4.2 Subspaces

It is often the case that some vector space of interest is contained within a larger vector space whose properties are known. In this section we will show how to recognize when this is the case, we will explain how the properties of the larger vector space can be used to obtain properties of the smaller vector space, and we will give a variety of important examples.

We begin with some terminology.

Definition 1

A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

In general, to show that a nonempty set W with two operations is a vector space one must verify the ten vector space axioms. However, if W is a subspace of a known vector space V, then certain axioms need not be verified because they are "inherited" from V. For example, it is *not* necessary to verify that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ holds in W because it holds for all vectors in V including those in W. On the other hand, it *is* necessary to verify that W is closed under addition and scalar multiplication since it is possible that adding two vectors in W or multiplying a vector in W by a scalar produces a vector in V that is outside of W (**Figure 4.2.1**). Those axioms that are *not* inherited by W are

Axiom 1—Closure of W under addition

Axiom 4—Existence of a zero vector in W

Axiom 5—Existence of a negative in W for every vector in W

Axiom 6—Closure of W under scalar multiplication

so these must be verified to prove that it is a subspace of V. However, the next theorem shows that if Axiom 1 and Axiom 6 hold in W, then Axioms 4 and 5 hold in W as a consequence and hence need not be verified.

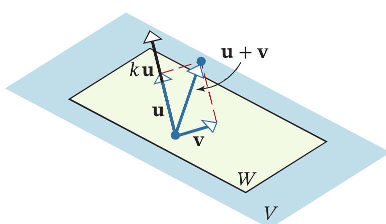


FIGURE 4.2.1 The vectors $\mathbf{u}$ and $\mathbf{v}$ are in W, but the vectors $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ are not.

Theorem 4.2.1

Subspace Test

If W is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.

(*a*) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.

(*b*) If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.

The Subspace Test states that W is a subspace of V if and only if it is closed under addition and scalar multiplication.

Proof If W is a subspace of V, then all the vector space axioms hold in W, including Axioms 1 and 6, which are precisely conditions (a) and (b).

Conversely, assume that conditions (a) and (b) hold. Since these are Axioms 1 and 6, and since Axioms 2, 3, 7, 8, 9, and 10 are inherited from V, we only need to show that Axioms 4 and 5 hold in W. For this purpose, let $\mathbf{u}$ be any vector in W. It follows from condition (b) that the product $k\mathbf{u}$ is also a vector in W for every scalar k. In particular, $0\mathbf{u} = \mathbf{0}$ and $(-1)\mathbf{u} = -\mathbf{u}$ are in W, which shows that Axioms 4 and 5 hold in W. ■

It is important to note that the first step in applying the Subspace Test to a set W is to confirm that the set is nonempty. This should be clear for all of the examples in this section, so we will omit its explicit verification.

EXAMPLE 1 | The Zero Subspace

If V is any vector space, and if $W = \{\mathbf{0}\}$ is the subset of V that consists of the zero vector only, then W is closed under addition and scalar multiplication since

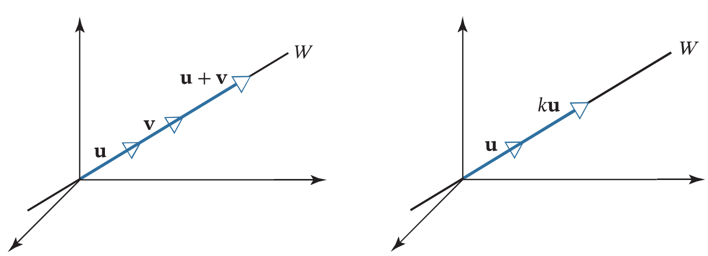
$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0}$$

for any scalar k. We call W the ***zero subspace*** of V.

Note that every vector space has at least two subspaces, itself and its zero subspace.

EXAMPLE 2 | Lines Through the Origin Are Subspaces of R^2 and of R^3

If W is a line through the origin of either R^2 or R^3, then adding two vectors on the line or multiplying a vector on the line by a scalar produces another vector on the line, so W is closed under addition and scalar multiplication (see **Figure 4.2.2** for an illustration in R^3).



(*a*) W is closed under addition. **(*b*)** W is closed under scalar multiplication.

FIGURE 4.2.2

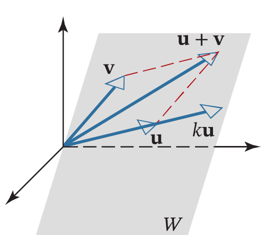


FIGURE 4.2.3 The vectors $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ both lie in the same plane as $\mathbf{u}$ and $\mathbf{v}$.

EXAMPLE 3 | Planes Through the Origin Are Subspaces of R^3

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a plane W through the origin of R^3, then it is evident geometrically that $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ also lie in the same plane W for any scalar k (**Figure 4.2.3**). Thus W is closed under addition and scalar multiplication.

Table 1 gives a list of subspaces of R^2 and of R^3 that we have encountered thus far. We will see later that these are the *only* subspaces of R^2 and of R^3.

TABLE 1

Subspaces of R^2	Subspaces of R^3
• $\{\mathbf{0}\}$ • Lines through the origin • R^2	• $\{\mathbf{0}\}$ • Lines through the origin • Planes through the origin • R^3

EXAMPLE 4 | A Subset of R^2 That Is Not a Subspace

Let W be the set of all points (x, y) in R^2 for which $x \geq 0$ and $y \geq 0$ (the shaded region in Figure 4.2.4). This set is not a subspace of R^2 because it is not closed under scalar multiplication. For example, $\mathbf{v} = (1, 1)$ is a vector in W, but $(-1)\mathbf{v} = (-1, -1)$ is not.

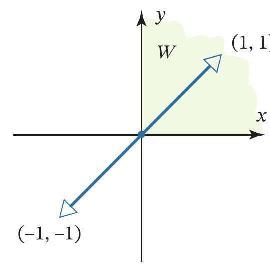


FIGURE 4.2.4 W is not closed under scalar multiplication.

EXAMPLE 5 | Subspaces of M_{nn}

We know from Theorem 1.7.2 that the sum of two symmetric $n \times n$ matrices is symmetric and that a scalar multiple of a symmetric $n \times n$ matrix is symmetric. Thus, the set of symmetric $n \times n$ matrices is closed under addition and scalar multiplication and hence is a subspace of M_{nn}. Similarly, the sets of upper triangular matrices, lower triangular matrices, and diagonal matrices are subspaces of M_{nn}.

EXAMPLE 6 | A Subset of M_{nn} That Is Not a Subspace

The set W of invertible $n \times n$ matrices is not a subspace of M_{nn}, failing on two counts—it is not closed under addition and not closed under scalar multiplication. We will illustrate this with an example in M_{22} that you can readily adapt to M_{nn}. Consider the matrices

$$U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$

The matrix $0U$ is the 2×2 zero matrix and hence is not invertible, and the matrix $U + V$ has a column of zeros so it also is not invertible.

EXAMPLE 7 | The Subspace $C(-\infty, \infty)$

CALCULUS REQUIRED

There is a theorem in calculus which states that a sum of continuous functions is continuous and that a constant times a continuous function is continuous. Rephrased in vector language, the set of continuous functions on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$. We will denote this subspace by $C(-\infty, \infty)$.

CALCULUS REQUIRED

EXAMPLE 8 | Functions with Continuous Derivatives

A function with a continuous derivative is said to be *continuously differentiable*. There is a theorem in calculus which states that the sum of two continuously differentiable functions is continuously differentiable and that a constant times a continuously differentiable function is continuously differentiable. Thus, the functions that are continuously differentiable on $(-\infty, \infty)$ form a subspace of $F(-\infty, \infty)$. We will denote this subspace by $C^1(-\infty, \infty)$, where the superscript emphasizes that the *first* derivatives are continuous. To take this a step further, the set of functions with m continuous derivatives on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$ as is the set of functions with derivatives of all orders on $(-\infty, \infty)$. We will denote these subspaces by $C^m(-\infty, \infty)$ and $C^\infty(-\infty, \infty)$, respectively.

EXAMPLE 9 | The Subspace of All Polynomials

Recall that a ***polynomial*** is a function that can be expressed in the form

$$p(x) = a_0 + a_1x + \cdots + a_nx^n \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are constants. It is evident that the sum of two polynomials is a polynomial and that a constant times a polynomial is a polynomial. Thus, the set W of all polynomials is closed under addition and scalar multiplication and hence is a subspace of $F(-\infty, \infty)$. We will denote this space by P_∞.

EXAMPLE 10 | The Subspace of Polynomials of Degree $\leq n$

Recall that the ***degree*** of a polynomial is the highest power of the variable that occurs with a nonzero coefficient. Thus, for example, if $a_n \neq 0$ in Formula (1), then that polynomial has degree n. It is *not* true that the set W of polynomials with positive degree n is a subspace of $F(-\infty, \infty)$ because that set is not closed under addition. For example, the polynomials

$$1 + 2x + 3x^2 \quad \text{and} \quad 5 + 7x - 3x^2$$

both have degree 2, but their sum has degree 1. What *is* true, however, is that for each nonnegative integer n the polynomials of degree *n or less* form a subspace of $F(-\infty, \infty)$. We will denote this space by P_n.

In this text we regard all constants to be polynomials of degree zero. Be aware, however, that some authors do not assign a degree to the constant 0.

The Hierarchy of Function Spaces

It is proved in calculus that polynomials are continuous functions and have continuous derivatives of all orders on $(-\infty, \infty)$. Thus, it follows that P_∞ is not only a subspace of $F(-\infty, \infty)$, as previously observed, but is also a subspace of $C^\infty(-\infty, \infty)$. We leave it for you to convince yourself that the vector spaces discussed in Examples 7 to 10 are "nested" one inside the other as illustrated in **Figure 4.2.5**.

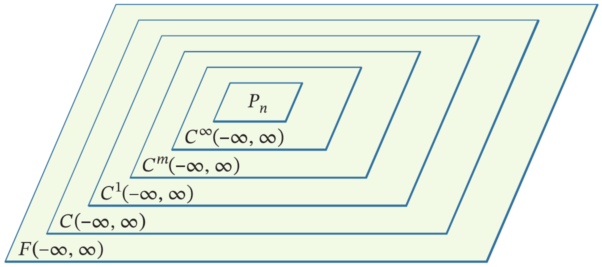


FIGURE 4.2.5

Remark In our previous examples we considered functions that were defined at all points of the interval $(-\infty, \infty)$. Sometimes we will want to consider functions that are only defined on some subinterval of $(-\infty, \infty)$, say the closed interval $[a, b]$ or the open interval (a, b). In such cases we will make an appropriate notation change. For example, $C[a, b]$ is the space of continuous functions on $[a, b]$ and $C(a, b)$ is the space of continuous functions on (a, b).

In the following examples we will illustrate how the Subspace Test can be applied to various nonempty subsets of R^n, M_{mn}, P_n, and $F(-\infty, \infty)$.

EXAMPLE 11 | Applying the Subspace Test in M_{22}

Determine whether the indicated set of matrices is a subspace of M_{22}.

(*a*) The set U consisting of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 2x & y \end{bmatrix} \tag{2}$$

(*b*) The set W consisting of all 2×2 matrices A such that

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{3}$$

Solution (*a*) If A and B are matrices in U, then they can be expressed in the form

$$A = \begin{bmatrix} a & 0 \\ 2a & b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & 0 \\ 2c & d \end{bmatrix}$$

for some real numbers a, b, c, and d. But

$$A + B = \begin{bmatrix} a+c & 0 \\ 2(a+c) & b+d \end{bmatrix}$$

is also a matrix in U since it is of form (2) with $x = a + c$ and $y = b + d$. Thus, U is closed under addition. Similarly, U is closed under scalar multiplication since

$$kA = \begin{bmatrix} ka & 0 \\ 2ka & kb \end{bmatrix}$$

is of form (2) with $x = ka$ and $y = kb$. These two results establish that U is a subspace of M_{22}.

Solution (*b*) The set W is not a subspace of M_{22}. To see that this is so, it suffices to show that W is *either* not closed under addition or not closed under scalar multiplication. To see that it is not closed under scalar multiplication, let

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

This is a vector in W since

$$A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$$

so A satisfies Equation (3). However, $2A$ does not satisfy Equation (3) since

$$(2A)\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2 & 0\\-2 & 0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\-2\end{bmatrix}$$

and hence is not a vector in W. This alone establishes that W is not a subspace of M_{22}. However, it is also true that W is not closed under addition. We leave the proof for the reader.

EXAMPLE 12 | Applying the Subspace Test in P_2

Determine whether the indicated set of polynomials is a subspace of P_2.

(*a*) The set U consisting of all polynomials of the form $\mathbf{p} = 1 + ax - ax^2$, where a is a real number.

(*b*) The set W consisting of all polynomials $\mathbf{p}$ in P_2 such that $\mathbf{p}(2) = 0$.

Solution (*a*) The set U is not a subspace of P_2 because it is not closed under addition. For example, the polynomials $\mathbf{p} = 1 + x - x^2$ and $\mathbf{q} = 1 + 2x - 2x^2$ are in U, but

$$\mathbf{p} + \mathbf{q} = 2 + 3x - 3x^2$$

is not. We leave it for you to verify that U is also not closed under scalar multiplication.

Solution (*b*) If $\mathbf{p}$ and $\mathbf{q}$ are polynomials in W, and k is any real number, then

$$(\mathbf{p} + \mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = 0 + 0 = 0$$

and

$$(k\mathbf{p})(2) = k \cdot \mathbf{p}(2) = k \cdot 0 = 0.$$

Since $\mathbf{p} + \mathbf{q}$ and $k\mathbf{p}$ are in W, it follows that W is a subspace of P_2.

Building Subspaces

The following theorem provides a useful way of creating a new subspace from known subspaces.

Theorem 4.2.2

If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

Note that the first step in proving Theorem 4.2.2 was to establish that W contained at least one vector. This is important, for otherwise the subsequent argument might be logically correct but meaningless.

Proof Let W be the intersection of the subspaces $W_1, W_2, \ldots, W_r$. This set is not empty because each of these subspaces contains the zero vector of V, and hence so does their intersection. Thus, it remains to show that W is closed under addition and scalar multiplication.

To prove closure under addition, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in W. Since W is the intersection of $W_1, W_2, \ldots, W_r$, it follows that $\mathbf{u}$ and $\mathbf{v}$ also lie in each of these subspaces. Moreover, since these subspaces are closed under addition and scalar multiplication, they also all contain the vectors $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for every scalar k, and hence so does their intersection W. This proves that W is closed under addition and scalar multiplication. ■

Solution Spaces of Homogeneous Systems

The solutions of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns can be viewed as vectors in R^n. The following theorem provides an important insight into the geometric structure of the solution set.

Theorem 4.2.3

The solution set of a homogeneous system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of R^n.

Proof Let W be the solution set of the system. The set W is not empty because it contains at least the trivial solution $\mathbf{x} = \mathbf{0}$.

To show that W is a subspace of R^n, we must show that it is closed under addition and scalar multiplication. To do this, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be vectors in W. Since these vectors are solutions of $A\mathbf{x} = \mathbf{0}$, we have

$$A\mathbf{x}_1 = \mathbf{0} \quad \text{and} \quad A\mathbf{x}_2 = \mathbf{0}$$

It follows from these equations and the distributive property of matrix multiplication that

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so W is closed under addition. Similarly, if k is any scalar then

$$A(k\mathbf{x}_1) = kA\mathbf{x}_1 = k\mathbf{0} = \mathbf{0}$$

so W is also closed under scalar multiplication. ■

Because the solution set of a homogeneous system in n unknowns is actually a subspace of R^n, we will generally refer to it as the ***solution space*** of the system.

EXAMPLE 13 | Solution Spaces of Homogeneous Systems

In each part the solution of the linear system is provided. Give a geometric description of the solution set.

$$(a)\ \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (b)\ \begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(c)\ \begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (d)\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution (*a*) The solutions are

$$x = 2s - 3t, \quad y = s, \quad z = t$$

from which it follows that

$$x = 2y - 3z \quad \text{or} \quad x - 2y + 3z = 0$$

This is the equation of a plane through the origin that has $\mathbf{n} = (1, -2, 3)$ as a normal.

Solution (*b*) The solutions are

$$x = -5t, \quad y = -t, \quad z = t$$

which are parametric equations for the line through the origin that is parallel to the vector $\mathbf{v} = (-5, -1, 1)$.

Solution (*c*) The only solution is $x = 0, y = 0, z = 0$, so the solution space consists of the single point $\{\mathbf{0}\}$.

Solution (*d*) This linear system is satisfied by all real values of x, y, and z, so the solution space is all of R^3.

Remark Whereas the solution set of every *homogeneous* system of m equations in n unknowns is a subspace of R^n, it is *never* true that the solution set of a *nonhomogeneous* system of m equations in n unknowns is a subspace of R^n. There are two possible scenarios: first, the system may not have any solutions at all, and second, if there are solutions, then the solution set will not be closed under either addition or scalar multiplication (Exercise 22).

The Linear Transformation Viewpoint

Theorem 4.2.3 can be viewed as a statement about matrix transformations by letting $T_A : R^n \to R^m$ be multiplication by the coefficient matrix A. From this point of view the solution space of $A\mathbf{x} = \mathbf{0}$ is the set of vectors in R^n that T_A maps into the zero vector in R^m. This set is sometimes called the ***kernel*** of the transformation, so with this terminology Theorem 4.2.3 can be rephrased as follows.

Theorem 4.2.4

If A is an $m \times n$ matrix, then the kernel of the matrix transformation $T_A : R^n \to R^m$ is a subspace of R^n.

Exercise Set 4.2

In Exercises 1–2, use the Subspace Test to determine which of the sets are subspaces of R^3.

1. a. All vectors of the form $(a, 0, 0)$.
   b. All vectors of the form $(a, 1, 1)$.
   c. All vectors of the form (a, b, c), where $b = a + c$.

2. a. All vectors of the form (a, b, c), where $b = a + c + 1$.
   b. All vectors of the form $(a, b, 0)$.
   c. All vectors of the form (a, b, c) for which $a + b = 7$.

In Exercises 3–4, use the Subspace Test to determine which of the sets are subspaces of M_{nn}.

3. a. The set of all diagonal $n \times n$ matrices.
   b. The set of all $n \times n$ matrices A such that $\det(A) = 0$.
   c. The set of all $n \times n$ matrices A such that $\text{tr}(A) = 0$.
   d. The set of all symmetric $n \times n$ matrices.

4. a. The set of all $n \times n$ matrices A such that $A^T = -A$.
   b. The set of all $n \times n$ matrices A for which $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
   c. The set of all $n \times n$ matrices A such that $AB = BA$ for some fixed $n \times n$ matrix B.
   d. The set of all invertible $n \times n$ matrices.

In Exercises 5–6, use the Subspace Test to determine which of the sets are subspaces of P_3.

5. a. All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.
   b. All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.

6. a. All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers.
   b. All polynomials of the form $a_0 + a_1x$, where a_0 and a_1 are real numbers.

In Exercises 7–8, use the Subspace Test to determine which of the sets are subspaces of $F(-\infty, \infty)$.

7. **a.** All functions f in $F(-\infty, \infty)$ for which $f(0) = 0$.

b. All functions f in $F(-\infty, \infty)$ for which $f(0) = 1$.

8. **a.** All functions f in $F(-\infty, \infty)$ for which $f(-x) = f(x)$.

b. All polynomials of degree 2.

In Exercises 9–10, use the Subspace Test to determine which of the sets are subspaces of R^∞.

9. **a.** All sequences $\mathbf{v}$ in R^∞ of the form $\mathbf{v} = (v, 0, v, 0, v, 0, \ldots)$.

b. All sequences $\mathbf{v}$ in R^∞ of the form $\mathbf{v} = (v, 1, v, 1, v, 1, \ldots)$.

10. **a.** All sequences $\mathbf{v}$ in R^∞ of the form

$$\mathbf{v} = (v, 2v, 4v, 8v, 16v, \ldots)$$

b. All sequences in R^∞ whose components are 0 from some point on.

In Exercises 11–12, use the Subspace Test to determine which of the sets are subspaces of M_{22}.

11. **a.** All matrices of the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$.

b. All matrices of the form $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$.

c. All 2×2 matrices A such that

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

12. **a.** All 2×2 matrices A such that

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b. All 2×2 matrices A such that

$$A\begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix}A$$

c. All 2×2 matrices A for which $\det(A) = 0$.

In Exercises 13–14, use the Subspace Test to determine which of the sets are subspaces of R^4.

13. **a.** All vectors of the form (a, a^2, a^3, a^4).

b. All vectors of the form $(a, 0, b, 0)$.

14. **a.** All vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where

$$A = \begin{bmatrix} 0 & -1 & 0 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

b. All vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where A is as in part (a).

In Exercises 15–16, use the Subspace Test to determine which of the sets are subspaces of P_∞.

15. **a.** All polynomials of degree less than or equal to 6.

b. All polynomials of degree equal to 6.

c. All polynomials of degree greater than or equal to 6.

16. **a.** All polynomials with even coefficients.

b. All polynomials whose coefficients sum to 0.

c. All polynomials of even degree.

17. **(*Calculus Required*)** Which of the following are subspaces of R^∞?

a. All sequences of the form $\mathbf{v} = (v_1, v_2, \ldots, v_n, \ldots)$ such that $\lim_{n\to\infty} v_n = 0$.

b. All convergent sequences (that is, all sequences of the form $\mathbf{v} = (v_1, v_2, \ldots, v_n, \ldots)$ such that $\lim_{n\to\infty} v_n$ exists).

c. All sequences of the form $\mathbf{v} = (v_1, v_2, \ldots, v_n, \ldots)$ such that $\sum_{n=1}^{\infty} v_n = 0$.

d. All sequences of the form $\mathbf{v} = (v_1, v_2, \ldots, v_n, \ldots)$ such that $\sum_{n=1}^{\infty} v_n$ converges.

18. A line L through the origin in R^3 can be represented by parametric equations of the form $x = at$, $y = bt$, and $z = ct$. Use these equations to show that L is a subspace of R^3 by showing that if $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$ are points on L and k is any real number, then $k\mathbf{v}_1$ and $\mathbf{v}_1 + \mathbf{v}_2$ are also points on L.

19. Determine whether the solution space of the system $A\mathbf{x} = \mathbf{0}$ is a line through the origin, a plane through the origin, or the origin only. If it is a plane, find an equation for it. If it is a line, find parametric equations for it.

a. $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}$ **d.** $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix}$

20. **(*Calculus required*)** Show that the following sets of functions are subspaces of $F(-\infty, \infty)$.

a. All continuous functions on $(-\infty, \infty)$.

b. All differentiable functions on $(-\infty, \infty)$.

c. All differentiable functions on $(-\infty, \infty)$ that satisfy $\mathbf{f}' + 2\mathbf{f} = \mathbf{0}$.

21. **(*Calculus required*)** Show that the set of continuous functions $\mathbf{f} = f(x)$ on $[a, b]$ such that

$$\int_a^b f(x)\,dx = 0$$

is a subspace of $C[a, b]$.

22. Show that the solution vectors of a consistent nonhomogeneous system of m linear equations in n unknowns do not form a subspace of R^n.

23. If T_A is multiplication by a matrix A with three columns, then the kernel of T_A is one of four possible geometric objects. What are they? Explain how you reached your conclusion.

24. Consider the following subsets of P_3: V consists of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ such that $a_0 + a_3 = 0$ and W consists of all polynomials $\mathbf{p}$ such that $\mathbf{p}(1) = 0$.

a. Use the Subspace Test to show that V and W are subspaces of P_3.

b. Show that the set of all polynomials

$$\mathbf{p} = a_0 + a_1x + a_2x^2 + a_3x^3$$

such that $a_0 + a_3 = 0$ *and* $\mathbf{p}(1) = 0$ is a subspace of P_3 *without* using the Subspace Test.