathscr{A}(\alpha) = 0\}$ 是 V 的子空间，称为 $\mathscr{A}$ 的核

W 中的零元

def: $\dim(Im\,\mathscr{A})$ 为 $\mathscr{A}$ 的秩，记为 $r(\mathscr{A})$

$\dim(ker\,\mathscr{A})$ 为 $\mathscr{A}$ 的零度

example: $A \in F^{m\times n}$ $\quad \mathscr{A}: F^n \to F^m$, $\alpha \mapsto A\alpha$

$ker(\mathscr{A}) = \{\alpha \mid A\alpha = 0\} = \{AX=0$ 的解空间$\}$ $\quad \dim = n - r(A)$.

$Im(\mathscr{A}) = \{A\alpha \mid \alpha \in F^n\} = \langle A\varepsilon_1, \cdots, A\varepsilon_n \rangle$ $\quad \dim = r(A)$

theorem: 秩—零度定理

set $\dim V = n$，基为 $\xi_1 \sim \xi_n$

$\dim U = m$，基为 $\eta_1 \sim \eta_m$

$\phi \in Hom(V, U)$，其中 $\phi(\xi_1 \sim \xi_n) = (\eta_1 \sim \eta_m) A_{m\times n}$

令 $\tau_1: V \to F^n$ $\qquad \tau_2: U \to F^m$

$\alpha = \sum a_i \xi_i \mapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\qquad \beta = \sum b_i \eta_i \mapsto \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

（以 ξ_i 为基时可用 $\begin{smallmatrix} a_1 \\ \vdots \\ a_n \end{smallmatrix}$ 坐标表示） （以 η_i 为基时可用 $\begin{smallmatrix} b_1 \\ \vdots \\ b_m \end{smallmatrix}$ 坐标表表示）

1) 则此图可交换：$\tau_2\phi = \mathscr{A}\cdot\tau_1$

$$\begin{tikzcd} \alpha \quad V \arrow[r, "\phi"] \arrow[d, "\tau_1"'] & U \quad \beta \arrow[d, "\tau_2"] \\ F^n \arrow[r, "\mathscr{A}"] & F^m \end{tikzcd}$$

2) $\tau_1(ker\,\phi) = ker\,\mathscr{A}$

3) $\dim ker\,\phi = n - r(A)$

4) $\tau_2(Im\,\phi) = Im(\mathscr{A})$

5) $\dim(Im\,\phi) = r(A)$

1) 令 $\alpha = \sum a_i \xi_i$，则 $\phi(\alpha) = \sum a_i \phi(\xi_i) = (\eta_1 \sim \eta_m) A_{m\times n} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

故 $(\tau_2\phi)(\alpha) = A_{m\times n} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

$\tau_1(\alpha) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\quad (\mathscr{A}\tau_1)(\alpha) = A_{m\times n} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

$$\begin{tikzcd} \alpha = \sum a_i \xi_i \arrow[r, mapsto, "\phi"] \arrow[d, mapsto, "\tau_1"'] & \sum a_i \phi(\xi_i) = (\eta_1 \sim \eta_m) A_{m\times n} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \arrow[d, mapsto, "\tau_2"] \\ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \arrow[r, "\mathscr{A}"] & A_{m\times n} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \end{tikzcd}$$

2) 要证 $\tau_1(ker\,\phi) = ker\,\mathscr{A}$，即证两个空间相互包含即可

$\subset$: $\forall \alpha \in ker\,\phi$ $\ \phi(\alpha) = 0$，而 $\mathscr{A}(\tau_1\alpha) = \tau_2\phi(\alpha) = 0$

$\supset$: $\forall \beta \in ker\,\mathscr{A}$ $\ \mathscr{A}(\beta) = 0$，而 τ_1 为同构映射 $\exists x \in V$ s.t. $\tau_1(x) = \beta$ $\ \mathscr{A}(\beta) = \mathscr{A}(\tau_1 x) = \tau_2\phi(x) = 0$

而 τ_2 也为同构，$\phi(x) = 0$，$x \in ker\,\phi$，$\tau_1(x) = \alpha$，故 $\alpha \in \tau_1(ker\,\phi)$

3) 由于 τ_1 为同构可得 $\dim ker\,\phi = \dim ker\,\mathscr{A} = n - r(A)$

5) 由于 τ_2 为同构得 $\dim Im\,\phi = \dim Im\,\mathscr{A} = r(A)$

4) 要证 $\tau_2(Im\,\phi) = Im(\mathscr{A})$，即证左右相互包含

$\tau_2 Im\,\phi \subseteq Im(\mathscr{A})$: $\forall \alpha \in Im\,\phi$，$\exists x \in V$ s.t. $\phi(x) = \alpha$ $\quad \tau_2\phi(x) = \mathscr{A}\tau_1(x)$

$\tau_2\phi(x) \in Im\,\mathscr{A}$ $\quad \tau_2 Im\,\phi \subset Im\,\mathscr{A}$

$\tau_2 Im\,\phi \supseteq Im(\mathscr{A})$: $\forall \beta \in Im\,\mathscr{A}$ $\ \exists a \in V$ s.t. $\tau_1(a) = y$ $\ \mathscr{A}(y) = \beta$

$\beta = \mathscr{A}(\tau_1 a) = \tau_2\phi(a) \in \tau_2 Im\,\phi(a)$ $\quad Im\,\mathscr{A} \subset \tau_2 Im\,\phi$

(秩—零度定理)

theorem: $\dim \mathrm{Im}\,\phi + \dim \ker\phi = \dim V$

corollary: $\ker\phi = V \iff \phi = 0 \iff \mathrm{Im}\,\phi = 0$

ker ϕ=0 means 满足 $AX=0$ 的 X 为零向量 即列向量组线性无关

$\ker\phi = 0 \iff \phi$ 为单射 $\iff r(A_{m\times n}) = n \iff A$ 列满秩

$\mathrm{Im}\,\phi = U \iff \phi$ 为满射 $\iff r(A_{m\times n}) = m \iff A$ 行满秩

use 秩零度定理 proof 很简便。

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "T_1"'] & U \arrow[d, "T_2"] \\ F^n \arrow[r, "\mathcal{A}"] & F^m \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\phi"] \arrow[d, "T_1"'] & \phi(x) \arrow[d, "T_2"] \\ \alpha \arrow[r, "\mathcal{A}"] & A\alpha \end{tikzcd}$$

$\dim V = 5 \quad \varepsilon_1 \sim \varepsilon_5$

$\dim U = 4 \quad \eta_1 \sim \eta_4$

$\mathcal{A} \in \mathrm{Hom}(V, U)$

$\sigma: \mathrm{Hom}(V, U) \longrightarrow M_{m\times n}(F)$

$\phi \longmapsto A_{m\times n}$

$T: V \longrightarrow F^n$

$\alpha = \sum a_i\varepsilon_i \longmapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

theorem: (秩—零度定理) set $\dim V = n$, $\mathcal{A} \in \mathrm{Hom}(V, U)$ 则 $\dim \ker\mathcal{A} + \dim \mathrm{Im}\,\mathcal{A} = n$

proof

假设 $\dim \ker\mathcal{A} = s$ 且 $\varepsilon_1 \sim \varepsilon_s$ 为 $\ker\mathcal{A}$ 的一组基向量，将其扩充为 V 中的一组基 $\varepsilon_1 \sim \varepsilon_s, \varepsilon_{s+1}, \cdots \varepsilon_n$

下证 $\varepsilon_{s+1}, \cdots, \varepsilon_n$ 为 $\mathrm{Im}\,\mathcal{A}$ 的一组基向量

线性无关：

$a_{s+1}\mathcal{A}(\varepsilon_{s+1}) + \cdots + a_n\mathcal{A}(\varepsilon_n) = 0 \iff \mathcal{A}(a_{s+1}\varepsilon_{s+1} + \cdots + a_n\varepsilon_n) = 0$ 故 $a_{s+1}\varepsilon_{s+1} + \cdots + a_n\varepsilon_n$ 可用 $\varepsilon_1 \sim \varepsilon_s$ 线性表出 而 $\varepsilon_{s+1} \sim \varepsilon_n$ 为 V 的一组基，故 $\varepsilon_i\ (i=1 \sim n) \neq 0$，$a_{s+1} = \cdots = a_n = 0$

故 $\varepsilon_{s+1} \sim \varepsilon_n$ 线性无关

线性表出：

$\forall \beta \in \mathrm{Im}\,\mathcal{A}$，$\exists \alpha \in V$ s.t. $\mathcal{A}(\alpha) = \beta$，而 $\alpha \in V$，有 $\alpha = \sum_{i=1}^{n} a_i\varepsilon_i$ $\quad \mathcal{A}(\alpha) = \sum_{i=1}^{n} a_i\mathcal{A}(\varepsilon_i) = \sum_{i=1}^{s} a_i\mathcal{A}(\varepsilon_i) + \sum_{j=s+1}^{n} a_j\mathcal{A}(\varepsilon_j) = \ker\mathcal{A} + \mathrm{Im}\,\mathcal{A} = \mathrm{Im}\,\mathcal{A} = \sum_{j=s+1}^{n} a_j\mathcal{A}(\varepsilon_j)$

故 $\forall \beta \in \mathrm{Im}\,\mathcal{A}$ 均可由 $\varepsilon_{s+1} \sim \varepsilon_n$ 线性表出

corollary: 有限维线性空间上的线性变换，它为单射 $\iff$ 它为满射

example:

1、$\dim V = n$, $\dim U = m$, $\mathcal{A} \in \mathrm{Hom}(V, U)$, proof: $\exists$ V 的基 $\varepsilon_1 \sim \varepsilon_n$ U 的基 $\eta_1 \sim \eta_m$ s.t. $\mathcal{A}(\varepsilon_1 \sim \varepsilon_n) = (\eta_1 \sim \eta_m)\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$

proof

2、set $A \in \mathbb{R}^{n\times n}$，满足 $A^2 = A$，proof: A 相似于对角矩阵 $\mathrm{diag}\{1, 1, \cdots, 1, 0, \cdots, 0\}$

(通过 秩零度定理证明，interesting)

证明更为巧妙，但是需要了解商空间概念。

线性变换的不变子空间

def: 设 $\mathscr{A}\in Hom(V,V)$，W是V的子空间，若 $\forall\alpha\in W$，有 $\mathscr{A}(\alpha)\in W$，则称W为V的不变子空间，简称 $\mathscr{A}$—子空间

几类不变子空间：平凡的不变子空间：零空间和整个空间 V
数乘变换
线性变换的 $\ker\mathscr{A}$ 与 $Im\,\mathscr{A}$

def: if $\mathscr{A}\in Hom(V,V)$，W为V的不变子空间，构造线性变换

$$\mathscr{A}|_W: W\to W,\quad \alpha\mapsto\mathscr{A}(\alpha)$$

称 $\mathscr{A}|_W$ 为导出变换 / $\mathscr{A}$在W上限制变换

lemma: set $\mathscr{A}\in Hom(V,V)$ $W=\langle\alpha_1\sim\alpha_s\rangle$ 是V的子空间，W为V的不变子空间 $\Longleftrightarrow$ $\mathscr{A}(\alpha_i)\in W$ $\forall i\in\{1,2,\cdots,s\}$

验证不变子空间的例子：

1、$\mathscr{A}\in Hom(V,V)$ V_1与V_2为$\mathscr{A}$-子空间，proof: V_1+V_2为$\mathscr{A}$-子空间
$V_1\cap V_2$为$\mathscr{A}$-子空间

first step: $V_1\cap V_2$为子空间（非空，加、数乘封闭）

second step: $v\in V_1, v\in V_2$ $\mathscr{A}(v)\in V_1$ $\mathscr{A}(v)\in V_2$ $\mathscr{A}(v)\in V_1\cap V_2$

2、$\mathscr{A}\in Hom(V,V)$ $\mathscr{B}\in Hom(V,V)$ 且 $\mathscr{A}\mathscr{B}=\mathscr{B}\mathscr{A}$，则 $\boxed{\begin{array}{l}Im\,\mathscr{A}\text{为}\mathscr{B}\text{—子空间}\\ \ker\mathscr{A}\text{为}\mathscr{B}\text{—子空间}\end{array}}$ → 结论可直接用

$\forall a\in V$ $\exists\ \alpha=\mathscr{A}(a)$

$\mathscr{B}(Im\,\mathscr{A})\in Im\,\mathscr{A}$ $Im\,\mathscr{A}=\{\overset{\mathscr{A}(a)}{\alpha}\mid a\in V\}$

$\ker\mathscr{A}=\{a\mid\mathscr{A}(a)=0\}$

3、$\mathscr{A}\in Hom(V,V)$ $f(x)\in F(x)$ 则 $f(\mathscr{A})\mathscr{A}=\mathscr{A}f(\mathscr{A})$ 且 $f(\mathscr{A})$的 Im，ker，特征子空间均为 $\mathscr{A}$—子空间

显然

$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$

$f(\mathscr{A})=a_nA^n+a_{n-1}A^{n-1}+\cdots+a_1A+a_0$

$\mathscr{A}f(\mathscr{A})=f(\mathscr{A})\mathscr{A}=a_nA^{n+1}+\cdots+a_1A^2+a_0A$

显然可交换

$V_\lambda=\{v\in V\mid\mathscr{A}(v)=\lambda v\}$，proof $\mathscr{A}(V_\lambda)\subseteq V_\lambda$

$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$

$f(\mathscr{A})\cancel{=}f(A)=a_nA^n+a_{n-1}A^{n-1}+\cdots+a_1A+a_0$

$Im\,f(\mathscr{A})=\{f(\mathscr{A})(\alpha)\mid\alpha\in V\}$ $\mathscr{A}[f(\mathscr{A})\alpha]=f(\mathscr{A})[\mathscr{A}\alpha]\in Im\,f(\mathscr{A})$

$\ker f(\mathscr{A})=\{\beta\mid f(\mathscr{A})(\beta)=0\}$ $0=\mathscr{A}f(\mathscr{A})\beta=f(\mathscr{A})\mathscr{A}(\beta)$ 故 $\mathscr{A}(\beta)\in\ker f(\mathscr{A})$

$V_\lambda=\{\gamma\mid f(\mathscr{A})(\gamma)=\lambda_0\gamma\}$ $\mathscr{A}f(\mathscr{A})(\gamma)=f(\mathscr{A})\mathscr{A}(\gamma)=\lambda_0\gamma$ $\in V_\lambda$

不变子空间的直和和矩阵化简的关系

prop： set $\dim V=n$ $\mathscr{A}\in Hom(V,V)$ 假设W是$\mathscr{A}$的r维不变子空间，基$(\varepsilon_1\sim\varepsilon_r)$ 将其扩充为V的一组基$(\varepsilon_1\sim\varepsilon_r,\varepsilon_{r+1}\sim\varepsilon_n)$

$\Updownarrow$

那么 $\mathscr{A}(\varepsilon_1\sim\varepsilon_r,\varepsilon_{r+1}\sim\varepsilon_n)=(\varepsilon_1\sim\varepsilon_r,\varepsilon_{r+1}\sim\varepsilon_n)\begin{pmatrix}A_r & B\\ 0 & C\end{pmatrix}=(\varepsilon_1\sim\varepsilon_r,\varepsilon_{r+1}\sim\varepsilon_n)\begin{pmatrix}a_{11}\cdots a_{r1} & a_{r+1,1}\cdots a_{n,1}\\ \vdots\quad\vdots & \vdots\\ a_{1r}\cdots a_{rr} & a_{r+1,r}\cdots a_{n,r}\\ 0\cdots 0 & a_{r+1,r+1}\cdots a_{n,r+1}\\ \vdots\quad\vdots & \vdots\\ 0\cdots 0 & a_{r+1,n}\cdots a_{nn}\end{pmatrix}$

$\mathscr{A}(\varepsilon_1)=a_{11}\varepsilon_1+a_{12}\varepsilon_2+\cdots+a_{1r}\varepsilon_r$ 故$\varepsilon_{r+1}\sim\varepsilon_n$前系数均为0

$\vdots$

$\mathscr{A}(\varepsilon_r)=a_{r1}\varepsilon_1+a_{r2}\varepsilon_2+\cdots+a_{rr}\varepsilon_r$

corollary： 1、$\dim V=n$，$\mathscr{A}\in Hom(V,V)$，假设W_1 W_2是$\mathscr{A}$的不变子空间，且$V=W_1\oplus W_2$，设W_1一组基为$\varepsilon_1\sim\varepsilon_r$，$W_2$的一组基$\varepsilon_{r+1}\sim\varepsilon_n$

$\Updownarrow$

那么 $\mathscr{A}(\varepsilon_1\sim\varepsilon_n)=(\varepsilon_1\sim\varepsilon_n)\begin{pmatrix}A_r & 0\\ 0 & B_{n-r}\end{pmatrix}$

2、$\dim V=n$，$\mathscr{A}\in Hom(V,V)$，假设W_i为$\mathscr{A}$的不变子空间 $(i=1\sim s)$ 且$V=W_1\oplus\cdots\oplus W_s$，设$W_i$的一组基为$\varepsilon_{i1}\sim\varepsilon_{it_i}$

$\Updownarrow$

$\mathscr{A}(\varepsilon_{11}\sim\varepsilon_{1t_1},\cdots,\varepsilon_{s1}\sim\varepsilon_{st_s})=(\varepsilon_{11}\sim\varepsilon_{1t_1},\cdots,\varepsilon_{s1}\sim\varepsilon_{st_s})\begin{pmatrix}A_{t_1} & 0 & \cdots & 0\\ 0 & A_{t_2} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & A_{t_s}\end{pmatrix}$

lemma： set $\mathscr{A}\in Hom(V,V)$ $\mathscr{B}\in Hom(V,V)$ $\mathscr{A}\mathscr{B}=\mathscr{B}\mathscr{A}$ ✓

记 $V_\lambda=\{\alpha\in V\mid \exists m\ s.t.\ (\mathscr{A}-\lambda\mathscr{E})^m(\alpha)=0\}$

proof： 1) V_λ为V的子空间

2) V_λ为$\mathscr{A}$-子空间

3) V_λ为$\mathscr{B}$-子空间

1) $\alpha=0$时 $(\mathscr{A}-\lambda\mathscr{E})^m(\alpha)=0$ 故$0\in V_\lambda$，V_λ非空集合 非空

m与n不一定相同

$\forall\alpha,\beta\in V_\lambda$ $\exists m,n\in N^*$ s.t. $(\mathscr{A}-\lambda\mathscr{E})^m\alpha=0$ $(\mathscr{A}-\lambda\mathscr{E})^n\beta=0$ 加法封闭

$p=\max(m,n)$ 则 $(\mathscr{A}-\lambda\mathscr{E})^p(\alpha)=0$ $(\mathscr{A}-\lambda\mathscr{E})^p(\beta)=0$ $\Rightarrow(\mathscr{A}-\lambda\mathscr{E})^p(\alpha+\beta)=0$ $\alpha+\beta\in V_\lambda$

数乘封闭显然，故V_λ为V的子空间

2) 要证 $\forall\alpha\in V_\lambda$ $\mathscr{A}(\alpha)\in V_\lambda$，设$\alpha\in V_\lambda$ $\exists m$ s.t. $(\mathscr{A}-\lambda\mathscr{E})^m(\alpha)=0$

$\mathscr{A}(\alpha)\in V_\lambda\Leftarrow\exists n$ s.t. $(\mathscr{A}-\lambda\mathscr{E})^n(\mathscr{A}(\alpha))=0$ m.n不一定相等

由数学归纳法易得 $\mathscr{A}$与$(\mathscr{A}-\lambda\mathscr{E})^m$可交换 （m=1时显然成立，不妨设m=k-1时成立，下证m=k时成立）

$(\mathscr{A}-\lambda\mathscr{E})^n\mathscr{A}(\alpha)=\mathscr{A}[(\mathscr{A}-\lambda\mathscr{E})^n\alpha]=\mathscr{A}0=0$

$\mathscr{A}(\mathscr{A}-\lambda\mathscr{E})^k=\mathscr{A}\underbrace{(\mathscr{A}-\lambda\mathscr{E})\cdots(\mathscr{A}-\lambda\mathscr{E})}_{k个}$

$(\mathscr{A}-\lambda\mathscr{E})^k\mathscr{A}=\underbrace{(\quad)\cdots(\quad)}_{k个}\mathscr{A}=\mathscr{A}\underbrace{(\quad)\cdots(\quad)}_{k个}$ ◻

3) 显然

$\mathscr{A}\in Hom(V,V)$ $\mathscr{B}\in Hom(V,V)$，W为$\mathscr{A}$和$\mathscr{B}$一子空间

proof： W为$\mathscr{A}+\mathscr{B}$一子空间

W为$\mathscr{A}\mathscr{B}$一子空间

$\forall\alpha\in W$ $\mathscr{A}(\alpha)\in W$

$\beta\in W$ $\mathscr{B}(\beta)\in W$

$(\mathscr{A}+\mathscr{B})(\alpha)=\mathscr{A}(\alpha)+\mathscr{B}(\alpha)\in W$ $\mathscr{A}+\mathscr{B}$子空间

$(\mathscr{A}\mathscr{B})(\alpha)=\mathscr{A}(\mathscr{B}(\alpha))\in W$ $\mathscr{A}\mathscr{B}$子空间

$=\mathscr{B}(\mathscr{A}(\alpha))$

由$\mathscr{A}\mathscr{B}=\mathscr{B}\mathscr{A}$可知 $Im\mathscr{A}$为$\mathscr{B}$一子空间

$Ker\mathscr{A}$为$\mathscr{B}$一子空间

set $\dim V=3$，$\varepsilon_1\sim\varepsilon_3$为一组基，$\mathscr{A}\in Hom(V,V)$

$\mathscr{A}(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(\varepsilon_1,\varepsilon_2,\varepsilon_3)\begin{pmatrix}3&1&-1\\2&2&-1\\2&2&0\end{pmatrix}$，proof $\langle\varepsilon_3,\varepsilon_1+\varepsilon_2-2\varepsilon_3\rangle$为$\mathscr{A}$一子空间

$\mathscr{A}(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(\varepsilon_1,\varepsilon_2,\varepsilon_3)\begin{pmatrix}3&1&-1\\2&2&-1\\2&2&0\end{pmatrix}\longrightarrow$ 经过映射后的坐标

即证 $\mathscr{A}(\varepsilon_1)\in\varepsilon_1$ $\mathscr{A}\varepsilon_2\in\varepsilon_2$ $\mathscr{A}(\varepsilon_3)\in\varepsilon_3$

$\mathscr{A}\langle\varepsilon_3,\varepsilon_1+\varepsilon_2-2\varepsilon_3\rangle\in\langle\varepsilon_3,\varepsilon_1+\varepsilon_2-2\varepsilon_3\rangle$

$\mathscr{A}(\varepsilon_3)=\begin{pmatrix}3&1&-1\\2&2&-1\\2&2&0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\-1\\0\end{pmatrix}$

$\mathscr{A}(\varepsilon_1+\varepsilon_2-2\varepsilon_3)=\begin{pmatrix}3&1&-1\\2&2&-1\\2&2&0\end{pmatrix}\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}1\\1\\-2\end{pmatrix}$

$\mathscr{A}(\varepsilon_3)$与$\mathscr{A}(\varepsilon_1+\varepsilon_2-2\varepsilon_3)$能否用$\varepsilon_3$与$\varepsilon_1+\varepsilon_2-2\varepsilon_3$线性表出

ε_3经过$\mathscr{A}$的坐标为$\begin{pmatrix}-1\\-1\\0\end{pmatrix}$ $\begin{pmatrix}0&1\\0&1\\1&-2\end{pmatrix}x_1=\begin{pmatrix}-1\\-1\\0\end{pmatrix}$ $x_1=\begin{pmatrix}-2\\-1\end{pmatrix}$

$\varepsilon_1+\varepsilon_2-2\varepsilon_3$经过$\mathscr{A}$坐标为$\begin{pmatrix}6\\6\\4\end{pmatrix}$ $\begin{pmatrix}0&1\\0&1\\1&-2\end{pmatrix}x_2=\begin{pmatrix}6\\6\\4\end{pmatrix}$ $x_2=\begin{pmatrix}16\\6\end{pmatrix}$

根子空间分解定理（巧妙的）

特征子空间 $V_{\lambda_i} = \{\xi \mid \mathcal{A}\xi = \lambda_i\xi,\ \xi \in V\text{总}\} = \ker(\mathcal{A}-\lambda_i\mathcal{E})$

根子空间 $V^{\lambda_i} = \{\xi \mid (\mathcal{A}-\lambda_i\mathcal{E})^{r_i}\xi = 0,\ \xi \in V\text{总}\} = \ker(\mathcal{A}-\lambda_i\mathcal{E})^{r_i}$

而 $\ker(\mathcal{A}-\lambda_i\mathcal{E}) \subset \ker(\mathcal{A}-\lambda_i\mathcal{E})^{r_i}$　$r_i \geq 1$　故

$V_{\lambda_i} \subseteq V^{\lambda_i}$

初学时的 question：

1、特征子空间与根子空间包含性的 proof

2、多项式与线性变换的同构性 proof

3、$u_i(\mathcal{A})$ 与 $f_i(\mathcal{A})$ 可交换的 proof

根子空间分解定理 def：

线性变换 $\mathcal{A}$ 的特征多项式为 $f(\lambda)$，且 $f(\lambda) = (\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots(\lambda-\lambda_s)^{r_s}$

则 $V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$　$V^{\lambda_i} = \{\xi \mid (\mathcal{A}-\lambda_i\varepsilon)^{r_i}(\xi) = 0,\ \xi \in V\} = \ker(\mathcal{A}-\lambda_i\mathcal{E})^{r_i}$

线性变换 $\mathcal{A}$ 的根子空间 V^{λ_i}

特征子空间 V_{λ_i} 与根子空间 V^{λ_i} 的关系：$V_{\lambda_i} \subset V^{\lambda_i}$

proof：令 $f_i(\lambda) = \dfrac{f(\lambda)}{(\lambda-\lambda_i)^{r_i}} = (\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots(\lambda-\lambda_{i-1})^{r_{i-1}}(\lambda-\lambda_{i+1})^{r_{i+1}}\cdots(\lambda-\lambda_s)^{r_s}$　$\Rightarrow f_i(\lambda)(\lambda-\lambda_i)^{r_i} = f(\lambda)$

则 $(f_1(\lambda), f_2(\lambda), \cdots, f_s(\lambda)) = 1$

W_i 为线性变换 $\mathcal{A}$ 一子空间

☆：构造 $W_i = f_i(\mathcal{A})V = \mathrm{Im}\, f_i(\mathcal{A})$　而 $f_i(\mathcal{A})\mathcal{A} = \mathcal{A}f_i(\mathcal{A})$　（数学归纳法 proof）

下证 $W_i \subseteq V^{\lambda_i}$：

$\forall \alpha \in V^{\lambda_i}$，$(\mathcal{A}-\lambda_i\varepsilon)^{r_i}\underbrace{f_i(\mathcal{A})(\alpha)}_{\in \mathrm{Im}\, f_i(\mathcal{A})} = f(\mathcal{A})\alpha = \mathcal{O}\alpha = 0 \Rightarrow \mathrm{Im}\, f_i(\mathcal{A}) \subseteq V^{\lambda_i}$，故 $W_i \subseteq V^{\lambda_i}$

由 Hamilton—Cayley 定理有 $(\mathcal{A}-\lambda_i\varepsilon)^{r_i}W_i = f(\mathcal{A})V = 0$

主要 proof：

1) $V = W_1 + W_2 + \cdots + W_s$

$V = W_1 + W_2 + \cdots + W_s$ proof：

由于 $(f_1(\lambda), f_2(\lambda), \cdots, f_s(\lambda)) = 1$

$\Downarrow$

$\exists u_1, u_2, \cdots, u_s$ s.t. $u_1f_1 + \cdots + u_sf_s = 1$ 则（由多项式与线性变换同构可得）

$u_1(\mathcal{A})f_1(\mathcal{A}) + \cdots + u_s(\mathcal{A})f_s(\mathcal{A}) = \mathcal{E}$

$\forall \alpha \in V$，$u_1(\mathcal{A})f_1(\mathcal{A})(\alpha) + \cdots + u_s(\mathcal{A})f_s(\mathcal{A})(\alpha) = \alpha$

令 $\alpha_i := u_i(\mathcal{A})f_i(\mathcal{A})(\alpha)$，$\boxed{u_i(\mathcal{A})\text{与}f_i(\mathcal{A})\text{可交换}}$ 多项式可交换为显然的

$\alpha_i = u_i(\mathcal{A})f_i(\mathcal{A})(\alpha) = f_i(\mathcal{A})\,u_i(\mathcal{A})(\alpha) \in \mathrm{Im}(f_i(\mathcal{A}))$

则 $\alpha_i \in \mathrm{Im}(f_i(\mathcal{A})) = W_i$

故 $\alpha = \alpha_1 + \cdots + \alpha_s$，从而 $V = W_1 + W_2 + \cdots + W_s$

2) $W_1 \oplus W_2 \oplus \cdots \oplus W_s$　$\exists \alpha \in V$ s.t. $f_i(\mathcal{A})\alpha = \beta_i$

设 $\beta_1 + \beta_2 + \cdots + \beta_s = 0$，下证 $\forall i$，$\beta_i = 0$

$\beta_i \in W_i \Rightarrow (\mathcal{A}-\lambda_i\mathcal{E})^{r_i}\underbrace{(f_i(\mathcal{A}))\alpha}_{\beta_i} = f(\mathcal{A})\alpha = 0$　$\beta_i \in \ker(\mathcal{A}-\lambda_i\mathcal{E})^{r_i} = V^{\lambda_i}$

题目条件，$W_i \subseteq V^{\lambda_i}$

$\Rightarrow f_j(\mathcal{A})(\beta_i) = 0$　$(i \neq j)$ $\longrightarrow$ 特征多项式 reason

$\Rightarrow f_i(\mathcal{A})(\beta_1 + \beta_2 + \cdots + \beta_s) = f_i(\mathcal{A})(0) = 0 = f_i(\mathcal{A})(\beta_i)$

而 $((\lambda-\lambda_i)^{r_i}, f_i(\lambda)) = 1 \Rightarrow (u(\lambda)(\lambda-\lambda_i)^{r_i} + v(\lambda)f_i(\lambda)) = 1$

$\Rightarrow u(\mathcal{A})(\mathcal{A}-\lambda_i)^{r_i} + v(\mathcal{A})f_i(\mathcal{A}) = \mathcal{E}$

$\Rightarrow u(\mathcal{A})\underbrace{(\mathcal{A}-\lambda_i)^{r_i}(\beta_i)}_{0} + v(\mathcal{A})\underbrace{f_i(\mathcal{A})(\beta_i)}_{0} = \beta_i$

$\beta_i = 0$，故 $V_1 \oplus V_2 \oplus \cdots\cdots \oplus V_s$，同时 $W_1 \oplus W_2 \oplus \cdots\cdots \oplus W_s$

3) $V^{\lambda_i} \subseteq W_i$

其中 $W_i = f_i(\mathcal{A})V = \mathrm{Im}\, f_i(\mathcal{A})$　$V^{\lambda_i} = \{\xi \mid (\mathcal{A}-\lambda_i\mathcal{E})^{r_i}\xi = 0,\ \xi \in V\}$

$\forall \alpha \in V^{\lambda_i}$ 则 $\alpha \in V$，从而 $\alpha = \alpha_1 + \cdots + \alpha_s$　$\alpha_i \in W_i \subseteq V^{\lambda_i}$

则 $0 = \alpha_1 + \cdots + (\alpha_i - \alpha) + \cdots + \alpha_s$　其中 $\alpha_i - \alpha \in V^{\lambda_i}$

由 2) 可得 $V_1 \oplus V_2 \oplus \cdots \oplus V_s$　故 $\alpha_i - \alpha = 0$　$\alpha = \alpha_i \in W_i$

故 $V^{\lambda_i} \subseteq W_i$

若尔当标准型

theorem（根子空间分解定理）：设线性变换 $\mathscr{A}$ 的特征多项式为 $f(\lambda)$，它可分解为一次因式乘积 $f(\lambda)=(\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots\cdots(\lambda-\lambda_s)^{r_s}$

，则 V 可分解为不变子空间的直和 $V=V_1\oplus V_2\oplus\cdots\cdots\oplus V_s$，$V_i=\{\xi\mid(\mathscr{A}-\lambda_i\mathscr{E})^{r_i}\xi=0,\ \xi\in V\}$

若尔当块 def：形式为 $J(\lambda,t)=\begin{pmatrix}\lambda&0&\cdots&0&0&0\\1&\lambda&\cdots&0&0&0\\\vdots&\vdots&&\vdots&\vdots&\vdots\\0&0&\cdots&1&\lambda&0\\0&0&\cdots&0&1&\lambda\end{pmatrix}_{t\times t}$ 矩阵 is called 若尔当块（λ为复数），

由若干个 Jordan 块组成的准对角矩阵 called 若尔当形矩阵 $\begin{pmatrix}A_1&&&\\&A_2&&\\&&\ddots&\\&&&A_s\end{pmatrix}$，其中 $A_i=\begin{pmatrix}\lambda_i&&&&\\1&\lambda_i&&&\\&\ddots&\ddots&&\\&&1&\lambda_i&\\&&&1&\lambda_i\end{pmatrix}_{k_i\times k_i}$ $\lambda_1\sim\lambda_s$ 有一些可以相等

幂零线性变换 def：（特殊的一种线性变换）设V为数域P上的线性空间，$\mathscr{A}$为V的线性变换，$\exists\ k\in N$ s.t. $\mathscr{A}^k=0$，$\mathscr{A}^{k-1}\neq0$

$\mathscr{A}$ is called 幂零线性变换，k is called 幂零指数

循环子空间 def：（特殊的一类子空间）设V为数域P上的线性空间，$\mathscr{B}$为V的线性变换，对 $0\neq\eta\in V$，$\exists\ k$ s.t. $\eta,\mathscr{B}\eta,\mathscr{B}^2\eta\cdots\cdots\mathscr{B}^{k-1}\eta$ 线性无关 但 $\eta,\mathscr{B}\eta,\cdots,\mathscr{B}^{k-1}\eta,\mathscr{B}^k\eta$ 线性相关

由 $\eta,\mathscr{B}\eta,\cdots,\mathscr{B}^{k-1}\eta$ 生成的子空间 L is called $\mathscr{B}$循环子空间，$\eta\sim\mathscr{B}^{k-1}\eta$ is called L的循环基

theorem：设$\mathscr{A}$是复数域 V 上线性变换，则在V中必定$\exists$一组基 s.t. $\mathscr{A}$在这组基下的矩阵为若尔当型矩阵 (proof)

设$\mathscr{A}$的特征多项式为 $f(\lambda)=(\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots\cdots(\lambda-\lambda_s)^{r_s}$　$\lambda_1\sim\lambda_s$ 为不同的根

由根子空间分解定理得 $V=V_1\oplus V_2\oplus\cdots\oplus V_s$　$V_i=\{\xi\mid(\mathscr{A}-\lambda_i\mathscr{E})^{r_i}\xi=0,\ \xi\in V\}$

（计能够proof 在每个V_i上都有一组基 s.t. $\mathscr{A}|_{V_i}$ 在该基下矩阵为 Jordan 型矩阵

则 theorem 得证）

在V_i上有 $(\mathscr{A}-\lambda_i\mathscr{E})^{r_i}\xi=0$　作 $\mathscr{B}=(\mathscr{A}-\lambda_i\mathscr{E})$　故 $\mathscr{B}^{r_i}=0$　$\exists V_i$的基 s.t. $\mathscr{B}$的矩阵为 Jordan 矩阵

$\mathscr{A}=\mathscr{B}+\lambda_i\mathscr{E}$，在该下矩阵为：$\begin{pmatrix}A_1&&&\\&A_2&&\\&&\ddots&\\&&&A_n\end{pmatrix}$，其中 $A_i=\begin{pmatrix}\lambda_i&&&&\\1&\lambda_i&&&\\&\ddots&\ddots&&\\&&&\lambda_i&\\&&&1&\lambda_i\end{pmatrix}_{l_i\times l_i}$

lemma：n 维线性空间V上$\mathscr{B}$ 满足 $\mathscr{B}^k=0$，k为正整数，称$\mathscr{B}$为 V上幂零线性变换，对于$\mathscr{B}$　V中必有下列形式一组元素作为基

$$\begin{matrix}\alpha_1&\alpha_2&\cdots&\alpha_s\\ \mathscr{B}\alpha_1&\mathscr{B}\alpha_2&\cdots&\mathscr{B}\alpha_s\\ \vdots&\vdots&&\vdots\\ \mathscr{B}^{k_1-1}\alpha_1&\mathscr{B}^{k_2-1}\alpha_2&\cdots&\mathscr{B}^{k_s-1}\alpha_s\\ (\mathscr{B}^{k_1}\alpha_1=0)&(\mathscr{B}^{k_2}\alpha_2=0)&\cdots&(\mathscr{B}^{k_s}\alpha_s=0)\end{matrix}$$

$\mathscr{B}$在这组基下矩阵为 $\begin{pmatrix}B_1&&\\&\ddots&\\&&B_s\end{pmatrix}$　其中 $B_i=\begin{pmatrix}0&&&\\1&0&&\\&1&\ddots&\\&&\ddots&0\\&&1&0\end{pmatrix}_{k_i\times k_i}$　B_i 为$\mathscr{B}|_{W_i}$ 上矩阵

theorem：1、每个n级复矩阵 A 都与一个 Jordan 矩阵相似

2、一个线性空间V，$\mathscr{A}\in Hom(V,V)$，$\exists$ V的一组基 s.t. $\mathscr{A}$在这组基下为 Jordan 矩阵

proof：$V=V_1\oplus V_2\oplus\cdots\oplus V_s$　$V_i=\ker(\mathscr{A}-\lambda_i\mathscr{E})^{r_i}$　令$\mathscr{B}_i=\mathscr{A}-\lambda_i\mathscr{E}$　则 $V_i=\ker\mathscr{B}_i^{r_i}$ $\Rightarrow$ V_i 为 $\mathscr{B}_i|_{V_i}$ 的幂零变换

$V_i=V_{i1}\oplus\cdots\oplus V_{it_i}$　V_{ij} 为$\mathscr{B}_i$的循环子空间，且$\mathscr{B}_i$在V_i的组合循环基下矩阵为 $\begin{pmatrix}J(0,k_{i1})&&\\&\ddots&\\&&J(0,k_{it_i})\end{pmatrix}$　$\mathscr{A}\sim V_i$ $\begin{pmatrix}J(\lambda_i,k_{i1})&&\\&\ddots&\\&&J(\lambda_i,k_{it_i})\end{pmatrix}$

故$\mathscr{A}$在 V($V_1\sim V_s$的组合基)下矩阵为 $\begin{pmatrix}J(\lambda_1,k_{11})&&&\\&\ddots&&\\&&J(\lambda_1,k_{1t_1})&\\&&&\ddots\\&&&&J(\lambda_s,k_{s t_s})\end{pmatrix}$

$V=V_1\oplus\cdots\oplus V_s$　$V_i=\ker(\mathscr{A}-\lambda_i\mathscr{E})^{r_i}$　$V_{\lambda_i}\subseteq V_i=V_{i1}\oplus\cdots\oplus V_{is_i}$　V_{ij}为 $\mathscr{A}-\lambda_i\mathscr{E}$ 循环子空间

$\mathscr{A}-\lambda_i\mathscr{E}|_{V_{ij}}$：取$\alpha_1\cdots\alpha_{k_i,1}$，则矩阵为 $\begin{pmatrix}0&&&\\1&0&&\\&1&\ddots&\\&&\ddots&\\&&1&0\end{pmatrix}_{k_{ij}}$

$\mathscr{A}|_{V_{ij}}$：- - -　$\begin{pmatrix}\lambda_i&&&\\1&\lambda_i&&\\&\ddots&\ddots&\\&&1&\lambda_i\end{pmatrix}_{k_{ij}}$

theorem：$\mathscr{A}$中对应λ_i的 Jordan 块数

$=V_i$中分解的循环子空间的个数（记为s_i）

$=$对应于λ_i的线性无关的特征向量个数

$=\dim V_{\lambda_i}$

$=$几何重数

设 V_{ij} 为 $\mathscr{A}-\lambda_i\mathscr{E}$的循环子空间，取$\varepsilon_1\sim\varepsilon_t$ 为V_{ij} 的循环基

$\Rightarrow(\mathscr{A}-\lambda_i\mathscr{E})(\varepsilon_1,\cdots,\varepsilon_t)=(\varepsilon_1\sim\varepsilon_t)\begin{pmatrix}0&0&\cdots&0&0\\1&0&\cdots&0&0\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&0&0\\0&0&\cdots&1&0\end{pmatrix}$

$\mathscr{A}\varepsilon_t=\lambda\varepsilon_t$ $\Rightarrow$ ε_t为V_{ij}上的唯一特征向量

$\Rightarrow\mathscr{A}(\varepsilon_1,\cdots,\varepsilon_t)=(\varepsilon_1,\cdots,\varepsilon_t)\begin{pmatrix}\lambda_i&0&\cdots&0&0\\1&\lambda_i&\cdots&0&0\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&\lambda_i&0\\0&0&\cdots&1&\lambda_i\end{pmatrix}$

$V_{\lambda_i}\subseteq V_i$ $\Rightarrow$ s_i 即为 $\dim V_{\lambda_i}$

$\mathscr{A}(r_1, r_2, \cdots, r_n) = (r_1, r_2, \cdots, r_n) J$

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{pmatrix}$$

$$J(\lambda, t) = \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}_{t \times t}$$

对角线 阶数

theorem：$\mathscr{A}$为复数域上 - - - - -

推论：
- 几何重数 — J中以λ_i为主对角线元素Jordan子块个数
- 代数重数 — J中以λ_i - - - - 的阶数之和

（J为n级复矩阵A的Jordan标准型，矩阵A线性无关向量个数=J所含Jordan块个数）