Dimension

Question:

12. Find a standard basis vector for R^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^3.

a. $\mathbf{v}_1 = (-1, 2, 3)$, $\mathbf{v}_2 = (1, -2, -2)$

b. $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (3, 1, -2)$

Solution:

(a) Either $(1,0,0)$ or $(0,1,0)$ can be used since neither is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ (e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix} -1 & 1 & 1 \\ 2 & -2 & 0 \\ 3 & -2 & 0 \end{vmatrix} = 2 \neq 0$ then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.6.4)

(b) Any of the three standard basis vector for R^3 can be used since none of them is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ (e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix} 1 & 3 & 1 \\ -1 & 1 & 0 \\ 0 & -2 & 0 \end{vmatrix} = 2 \neq 0$ then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.6.4)

Question:

13. Find standard basis vectors for R^4 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^4.

$$\mathbf{v}_1 = (1, -4, 2, -3), \quad \mathbf{v}_2 = (-3, 8, -4, 6)$$

Solution:

The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 + k_6\mathbf{e}_4 = \mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcrcrcl} k_1 & - & 3k_2 & + & k_3 & & & & & & & = & 0 \\ -4k_1 & + & 8k_2 & & & + & k_4 & & & & & = & 0 \\ 2k_1 & - & 4k_2 & & & & & + & k_5 & & & = & 0 \\ -3k_1 & + & 6k_2 & & & & & & & + & k_6 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -2 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 \end{bmatrix}$.

Based on the leading entries in the first, second, fourth, and fifth columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are linearly independent. Since $\dim(R^4) = 4$, it follows by Theorem 4.6.4 that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form a basis for R^4. (The answer is not unique.)

Question:

17. Find a basis for the subspace of R^3 that is spanned by the vectors

$$\mathbf{v}_1 = (1,0,0),\ \ \mathbf{v}_2 = (1,0,1),\ \ \mathbf{v}_3 = (2,0,1),\ \ \mathbf{v}_4 = (0,0,-1)$$

Solution:

The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{ccccccccc} 1k_1 & + & 1k_2 & + & 2k_3 & + & 0k_4 & = & 0 \\ 0k_1 & + & 0k_2 & + & 0k_3 & + & 0k_4 & = & 0 \\ 0k_1 & + & 1k_2 & + & 1k_3 & - & 1k_4 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

For arbitrary values of s and t, we have $k_1 = -s - t$, $k_2 = -s + t$, $x_3 = s$, $k_4 = t$.

Letting $s = 1$ and $t = 0$ allows us to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

Letting $s = 0$ and $t = 1$ allows us to express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_4 = \mathbf{v}_1 - \mathbf{v}_2$.

By part (b) of Theorem 4.6.3, $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Based on the leading entries in the first two columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. (The answer is not unique.)