National University of Computer and Emerging Sciences
Karachi Campus

**Discrete Structures (CS1005)**

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**Course Instructors:**

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# Sessional-I Exam

**Total Time (Hrs): 1**
**Total Marks: 24**
**Total Questions: 2**

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**Attempt all the questions in the given sequence.**

***CLO # 2: Construct formal logic proofs and/or informal, but rigorous, logical reasoning to real problems, such as predicting the behavior of software or solving problems such as puzzles.***

**Propositional Logic, Predicate Logic and Rules of Inference. [ 6 x 2 = 12 Marks]** Time: 30 Minutes

**Q1: a.** Let p and q be the propositions:

p: The event is scheduled to start on time.
q: There is a delay in the event's start.
r: "The project is completed."
s: "The report is submitted."
t: "The project is approved."

Express the following statement using logical connectives (including negation):

i) The event is not scheduled to start on time and either the event is scheduled to start on time or there is no delay in the event's start.
**Solution:** $\neg p \wedge ( p \vee \neg q )$

ii) The event is scheduled to start on time if and only if there is no delay in the event's start.
**Solution:** $p \leftrightarrow \neg q$

iii) The project is completed and the report is submitted is a sufficient and the necessary for the project to be approved.
$(r \wedge s) \leftrightarrow t$

iv) Submitting the report is sufficient for the project to be completed.
$S \rightarrow r$

**b.** Show that $\neg [(p \rightarrow q) \wedge (\neg q \vee r)] \equiv p \wedge (\neg q \vee \neg r)$ using laws of logical equivalence and justify steps.

$\neg[(\neg p \vee q) \wedge (\neg q \vee r)]$

**Apply De Morgan's Law to the negation of the conjunction:**

$\neg[(\neg p \vee q) \wedge (\neg q \vee r)] \Rightarrow \neg(\neg p \vee q) \vee \neg(\neg q \vee r)$

$(p \wedge \neg q) \vee (q \wedge \neg r)$

**Apply Distribute law and simplify, now distribute the terms:**

$p \wedge (\neg q \vee \neg r)$

**c.** Determine the truth value of each of the following statements, where is the universe of discourse is Z.

i) $\exists x \forall y (x^2 < y + 1)$

**False: x !=0, we can always find a y that makes the inequality false, such as $y=x^2-1$. Thus, no such x exists.**

ii) $\forall x \exists y (x^2 + y^2 < 12)$

**False, because for $|x| \geq 4$, no such y exists.**

**d.** Consider the following statement: "There is a tourist who has taken a flight on every airline in the world."

i) Express the above statement mathematically using the appropriate universal and existential quantifiers along with the appropriate propositional functions.

- **Let *P(t,f)* be "*t* has taken *f* " and *Q*(*f*,*a*) be "*f* is a flight on *a* ."**
- **The domain of *t* is all Tourist, the domain of *f* is all flights, and the domain of *a* is all airlines.**
- **Then the statement can be expressed as:**
  **$\exists t \exists f \ \forall a \ (P(t,f) \land Q(f,a))$**

iii) Negate the resultant nested quantifiers using De Morgan's Laws.

$\exists t \ \forall a \ \exists f \ (P(t,f) \land Q(f,a))$

Part 1: Use quantifiers to express the statement that "There does not exist a Tourist who has taken a flight on every airline in the world."

Solution: $\neg\exists t \ \forall a \ \exists f \ (P(t,f) \land Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg\exists t \ \forall a \ \exists f \ (P(t,f) \land Q(f,a))$
2. $\forall t \neg \forall a \ \exists f \ (P(t,f) \land Q(f,a))$ by De Morgan's for $\exists$
3. $\forall t \ \exists a \neg \exists f \ (P(t,f) \land Q(f,a))$ by De Morgan's for $\forall$
4. $\forall t \ \exists a \ \forall f \neg (P(t,f) \land Q(f,a))$ by De Morgan's for $\exists$
5. $\forall t \ \exists a \ \forall f (\neg P(t,f) \lor \neg Q(f,a))$ by De Morgan's for $\land$.

**e.** Write each of the following arguments in symbolic form. Then establish the validity of the argument or give a counter example to show that it is invalid.

"If Dr. Sana gets the supervisor's position (p) and works hard (q), then she'll get a raise (r)." "She gets the raise only if she'll buy a new car (s)." "She has not purchased a new car."

Therefore, "either Dr. Sana did not get the supervisor's position or she did not work hard."

$(((p \land q) \to r) \land (r \to s) \land \neg s) \to (\neg p \lor \neg q)$

| | | |
|---|---|---|
| (1) | $\neg s$ | Premise |
| (2) | $r \to s$ | Premise |
| (3) | $\neg r$ | Steps (1), (2) and Modus Tollens |
| (4) | $(p \land q) \to r$ | Premise |
| (5) | $\neg(p \land q)$ | Steps (3), (4) and Modus Tollens |
| (6) | $\therefore \neg p \lor \neg q$ | Step (5) and $\neg(p \land q) \iff \neg p \lor \neg q$. |

**f.** Using truth tables, prove that the following statement is a tautology, contingency or Contradiction.

$(((\neg s \rightarrow \neg h) \wedge (s \rightarrow w) \wedge (\neg w \wedge s)) \rightarrow \neg h)$

| | |
|---|---|
| $((\underline{h \rightarrow s}) \wedge (s \rightarrow w) \wedge (\neg w \wedge s))$ | Contrapositive |
| $((\underline{h \rightarrow s) \wedge (s \rightarrow w}) \wedge (\neg w \wedge s))$ | Hypothetical Syllogism |
| $((h \rightarrow w) \wedge (\neg w \wedge s))$ | Simplification |
| $((h \rightarrow w) \wedge \underline{(\neg w)})$ | Modus Tollen |
| $\neg h$ | |

***CLO # 1: Explain the key concepts of Discrete Structures such as Mathematical Logic, Sets, Permutations, Relations, Graphs and Trees etc.***

**Sets Theory and Functions:** **[6 x 2 = 12 Marks]** Time: 30 Minutes

**Q2: a.** i) Sets are stored in an unordered fashion in memory. Why we impose (any) fixed ordering on elements of U in the bit-vector representation? Also, find the characteristic bit-vector of the set A = {DS, DM}, if U= {DM, Cal, DS, Bio, Phy, Pro} (in order)?

**Solution: A fixed ordering ensures that each element of U always maps to the same bit position in the bit vector, allowing for a standardized and consistent representation of sets. It also enable efficient set operations, and maintain compactness in memory.**

| DM | Cal | DS | Bio | Phy | Pro |
|---|---|---|---|---|---|
| **1** | **0** | **1** | **0** | **0** | **0** |

ii) Formally define the multiset difference. Also, find the multiset difference A — B, if A= {(Ayesha, 5), (Khan, 2), (Ali, 1), (Imadad, 1)}, and B= {Ayesha, 3), (Khan, 3), (Imdad, 1)}.

The difference of a multiset $B$ from a multiset $A$ is a multiset $C$ such that

$$\forall x \in U,\ m_C(x) = max(m_A(x) - m_B(x), 0)$$

denoted as $C = A \setminus B$

**{(Ayesha, 2), (Ali, 1)}** **by definition For Khan: max ((3-2) , 0) = max (-1,0) = 0 is not considered**
**For Imdad: max ((1-1) , 0) = max (0,0) = 0 is not considered**

**b.** Prove using logical equivalences and set builder notation that A ∩ (B ∪ C) ⊆ (A ∩ B) ∪ (A ∩ C). Provide a step-by-step proof to demonstrate that the left-hand side of the expression is a subset of the right-hand side. Use logical equivalences and set builder notation in your explanation.

**The below solution is for** that A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C). Use it for ⊆

| | |
|---|---|
| $x \in A \cap (B \cup C)$ | ▷ LHS |
| $\equiv x \in A \wedge x \in (B \cup C)$ | ▷ Intersection |
| $\equiv x \in A \wedge (x \in B \vee x \in C)$ | ▷ Union |
| $\equiv (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$ | ▷ Distributive Law |
| $\equiv x \in (A \cap B) \vee x \in (A \cap C)$ | ▷ Intersection |
| $\equiv x \in (A \cap B) \cup (A \cap C)$ | ▷ Union |

**c.** Provide the name of the set definition for each of the following.

i) $\forall x\ (x \in A \rightarrow x \in B) \land \exists x\ (x \in B \land x \notin A)$ $A \subset B$ or $A \subsetneq B$

ii) $\forall x\ (x \in A \rightarrow x \in B)$ $A \subseteq B$

iii) $\forall x\ (x \in A \leftrightarrow x \in B)$ **A = B**

iv) $\{x \mid x \in A \land x \notin B\}$ A - B

**d.** Consider the following two situations:

i) Mark is buying his favorite comic books for $2 each. He starts with $187 in his account. Write a function that represents the amount of money Mark has left in his account after buying x comic books. Label the variables.

**f(x) = 187 – 2 x**

ii) Annie is helping her mom collect canned goods. Her mom already has 91 cans, and Annie is collecting 10 cans each week. Write a function that represents the total number of cans collected after y weeks. Label the variables.

**f(x) = 91 + 10 x**

**e.** i) $\forall\ y \in Y\ \exists x \in X\ f(x) = y$ is the definition of the surjective(onto) function. Determine whether the function $f(x)=x^3$ from R→R is surjective?

The function $f(x) = x^3$ is **surjective**, because for every $y \in \mathbb{R}$, there exists an $x \in \mathbb{R}$ (specifically $x = \sqrt[3]{y}$) such that $f(x) = y$.

ii) Formally define an injective function. Prove whether the function f(x)= 2x + 3 from R→R is injective.

A function $f : X \mapsto Y$ is **one-to-one** (or **injective**) iff

$$\forall\ x_1, x_2 \in X\ \ (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$$

**f(x1) = f (x2) → 2(x1) + 3 = 2(x2) + 3 → x1 = x2**
**Hence, f is one-to-one**

**f.** For the following functions, determine whether the function f is Injective (one-to-one), Surjective(onto), both injective and surjective or neither injective nor surjective. Provide a justification for your answer.

i) f: {1,2,3} → {a, b} be defined by f(1)=a, f(2)=a, f(3)=b.

Since every element in the codomain $\{a, b\}$ has a pre-image in the domain $\{1, 2, 3\}$, the function is **surjective**. The function $f : \{1, 2, 3\} \rightarrow \{a, b\}$ is **not injective** but **surjective**.

ii) f: N→N be defined by $f(x)=x^2$.

**Since, every element in N is not the square of some natural number, so the function is** not surjective**.**
The function $f : \mathbb{N} \rightarrow \mathbb{N}$, defined by $f(x) = x^2$, is **injective** but **not surjective**.