# Dimension

## Number of Vectors in a Basis

**Theorem 4.6.1**

All bases for a finite-dimensional vector space have the same number of vectors.

**Theorem 4.6.2**

Let $V$ be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for $V$.

(a) If a set in $V$ has more than $n$ vectors, then it is linearly dependent.

(b) If a set in $V$ has fewer than $n$ vectors, then it does not span $V$.

**Definition 1**

The ***dimension*** of a finite-dimensional vector space $V$ is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for $V$. In addition, the zero vector space is defined to have dimension zero.

**EXAMPLE 1 | Dimensions of Some Familiar Vector Spaces**

$\dim(R^n) = n$ **[The standard basis has $n$ vectors.]**

$\dim(P_n) = n + 1$ **[The standard basis has $n + 1$ vectors.]**

$\dim(M_{mn}) = mn$ **[The standard basis has $mn$ vectors.]**

Engineers often use the term ***degrees of freedom*** as a synonym for dimension.

## EXAMPLE 2 | Dimension of Span($S$)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ then every vector in span($S$) is expressible as a linear combination of the vectors in $S$. Thus, if the vectors in $S$ are *linearly independent*, they automatically form a basis for span($S$), from which we can conclude that

$$\dim\left[\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}\right] = r$$

In words, the dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

## EXAMPLE 3 | Dimension of a Solution Space

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\
5x_3 + 10x_4 \qquad\quad + 15x_6 &= 0 \\
2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

**Solution** In Example 6 of Section 1.2 we found the solution of this system to be

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

which can be written in vector form as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3r - 4s - 2t, r, -2s, s, t, 0)$$

or, alternatively, as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

This shows that the vectors

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

span the solution space. We leave it for you to check that these vectors are linearly independent by showing that none of them is a linear combination of the other two (but see the remark that follows). Thus, the solution space has dimension 3.

## Theorem 4.6.3

### Plus/Minus Theorem

Let $S$ be a nonempty set of vectors in a vector space $V$.

(*a*) If $S$ is a linearly independent set, and if $\mathbf{v}$ is a vector in $V$ that is outside of span($S$), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into $S$ is still linearly independent.

(*b*) If $\mathbf{v}$ is a vector in $S$ that is expressible as a linear combination of other vectors in $S$, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from $S$, then $S$ and $S - \{\mathbf{v}\}$ span the same space; that is,

$$\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$$

## EXAMPLE 5 | Bases by Inspection

(*a*) Explain why the vectors $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for $R^2$.

(*b*) Explain why the vectors $\mathbf{v}_1 = (2, 0, -1)$, $\mathbf{v}_2 = (4, 0, 7)$, and $\mathbf{v}_3 = (-1, 1, 4)$ form a basis for $R^3$.

**Solution (*a*)** Since neither vector is a scalar multiple of the other, the two vectors form a linearly independent set in the two-dimensional space $R^2$, and hence they form a basis by Theorem 4.6.4.

**Solution (*b*)** The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a linearly independent set in the *xz*-plane (why?). The vector $\mathbf{v}_3$ is outside of the *xz*-plane, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent. Since $R^3$ is three-dimensional, Theorem 4.6.4 implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for the vector space $R^3$.

## Theorem 4.6.5

Let $S$ be a finite set of vectors in a finite-dimensional vector space $V$.

(*a*) If $S$ spans $V$ but is not a basis for $V$, then $S$ can be reduced to a basis for $V$ by removing appropriate vectors from $S$.

(*b*) If $S$ is a linearly independent set that is not already a basis for $V$, then $S$ can be enlarged to a basis for $V$ by inserting appropriate vectors into $S$.

**Theorem 4.6.6**

If $W$ is a subspace of a finite-dimensional vector space $V$, then:

(*a*) $W$ is finite-dimensional.

(*b*) $\dim(W) \le \dim(V)$.

(*c*) $W = V$ if and only if $\dim(W) = \dim(V)$.

## Question:

*In Exercises 1–6, find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.*

**2.** $$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

## Solution:

The augmented matrix of the linear system $\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$. The general solution is $x_1 = -\frac{1}{4}s$, $x_2 = -\frac{1}{4}s - t$, $x_3 = s$, $x_4 = t$. In vector form

$$(x_1, x_2, x_3, x_4) = \left(-\tfrac{1}{4}s, -\tfrac{1}{4}s - t, s, t\right) = s\left(-\tfrac{1}{4}, -\tfrac{1}{4}, 1, 0\right) + t(0, -1, 0, 1)$$

therefore the solution space is spanned by vectors $\mathrm{v}_1 = \left(-\frac{1}{4}, -\frac{1}{4}, 1, 0\right)$ and $\mathrm{v}_2 = (0, -1, 0, 1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.4.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is $2$.

## Question:

In each part, find a basis for the given subspace of $R^3$, and state its dimension.

**a.** The plane $3x - 2y + 5z = 0$.

**b.** The plane $x - y = 0$.

**c.** The line $x = 2t, y = -t, z = 4t$.

**d.** All vectors of the form $(a, b, c)$, where $b = a + c$.

## Solution:

**(a)** If we let $y = s$ and $z = t$ be arbitrary values, we can solve the plane equation for $x$ : $x = \frac{2}{3}s - \frac{5}{3}t$. Expressing the solution in vector form $(x,y,z) = \left(\frac{2}{3}s - \frac{5}{3}t, s, t\right) = s\left(\frac{2}{3},1,0\right) + t\left(-\frac{5}{3},0,1\right)$. By Theorem 4.4.2(c), $\left\{\left(\frac{2}{3},1,0\right),\left(-\frac{5}{3},0,1\right)\right\}$ is linearly independent since neither vector in the set is a scalar multiple of the other. A basis for the subspace is $\left\{\left(\frac{2}{3},1,0\right),\left(-\frac{5}{3},0,1\right)\right\}$. The dimension of the subspace is $2$.

**(b)** If we let $y = s$ and $z = t$ be arbitrary values, we can solve the plane equation for $x$ : $x = s$. Expressing the solution in vector form $(x,y,z) = (s,s,t) = s(1,1,0) + t(0,0,1)$. By Theorem 4.4.2(c), $\{(1,1,0),(0,0,1)\}$ is linearly independent since neither vector in the set is a scalar multiple of the other. A basis for the subspace is $\{(1,1,0),(0,0,1)\}$. The dimension of the subspace is $2$.

**(c)** In vector form, $(x,y,z) = (2t,-t,4t) = t(2,-1,4)$. By Theorem 4.4.2(b), the vector $(2,-1,4)$ forms a linearly independent set since it is not the zero vector. A basis for the subspace is $\{(2,-1,4)\}$. The dimension of the subspace is $1$.

**(d)** The subspace contains all vectors $(a,a+c,c) = a(1,1,0) + c(0,1,1)$ thus we can express it as as $\operatorname{span}(S)$ where $S = \{(1,1,0),(0,1,1)\}$. By Theorem 4.4.2(c), $S$ is linearly independent since neither vector in the set is a scalar multiple of the other. Consequently, $S$ forms a basis for the given subspace. The dimension of the subspace is $2$.