*In Exercises **1–2**, find all the minors and cofactors of the matrix A.*

1. $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$

- Minor by hiding rows & column and finding determinant.
- Cofactor by multiplying $(-1)^{i+j}$ with Minor (M_{ij})

$$M_{11} = \begin{bmatrix} 7 & -1 \\ 1 & 4 \end{bmatrix} = 28 - (-1) = 29$$

$$C_{11} = (-1)^{1+1} \cdot M_{11} = (-1)^2 \cdot 29 = 29$$

$$M_{12} = \begin{bmatrix} 6 & -1 \\ -3 & 4 \end{bmatrix} = 24 - 3 = 21$$

$$C_{12} = (-1)^{1+2} \cdot M_{12} = (-1) \cdot 21 = -21$$

Find for others by above method.

3. Let

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

Find

a. M_{13} and C_{13}.

$$M_{13} = \begin{bmatrix} 0 & 0 & 3 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{bmatrix}$$

$\lfloor 4 \quad 1 \quad 2 \rfloor$

$$M_{13} = 0 \begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - 0 \begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 3 \begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$$

$$M_{13} = 3(4-4) = \boxed{0}$$ Ans

$$C_{13} = (-1)^{1+3} \cdot M_{13} = (-1)^4 \cdot 0 = \boxed{0}$$ Ans.

*In Exercises **5–8**, evaluate the determinant of the given matrix. If the matrix is invertible, use Equation (2) to find its inverse.*

5. $\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$ $A^{-1} = \frac{1}{\det(A)} \times \operatorname{adj}(A)$

- If $\det = 0$, not invertible

$\det(A) = 12 - (-10) = 22 \rightarrow$ Invertible

$$A^{-1} = \frac{1}{22} \cdot \begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} 2/11 & -5/22 \\ 1/11 & 3/22 \end{bmatrix}}$$ Ans

6. $\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$

$\det(A) = 8 - 8 = \boxed{0}$ Not invertible!

*In Exercises **9–14**, use the arrow technique of Figure 2.1.1 to evaluate the determinant.*

9. $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix}$

$\det = (a-3)\cdot(a-2) - (-15) = a^2 - 5a + 6 + 15$

$\det = (a-3)\cdot(a-2) - (-15) = a^2 - 5a + 6 + 15$

$= \boxed{a^2 - 5a + 21}$ Ans.

11. $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}$

- Repeat 1st & 2nd column and then make arrows.

11. $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} \begin{matrix} -2 & 1 \\ 3 & 5 \\ 1 & 6 \end{matrix}$

$\det = (-20 - 7 + 72) - (20 + 84 + 6)$

$\det = 45 - 110 = \boxed{-65}$ Ans

*In Exercises **15–18**, find all values of λ for which* $\det(A) = 0$.

15. $A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$

$\det(A) = (\lambda - 2)(\lambda + 4) - (-5)$

$0 = \lambda^2 + 2\lambda - 8 + 5$

$\lambda^2 + 2\lambda - 3 = 0$

$\lambda^2 - 3\lambda + \lambda - 3 = 0$

$\lambda(\lambda - 3) + 1(\lambda - 3) = 0$

$\boxed{\lambda = -1}$ $\boxed{\lambda = 3}$

16. $A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$

$\lambda - 4 \begin{vmatrix} \lambda & 2 \\ 3 & \lambda - 1 \end{vmatrix} - 0 + 0 = 0$

$\lambda - 4\,(\lambda^2 - \lambda - 6) = 0$

$\boxed{\lambda = 4}$ $\lambda^2 - 3\lambda + 2\lambda - 6 = 0$

$$\lambda(\lambda-3)+2(\lambda-3)=0$$

$$\boxed{\lambda=-2} \quad \boxed{\lambda=3}$$

19. Evaluate the determinant in Exercise 13 by a cofactor expansion along

13. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix}$

$+ \quad - \quad + \rightarrow$ First row & colm

$- \quad + \quad - \rightarrow$ Second " "

$+ \quad - \quad + \rightarrow$ Third " "

a. the first row.

$$3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 0\begin{vmatrix} 2 & 5 \\ 1 & -4 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix} = 3(4-45)$$

$$= \boxed{-123}$$

c. the second row.

$$2\begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} - 1\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} + 5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix}$$

$$2(0-0)+(-1)(-12-0)-5(27-0) = -123$$

f. the third column.

$+ \quad - \quad +$

13. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix}$

$$0\begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix} - 5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix}$$

$$0 - 5(27) - 4(-3) = -123$$

In Exercises 21–26, evaluate $\det(A)$ *by a cofactor expansion along a row or column of your choice.*

21. $A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$

second column → $- \quad + \quad -$

$$-0\begin{vmatrix}2 & 1\\ -1 & 5\end{vmatrix} + 5\begin{vmatrix}-3 & 7\\ -1 & 5\end{vmatrix} - 0\begin{vmatrix}-3 & 7\\ 2 & 1\end{vmatrix}$$

$$-0 + 5(-15+7) - 0 = \boxed{-40}$$

In Exercises 27–32, evaluate the determinant of the given matrix by inspection.

27. $\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Diagonal matrix so multiply diagonal entries

$(1)\cdot(-1)\cdot(1) = \boxed{-1}$ Ans

31. $\begin{bmatrix}1 & 2 & 7 & -3\\ 0 & 1 & -4 & 1\\ 0 & 0 & 2 & 7\\ 0 & 0 & 0 & 3\end{bmatrix}$

Upper Triangular matrix, multiply diagonal entries.

$(1)\cdot(1)\cdot(2)\cdot(3) = \boxed{6}$ Ans

2.2

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In Exercises 1–4, verify that $\det(A) = \det(A^T)$.

1. $A = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$

$\det(A) = -8 - 3 = \boxed{-5}$

$A^T = \begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix}$

$\det(A^T) = -8 - 3 = \boxed{-11}$

3. $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{bmatrix}$

$\det(A) = 2(12+12) - (-1)(6-20) + 3(-3-10)$

$= 2(24) + 1(-14) + 3(-13)$

$= \boxed{-5}$

$A^T = \begin{vmatrix} 2 & 1 & 5 \\ -1 & 2 & -3 \\ 3 & 4 & 6 \end{vmatrix}$

$\det(A^T) = 2(12+12) - 1(-6+9) + 5(-4-6)$

$= 2(24) - 1(3) + 5(-10) = \boxed{-5}$ Hence, Equal!

In Exercises 9–14, evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then using some combination of row operations and cofactor expansion.

9. $\begin{bmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{bmatrix}$

Take commons & take -1 as common when exchanging rows.

$$\begin{bmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{bmatrix} = (3)\begin{bmatrix} 1 & -2 & 3 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{bmatrix}$$

$$(3)\begin{bmatrix} 1 & -2 & 3 \\ \boxed{0 \quad 3 \quad 4} \\ \boxed{0 \quad 1 \quad 5} \end{bmatrix} = (3)(-1)\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 3 & 4 \end{bmatrix}$$

$$(3)(-1)\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -11 \end{bmatrix} = (3)(-1)(-11)\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{33}$ Ans.

11. $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$$(-1)\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} = (-1)\cdot(-1)\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ \boxed{0 \quad 2 \quad 1 \quad 0} \\ \boxed{0 \quad 1 \quad 2 \quad 3} \end{bmatrix}$$

$$(-1)\cdot(-1)\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 0 \end{bmatrix} = (-1)\cdot(-1)\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$(-1)\cdot(-1)\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix} = (-1)\cdot(-1)\cdot(6)\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(-1)\cdot(-1)\cdot(6) = \boxed{6} \text{ Ans.}$$

In Exercises **15–22**, *evaluate the determinant, given that*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

15. $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$

$$(-1)\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = (-1)\cdot(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= (-1)\cdot(-1)\cdot(-6) = \boxed{-6} \text{ Ans.}$$

19. $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$

$-R_3 + R_1$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \boxed{-6} \text{ Ans.}$$

21. $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$

$$(-3)\begin{vmatrix} a & b & c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$ Now $4R_2 + R_3$

$$(-3)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-3)(-6) = \boxed{18}$$ Ans

23. Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

$$\begin{array}{l} -aR_1+R_2 \\ -a^2R_1+R_3 \end{array}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = -(b+a)R_2+R_3 \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(c-a)(b+a) \end{vmatrix}$$

$(1)\cdot(b-a)\cdot(c^2-a^2-(c-a)(b+a))$

$(1)\cdot(b-a)\cdot(c-a)(c+\cancel{a}-b-\cancel{a})$

$\boxed{(b-a)\cdot(c-a)(c-b)}$ Ans.

2.3

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1) $\det(KA) = K^n \det(A)$
   where n is the no. of columns.

2) $\det(A+B) \neq \det(A) + \det(B)$

   ↳ But the last row of both matrix can be added to form a new matrix which will produce the same determinant

<u>Example</u>

$$\det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$\det(C) = \det(A) + \det(B)$

3) $\det(AB) = \det(A) \cdot \det(B)$ → Both of A & B must be square matrices.

4) A square matrix is invertible only when its determinant is not equal to 0.

5) If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$

   ↳ $\det(AA^{-1}) = \det(I)$

EXAMPLE 6 | Adjoint of a 3 × 3 Matrix

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

As noted in Example 5, the cofactors of A are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

so the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

and the adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

- Find minors
- Find cofactors through them.
- Take transpose to find adjoint.

$$\boxed{A^{-1} = \frac{1}{\det(A)} \times \text{adj}(A)}$$

Cramer's Rule

Q

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 5 \\ 4 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3000 \\ 2400 \\ 3600 \end{bmatrix}$$

$$Ax = B$$

$$\begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 5 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 3000 \\ 2400 \\ 3600 \end{bmatrix}$$

$$|A| = \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{vmatrix} 3 & 4 & 1 \\ 2 & 3 & 5 \\ 4 & 3 & 1 \end{vmatrix} \end{array}$$

$i - j + k$

$$|A| = 3\begin{vmatrix} 3 & 5 \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}$$

$$|A| = 3(3-15) - 4(2-20) + 1(6-12)$$

$$|A| = -36 + 72 - 6 = \boxed{30}$$

$|A| \neq 0$ so can be solved.

Now, for A_1, change the first column with B and so on.

$$|A_1| = \begin{vmatrix} 3000 & 4 & 1 \\ 2400 & 3 & 5 \\ 3600 & 3 & 1 \end{vmatrix}$$

$$|A_1| = 3000(3-15) - 4(2400 - 18000) + 1(7200 - 10800)$$

$$\boxed{|A_1| = 22800}$$

$$|A_2| = \begin{vmatrix} 3 & 3000 & 1 \\ 2 & 2400 & 5 \\ 4 & 3600 & 1 \end{vmatrix} = 4800$$

$$|A_3| = 2400$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{22800}{30} = 760$$

$$x_2 = \frac{4800}{30} = 160$$

$$x_3 = \frac{2400}{30} = 80$$

In Exercises 1–4, verify that $\det(kA) = k^n \det(A)$.

1. $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$; $k = 2$

$$\det(kA) = \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix} = -16 - 24 = \boxed{-40}$$

$$k^n \det(A) = 2^2 \cdot \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} = 4 \cdot (-4 - 6) = \boxed{-40}$$

3. $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}$; $k = -2$

$$\det(kA) = \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -10 \end{vmatrix}$$

$$= -4(40-16) - 2(60-4) - 6(48-8)$$

$$= \boxed{-448}$$

$$k^n \det(A) = -2^3 \cdot \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix}$$

$$= -8 \cdot [2(10-4) + 1(15-1) + 3(12-2)]$$

$$= -8 \cdot [56]$$

$$= \boxed{-448}$$

In Exercises 5–6, verify that $\det(AB) = \det(BA)$ *and determine whether the equality* $\det(A+B) = \det(A) + \det(B)$ *holds.*

5. $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$

$$A \cdot B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{bmatrix}$$

$$\det(A \cdot B) = -170$$

$$B \cdot A = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{bmatrix}$$

$$\det(BA) = -170$$

$\det(A+B) = \det(A) + \det(B)$

$$(A+B) = \begin{bmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{bmatrix} \qquad \det(A+B) = -30$$

$\det(A) = 10$, $\det(B) = -17$

$$\boxed{-30 \neq 10 - 17}$$

In Exercises **7–14**, *use determinants to decide whether the given matrix is invertible.*

7. $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

$\det(A) = -1$ Since $\det(A) \neq 0$, so it is invertible.

In Exercises **15–18**, *find the values of k for which the matrix A is invertible.*

17. $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix}$

$1(2-18) - 2(6-6k) + 4(9-k) \neq 0$

$-16 - 12 + 12k + 36 - 4k \neq 0$

$8 + 8k \neq 0$

$8(1+k) \neq 0$

$\boxed{k \neq -1}$ All values except -1

In Exercises **19–23**, *decide whether the matrix is invertible, and if so, use the adjoint method to find its inverse.*

21. $A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

$\det(A) = \boxed{4}$ so Invertible

$M_{11} = \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 2 \qquad C_{11} = (-1)^{1+1} \cdot (2) = 2$

Same step for all the cofactors.

$C_{12} = 0, \quad C_{13} = 0, \quad C_{21} = 6, \quad C_{22} = 4, \quad C_{23} = 0$

$C_{31} = 4, \quad C_{32} = 6, \quad C_{33} = 2$

$$\text{adj}(A) = \begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4} \times \begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & 3/2 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1/2 \end{bmatrix}$$ Ans.

In Exercises 24–29, solve by Cramer's rule, where it applies.

25.
$$\begin{aligned} 4x + 5y \phantom{{}+2z} &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

$$A = \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix} \qquad \boxed{\det(A) = -132}$$

Can be solved.

$$B = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \qquad AX = B$$

$$|A_1| = \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -36$$

$$|A_2| = \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = -24$$

$$|A_3| = \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = 12$$

$$x = \frac{|A_1|}{|A|} = \frac{-36}{-132} = \frac{3}{11}$$

$$y = \frac{|A_2|}{|A|} = \frac{-24}{-132} = \frac{2}{11}$$

$$z = \frac{|A_3|}{|A|} = \frac{12}{-132} = \frac{-1}{11}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/11 \\ 2/11 \\ -1/11 \end{bmatrix}} \quad \text{Ans.}$$

31. Use Cramer's rule to solve for the unknown y without solving for the unknowns x, z, and w.

$$\begin{aligned} 4x + y + z + w &= 6 \\ 3x + 7y - z + w &= 1 \\ 7x + 3y - 5z + 8w &= -3 \\ x + y + z + 2w &= 3 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 6 \\ 1 \\ -3 \\ 3 \end{bmatrix}}_{B}$$

$A \quad X = B$

$\det(A) = -424$ → can be solved!

For Y so solve for $|A_2|$

$$|A_2| = \begin{vmatrix} 4 & 6 & 1 & 1 \\ 3 & 1 & -1 & 1 \\ 7 & -3 & -5 & 8 \\ 1 & 3 & 1 & 2 \end{vmatrix} = 0$$

$$\frac{|A_2|}{|A|} = \frac{0}{-424} = \boxed{0} \qquad \boxed{Y = 0} \text{ Ans.}$$

Let V be an arbitrary nonempty set of objects for which two operations are defined: addition and multiplication by numbers called ***scalars***. By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There exists an object in V, called the ***zero vector***, that is denoted by $\mathbf{0}$ and has the property that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

Steps to Show That a Set with Two Operations Is a Vector Space

Step 1. Identify the set V of objects that will become vectors.

Step 2. Identify the addition and scalar multiplication operations on V.

Step 3. Verify Axioms 1 and 6; that is, adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V.

Axiom 1 is called ***closure under addition***, and Axiom 6 is called ***closure under scalar multiplication***.

Step 4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$:

$$\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2), \quad k\mathbf{u}=(0,ku_2)$$

a. Compute $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u}=(-1,2), \mathbf{v}=(3,4)$, and $k=3$.

$u+v = (-1+3, 2+4) = \boxed{(2, 6)}$

$ku = 3u = (0, 3\cdot 2) = \boxed{(0, 6)}$

b. In words, explain why V is closed under addition and scalar multiplication.

Because V is a set of ordered pair of real numbers and u+v and k u satisfy it.

c. Since addition on V is the standard addition operation on R^2, certain vector space axioms hold for V because they are known to hold for R^2. Which axioms are they?

Axioms 1 – 5

d. Show that Axioms 7, 8, and 9 hold.

<u>Axiom 7</u> :- $k(u+v) = ku + kv$

$k(u_1+v_1, u_2+v_2) = k(u_1,u_2) + k(v_1,v_2)$

$(0, k\cdot(u_2+v_2)) = (0, ku_2) + (0, kv_2)$

$$(0,\ k\cdot(u_2+v_2)) = (0,\ ku_2) + (0,\ kv_2)$$
$$(0,\ 3\cdot(2+4)) = (0,\ 3\cdot 2) + (0,\ 3\cdot 4)$$
$$(0,\ 18) = (0,\ 6) + (0,\ 12)$$
$$\boxed{(0,18) = (0,18)}$$

Axiom 8 :- $(k+m)u = ku + mu$

$$(k+m)(u_1,u_2) = k(u_1,u_2) + m(u_1,u_2)$$
$$(0,\ (k+m)u_2) = (0,\ ku_2) + (0,\ mu_2)$$
$$\boxed{ku_2 + mu_2 = ku_2 + mu_2}$$

Axiom 9 :- $k(mu) = (km)u$

$$k(m(u_1,u_2)) = (km)\cdot(u_1,u_2)$$
$$k(mu_1,\ mu_2) = (0,\ kmu_2)$$
$$\boxed{(0,\ kmu_2) = (0,\ kmu_2)}$$

e. Show that Axiom 10 fails and hence that V is not a vector space under the given operations.

Axiom 10 :- $1\cdot u = u$

$$1\cdot(-1,2) = (-1,2)$$
$$(0,\ 1\cdot 2) = (-1,\ 2) \qquad \because\ k(u_1,u_2) = (0,\ ku_2)$$
$$\boxed{(0,\ 2) \neq (-1,\ 2)}$$

In Exercises 3–12, determine whether each set equipped with the given operations is a vector space. For those that are not vector spaces identify the vector space axioms that fail.

9. The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

with the standard matrix addition and scalar multiplication.

Axiom 1: the sum of two diagonal 2×2 matrices is also a diagonal 2×2 matrix.

Axiom 2: follows from part (a) of Theorem 1.4.1.

Axiom 3: follows from part (b) of Theorem 1.4.1.

Axiom 4: taking $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; follows from part (a) of Theorem 1.4.2.

Axiom 5: let the negative of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be $\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$;
follows from part (c) of Theorem 1.4.2 and Axiom 2.

Axiom 6: the scalar multiple of a diagonal 2×2 matrix is also a diagonal 2×2 matrix.

Axiom 7: follows from part (h) of Theorem 1.4.1.

Axiom 8: follows from part (j) of Theorem 1.4.1.

Axiom 9: follows from part (l) of Theorem 1.4.1.

Axiom 10: $1\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for all real a and b.

This is a vector space – all axioms hold.

*In Exercises **1–6**, find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.*

1.
$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0 \end{aligned}$$

$$\begin{array}{c} \\ 2R_1+R_2 \\ R_1+R_3 \end{array} \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{x_3 = t}$

$\boxed{x_2 = 0}$

$x_1 + x_2 - x_3 = 0$

$x_1 + 0 - t = 0$

$\boxed{x_1 = t}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \text{Basis}$$

Hence dimension = 1

6.
$$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

$$\begin{array}{c} \\ -3R_1+R_2 \\ -4R_1+R_3 \\ -6R_1+R_4 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & -2 \\ 4 & 3 & -1 \\ 6 & 5 & 1 \end{bmatrix} = -R_2 \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & -1 & -5 \\ 0 & -1 & -5 \end{bmatrix}$$

$$\begin{array}{l} \\ \\ R_2+R_3 \\ R_2+R_4 \end{array}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & -1 & -5 \\ 0 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{z = t}$ $\quad y + 5z = 0 \quad \boxed{y = -5t}$ $\quad x + y + z = 0$, $\quad x - 5t + t = 0$, $\quad \boxed{x = 4t}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} \rightarrow \text{Dimension} = 1$$

7. In each part, find a basis for the given subspace of R^3, and state its dimension. $\rightarrow 3\times3$

a. The plane $3x - 2y + 5z = 0$.

$$\tfrac{1}{3}R_1 \begin{bmatrix} 3 & -2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2/3 & 5/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{y = s}$ $\boxed{z = t}$ $\quad x - \frac{2}{3}y + \frac{5}{3}z = 0$

$$\boxed{x = \frac{2}{3}s - \frac{5}{3}t}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{s\begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix}}_{\text{Dimension} = 2}$$

b. The plane $x - y = 0$.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{z = t}$ $\boxed{y = s}$

$x - y = 0$

$\boxed{x = s}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{Dimension} = 2}$$

c. The line $x = 2t, y = -t, z = 4t$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \longrightarrow \text{Dimension} = 1$$

d. All vectors of the form (a, b, c), where $b = a + c$.

$(a, a+c, c)$

$\boxed{a(1, 1, 0) + c(0, 1, 1)}$

$\downarrow$

Dimensions = 2

10. Find the dimension of the subspace of P_3 consisting of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

$0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

17. Find a basis for the subspace of R^3 that is spanned by the vectors

$$\mathbf{v}_1 = (1,0,0),\ \mathbf{v}_2 = (1,0,1),\ \mathbf{v}_3 = (2,0,1),\ \mathbf{v}_4 = (0,0,-1)$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{x_4 = t}$ $\boxed{x_3 = s}$

$x_2 + x_3 - x_4 = 0$

$x_2 + s - t = 0$

$\boxed{x_2 = -s + t}$

$x_1 + x_2 + 2x_3 = 0$

$x_1 - s + t + 2s = 0$

$x_1 + s + t = 0$

$\boxed{x_1 = -s - t}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

19. In each part, let $T_A : R^3 \to R^3$ be multiplication by A and find the dimension of the subspace of R^3 consisting of all vectors $\mathbf{x}$ for which $T_A(\mathbf{x}) = \mathbf{0}$.

a. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$\begin{array}{c} \\ -R_1+R_2 \\ -R_1+R_3 \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \\ -R_2 \\ \\ \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{array}{c} \\ \\ R_2+R_3 \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{x_3 = t}$ $\quad x_2 - t = 0 \quad x_1 + x_2 = 0$

$\boxed{x_2 = t}$ $\quad \boxed{x_1 = -t}$

$(-t, t, t) = \boxed{t(-1, 1, 1)} \rightarrow$ Dimension $= 1$

20. In each part, let T_A be multiplication by A and find the dimension of the subspace R^4 consisting of all vectors $\mathbf{x}$ for which $T_A(\mathbf{x}) = \mathbf{0}$.

a. $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix}$

$$\begin{array}{c} \\ R_1+R_2 \end{array} \begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ \tfrac{1}{4}R_2 \end{array} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 4 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1/2 & -1/4 \end{bmatrix}$$

$\boxed{x_4 = t}$ $\quad \boxed{x_3 = s}$ $\quad x_2 - \frac{1}{2}s - \frac{1}{4}t = 0$

$$\boxed{x_2 = \frac{1}{2}s + \frac{1}{4}t}$$

$x_1 + 2x_3 - x_4 = 0$

$x_1 + 2x_3 - x_4 = 0$

$x_1 + 2s - t = 0$

$\boxed{x_1 = -2s + t}$

$\boxed{(x_1, x_2, x_3, x_4) = s\left(-2, 1/2, 1, 0\right) + t\left(1, 1/4, 0, 1\right)}$

2 dimensions.

b. $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$$R_1 + R_2 \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\boxed{x_4 = t}$ $x_3 + x_4 = 0$ $\boxed{x_2 = -t}$ $x_1 + x_4 = 0$

$\boxed{x_3 = -t}$ $\boxed{x_1 = -t}$

$\boxed{(x_1, x_2, x_3, x_4) = t\left(-1, -1, -1, 1\right)}$

Dimension = 1

*In Exercises **1–4**, confirm by multiplication that* **x** *is an eigenvector of* A*, and find the corresponding eigenvalue.*

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$Ax = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\downarrow$$

$$-1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \boxed{-1\,x}$$

Proves that it is eigen vector with eigen value = -1

3. $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$A \cdot x = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$$

$$\downarrow$$

$$5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \boxed{5 \cdot x}$$

Vector with $\lambda = 5$

*In each part of Exercises **5–6**, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.*

5. a. $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$$\det(\lambda I - A) = 0$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-3) - 8 = 0$$

$$\lambda^2 - 4\lambda + 3 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\lambda^2 - 5\lambda + \lambda - 5 = 0$$

$$\lambda(\lambda - 5) + 1(\lambda - 5) = 0$$

$$\boxed{\lambda = -1} \quad \boxed{\lambda = 5}$$

For $\lambda = -1$

$$\begin{bmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \xrightarrow{2R_1 + R_2}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \boxed{x_2 = t} \quad \boxed{x_1 = -2t}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}} \text{ Ans}$$

In Exercises **7–12**, *find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.*

7. $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} \lambda-4 & 0 & -1 \\ 2 & \lambda-1 & 0 \\ 2 & 0 & \lambda-1 \end{vmatrix} = 0$$

$$(\lambda-4)\left[(\lambda-1)^2 - 0\right] - 0 - 1\left[0 - (2\lambda-2)\right] = 0$$

$$(\lambda-4)(\lambda^2 - 2\lambda + 1) + 2\lambda - 2 = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda - 4\lambda^2 + 8\lambda - 4 + 2\lambda - 2 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$\boxed{\lambda = 1}$, $\boxed{\lambda = 3}$, $\boxed{\lambda = 2}$

For $\lambda = 1$

$$\begin{bmatrix} \lambda-4 & 0 & -1 \\ 2 & \lambda-1 & 0 \\ 2 & 0 & \lambda-1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & -1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_1} \begin{bmatrix} 1 & 0 & 1/3 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[-2R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 0 & -2/3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{3}{2}R_2} \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{x_2 = t}$ $\boxed{x_3 = 0}$ $\boxed{x_1 = 0}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$ Ans.

For $\lambda = 2$

$$\xrightarrow{-\frac{1}{2}R_1} \begin{bmatrix} -2 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \cdots & \cdots & \cdots \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \begin{bmatrix} \cdots & \cdots & \cdots \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{x_3 = t}$ $\quad x_2 - t = 0 \quad x_1 + \frac{1}{2}t = 0$

$\boxed{x_2 = t}$ $\quad \boxed{x_1 = -\frac{1}{2}t}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}}$$ Ans.

For $\lambda = 3$

$$-R_1 \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{x_3 = t}$ $\quad x_2 - t = 0 \quad x_1 + t = 0$

$\boxed{x_2 = t}$ $\quad \boxed{x_1 = -t}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$ Ans.

In Exercises 13–14, find the characteristic equation of the matrix by inspection.

13. $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$ $\rightarrow$ Lower Triangular Matrix.

Take diagonal entries directly.

$\boxed{(\lambda - 3)(\lambda - 7)(\lambda - 1) = 0}$ Ans.

In Exercises 15–16, find the eigenvalues and a basis for each eigenspace of the linear operator defined by the stated formula. [Suggestion: *Work with the standard matrix for the operator.*]

15. $T(x, y) = (x + 4y, 2x + 3y)$

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda - 3) - 8 = 0$$

$$\lambda^2 - 4\lambda + 3 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\boxed{\lambda = -1} \quad \boxed{\lambda = 5}$$

For $\lambda = -1$

$$-\tfrac{1}{2}R_1 \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} = {}_{2R_1 + R_2}\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{x_2 = t}$$

$$\boxed{x_1 = -2t}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$ Ans.

For $\lambda = 5$

$$\tfrac{1}{4}R_1 \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} = {}_{2R_1 + R_2}\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$ Ans.

EXAMPLE 1 | Finding a Matrix P That Diagonalizes a Matrix A

Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution In Example 7 of the preceding section we found the characteristic equation of A to be

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and we found the following bases for the eigenspaces:

$$\lambda = 2: \quad \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \qquad \lambda = 1: \quad \mathbf{p}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

There are three basis vectors in total, so the matrix

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalizes A. As a check, you should verify that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE 2 | A Matrix That Is Not Diagonalizable

Show that the following matrix is not diagonalizable:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

Solution The characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2$$

so the characteristic equation is

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and the distinct eigenvalues of A are $\lambda = 1$ and $\lambda = 2$. We leave it for you to show that bases for the eigenspaces are

$$\lambda = 1: \quad \mathbf{p}_1 = \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}; \qquad \lambda = 2: \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since A is a 3×3 matrix and there are only two basis vectors in total, A is not diagonalizable.

Alternative Solution If you are concerned only in determining whether a matrix is diagonalizable and not with actually finding a diagonalizing matrix P, then it is not necessary to compute bases for the eigenspaces—it suffices to find the dimensions of the eigenspaces. For this example, the eigenspace corresponding to $\lambda = 1$ is the solution space of the system

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 3 & -5 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the coefficient matrix has rank 2 (verify), the nullity of this matrix is 1 by Theorem 4.9.2, and hence the eigenspace corresponding to $\lambda = 1$ is one-dimensional.

The eigenspace corresponding to $\lambda = 2$ is the solution space of the system

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & -5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This coefficient matrix also has rank 2 and nullity 1 (verify), so the eigenspace corresponding to $\lambda = 2$ is also one-dimensional. Since the eigenspaces produce a total of two basis vectors, and since three are needed, the matrix A is not diagonalizable.

EXAMPLE 4 | Diagonalizability of Triangular Matrices

From Theorem 5.1.2, the eigenvalues of a triangular matrix are the entries on its main diagonal. Thus, a triangular matrix with distinct entries on the main diagonal is diagonalizable. For example,

$$A = \begin{bmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

is a diagonalizable matrix with eigenvalues $\lambda_1 = -1, \lambda_2 = 3, \lambda_3 = 5, \lambda_4 = -2$.

Question:

9. Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

a. Find the eigenvalues of A.

b. For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

c. Is A diagonalizable? Justify your conclusion.

Solution:

(a) Cofactor expansion along the second column yields

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & 0 & -1 \\ -2 & \lambda - 3 & -2 \\ -1 & 0 & \lambda - 4 \end{vmatrix} = (\lambda - 3)\begin{vmatrix} \lambda - 4 & -1 \\ -1 & \lambda - 4 \end{vmatrix} = (\lambda - 3)\left[(\lambda - 4)^2 - 1\right] = (\lambda - 3)^2(\lambda - 5)$$

therefore A has eigenvalues 3 (with algebraic multiplicity 2) and 5.

Question:

9. Let

$$A=\begin{bmatrix}4&0&1\\2&3&2\\1&0&4\end{bmatrix}$$

a. Find the eigenvalues of A.

b. For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

c. Is A diagonalizable? Justify your conclusion.

Solution:

(a) Cofactor expansion along the second column yields

$$\det(\lambda I - A)=\begin{vmatrix}\lambda-4&0&-1\\-2&\lambda-3&-2\\-1&0&\lambda-4\end{vmatrix}=(\lambda-3)\begin{vmatrix}\lambda-4&-1\\-1&\lambda-4\end{vmatrix}=(\lambda-3)\left[(\lambda-4)^2-1\right]=(\lambda-3)^2(\lambda-5)$$

therefore A has eigenvalues 3 (with algebraic multiplicity 2) and 5.

(b) The reduced row echelon form of $3I-A$ is $\begin{bmatrix}1&0&1\\0&0&0\\0&0&0\end{bmatrix}$, consequently $\operatorname{rank}(3I-A)=1$.

The reduced row echelon form of $5I-A$ is $\begin{bmatrix}1&0&-1\\0&1&-2\\0&0&0\end{bmatrix}$, consequently $\operatorname{rank}(5I-A)=2$.

(c) Based on part (b), the geometric multiplicities of the eigenvalues $\lambda=3$ and $\lambda=5$ are $3-1=2$ and $3-2=1$, respectively. Since these are equal to the corresponding algebraic multiplicities, by Theorem 5.2.4(b) A is diagonalizable.

Eigenvalues of Powers of a Matrix

Theorem 5.2.3

If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

EXAMPLE 5 | Eigenvalues and Eigenvectors of Matrix Powers

In Example 2 we found the eigenvalues and corresponding eigenvectors of the matrix

$$A=\begin{bmatrix}1&0&0\\1&2&0\\-3&5&2\end{bmatrix}$$

Do the same for A^7.

Solution We know from Example 2 that the eigenvalues of A are $\lambda=1$ and $\lambda=2$, so the eigenvalues of A^7 are $\lambda=1^7=1$ and $\lambda=2^7=128$. The eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$ obtained in Example 1 corresponding to the eigenvalues $\lambda=1$ and $\lambda=2$ of A are also the eigenvectors corresponding to the eigenvalues $\lambda=1$ and $\lambda=128$ of A^7.

EXAMPLE 6 | Powers of a Matrix

Use (3) to find A^{13}, where

$$A=\begin{bmatrix}0&0&-2\\1&2&1\\1&0&3\end{bmatrix}$$

Solution We showed in Example 1 that the matrix A is diagonalized by

$$P=\begin{bmatrix}-1&0&-2\\0&1&1\\1&0&1\end{bmatrix}$$

and that

$$D=P^{-1}AP=\begin{bmatrix}2&0&0\\0&2&0\\0&0&1\end{bmatrix}$$

Thus, it follows from (3) that

$$A^{13}=PD^{13}P^{-1}=\begin{bmatrix}-1&0&-2\\0&1&1\\1&0&1\end{bmatrix}\begin{bmatrix}2^{13}&0&0\\0&2^{13}&0\\0&0&1^{13}\end{bmatrix}\begin{bmatrix}1&0&2\\1&1&1\\-1&0&-1\end{bmatrix} \tag{4}$$

$$=\begin{bmatrix}-8190&0&-16382\\8191&8192&8191\\8191&0&16383\end{bmatrix}$$

Question:

Let

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Confirm that P diagonalizes A, and then compute each of the following powers of A.

a. A^{1000} **b.** A^{-1000} **c.** A^{2301} **d.** A^{-2301}

Solution:

After calculating $P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$, we verify that

$$P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$ is a diagonal matrix therefore

P diagonalizes A.

(a) $$A^{1000} = PD^{1000}P^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} (-1)^{1000} & 0 & 0 \\ 0 & (-1)^{1000} & 0 \\ 0 & 0 & 1^{1000} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There is some terminology that is related to these ideas. If λ_0 is an eigenvalue of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the ***geometric multiplicity*** of λ_0, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the ***algebraic multiplicity*** of λ_0. The following theorem, which we state without proof, summarizes the preceding discussion.

Theorem 5.2.4

Geometric and Algebraic Multiplicity

If A is a square matrix, then:

(*a*) For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.

(*b*) A is diagonalizable if and only if its characteristic polynomial can be expressed as a product of linear factors, and the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.

Q1(a) Find a matrix P that orthogonally diagonalizes the matrix A,

$$A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

Q1(b) Also, determine the QR d...

$\det|\lambda I - A| = 0$

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix} = 0$$

$$\begin{vmatrix} \lambda+3 & -1 & -2 \\ -1 & \lambda+3 & -2 \\ -2 & -2 & \lambda \end{vmatrix} = 0$$

$$(\lambda+3)\left[\lambda^2+3\lambda-4\right] - (-1)\left[-\lambda-4\right] - 2\left[2-(-2\lambda-6)\right] = 0$$

$$(\lambda+3)(\lambda^2+3\lambda-4) + (-\lambda-4) - 2(8+2\lambda) = 0$$

$$\lambda^3+3\lambda^2-4\lambda+3\lambda^2+9\lambda-12-\lambda-4-16-4\lambda = 0$$

$$\lambda^3+6\lambda^2-32=0$$

$\boxed{\lambda = 2}$ $\boxed{\lambda = -4}$

For $\lambda = 2$

$$\begin{bmatrix} 2+3 & -1 & -2 \\ -1 & 2+3 & -2 \\ -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{1}{5}R_1 \begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix} = \begin{array}{c} \\ R_1+R_2 \\ 2R_1+R_3 \end{array} \begin{bmatrix} 1 & -1/5 & -2/5 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$

$$\frac{5}{24}R_2 \begin{bmatrix} 1 & -1/5 & -2/5 \\ 0 & 24/5 & -12/5 \\ 0 & -12/5 & 6/5 \end{bmatrix} = \begin{array}{c} \frac{1}{5}R_2+R_1 \\ \\ \frac{12}{5}R_2+R_3 \end{array} \begin{bmatrix} 1 & -1/5 & -2/5 \\ 0 & 1 & -1/2 \\ 0 & -12/5 & 6/5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{x_3 = t}$ $x_2 - \frac{1}{2}t = 0$ $x_1 - \frac{1}{2}t = 0$

$\boxed{x_2 = \frac{1}{2}t}$ $\boxed{x_1 = \frac{1}{2}t}$

$$\boxed{x_2 = \frac{1}{2}t} \qquad \boxed{x_1 = \frac{1}{2}t}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

<u>For $\lambda = -4$</u>

$$\begin{bmatrix} -4+3 & -1 & -2 \\ -1 & -4+3 & -2 \\ -2 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-R_1 \begin{bmatrix} -1 & -1 & -2 \\ -1 & -1 & -2 \\ -2 & -2 & -4 \end{bmatrix} = \begin{matrix} \\ R_1+R_2 \\ 2R_1+R_3 \end{matrix} \begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & -2 \\ -2 & -2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{x_3 = t} \qquad \boxed{x_2 = s} \qquad x_1 + s + 2t = 0$$

$$\boxed{x_1 = -s - 2t}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1/2 & -1 & -2 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D = P^{-1} A P = \begin{bmatrix} 1/3 & 1/3 & 2/3 \\ -1/6 & 5/6 & -1/3 \\ -1/3 & -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -1 & -2 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

In Exercises **1–4**, *show that A and B are not similar matrices.*

1. $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$

$\det(A) \neq \det(B)$

$$\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \neq \begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix}$$

$2 - 3 \neq -2 - 0$

$\boxed{-1 \neq -2}$ Proved!

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$1(1-0) - 2(0-0) + 3(0-0) \neq 1(1-0) - 2(1/2 - 0) + 0$

$1 \neq 1 - 1$

$\boxed{1 \neq 0}$ Proved!

In Exercises **5–8**, *find a matrix P that diagonalizes A, and check your work by computing* $P^{-1}AP$.

5. $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda - 1 & 0 \\ -6 & \lambda + 1 \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda + 1) - 0 = 0$$

$\boxed{\lambda = 1}$ $\boxed{\lambda = -1}$

For $\lambda = 1$

$$\begin{bmatrix} 0 & 0 \\ -6 & 2 \end{bmatrix} = \frac{-1}{6} R_1 \begin{bmatrix} -6 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

$\boxed{x_2 = t}$ $x_1 - \frac{1}{3} t = 0$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

$\boxed{x_1 = \frac{1}{3} t}$

For $\lambda = -1$

$$-\frac{1}{2} R_1 \begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix} = {}_{6R_1 + R_2} \begin{bmatrix} 1 & 0 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\boxed{x_2 = t}$ $\boxed{x_1 = 0}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} 1/3 & 0 \\ 1 & 1 \end{bmatrix}}$$

$D = P^{-1} A P$

$D = P^{-1} A P$

$$D = \begin{bmatrix} 3 & 0 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 0 \\ 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

Verified!

7. $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 2 & 0 & 2 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 3 \end{vmatrix} = 0$$

$$(\lambda - 2)\left[(\lambda - 3)^2\right] - 0 + 2[0] = 0$$

$$(\lambda - 2)(\lambda - 3)^2 = 0$$

$\boxed{\lambda = 2}$ $\boxed{\lambda = 3}$ $\boxed{\lambda = 3}$

For $\lambda = 2$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \overset{-R_1}{=} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$-R_2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} \underset{-2R_2 + R_3}{=} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{x_1 = t}$ $\boxed{x_2 = 0}$ $\boxed{x_3 = 0}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 3$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = t$

$x_2 = s$

$x_1 + 2t = 0$

$\boxed{x_1 = -2t}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$D = P^{-1} A P$

$$D = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

Verified!

10. Follow the directions in Exercise 9 for the matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

a. Find the eigenvalues of A.

b. For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

c. Is A diagonalizable? Justify your conclusion.

a)
$$\begin{vmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & -1 & \lambda - 2 \end{vmatrix} = 0$$

$$(\lambda - 3)\left[(\lambda - 2)^2 - 0\right] - 0 + 0 = 0$$

$$(\lambda - 3)(\lambda - 2)^2 = 0$$

$$(\lambda-3)\left[(\lambda-2)-0\right]-0+0=0$$

$$(\lambda-3)(\lambda-2)^2=0$$

$\boxed{\lambda=3}$ $\boxed{\lambda=2}$ $\boxed{\lambda=2}$

b) For $\lambda=3$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \overset{-R_1}{=} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-R_1+R_2 \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$

Rank $=2$

For $\lambda=2$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \overset{-R_1}{=} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$

Rank $=2$

c) For $\lambda=3$, Nullity $=1$ which means geometric multiplicity is 1.
For $\lambda=2$, Nullity $=1$ which means geometric multiplicity is 1.

But the algebraic multiplicities are 1 & 2 simultaneously. Since they are not same, matrix is not diagonalizable.

In Exercises 11–14, find the geometric and algebraic multiplicity of each eigenvalue of the matrix A, and determine whether A is diagonalizable. If A is diagonalizable, then find a matrix P that diagonalizes A, and find $P^{-1}AP$.

11. $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{vmatrix} = 0$$

$(\lambda+1)\left[(\lambda-4)(\lambda-3)\right] + 4\left[3\lambda - 9 - 0\right] + 2\left[-3 - 3\lambda + 12\right] = 0$

$(\lambda+1)(\lambda^2 - 3\lambda - 4\lambda + 12) + 12\lambda - 36 + 18 - 6\lambda = 0$

$(\lambda+1)(\lambda^2 - 7\lambda + 12) + 6\lambda - 18 = 0$

$\lambda^3 - 7\lambda^2 + 12\lambda + \lambda^2 - 7\lambda + 12 + 6\lambda - 18 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\boxed{\lambda = 1}$ $\boxed{\lambda = 3}$ $\boxed{\lambda = 2}$

For $\lambda = 1$

$$\tfrac{1}{2}R_1 \begin{bmatrix} 2 & -4 & 2 \\ 3 & -3 & 0 \\ 3 & -1 & -2 \end{bmatrix} = \begin{array}{c} \\ -3R_1+R_2 \\ -3R_1+R_3 \end{array} \begin{bmatrix} 1 & -2 & 1 \\ 3 & -3 & 0 \\ 3 & -1 & -2 \end{bmatrix}$$

$$\frac{1}{3}R_2 \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & 5 & -5 \end{bmatrix} = \begin{array}{c} 2R_2+R_1 \\ \\ -5R_2+R_3 \end{array} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 5 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underline{\lambda = 1}$
Algebraic = 1
Geometric = 1

For $\lambda = 3$

$$\frac{1}{4}R_1 \begin{bmatrix} 4 & -4 & 2 \\ 3 & -1 & 0 \\ 3 & -1 & 0 \end{bmatrix} = \begin{array}{c} \\ -3R_1+R_2 \\ -3R_1+R_3 \end{array} \begin{bmatrix} 1 & -1 & 1/2 \\ 3 & -1 & 0 \\ 3 & -1 & 0 \end{bmatrix}$$

$$\frac{1}{2}R_2 \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 2 & -3/2 \\ 0 & 2 & -3/2 \end{bmatrix} = \begin{array}{c} R_2+R_1 \\ \\ -2R_2+R_3 \end{array} \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & -3/4 \\ 0 & 2 & -3/2 \end{bmatrix}$$

$$-2R_3 \begin{bmatrix} 1 & 0 & -5/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} =$$

$\underline{\lambda = 3}$
Algebraic = 1
Geometric = 1

⋮

Working for $\lambda = 2$, Algebraic = 1
Geometric = 1

So Diagonalizable solve further..

In each part of Exercises **15–16**, *the characteristic equation of a matrix A is given. Find the size of the matrix and the possible dimensions of its eigenspaces.*

15. a. $(\lambda - 1)(\lambda + 3)(\lambda - 5) = 0$

3 × 3 matrix

$\boxed{\lambda = 1}$ $\boxed{\lambda = -3}$ $\boxed{\lambda = 5}$

All three will have 1 dimension.

b. $\lambda^2(\lambda-1)(\lambda-2)^3 = 0$

6 × 6 matrix because λ has a power of 6.

$\lambda = 0$ will have possibly 1 or 2 dimensions

$\lambda = 1$ " " " 1 dimension

$\lambda = 2$ " " " 1, 2 or 3 dimensions

In Exercises 17–18, use the method of Example 6 to compute the matrix A^{10}.

17. $A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$ → $A = PDP^{-1}$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda & -3 \\ -2 & \lambda+1 \end{vmatrix} = 0$$

$\lambda(\lambda+1) - 6 = 0$

$\lambda^2 + \lambda - 6 = 0$

$\boxed{\lambda = 2}$ $\boxed{\lambda = -3}$

For $\lambda = 2$

$$\tfrac{1}{2}R_1 \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} = {}_{2R_1+R_2}\begin{bmatrix} 1 & -3/2 \\ -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix} \quad \boxed{x_2 = t} \quad x_1 - \frac{3}{2}t = 0$$

$$\boxed{x_1 = \frac{3}{2}t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

<u>For $\lambda = -3$</u>

$$-\frac{1}{3}R_1 \begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} = {}_{2R_1+R_2} \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$\boxed{x_2 = t}$ $\boxed{x_1 = -t}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$D = P^{-1} A P$$

$$D = \begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\boxed{D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}}$$

$$A^{10} = P D^{10} P^{-1}$$

$$A^{10} = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2^{10} & 0 \\ 0 & (-3)^{10} \end{bmatrix} \cdot \begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix}$$

$$\boxed{A^{10} = \begin{bmatrix} 24234 & -34815 \\ -23210 & 35839 \end{bmatrix}}$$ Ans.

19. Let

$$A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}$$

Confirm that P diagonalizes A, and then compute A^{11}.

$$D = P^{-1} A P$$

$$D = \begin{bmatrix} 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 7 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}$$

$$\boxed{D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Shown!

$$A^{11} = P \, D^{11} \, P^{-1}$$

$$\boxed{A^{11} = \begin{bmatrix} -1 & 10237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{bmatrix}}$$

Ans.

In Exercises 1–2, find the cosine of the angle between the vectors with respect to the Euclidean inner product.

1. a. $\mathbf{u} = (1, -3)$, $\mathbf{v} = (2, 4)$

$$\cos\theta = \frac{\langle u, v\rangle}{|u|\,|v|}$$

$$\cos\theta = \frac{2 - 12}{\sqrt{10} \times \sqrt{20}} = \frac{-10}{\sqrt{200}} = \frac{-10}{\sqrt{2 \times 100}} = \frac{-\cancel{10}}{\cancel{10}\sqrt{2}} = \boxed{\frac{-1}{\sqrt{2}}}$$

c. $\mathbf{u} = (1, 0, 1, 0)$, $\mathbf{v} = (-3, -3, -3, -3)$

$$\cos\theta = \frac{-3 - 3}{\sqrt{2} \times \sqrt{36}} = \frac{-\cancel{6}}{\sqrt{2} \times \cancel{6}} = \boxed{\frac{-1}{\sqrt{2}}}$$

In Exercises 3–4, find the cosine of the angle between the vectors with respect to the standard inner product on P_2.

3. $\mathbf{p} = -1 + 5x + 2x^2$, $\mathbf{q} = 2 + 4x - 9x^2$

$$\cos\theta = \frac{-2 + 20 - 18}{\sqrt{30} \times \sqrt{101}} = \boxed{0}$$

In Exercises 5–6, find the cosine of the angle between A and B with respect to the standard inner product on M_{22}.

5. $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

$$\cos\theta = \frac{6 + 12 + 1 + 0}{5\sqrt{2} \times \sqrt{14}} = \frac{19}{5\sqrt{28}} = \frac{19}{5\sqrt{7 \times 4}} = \boxed{\frac{19}{10\sqrt{7}}}$$

Ans

In Exercises 7–8, determine whether the vectors are orthogonal with respect to the Euclidean inner product.

7. a. $\mathbf{u} = (-1, 3, 2)$, $\mathbf{v} = (4, 2, -1)$

- When $\langle c_1, c_2\rangle = \langle c_1, c_3\rangle = \langle c_2, c_3\rangle = 0$, it is orthogonal.

- when $|c_1| = |c_2| = |c_2| = 1$, it is orthonormal

$\rightarrow$ $\langle u, v \rangle = \langle (-1, 3, 2), (4, 2, -1) \rangle$

$= -4 + 6 - 2 = \boxed{0}$ Orthogonal

c. $\mathbf{u} = (a, b)$, $\mathbf{v} = (-b, a)$

$\langle u, v \rangle = \langle (a, b), (-b, a) \rangle = -ab + ab = \boxed{0}$

Orthogonal

8. a. $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (0, 0, 0)$

$\langle u, v \rangle = \langle (u_1, u_2, u_3), (0, 0, 0) \rangle = \boxed{0}$

In Exercises 9–10, show that the vectors are orthogonal with respect to the standard inner product on P_2.

9. $\mathbf{p} = -1 - x + 2x^2$, $\mathbf{q} = 2x + x^2$

$\langle p, q \rangle = \langle (1, -1, 2), (0, 2, 1) \rangle$

$= 0 - 2 + 2 = \boxed{0}$

In Exercises 11–12, show that the matrices are orthogonal with respect to the standard inner product on M_{22}.

11. $U = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $V = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

$\langle u, v \rangle = \langle (2, 1, -1, 3), (-3, 0, 0, 2) \rangle$

$= -6 + 0 + 0 + 6 = \boxed{0}$

17. Do there exist scalars k and l such that the vectors

$$\mathbf{p}_1 = 2 + kx + 6x^2, \quad \mathbf{p}_2 = l + 5x + 3x^2, \quad \mathbf{p}_3 = 1 + 2x + 3x^2$$

are mutually orthogonal with respect to the standard inner product on P_2?

$$\langle p_1, p_2 \rangle = \langle p_2, p_3 \rangle = \langle p_1, p_3 \rangle = 0$$

$$\langle p_1, p_2 \rangle = \langle (2, k, 6), (l, 5, 3) \rangle$$

$$\boxed{0 = 2l + 5k + 18} \quad \text{— } ①$$

$$\langle p_2, p_3 \rangle = \langle (l, 5, 3), (1, 2, 3) \rangle$$

$$0 = l + 10 + 9$$

$$0 = l + 19$$

$$\boxed{l = -19}$$

$$\langle p_1, p_3 \rangle = \langle (2, k, 6), (1, 2, 3) \rangle$$

$$0 = 2 + 2k + 18$$

$$0 = 2k + 20$$

$$2k = -20$$

$$\boxed{k = -10}$$

Now, substitute k & l in eq ①

$$2l + 5k + 18 = 0$$

$$2(-19) + 5(-10) + 18 = 0$$

$$-38 - 50 + 18 = 0$$

$$\boxed{-70 \neq 0} \longrightarrow \text{So these scalars don't exist.}$$

18. Show that the vectors

$$\mathbf{u} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$$

are orthogonal with respect to the inner product on R^2 that is generated by the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

[See Formulas (5) and (6) of Section 6.1.]

$$\left\langle \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} , \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right\rangle$$

$$\left\langle \begin{bmatrix} 9 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\rangle = 18 - 18 = \boxed{0}$$

19. Let P_2 have the evaluation inner product at the points

$$x_0 = -2, \quad x_1 = 0, \quad x_2 = 2$$

Show that the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal with respect to this inner product.

$$\langle p, q \rangle = p(-2) \cdot q(-2) + p(0) \cdot q(0) + p(2) \cdot q(2)$$

$\boxed{\langle p, q \rangle = 0}$ Ans.

1. In each part, determine whether the set of vectors is orthogonal and whether it is orthonormal with respect to the Euclidean inner product on R^2.

a. $(0,1),\ (2,0)$

- When $\langle c_1, c_2\rangle = \langle c_1, c_3\rangle = \langle c_2, c_3\rangle = 0$, it is orthogonal.
- when $|c_1| = |c_2| = |c_3| = 1$, it is orthonormal.

$\rightarrow \langle (0,1), (2,0)\rangle = 0 + 0 = \boxed{0}$ Orthogonal

$|c_1| = \sqrt{0^2+1^2} = 1$

$|c_2| = \sqrt{2^2+0^2} = 2$ $\rightarrow$ Not orthonormal.

2. In each part, determine whether the set of vectors is orthogonal and whether it is orthonormal with respect to the Euclidean inner product on R^3.

a. $\underbrace{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}_{c_1}, \underbrace{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)}_{c_2}, \underbrace{\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}_{c_3}$

$\rightarrow \langle c_1, c_2\rangle = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0$

$\langle c_1, c_3\rangle = \frac{-1}{\sqrt{4}} + 0 + \frac{1}{\sqrt{4}} = 0$

$\langle c_2, c_3\rangle = \frac{-1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = \frac{-2}{\sqrt{6}}$

$\rightarrow$ Not orthogonal

Not orthogonal, so no need to check further. It will not be orthonormal as well.

3. In each part, determine whether the set of vectors is orthogonal with respect to the standard inner product on P_2 (see Example 7 of Section 6.1).

a. $p_1(x) = \frac{2}{3} - \frac{2}{3}x + \frac{1}{3}x^2,\ p_2(x) = \frac{2}{3} + \frac{1}{3}x - \frac{2}{3}x^2,$

$p_3(x) = \frac{1}{3} + \frac{2}{3}x + \frac{2}{3}x^2$

$\langle p_1, p_2\rangle = \langle \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\rangle$

$\langle p_1, p_2\rangle = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$

$\langle p_1, p_3\rangle = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$

$\langle p_2, p_3\rangle = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$

$\rightarrow$ Orthogonal

4. In each part, determine whether the set of vectors is orthogonal with respect to the standard inner product on M_{22} (see Example 6 of Section 6.1).

a. $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{c_1}, \underbrace{\begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}}_{c_2}, \underbrace{\begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}}_{c_3}, \underbrace{\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}}_{c_4}$

$\langle c_1, c_2 \rangle = 0 + 0 + 0 + 0 = 0$

$\langle c_1, c_3 \rangle = 0 + 0 + 0 + 0 = 0$

$\langle c_1, c_4 \rangle = 0$

$\langle c_2, c_3 \rangle = 0 + \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$

$\langle c_2, c_4 \rangle = 0 + \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$

$\langle c_3, c_4 \rangle = 0 + \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$

$\rightarrow$ Orthogonal

In Exercises 5–6, show that the column vectors of A form an orthogonal basis for the column space of A with respect to the Euclidean inner product, and then find an orthonormal basis for that column space.

5. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$

$c_1 = (1, 0, -1) \quad c_2 = (2, 0, 2) \quad c_3 = (0, 5, 0)$

$\langle c_1, c_2 \rangle = 2 + 0 - 2 = 0$

$\langle c_1, c_3 \rangle = 0 + 0 + 0 = 0$

$\langle c_2, c_3 \rangle = 0 + 0 + 0 = 0$

$\rightarrow$ Orthogonal basis shown.

$$\frac{c_1}{|c_1|} = \frac{(1, 0, -1)}{\sqrt{1^2 + 1^2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$$\frac{c_2}{|c_2|} = \frac{(2, 0, 2)}{\sqrt{4 + 4}} = \left(\frac{2}{\sqrt{8}}, 0, \frac{2}{\sqrt{8}}\right)$$

$$\frac{c_3}{|c_3|} = \frac{(0, 5, 0)}{\sqrt{5^2}} = (0, 1, 0)$$

$$\boxed{\begin{bmatrix} 1/\sqrt{2} & 2/\sqrt{8} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 2/\sqrt{8} & 0 \end{bmatrix}}$$ Ans.

7. Verify that the vectors

$$\mathbf{v}_1 = \left(-\tfrac{3}{5}, \tfrac{4}{5}, 0\right),\quad \mathbf{v}_2 = \left(\tfrac{4}{5}, \tfrac{3}{5}, 0\right),\quad \mathbf{v}_3 = (0, 0, 1)$$

form an orthonormal basis for R^3 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*b*) to express the vector $\mathbf{u} = (1, -2, 2)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$\left.\begin{aligned} |v_1| &= \sqrt{\tfrac{9}{25} + \tfrac{16}{25}} = 1 \\ |v_2| &= \sqrt{\tfrac{16}{25} + \tfrac{9}{25}} = 1 \\ |v_3| &= \sqrt{0+0+1} = 1 \end{aligned}\right\} \rightarrow \text{Shown orthonormal.}$$

$$\langle u, v_1 \rangle = (1, -2, 2) \cdot \left(\tfrac{-3}{5}, \tfrac{4}{5}, 0\right) = -11/5$$

$$\langle u, v_2 \rangle = (1, -2, 2) \cdot \left(\tfrac{4}{5}, \tfrac{3}{5}, 0\right) = -2/5$$

$$\langle u, v_3 \rangle = (1, -2, 2) \cdot (0, 0, 1) = 2$$

Linear Combination

$$\langle u, v_1 \rangle\, v_1 + \langle u, v_2 \rangle \cdot v_2 + \langle u, v_3 \rangle \cdot v_3$$

$$\boxed{\tfrac{-11}{5} v_1 - \tfrac{2}{5} v_2 + 2 v_3} \quad \text{Ans}$$

In Exercises ***11–14****, find the coordinate vector* $(\mathbf{u})_S$ *for the vector* $\mathbf{u}$ *and the basis* S *that were given in the stated exercise.*

11. Exercise 7

$$\langle u, v_1 \rangle = -11/5 \qquad \langle u, v_2 \rangle = -\tfrac{2}{5} \qquad \langle u, v_3 \rangle = 2$$

$$\boxed{(u)_S = \left(-11/5,\ -2/5,\ 2\right)} \quad \text{Ans}$$

9. Verify that the vectors

$$\mathbf{v}_1 = (2, -2, 1),\quad \mathbf{v}_2 = (2, 1, -2),\quad \mathbf{v}_3 = (1, 2, 2)$$

form an orthogonal basis for R^3 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*a*) to express the vector $\mathbf{u} = (-1, 0, 2)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Theorem 6.3.2

(*a*) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n \tag{3}$$

(*b*) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n \tag{4}$$

$$\left.\begin{aligned} \langle v_1, v_2 \rangle &= 4 - 2 - 2 = 0 \\ \langle v_2, v_3 \rangle &= 2 + 2 - 4 = 0 \\ \langle v_1, v_3 \rangle &= 2 - 4 + 2 = 0 \end{aligned}\right\} \rightarrow \text{orthogonal}$$

$$\langle u, v_1 \rangle = (-1, 0, 2) \cdot (2, -2, 1) = -2 + 0 + 2 = 0$$

$$\langle u, v_2 \rangle = (-1, 0, 2) \cdot (2, 1, -2) = -2 + 0 - 4 = -6$$

$$\langle u, v_3 \rangle = (-1, 0, 2) \cdot (1, 2, 2) = -1 + 0 + 4 = 3$$

$$u = \frac{\langle u, v_1 \rangle}{|v_1|^2} \cdot v_1 + \frac{\langle u, v_2 \rangle}{|v_2|^2} \cdot v_2 + \frac{\langle u, v_3 \rangle}{|v_3|^2} \cdot v_3$$

$$u = 0\, v_1 - \frac{6}{\sqrt{4+1+4}^{\cancel{2}}} \cdot v_2 + \frac{3}{\sqrt{1+4+4}^{\cancel{2}}} \cdot v_3$$

$$\boxed{u = 0\, v_1 - \frac{2}{3} v_2 + \frac{1}{3} \cdot v_3}$$ Ans.

13. Exercise 9

$$\boxed{u_3 = \left(0, \frac{-2}{3}, \frac{1}{3}\right)}$$ Ans.

The Gram-Schmidt Process

To convert basis into orthogonal basis

Step 1: $v_1 = u_1$

Step 2: $v_2 = u_2 - \frac{(u_2, v_1)}{|v_1|^2} \cdot v_1$

Step 3: $v_3 = u_3 - \frac{(u_3, v_1)}{|v_1|^2} v_1 - \frac{(u_3, v_2)}{|v_2|^2} \cdot v_2$

Step 4: $v_4 = u_4 - \frac{(u_4, v_1)}{|v_1|^2} \cdot v_1 - \frac{(u_4, v_2)}{|v_2|^2} \cdot v_2 - \frac{(u_4, v_3)}{|v_3|^2} \cdot v_3$

Optional Step :- To convert the orthogonal basis into an orthonormal basis $(q_1, q_2, \ldots, q_k)$, normalize the orthogonal basis vectors.

In Exercises 27–28, let R^2 have the Euclidean inner product and use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ into an orthonormal basis. Draw both sets of basis vectors in the xy-plane.

27. $\mathbf{u}_1 = (1, -3)$, $\mathbf{u}_2 = (2, 2)$

$v_1 = u_1$

$$\boxed{v_1 = (1, -3)}$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{|v_1|^2} \cdot v_1$$

$\langle u_2, v_1 \rangle = (2, 2) \cdot (1, -3) = 2 - 6 = -4$

$|v_1| = \sqrt{1 + 9} = \sqrt{10}$

$$v_2 = (2, 2) + \frac{4}{(\sqrt{10})^{\cancel{2}}} \cdot (1, -3)$$

$$v_2 = (2, 2) + \left(\frac{4}{10}, \frac{-12}{10}\right) = \boxed{(12/5,\ 4/5)}$$

$$q_1 = \frac{v_1}{|v_1|} = \frac{(1, -3)}{\sqrt{1+9}} = \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$$

$$q_2 = \frac{v_2}{|v_2|} = \frac{(12/5,\ 4/5)}{4\sqrt{10}/5} = \left(\frac{3\sqrt{10}}{10}, \frac{\sqrt{10}}{10}\right)$$

In Exercises 29–30, let R^3 have the Euclidean inner product and use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an orthonormal basis.

29. $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (-1, 1, 0)$, $\mathbf{u}_3 = (1, 2, 1)$

$$v_1 = (1, 1, 1)$$

$$v_2 = (-1, 1, 0) - \frac{\langle u_2, v_1 \rangle}{|v_1|^2} \cdot v_1$$

$$\langle u_2, v_1 \rangle = (-1, 1, 0) \cdot (1, 1, 1) = 0$$

$$|v_1| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$v_2 = (-1, 1, 0) - 0$$

$$\boxed{v_2 = (-1, 1, 0)}$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{|v_1|^2} \cdot v_1 - \frac{\langle u_3, v_2 \rangle}{|v_2|^2} \cdot v_2$$

$$\langle u_3, v_1 \rangle = (1, 2, 1) \cdot (1, 1, 1) = 4$$

$$|v_1| = \sqrt{3}$$

$$\langle u_3, v_2 \rangle = (1, 2, 1) \cdot (-1, 1, 0) = 1$$

$$|v_2| = \sqrt{2}$$

$$v_3 = (1, 2, 1) - \frac{4}{(\cancel{\sqrt{3}})^{\cancel{2}}} \cdot (1, 1, 1) - \frac{1}{(\cancel{\sqrt{2}})^{\cancel{2}}} \cdot (-1, 1, 0)$$

$$v_3 = (1, 2, 1) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) - \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$$

$$v_3 = \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$$

$$q_1 = \frac{v_1}{|v_1|} = \frac{(1, 1, 1)}{\sqrt{3}} = \boxed{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

$$q_2 = \frac{v_2}{|v_2|} = \frac{(-1, 1, 0)}{\sqrt{2}} = \boxed{\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}$$

$$q_3 = \frac{v_3}{|v_3|} = \frac{(1/6, 1/6, -1/3)}{\sqrt{(\frac{1}{6})^2+(\frac{1}{6})^2+(-\frac{1}{3})^2}} = \boxed{\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{-\sqrt{6}}{3}\right)}$$

31. Let R^4 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ into an orthonormal basis.

$$\mathbf{u}_1 = (0,2,1,0), \quad \mathbf{u}_2 = (1,-1,0,0),$$
$$\mathbf{u}_3 = (1,2,0,-1), \quad \mathbf{u}_4 = (1,0,0,1)$$

$\rightarrow$ $v_1 = u_1$

$v_1 = (0, 2, 1, 0)$

$\rightarrow$ $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{|v_1|^2} \cdot v_1$

$\langle u_2, v_1 \rangle = (1,-1,0,0) \cdot (0,2,1,0) = 0-2+0+0$
$= \boxed{-2}$

$$v_2 = (1,-1,0,0) - \frac{(-2)}{\left(\sqrt{0^2+2^2+1^2+0^2}\right)^2} \cdot (0,2,1,0)$$

$$v_2 = (1,-1,0,0) - \frac{(0,-4,-2,0)}{5}$$

$$v_2 = (1,-1,0,0) - \left(0, -\frac{4}{5}, -\frac{2}{5}, 0\right)$$

$$v_2 = \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$\rightarrow$ $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{|v_1|^2} \cdot v_1 - \frac{\langle u_3, v_2 \rangle}{|v_2|^2} \cdot v_2$

$\langle u_3, v_1 \rangle = (1,2,0,-1) \cdot (0,2,1,0) = 4$

$\langle u_3, v_2 \rangle = (1,2,0,-1) \cdot \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) = \frac{3}{5}$

$$v_3 = (1,2,0,-1) - \frac{4}{5} \cdot (0,2,1,0) - \frac{3/5}{6/5} \cdot (1, -1/5, 2/5, 0)$$

$$v_3 = (1,2,0,-1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{1}{2}, \frac{-1}{10}, \frac{1}{5}, 0\right)$$

$$v_3 = \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$v_4 = u_4 - \frac{\langle u_4, v_1 \rangle}{|v_1|^2} \cdot v_1 - \frac{\langle u_4, v_2 \rangle}{|v_2|^2} \cdot v_2 - \frac{\langle u_4, v_3 \rangle}{|v_3|^2} \cdot v_3$$

$$v_4 = u_4 - \frac{\langle u_4, v_1\rangle}{|v_1|^2}\cdot v_1 - \frac{\langle u_4, v_2\rangle}{|v_2|^2}\cdot v_2 - \frac{\langle u_4, v_3\rangle}{|v_3|^2}\cdot v_3$$

$\langle u_4, v_1\rangle = (1,0,0,1)\cdot(0,2,1,0) = 0$

$\langle u_4, v_2\rangle = (1,0,0,1)\cdot(1,-1/5,2/5,0) = 1$

$\langle u_4, v_3\rangle = (1,0,0,1)\cdot(1/2,1/2,-1,-1) = -1/2$

$$v_4 = (1,0,0,1) - 0 - \frac{1}{6/5}\cdot(1,-1/5,2/5,0) - \frac{-1/\cancel{2}}{5/\cancel{2}}\cdot(1/2, 1/2, -1, -1)$$

$$v_4 = (1,0,0,1) - \left(\frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, 0\right) - \left(\frac{-1}{10}, \frac{-1}{10}, \frac{1}{5}, \frac{1}{5}\right)$$

$$v_4 = \left(\frac{4}{15}, \frac{4}{15}, \frac{-8}{15}, \frac{4}{5}\right)$$

$$q_1 = \frac{v_1}{|v_1|} = \frac{(0,2,1,0)}{\sqrt{5}} = \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$q_2 = \frac{v_2}{|v_2|} = \frac{(1,-1/5,2/5,0)}{\sqrt{6/5}} = \left(\frac{\sqrt{30}}{6}, \frac{-\sqrt{30}}{30}, \frac{\sqrt{30}}{15}, 0\right)$$

$$q_3 = \frac{(1/2, 1/2, -1, -1)}{\sqrt{5/2}} = \left(\frac{\sqrt{10}}{10}, \frac{\sqrt{10}}{10}, -\frac{\sqrt{10}}{5}, -\frac{\sqrt{10}}{5}\right)$$

$$q_4 = \frac{(4/15, 4/15, -8/15, 4/5)}{\sqrt{16/15}} = \left(\frac{\sqrt{15}}{15}, \frac{\sqrt{15}}{15}, \frac{-2\sqrt{15}}{15}, \frac{\sqrt{15}}{5}\right)$$

44. Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} 6 & 1 & -5 \\ 2 & 1 & 1 \\ -2 & -2 & 5 \\ 6 & 8 & -7 \end{bmatrix}$$

$u_1 = (6, 2, -2, 6)$ $\quad u_2 = (1, 1, -2, 8)$ $\quad u_3 = (-5, 1, 5, -7)$

$\boxed{v_1 = (6, 2, -2, 6)}$

$\langle u_2, v_1\rangle = (1,1,-2,8)\cdot(6,2,-2,6) = 6 + 2 + 4 + 48 = 60$

$|v_1| = \sqrt{36+4+4+36} = \sqrt{80}$

$$v_2 = (1, 1, -2, 8) - \frac{6\cancel{0}}{(\sqrt{8\cancel{0}})^{\cancel{2}}}\cdot(6, 2, -2, 6)$$

$v_2 = (1, 1, -2, 8) - \left(\frac{9}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{9}{2}\right)$

$$\boxed{v_2 = \left(\frac{-7}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{7}{2}\right)}$$

$\langle u_3, v_1 \rangle = (-5, 1, 5, -7) \cdot (6, 2, -2, 6) = -80$

$|v_1| = \sqrt{80}$

$\langle u_3, v_2 \rangle = (-5, 1, 5, -7) \cdot \left(\frac{-7}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{7}{2}\right) = -10$

$|v_2| = 5$

$V_3 = (-5, 1, 5, -7) + \frac{\cancel{80}}{\cancel{80}} \cdot (6, 2, -2, 6) + \frac{10}{5^2} \cdot \left(\frac{-7}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{7}{2}\right)$

$V_3 = (1, 3, 3, -1) + \left(-\frac{7}{5}, \frac{-1}{5}, \frac{-1}{5}, \frac{7}{5}\right)$

$$\boxed{V_3 = \left(-\frac{2}{5}, \frac{14}{5}, \frac{14}{5}, \frac{2}{5}\right)}$$

*In Exercises **45–48**, we obtained the column vectors of Q by applying the Gram–Schmidt process to the column vectors of A. Find a QR-decomposition of the matrix A.*

45. $A = \begin{bmatrix} \underbrace{\boxed{\begin{matrix}1\\2\end{matrix}}}_{u_1} & \underbrace{\begin{matrix}-1\\3\end{matrix}}_{u_2} \end{bmatrix}, \quad Q = \begin{bmatrix} \underbrace{\begin{matrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{matrix}}_{q_1} & \underbrace{\begin{matrix}-\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{matrix}}_{q_2} \end{bmatrix}$

$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle \end{bmatrix}$

$\langle u_1, q_1 \rangle = (1, 2) \cdot (1/\sqrt{5}, 2/\sqrt{5}) = \sqrt{5}$

$\langle u_2, q_1 \rangle = (-1, 3) \cdot (1/\sqrt{5}, 2/\sqrt{5}) = \sqrt{5}$

$\langle u_2, q_2 \rangle = (-1, 3) \cdot (-2/\sqrt{5}, 1/\sqrt{5}) = \sqrt{5}$

$$\boxed{R = \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}}$$

47. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$

$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix}$

$\langle u_1, q_1 \rangle = (1, 0, 1) \cdot (1/\sqrt{2}, 0, 1/\sqrt{2}) = \sqrt{2}$

$\langle u_1, q_1 \rangle = (1, 0, 1) \cdot (1/\sqrt{2}, 0, 1/\sqrt{2}) = \sqrt{2}$

$\langle u_2, q_1 \rangle = (0, 1, 2) \cdot (1/\sqrt{2}, 0, 1/\sqrt{2}) = \sqrt{2}$

$\langle u_3, q_1 \rangle = (2, 1, 0) \cdot (1/\sqrt{2}, 0, 1/\sqrt{2}) = \sqrt{2}$

$\langle u_2, q_2 \rangle = (0, 1, 2) \cdot (-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = \sqrt{3}$

$\langle u_3, q_2 \rangle = (2, 1, 0) \cdot \quad '' \quad = -\sqrt{3}/3$

$\langle u_3, q_3 \rangle = (2, 1, 0) \cdot (1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}) = \frac{2\sqrt{6}}{3}$

$$\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\sqrt{3}/3 \\ 0 & 0 & 2\sqrt{6}/3 \end{bmatrix}$$ Equivalent to $$= \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{4}{\sqrt{6}} \end{bmatrix}.$$

49. Find a QR-decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

→ First apply Gram-Schmidt

→ Then, find orthonormal basis

→ Then, apply QR

Definition 1

A square matrix A is said to be ***orthogonal*** if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^TA = I \quad (1)$$

A matrix transformation $T_A: R^n \to R^n$ is said to be an ***orthogonal transformation*** or an ***orthogonal operator*** if A is an orthogonal matrix.

In each part of Exercises 1–4, determine whether the matrix is orthogonal, and if so find it inverse.

1. a. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So orthogonal

Fining inverse

$$-R_2 \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

Inverse.

$\boxed{\text{Can be done with calculator directly}}$

4. a. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -5/6 & 1/6 & 1/6 \\ 1/2 & 1/6 & 1/6 & -5/6 \\ 1/2 & 1/6 & -5/6 & 1/6 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -5/6 & 1/6 & 1/6 \\ 1/2 & 1/6 & 1/6 & -5/6 \\ 1/2 & 1/6 & -5/6 & 1/6 \end{bmatrix} \cdot$$

$A^T A = I_4$

$A A^T = I_4$

Finding Inverse

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -5/6 & 1/6 & 1/6 \\ 1/2 & 1/6 & 1/6 & -5/6 \\ 1/2 & 1/6 & -5/6 & 1/6 \end{bmatrix}$$

In Exercises 5–6, show that the matrix is orthogonal three ways: first by calculating $A^T A$, then by using part (b) of Theorem 7.1.1, and then by using part (c) of Theorem 7.1.1.

5. $A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$

1) $A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2) <u>By row vectors</u>

$u_1 = (4/5, 0, -3/5) \quad u_2 = (-9/25, 9/25, -12/25)$

$u_3 = (12/25, 3/25, 16/25)$

$\langle u_1, u_2 \rangle = 0$

$\langle u_1, u_3 \rangle = 0$

$\langle u_2, u_3 \rangle = 0$

3) <u>By colm vectors.</u>

$u_1 = (4/5, -9/25, 12/25), \quad u_2 = (0, 4/5, 3/5)$

$u_3 = (-3/5, -12/25, 16/25)$

$\langle u_1, u_2 \rangle = 0$

$\langle u_1, u_3 \rangle = 0$

$\langle u_2, u_3 \rangle = 0$

7.2

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In Exercises 1–6, find the characteristic equation of the given symmetric matrix, and then by inspection determine the dimensions of the eigenspaces.

1. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 4 \end{vmatrix} = 0$$

$(\lambda - 1)(\lambda - 4) - 4 = 0$

$\lambda^2 - 4\lambda - \lambda + \cancel{4} - \cancel{4} = 0$

$\lambda^2 - 5\lambda = 0$

$\lambda(\lambda - 5) = 0$

λ has a power of 2 so (2×2) matrix

For $\lambda = 0$, 1 dimension
For $\lambda = 5$, 1 dimension → So total 2 dimensions.

3. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{vmatrix} = 0$$

$(\lambda - 1)[(\lambda - 1)^2 - 1] + 1[-\lambda + 1 - 1] - 1[1 - (-\lambda + 1)] = 0$

$(\lambda - 1)(\lambda^2 - 2\lambda + \cancel{1} - \cancel{1}) - \lambda - [\cancel{1} + \lambda - \cancel{1}] = 0$

$\lambda^3 - 2\lambda^2 - \lambda^2 + \cancel{2\lambda} - \cancel{\lambda} - \cancel{\lambda} = 0$

$\lambda^3 - 3\lambda^2 = 0$

$\lambda^2(\lambda - 3) = 0$

$\boxed{\lambda = 0}$, $\boxed{\lambda = 0}$ $\boxed{\lambda = 3}$

$\lambda = 0$, 2 dimensions

$\lambda = 3$, 1 dimension

In Exercises 7–14, find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$.

7. $A = \begin{bmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{bmatrix}$

Orthogonally Diagonalizing an $n \times n$ Symmetric Matrix

Step 1. Find a basis for each eigenspace of A.

Step 2. Apply the Gram-Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.

Step 3. Form the matrix P whose columns are the vectors constructed in Step 2. This matrix will orthogonally diagonalize A, and the eigenvalues on the diagonal of $D = P^TAP$ will be in the same order as their corresponding eigenvectors in P.

$$\begin{vmatrix} \lambda - 6 & -2\sqrt{3} \\ -2\sqrt{3} & \lambda - 7 \end{vmatrix} = 0$$

$$(\lambda-6)(\lambda-7) - 12 = 0$$

$$\lambda^2 - 7\lambda - 6\lambda + 42 - 12 = 0$$

$$\lambda^2 - 13\lambda + 30 = 0$$

$\boxed{\lambda = 10}$ $\boxed{\lambda = 3}$

For $\lambda = 3$

$$-\tfrac{1}{3}R_1 \begin{bmatrix} -3 & -2\sqrt{3} \\ -2\sqrt{3} & -4 \end{bmatrix} = {}_{2\sqrt{3}R_1 + R_2}\begin{bmatrix} 1 & 2\sqrt{3}/3 \\ -2\sqrt{3} & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2\sqrt{3}/3 \\ 0 & 0 \end{bmatrix}$$

$x_2 = t$

$x_1 = -\dfrac{2\sqrt{3}}{3}t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -2\sqrt{3}/3 \\ 1 \end{bmatrix}$$

For $\lambda = 10$

$$\tfrac{1}{4}R_1 \begin{bmatrix} 4 & -2\sqrt{3} \\ -2\sqrt{3} & 3 \end{bmatrix} = {}_{2\sqrt{3}R_1 + R_2}\begin{bmatrix} 1 & -\sqrt{3}/2 \\ -2\sqrt{3} & 3 \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{3}/2 \\ 0 & 0 \end{bmatrix}$$

$x_2 = t$

$x_1 = \dfrac{\sqrt{3}}{2}t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} \sqrt{3}/2 \\ 1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} -2\sqrt{3}/3 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} \sqrt{3}/2 \\ 1 \end{bmatrix}$$

$v_1 = u_1$

$$\boxed{v_1 = \left(-2\sqrt{3}/3 \,,\; 1\right)} \longrightarrow \text{Normalize} \quad v_1 = \begin{bmatrix} \frac{-2\sqrt{7}}{7} \\ \frac{\sqrt{21}}{7} \end{bmatrix}$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{|v_1|^2} \cdot v_1$$

$$\langle u_2, v_1 \rangle = \left(\frac{\sqrt{3}}{2}, 1\right) \cdot \left(-\frac{2\sqrt{3}}{3}, 1\right) = -1 + 1 = 0$$

$$|v_1| = \sqrt{21}/3$$

$$v_2 = \left(\frac{\sqrt{3}}{2}, 1\right) - 0$$

$$\boxed{v_2 = \left(\frac{\sqrt{3}}{2}, 1\right)} \longrightarrow \quad v_2 = \begin{bmatrix} \sqrt{21}/7 \\ \frac{2\sqrt{7}}{7} \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} -2\sqrt{7}/7 & \sqrt{21}/7 \\ \sqrt{21}/7 & 2\sqrt{7}/7 \end{bmatrix}} \quad \text{Ans.}$$

solve further for $P^{-1}AP$ by calculator.

13. $A = \begin{bmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}$

$$\begin{vmatrix} \lambda+7 & -24 & 0 & 0 \\ -24 & \lambda-7 & 0 & 0 \\ 0 & 0 & \lambda+7 & -24 \\ 0 & 0 & -24 & \lambda-7 \end{vmatrix}$$

$$\lambda+7 \begin{bmatrix} \lambda-7 & 0 & 0 \\ 0 & \lambda+7 & -24 \\ 0 & -24 & \lambda-7 \end{bmatrix} - 24 \begin{bmatrix} -24 & 0 & 0 \\ 0 & \lambda+7 & -24 \\ 0 & -24 & \lambda-7 \end{bmatrix} + 0 - 0$$

$$(\lambda-7)\left[(\lambda+7)(\lambda-7) - 576\right] - 0 + 0$$

$$(\lambda-7)\left[\lambda^2 - \cancel{7\lambda} + \cancel{7\lambda} - 49 - 576\right]$$

$$(\lambda-7)(\lambda^2 - 625)$$

$(\lambda-7)(\lambda^2-625)$

$\boxed{\lambda^3-625\lambda-7\lambda^2+4375}$

$-24\left[(\lambda+7)(\lambda-7)-576\right]-0+0$

$-24\left[\lambda^2-625\right]$

$\boxed{-24\lambda^2+15000}$

$(\lambda+7)(\lambda^3-625\lambda-7\lambda^2+4375)-24(-24\lambda^2+15000)=0$

$\lambda^4-625\lambda^2-\cancel{7\lambda^3}+\cancel{4375}\lambda+\cancel{7\lambda^3}-4\cancel{375}\lambda-49\lambda^2+30625+$

$576\lambda^2-360000=0$

$\lambda^4-98\lambda^2-329375=0$

$\boxed{\lambda=25 \quad \lambda=25 \quad \lambda=-25 \quad \lambda=-25}$

For each λ, find basis then apply gram schmidt and normalize to find P.

Spectral Decomposition

If A is a symmetric matrix with real entries that is orthogonally diagonalized by

$$P=[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]$$

and if $\lambda_1,\lambda_2,\ldots,\lambda_n$ are the eigenvalues of A corresponding to the unit eigenvectors $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n$, then we know that $D=P^TAP$, where D is a diagonal matrix with the eigenvalues in the diagonal positions. It follows from this that the matrix A can be expressed as

$$A=PDP^T=[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}\begin{bmatrix}\mathbf{u}_1^T\\ \mathbf{u}_2^T\\ \vdots\\ \mathbf{u}_n^T\end{bmatrix}$$

$$=[\lambda_1\mathbf{u}_1 \quad \lambda_2\mathbf{u}_2 \quad \cdots \quad \lambda_n\mathbf{u}_n]\begin{bmatrix}\mathbf{u}_1^T\\ \mathbf{u}_2^T\\ \vdots\\ \mathbf{u}_n^T\end{bmatrix}$$

Multiplying out, we obtain the formula

$$\boxed{A=\lambda_1\mathbf{u}_1\mathbf{u}_1^T+\lambda_2\mathbf{u}_2\mathbf{u}_2^T+\cdots+\lambda_n\mathbf{u}_n\mathbf{u}_n^T}$$

which is called a ***spectral decomposition of A***.*

In Exercises 15–18, find the spectral decomposition of the matrix.

15. $\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$

$\det(\lambda I-A)=0$

| |

$\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = 0$$

$(\lambda - 3)^2 - 1 = 0$

$\lambda^2 - 6\lambda + 9 - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$\boxed{\lambda = 4}$ $\boxed{\lambda = 2}$

For $\lambda = 4$

$$R_1 + R_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$x_2 = t$
$x_1 = t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to u_2$$

For $\lambda = 2$

$$-R_1 \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = R_1 + R_2 \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_2 = t$
$x_1 = -t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to u_1$$

Gram Schmidt

$\boxed{v_1 = (-1, 1)}$

$\langle u_2, v_1 \rangle = (1, 1) \cdot (-1, 1) = -1 + 1 = 0$

$v_2 = u_2 - 0$

$\boxed{v_2 = (1, 1)}$

$$q_1 = \frac{v_1}{|v_1|} = \frac{(-1, 1)}{\sqrt{1^2 + 1^2}} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$q_2 = \frac{v_2}{|v_2|} = \frac{(1, 1)}{\sqrt{1^2 + 1^2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$$

$$A = 2\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + 4\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}}$$

17. $\begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

$$\begin{vmatrix} \lambda+3 & -1 & -2 \\ -1 & \lambda+3 & -2 \\ -2 & -2 & \lambda \end{vmatrix} = 0$$

$$(\lambda+3)\left[\lambda^2+3\lambda-4\right] + 1\left[-\lambda-4\right] - 2\left[2+2\lambda+6\right] = 0$$

$$\lambda^3+3\lambda^2-4\cancel{\lambda}+3\lambda^2+9\cancel{\lambda}-12-\cancel{\lambda}-4-16-4\cancel{\lambda}=0$$

$$\lambda^3+6\lambda^2-32=0$$

$$\boxed{\lambda=-4, \quad \lambda=-4, \quad \lambda=2}$$

For $\lambda=-4$

$$-R_1\begin{bmatrix} -1 & -1 & -2 \\ -1 & -1 & -2 \\ -2 & -2 & -4 \end{bmatrix} = \begin{array}{c} \\ R_1+R_2 \\ 2R_1+R_3 \end{array}\begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & -2 \\ -2 & -2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \quad x_3 = t \quad x_2 = s \qquad x_1+s+2t=0 \quad x_1=-s-2t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{u_1} + t\underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{u_2}$$

For $\lambda=2$

$$\downarrow u_1 \qquad \downarrow u_2$$

For $\lambda = 2$

$$\tfrac{1}{5}R_1 \begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix} \begin{matrix} = \\ R_1+R_2 \\ 2R_1+R_3 \end{matrix} \begin{bmatrix} 1 & -1/5 & -2/5 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$

$$\tfrac{5}{24}R_2 \begin{bmatrix} 1 & -1/5 & -2/5 \\ 0 & 24/5 & -12/5 \\ 0 & -12/5 & 6/5 \end{bmatrix} \begin{matrix} = \tfrac{1}{5}R_2+R_1 \\ \\ \tfrac{12}{5}R_2+R_3 \end{matrix} \begin{bmatrix} 1 & -1/5 & -2/5 \\ 0 & 1 & -1/2 \\ 0 & -12/5 & 6/5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = t \qquad x_2 = \frac{1}{2}t$

$x_1 - \frac{1}{2}t = 0$

$x_1 = \frac{1}{2}t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} \longrightarrow u_3$$

Applying Gram Schmidt

$u_1 = (-1, 1, 0) \qquad u_2 = (-2, 0, 1) \qquad u_3 = (1/2, 1/2, 1)$

$\boxed{v_1 = -1, 1, 0}$

$\langle u_2, v_1 \rangle = (-2, 0, 1) \cdot (-1, 1, 0) = 2$

$|v_1| = \sqrt{2}$

$v_2 = (-2, 0, 1) - \frac{2}{2} \cdot (-1, 1, 0)$

$v_2 = (-2, 0, 1) - (-1, 1, 0) = \boxed{(-1, -1, 1)}$

$\langle u_3, v_1 \rangle = (1/2, 1/2, 1) \cdot (-1, 1, 0) = 0$

$\langle u_3, v_2 \rangle = (1/2, 1/2, 1) \cdot (-1, -1, 1) = 0$

$$\langle u_3, v_2 \rangle = (1/2, 1/2, 1) \cdot (-1, -1, 1) = 0$$

$$v_3 = u_3 - 0 - 0$$

$$\boxed{v_3 = (1/2, 1/2, 1)}$$

Normalizing each

$$q_1 = \frac{v_1}{|v_1|} = \frac{(-1, 1, 0)}{\sqrt{2}} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \to u_1$$

$$q_2 = \frac{v_2}{|v_2|} = \frac{(-1, -1, 1)}{\sqrt{3}} = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \to u_2$$

$$q_3 = \frac{v_3}{|v_3|} = \frac{(1/2, 1/2, 1)}{\sqrt{6}/2} = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix} \to u_3$$

Can check $D = P^T A P$

$$D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A = \lambda_1 u_1 u_1^T + \dots \lambda_n u_n u_n^T$$

$$A = -4 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} - 4 \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$+ 2 \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} \sqrt{6}/6 & \sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix}$$

In Exercises 1–2, express the quadratic form in the matrix notation $\mathbf{x}^T A\mathbf{x}$*, where A is a symmetric matrix.*

Square term on main diagonal
Half of other terms on off-diagonal

1. a. $3x_1^2 + 7x_2^2$

$$Q = \underbrace{[x_1 \quad x_2]}_{x^T} \cdot \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

c. $9x_1^2 - x_2^2 + 4x_3^2 + \overset{3}{6x_1x_2} - \overset{-4}{8x_1x_3} + \overset{\frac{1}{2}}{x_2x_3}$

$$Q = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & 1/2 \\ -4 & 1/2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

c. $x_1^2 + x_2^2 - 3x_3^2 - 5x_1x_2 + 9x_1x_3$

$$Q = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & -5/2 & 9/2 \\ -5/2 & 1 & 0 \\ 9/2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In Exercises 3–4, find a formula for the quadratic form that does not use matrices.

3. $[x \quad y]\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Reverse process of finding Q.

→ Multiply off diagonal entries by 2.

$\boxed{2x^2 + 5y^2 - 6xy}$ Ans.

4. $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -2 & \frac{7}{2} & 1 \\ \frac{7}{2} & 0 & 6 \\ 1 & 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\boxed{-2x_1^2 + 3x_3^2 + 7x_1x_2 + 2x_1x_3 + 12x_2x_3} \quad \text{Ans}$$

*In Exercises **5–8**, find an orthogonal change of variables that eliminates the cross product terms in the quadratic form Q, and express Q in terms of the new variables.*

5. $Q = 2x_1^2 + 2x_2^2 - 2x_1x_2$

$$Q = x^T A x = y^T (P^T A P) y$$

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 2 \end{vmatrix} = 0$$

$$(\lambda - 2)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 4 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\boxed{\lambda = 3} \quad \boxed{\lambda = 1}$$

For $\lambda = 1$

$$-R_1 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = {}_{-R_1+R_2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_2 &= t \\ x_1 &= t \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 3$

$$-R_1 + R_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_2 &= t \\ x_1 &= -t \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Gram Schmidt

$v_1 = u_1$

$$\boxed{v_1 = (1, 1)}$$

$v_2 = u_2 - 0$

$$\boxed{v_2 = (-1, 1)}$$

$\langle u_2, v_1 \rangle = (-1, 1) \cdot (1, 1) = 0$

$$q_1 = \frac{(1,1)}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$q_2 = \frac{(-1,1)}{\sqrt{2}} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$Q = x^T A x = y^T (P^T A P) y$$

$$P^T A P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$Q = [y_1 \quad y_2] \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\boxed{Q = y_1^2 + 3y_2^2}$$

7. $Q = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

characteristic equation to find basis and then gram schmidt.

$$P = \begin{bmatrix} -2/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$[y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\boxed{y_1^2 + 4y_2^2 + 7y_3^2}$$ Ans.

Singular Value Decomposition (SVD)

→ Eigen Values Decomposition

→ $\det(\lambda I - A^T A) = 0$

→ $A = \underset{m \times m}{U} \; \underset{m \times n}{\Sigma} \; \underset{n \times n}{V^T}$

Formed by

u_1 $\qquad \sigma_1 = \sqrt{\lambda_1}$

u_2 $\qquad \sigma_2 = \sqrt{\lambda_2}$

u_3 $\qquad \sigma_3 = \sqrt{\lambda_3}$

$\vdots$ $\qquad \vdots$

u_n $\qquad \sigma_n = \sqrt{\lambda_n}$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & & 0 & 0 \\ 0 & 0 & 0 & & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_5 \end{bmatrix}$$

Q Solve the following (3×2) matrix by SVD.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

→ $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}$$

→ Now Apply Characteristic Equation

$\det(\lambda I - A^T A) = 0$

$$\det\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right) = 0$$

$$\begin{vmatrix}\lambda - 2 & -1\\ -1 & \lambda - 1\end{vmatrix} = 0$$

$\boxed{\lambda = 3}$, $\boxed{\lambda = 1}$

$\rightarrow$ Now, $\sigma_1 = \sqrt{\lambda_1}$ $\cdots$ $\sigma_n = \sqrt{\lambda_n}$

$\boxed{\sigma_1 = \sqrt{3}}$ $\quad \sigma_1 = \sqrt{1}$

$\boxed{\sigma_1 = 1}$

$\rightarrow$ For $\lambda = 3$

$$\begin{bmatrix}3-2 & -1\\ -1 & 3-2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$R_1 + R_2 \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$x_1 - x_2 = 0$ $\quad \boxed{x_2 = t}$

$x_1 - t = 0$

$\boxed{x_1 = t}$

$V = \frac{v_1}{|v_1|}$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = t\begin{bmatrix}1\\ 1\end{bmatrix} \rightarrow \text{Normalize} \quad \boxed{v_1 = \begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}} \quad \star$$

$\rightarrow$ For $\lambda = 1$

$$\begin{bmatrix}1-2 & -1\\ -1 & 1-2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$-R_1 \begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix} = R_1 + R_2 \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$ $\qquad$ $\boxed{x_2 = s}$

$x_1 + s = 0$

$\boxed{x_1 = -s}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \boxed{v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}} \quad \star$$

$$\rightarrow \quad u_i = \frac{1}{\sigma_i} \cdot A \cdot v_i$$

$$u_1 = \frac{1}{\sigma_1} \cdot A \cdot v_1 = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\boxed{u_1 = \begin{bmatrix} \sqrt{6}/3 \\ \sqrt{6}/6 \\ \sqrt{6}/6 \end{bmatrix}}$$

$$u_2 = \frac{1}{\sigma_2} \cdot A \cdot v_2$$

$$u_2 = \frac{1}{1} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\boxed{u_2 = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}}$$

$$u_1 = \begin{bmatrix} \sqrt{6}/3 \\ \sqrt{6}/6 \\ \sqrt{6}/6 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$\sqrt{6}\, u_1 = \sqrt{6} \begin{bmatrix} \sqrt{6}/3 \\ \sqrt{6}/6 \\ \sqrt{6}/6 \end{bmatrix} \qquad \sqrt{2} \cdot u_2 = \sqrt{2} \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$\begin{bmatrix} \\ \sqrt{6}/6 \\ \sqrt{6}/6 \end{bmatrix} \qquad \begin{bmatrix} \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$\boxed{\sqrt{6}\ u_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} \qquad \boxed{\sqrt{2}\cdot u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$$

Now Transpose

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{1}{2}R_1 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = -\frac{1}{2}R_2 + R_1 \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = 0$, $x_2 - x_3 = 0$, $\boxed{x_3 = t}$

$x_1 + t = 0$ $\qquad$ $x_2 - t = 0$

$\boxed{x_1 = -t}$ $\qquad$ $\boxed{x_2 = t}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{\text{Normalize}} u_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\boxed{U = \begin{bmatrix} \sqrt{6}/3 & 0 & -1/\sqrt{3} \\ \sqrt{6}/6 & \sqrt{2}/2 & 1/\sqrt{3} \\ \sqrt{6}/6 & -\sqrt{2}/2 & 1/\sqrt{3} \end{bmatrix}} \rightarrow 3 \times 3$$

$m \times m$

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

→ Now, V^T

$$\boxed{V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}} \longrightarrow \begin{matrix} 2 \times 2 \\ n \times n \end{matrix}$$

→ Now Σ for $m \times n \rightarrow 3 \times 2$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

→ Now $A = U \Sigma V^T$

$$\boxed{A = \begin{bmatrix} \sqrt{6}/3 & 0 & -1/\sqrt{3} \\ \sqrt{6}/6 & \sqrt{2}/2 & 1/\sqrt{3} \\ \sqrt{6}/6 & -\sqrt{2}/2 & 1/\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

Ans

In Exercises **1–4**, *find the distinct singular values of* A.

1. $A = [1 \quad 2 \quad 0]$

$$A^T = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\det(\lambda I - A^TA) = 0$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda - 1 & -2 & 0 \\ -2 & \lambda - 4 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = 0$$

$(\lambda - 1)[\lambda^2 - 4\lambda - 0] + 2[-2\lambda - 0] + 0 = 0$

$(\lambda - 1)(\lambda^2 - 4\lambda) - 4\lambda = 0$

$\lambda^3 - 4\lambda^2 - \lambda^2 + \cancel{4\lambda} - \cancel{4\lambda} = 0$

$\lambda^3 - 5\lambda^2 = 0$

$\lambda^2(\lambda - 5) = 0$

$\boxed{\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 5}$

$\sigma_i = \sqrt{\lambda_i}$

$\boxed{\sigma_1 = 0, \quad \sigma_2 = 0, \quad \sigma_3 = \sqrt{5}}$ Ans.

3. $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda - 5 & 0 \\ 0 & \lambda - 5 \end{vmatrix} = 0$$

$(\lambda - 5)^2 = 0$

$$(\lambda - 5)^2 = 0$$

$$\boxed{\lambda_1 = 5 \quad , \quad \lambda_2 = 5}$$

$$\boxed{\sigma_1 = \sqrt{5} \quad , \quad \sigma_2 = \sqrt{5}}$$ Ans.

In Exercises 5–12, find a singular value decomposition of A.

5. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 2 \end{vmatrix} = 0$$

$\boxed{\lambda_1 = 2}$ $\boxed{\lambda_2 = 2}$

$$\boxed{\sigma_1 = \sqrt{2} \quad , \quad \sigma_2 = \sqrt{2}}$$

For $\lambda = 2$

$$\begin{bmatrix} 2-2 & 0 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\boxed{x_2 = t}$ $\boxed{x_1 = s}$ $\quad s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Normalize

$$v_i = \frac{v}{|v|}$$

$$v_1 = \frac{(1, 0)}{\sqrt{1}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad v_2 = \frac{(0, 1)}{\sqrt{1}}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u_i = \frac{1}{\sigma_i} \cdot A \cdot v_i$$

$$u_1 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}$$

$$u_2 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}$$

$$\underline{u}$$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad m \times m$$

$$A = u \; \Sigma \; v^T$$

$$\boxed{A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \quad \text{Ans.}$$

7. $A = \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 4 & 0 \\ 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 24 \\ 24 & 52 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 16 & -24 \\ -24 & \lambda - 52 \end{vmatrix} = 0$$

$$(\lambda - 16)(\lambda - 52) - 576 = 0$$

$$\lambda^2 - 52\lambda - 16\lambda + 832 - 576 = 0$$

$$\lambda^2 - 68\lambda + 256 = 0$$

$\boxed{\lambda_1 = 64}$ $\boxed{\lambda_2 = 4}$

$\sigma_1 = \sqrt{64}$ $\sigma_2 = \sqrt{4}$

$\boxed{\sigma_1 = 8}$ $\boxed{\sigma_2 = 2}$

For $\lambda = 64$

$$\begin{bmatrix} 64-16 & -24 \\ -24 & 64-52 \end{bmatrix}$$

$$\frac{1}{48}R_1 \begin{bmatrix} 48 & -24 \\ -24 & 12 \end{bmatrix} = {}_{24R_1+R_2}\begin{bmatrix} 1 & -1/2 \\ -24 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \quad \boxed{x_2 = t} \quad x_1 - \frac{1}{2}t = 0 \quad \boxed{x_1 = \frac{1}{2}t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \rightarrow v_1 = \frac{(1/2,\ 1)}{\sqrt{(1/2)^2 + 1^2}} = \begin{bmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix}$$

For $\lambda = 4$

$$\begin{bmatrix} 4-16 & -24 \\ -24 & 4-52 \end{bmatrix} = \frac{-1}{12}R_1 \begin{bmatrix} -12 & -24 \\ -24 & -48 \end{bmatrix}$$

$$24R_1+R_2 \begin{bmatrix} 1 & 2 \\ -24 & -48 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \boxed{x_2 = t} \quad \boxed{x_1 = -2t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix} \rightarrow v_2 = \frac{(-2,\ 1)}{\sqrt{5}} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$u_i = \frac{1}{\sigma_i} \cdot A \cdot v_i$$

$$u_1 = \frac{1}{8} \cdot \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1/2 & 3/4 \\ 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix} = \begin{bmatrix} 2\sqrt{5}/5 \\ \sqrt{5}/5 \end{bmatrix}$$

$$u_2 = \frac{1}{2} \cdot \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -\sqrt{5}/5 \\ 2/\sqrt{5} \end{bmatrix}$$

$$A = U \quad \Sigma \quad V^T$$

$$A = \begin{bmatrix} -2/\sqrt{5} & -\sqrt{5}/5 \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5}/5 & 2\sqrt{5}/5 \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \quad \text{Ans}$$

9. $A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$

$$A^T A = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 9 & 9 \\ 9 & \lambda - 9 \end{vmatrix} = 0$$

$$(\lambda - 9)(\lambda - 9) - 81 = 0$$

$$\lambda^2 - 9\lambda - 9\lambda + \cancel{81} - \cancel{81} = 0$$

$$\lambda^2 - 18\lambda = 0$$

$$\lambda(\lambda - 18) = 0$$

$\boxed{\lambda_1 = 0}$ $\boxed{\lambda_2 = 18}$

$\boxed{\sigma_1 = 0}$ $\boxed{\sigma_2 = \sqrt{18}}$

For $\lambda = 0$

$$\begin{bmatrix} 0-9 & 9 \\ 9 & 0-9 \end{bmatrix} = \frac{-1}{9}R_1 \begin{bmatrix} -9 & 9 \\ 9 & -9 \end{bmatrix}$$

$$-9R_1+R_2 \begin{bmatrix} 1 & -1 \\ 9 & -9 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$\boxed{x_2 = t}$

$x_1 - t = 0$

$\boxed{x_1 = t}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow v_1 = \frac{(1,1)}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

For $\lambda = 18$

$$\frac{1}{9}R_1 \begin{bmatrix} 9 & 9 \\ 9 & 9 \end{bmatrix} = -9R_1+R_2 \begin{bmatrix} 1 & 1 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_2 = t \quad x_1 = -t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow v_2 = \frac{(-1,1)}{\sqrt{2}}$$

$$v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\boxed{V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

$$u_2 = \frac{1}{\sqrt{18}} \cdot \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$u_1 = \frac{1}{0} \cdot \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

cannot be computed so now take AA^T

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & -8 \\ 4 & 2 & -4 \\ -8 & -4 & 8 \end{bmatrix}$$

$$\begin{vmatrix} \lambda-8 & -4 & 8 \\ -4 & \lambda-2 & 4 \\ 8 & 4 & \lambda-8 \end{vmatrix} = 0$$

$$(\lambda-8)\left[(\lambda-2)(\lambda-8)-16\right] + 4\left[-4\lambda+\cancel{32}-\cancel{32}\right] - 8\left[-\cancel{16}-8\lambda+\cancel{16}\right] = 0$$

$$(\lambda-8)\left[\lambda^2-8\lambda-2\lambda+\cancel{16}-\cancel{16}\right] - 16\lambda + 64\lambda = 0$$

$$\lambda^3-8\lambda^2-2\lambda^2-8\lambda^2-\cancel{64}\lambda+\cancel{16\lambda}-\cancel{16\lambda}+\cancel{64}\lambda=0$$

$$\lambda^3 - 20\lambda^2 = 0$$

$$\lambda^2(\lambda-20) = 0$$

$$\boxed{\lambda=0, \quad \lambda=0, \quad \lambda=20}$$

For $\lambda=0$

$$-\frac{1}{8}R_1 \begin{bmatrix} -8 & -4 & 8 \\ -4 & -2 & 4 \\ 8 & 4 & -8 \end{bmatrix} \underset{-8R_1+R_3}{\overset{4R_1+R_2}{=}} \begin{bmatrix} 1 & 1/2 & -1 \\ -4 & -2 & 4 \\ 8 & 4 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad x_3 = t \qquad x_2 = s \qquad x_1 + \frac{1}{2}s - t = 0$$

$$\boxed{x_1 = -\frac{1}{2}s + t}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Normalize

$$u_1 = \begin{bmatrix} -\sqrt{5}/5 \\ 2\sqrt{5}/5 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

For $\lambda = 20$

$$\frac{1}{12}R_1 \begin{bmatrix} 12 & -4 & 8 \\ -4 & 18 & 4 \\ 8 & 4 & 12 \end{bmatrix} \underset{-8R_1+R_3}{\overset{4R_1+R_2}{=}} \begin{bmatrix} 1 & -1/3 & 2/3 \\ -4 & 18 & 4 \\ 8 & 4 & 12 \end{bmatrix}$$

$$\frac{3}{50}R_2 \begin{bmatrix} 1 & -3 & 2/3 \\ 0 & 50/3 & 20/3 \\ 0 & 20/3 & 20/3 \end{bmatrix} \underset{-\frac{20}{3}R_2+R_3}{\overset{3R_2+R_1}{=}} \begin{bmatrix} 1 & -3 & 2/3 \\ 0 & 1 & 2/5 \\ 0 & 20/3 & 20/3 \end{bmatrix} \underset{\frac{1}{4}R_3}{=} \begin{bmatrix} 1 & 0 & 28/15 \\ 0 & 1 & 2/5 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\begin{matrix} -\frac{28}{15}R_3+R_1 \\ -\frac{2}{5}R_3+R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 28/15 \\ 0 & 1 & 2/5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\sqrt{5}/5 & 1/\sqrt{2} \\ 0 & 2\sqrt{5}/5 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

11. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$ 3×2

$$A^T A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} \cdot & \cdot & -1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cdot & \cdot \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 3 & 0 \\ 0 & \lambda - 2 \end{vmatrix} = 0$$

$$(\lambda - 3)(\lambda - 2) = 0$$

$$\boxed{\lambda = 3} \quad \boxed{\lambda = 2}$$

$$\sigma_1 = \sqrt{3} \quad , \quad \sigma_2 = \sqrt{2}$$

For $\lambda = 3$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \boxed{x_2 = 0} \quad \boxed{x_1 = t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow v_1 = \frac{(1, 0)}{\sqrt{1}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda = 2$

$$-R_1 \quad \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_2 = t \\ x_1 = 0 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow v_2 = \frac{(0, 1)}{\sqrt{1}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u_1 = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\underline{u^T}$

$$\sqrt{3} R_1 \quad \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \end{bmatrix} \begin{bmatrix} 0 \\ \cdots \end{bmatrix}$$

$$\sqrt{3}R_1 \quad \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sqrt{2}R_2 \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = {-R_2 + R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$\boxed{x_3 = t}$ $\quad x_2 + x_3 = 0$ $\quad x_1 - 2x_3 = 0$

$\boxed{x_2 = -t}$ $\quad \boxed{x_1 = 2t}$

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$u_3 = \frac{(2, -1, 1)}{\sqrt{6}} = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$A = U \; \Sigma \; V^T$$

$$A = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Ans.}$$

$3 \times 3 \qquad 3 \times 2 \qquad 2 \times 2$