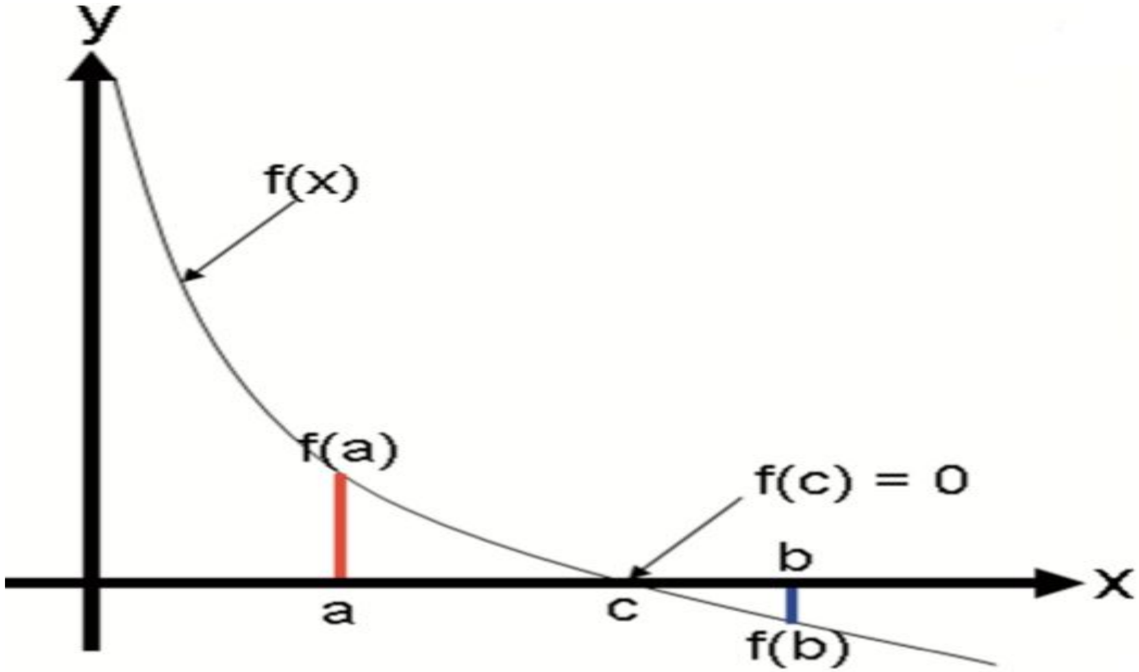
Lecture # 04 & 5

Root of equations in one variable

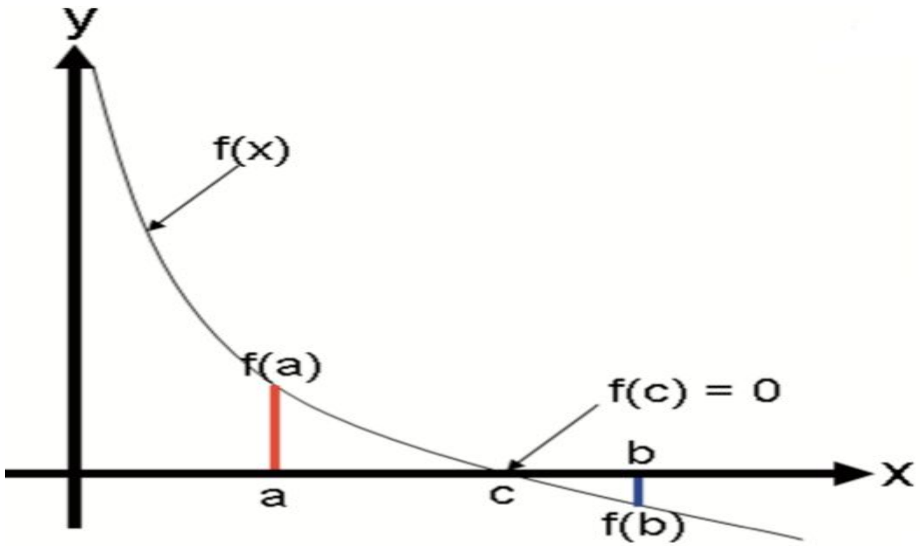
Bisection or Binary Search Method

Bisection OR Binary Search Method:

The **Intermediate Value Theorem** says that if $f(x)$ is a continuous function between a and b, and $\text{sign}(f(a)) \neq \text{sign}(f(b))$, then there must be a c, such that $a < c < b$ and $f(c) = 0$. This is illustrated in the following figure.



The **bisection method** uses the intermediate value theorem iteratively to find roots. Let $f(x)$ be a continuous function, and a and b be real scalar values such that $a < b$. Assume, without loss of generality, that $f(a) > 0$ and $f(b) < 0$. Then by the intermediate value theorem, there must be a root on the open interval (a, b). Now let $m = \frac{b+a}{2}$, the midpoint between and a and b. If $f(m) = 0$ or is close enough, then m is a root. If $f(m) > 0$, then m is an improvement on the left bound, a, and there is guaranteed to be a root on the open interval (m, b). If $f(m) < 0$, then m is an improvement on the right bound, b, and there is guaranteed to be a root on the open interval (a, m).



Bisection OR Binary Search Method:

The **bisection method** uses the intermediate value theorem iteratively to find roots.

In computer science, the process of dividing a set continually in half to search for the solution to a problem, as the bisection method does, is known as a *binary search* procedure.

Algorithm:

The steps to apply the bisection method to find the root of the equation $f(x) = 0$ are

1. Choose x_λ and x_u as two guesses for the root such that $f(x_\lambda)f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_λ and x_u.
2. Estimate the root, x_m, of the equation $f(x) = 0$ as the mid-point between x_λ and x_u as

$$x_m = \frac{x_\lambda + x_u}{2}$$

Algorithm:

3. Now check the following
   a) If $f(x_\lambda)f(x_m) < 0$, then the root lies between x_λ and x_m; then $x_\lambda = x_\lambda$ and $x_u = x_m$.
   b) If $f(x_\lambda)f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\lambda = x_m$ and $x_u = x_u$.
   c) If $f(x_\lambda)f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.
4. Find the new estimate of the root

$$x_m = \frac{x_\lambda + x_u}{2}$$

Find the absolute relative approximate error as

$$|\in_a| = \left| \frac{x_m^{\text{new}} - x_m^{\text{old}}}{x_m^{\text{new}}} \right| \times 100$$

where

x_m^{new} = estimated root from present iteration

x_m^{old} = estimated root from previous iteration

Algorithm:

5. Compare the absolute relative approximate error $|\epsilon_a|$ with the pre-specified relative error tolerance ϵ_s. If $|\epsilon_a| > \epsilon_s$, then go to Step 3, else stop the algorithm. Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Number of iterations for getting root:

$$n = \frac{\ln(\Delta x/\varepsilon)}{\ln 2}$$

Example:

Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$, and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4}.

Solution Because $f(1) = -5$ and $f(2) = 14$ the Intermediate Value Theorem that this continuous function has a root in $[1, 2]$.

For the first iteration of the Bisection method we use the fact that at the midpoint of $[1, 2]$ we have $f(1.5) = 2.375 > 0$. This indicates that we should select the interval $[1, 1.5]$

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364990235	1.365234375	1.365112305	−0.00194

The Bisection method, though conceptually clear, has significant drawbacks. It is relatively slow to converge (that is, N may become quite large before $|p - p_N|$ is sufficiently small), and a good intermediate approximation might be inadvertently discarded. However, the method has the important property that it always converges to a solution, and for that reason it is often used as a starter for the more efficient methods we will see later in this chapter.

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Example:

Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-4} using $a_1 = 1$ and $b_1 = 2$.

HINT:

$$n = \frac{\ln(\Delta x/\varepsilon)}{\ln 2}$$

Do Q 1,2,3,4,5,6,12 & 13 from Ex # 2.1

Root of equations in one variable

Fixed Point Iteration

Fixed Point Iteration:

The number p is a **fixed point** for a given function g if $g(p) = p$.

OR

$$x = \emptyset(x)$$

Algorithm:

To find a solution to $p = g(p)$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \le N_0$ do Steps 3–6.

Step 3 Set $p = g(p_0)$. (*Compute* p_i.)

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (*Update* p_0.)

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Algorithm:

Iteration Algorithm with the Form $x = g(x)$

To determine a root of $f(x) = 0$, given a value x_1 reasonably close to the root,

Rearrange the equation to an equivalent form $x = g(x)$.

Repeat
Set $x_2 = x_1$.
Set $x_1 = g(x_1)$
Until $|x_1 - x_2| <$ tolerance value

Note: The method may converge to a root different from the expected one, or it may diverge. Different rearrangements will converge at different rates.

Example

- Fixed Point Iteration

$$f(x) = x^2 - 2x - 3 = 0$$

(ans: $x = 3$ or -1)

Case a:

$$\begin{aligned} &x^2 - 2x - 3 = 0 \\ &\Rightarrow x^2 = 2x + 3 \\ &\Rightarrow x = \sqrt{2x+3} \\ &\Rightarrow g(x) = \sqrt{2x+3} \end{aligned}$$

Case b:

$$\begin{aligned} &x^2 - 2x - 3 = 0 \\ &\Rightarrow x(x-2) - 3 = 0 \\ &\Rightarrow x = \frac{3}{x-2} \\ &\Rightarrow g(x) = \frac{3}{x-2} \end{aligned}$$

Case c:

$$\begin{aligned} &x^2 - 2x - 3 = 0 \\ &\Rightarrow 2x = x^2 - 3 \\ &\Rightarrow x = \frac{x^2-3}{2} \\ &\Rightarrow g(x) = \frac{x^2-3}{2} \end{aligned}$$

So which one is better?

Case a

$$x_{i+1} = \sqrt{2x_i + 3}$$

1. $x_0 = 4$
2. $x_1 = 3.31662$
3. $x_2 = 3.10375$
4. $x_3 = 3.03439$
5. $x_4 = 3.01144$
6. $x_5 = 3.00381$

Converge!

Case b

$$x_{i+1} = \frac{3}{x_i - 2}$$

1. $x_0 = 4$
2. $x_1 = 1.5$
3. $x_2 = -6$
4. $x_3 = -0.375$
5. $x_4 = -1.263158$
6. $x_5 = -0.919355$
7. $x_6 = -1.02762$
8. $x_7 = -0.990876$
9. $x_8 = -1.00305$

Converge, but slower

Case c

$$x_{i+1} = \frac{x_i^2 - 3}{2}$$

1. $x_0 = 4$
2. $x_1 = 6.5$
3. $x_2 = 19.625$
4. $x_3 = 191.070$

Diverge!

(Fixed-Point Theorem)

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b).$$

Then for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a,b]$. ■

Do Q 1,2,3,4,5,6,9,10,11 & 14 from Ex # 2.2

Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in [1, 2], and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4}.

Solve by Fixed Point Iteration Method

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

(b) $x = g_2(x) = \left(\dfrac{10}{x} - 4x\right)^{1/2}$

(c) $x = g_3(x) = \dfrac{1}{2}(10 - x^3)^{1/2}$

(d) $x = g_4(x) = \left(\dfrac{10}{4 + x}\right)^{1/2}$

(e) $x = g_5(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

example, to obtain the function g described in part (c), we can manipulate the equation $x^3 + 4x^2 - 10 = 0$ as follows:

$$4x^2 = 10 - x^3, \quad \text{so} \quad x^2 = \frac{1}{4}(10 - x^3), \quad \text{and} \quad x = \pm\frac{1}{2}(10 - x^3)^{1/2}.$$

To obtain a positive solution, $g_3(x)$ is chosen. It is not important for you to derive the functions shown here, but you should verify that the fixed point of each is actually a solution to the original equation, $x^3 + 4x^2 - 10 = 0$.

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	−0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	−469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

The actual root is 1.365230013, as was noted in Example 1 of Section 2.1. Comparing the results to the Bisection Algorithm given in that example, it can be seen that excellent results have been obtained for choices (c), (d), and (e) (the Bisection method requires 27 iterations for this accuracy). It is interesting to note that choice (a) was divergent and that (b) became undefined because it involved the square root of a negative number.

Do Class Question

H.W:

Find the root of the transcendental equation $\boldsymbol{cosx - 3x + 1 = 0}$ correct up to seven decimal values.

Ans: 0.6071016

Newton's Raphson Method:

The Newton-Raphson algorithm is the **best-known** method of finding roots for a good reason:

- **It is simple and fast**.
- The only drawback of the method is that it uses the derivative $f'(x)$ of the function as well as the function $f(x)$ itself. Therefore, the Newton-Raphson method is usable only in problems where $f'(x)$ can be readily computed.

History:

Isaac Newton (1641–1727) was one of the most brilliant scientists of all time. The late 17th century was a vibrant period for science and mathematics and Newton's work touched nearly every aspect of mathematics. His method for solving was introduced to find a root of the equation $y^3 - 2y - 5 = 0$. Although he demonstrated the method only for polynomials, it is clear that he realized its broader applications.

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Formula:

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1.$$

Algorithm:

To find a solution to $f(x) = 0$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. (*Compute p_i.*)

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (*Update p_0.*)

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Example:

Consider the function $f(x) = \cos x - x = 0$. Approximate a root of f using **(a)** a fixed-point method, and **(b)** Newton's Method

Note that the variable in the trigonometric function is in radian measure, not degrees. This will always be the case unless specified otherwise.

n	p_n
0	0.7853981635
1	0.7071067810
2	0.7602445972
3	0.7246674808
4	0.7487198858
5	0.7325608446
6	0.7434642113
7	0.7361282565

Table 2.3 shows the results of fixed-point iteration with $p_0 = \pi/4$. The best we could conclude from these results is that $p \approx 0.74$.

(b) To apply Newton's method to this problem we need $f'(x) = -\sin x - 1$. Starting again with $p_0 = \pi/4$, we generate the sequence defined, for $n \geq 1$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f(p'_{n-1})} = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}.$$

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$$x_0 = 0.7853982$$

$$f(x_0) = f(0.7853982) = \cos(0.7853982) - 0.7853982 = -0.0782914$$

$$f'(x_0) = f'(0.7853982) = -\sin(0.7853982) - 1 = -1.7071068$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.7853982 - \frac{-0.0782914}{-1.7071068}$$

$$x_1 = 0.7395361$$

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$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f(p'_{n-1})} = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}.$$

Newton's Method

n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332