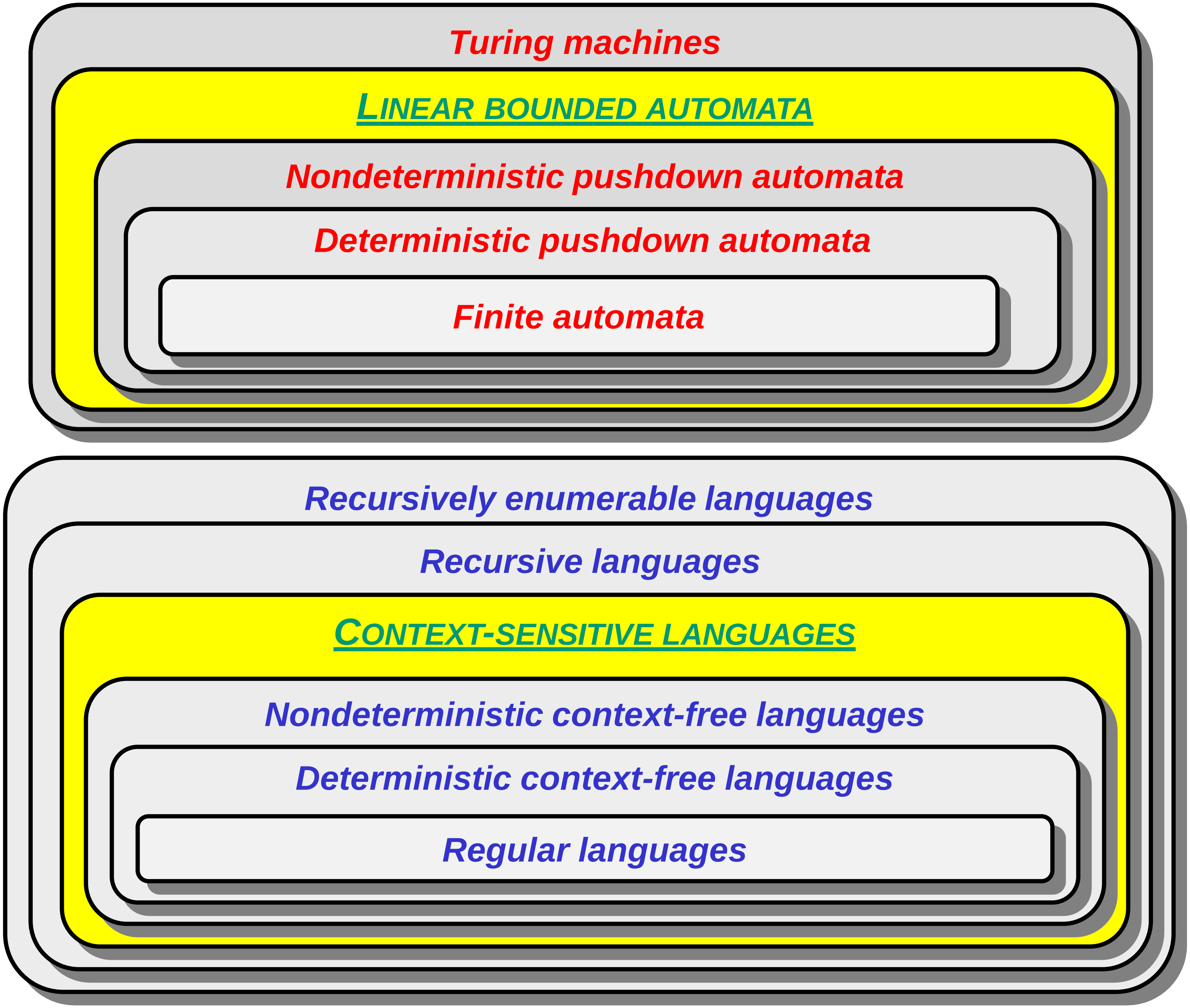

Turing machines
LINEAR BOUNDED AUTOMATA
Nondeterministic pushdown automata
Deterministic pushdown automata
Finite automata
Recursively enumerable languages
Recursive languages
CONTEXT-SENSITIVE LANGUAGES
Nondeterministic context-free languages
Deterministic context-free languages
Regular languages

Lecture 15

Note: LBA's and context-sensitive grammars/languages are only covered in the first edition of the textbook, not in the later ones ☹

Lecture 15

Context-sensitive grammar

$$\alpha_1 A \alpha_2 \rightarrow \alpha_1 \beta \alpha_2$$

$$\alpha \rightarrow \beta$$

$$|\,\alpha\,| \leq |\,\beta\,|, \qquad \alpha \neq \varepsilon\,,\ \beta \neq \varepsilon$$

Context-sensitive grammar

$$L = \{ a^n b^n c^n \mid n \geq 1 \}$$

1) $S \rightarrow a\ b\ c$
2) $S \rightarrow a\ S\ B\ c$
3) $c\ B \rightarrow W\ B$
4) $W\ B \rightarrow W\ X$
5) $W\ X \rightarrow B\ X$
6) $B\ X \rightarrow B\ c$
7) $b\ B \rightarrow b\ b$

$n = 1$

$$\boxed{S \Rightarrow a\ b\ c}$$

$n = 3$

$S \Rightarrow a\ S\ B\ c \Rightarrow a\ a\ S\ B\ c\ B\ c \Rightarrow a\ a\ a\ b\ c\ B\ c\ B\ c \Rightarrow a\ a\ a\ b\ W\ B\ c\ B\ c$
$\Rightarrow a\ a\ a\ b\ W\ X\ c\ B\ c \Rightarrow a\ a\ a\ b\ B\ X\ c\ B\ c \Rightarrow a\ a\ a\ b\ B\ c\ c\ B\ c \Rightarrow \ldots\ldots$
$\Rightarrow a\ a\ a\ b\ B\ c\ B\ c\ c \Rightarrow \ldots\ldots \Rightarrow a\ a\ a\ b\ B\ B\ c\ c\ c \Rightarrow a\ a\ a\ b\ \underline{b\ B}\ c\ c\ c$
$\Rightarrow a\ a\ a\ b\ \underline{b\ b}\ c\ c\ c$

Context-sensitive grammar

1) $S \to ACaB$
2) $Ca \to aaC$
3) $CB \to DB$
4) $CB \to E$
5) $aD \to Da$
6) $AD \to AC$
7) $aE \to Ea$
8) $AE \to \varepsilon$

---

1) $S \to [ACaB]$

2) $[Ca]a \to aa[Ca]$
$[Ca][aB] \to aa[CaB]$
$[CaB] \to a[aCB]$
$[ACa]a \to [Aa]a[Ca]$
$[ACa][aB] \to [Aa]a[CaB]$
$[ACaB] \to [Aa][aCB]$

3) $[aCB] \to [aDB]$

4) $[aCB] \to [aE]$

5) $a[Da] \to [Da]a$
$[aDB] \to [DaB]$
$a[DaB] \to [Da][aB]$
$[Aa][Da] \to [ADa]a$
$[Aa][DaB] \to [ADa][aB]$

6) $[ADa] \to [ACa]$

7) $[aE] \to [Ea]$
$a[Ea] \to [Ea]a$
$[Aa][Ea] \to [AEa]a$

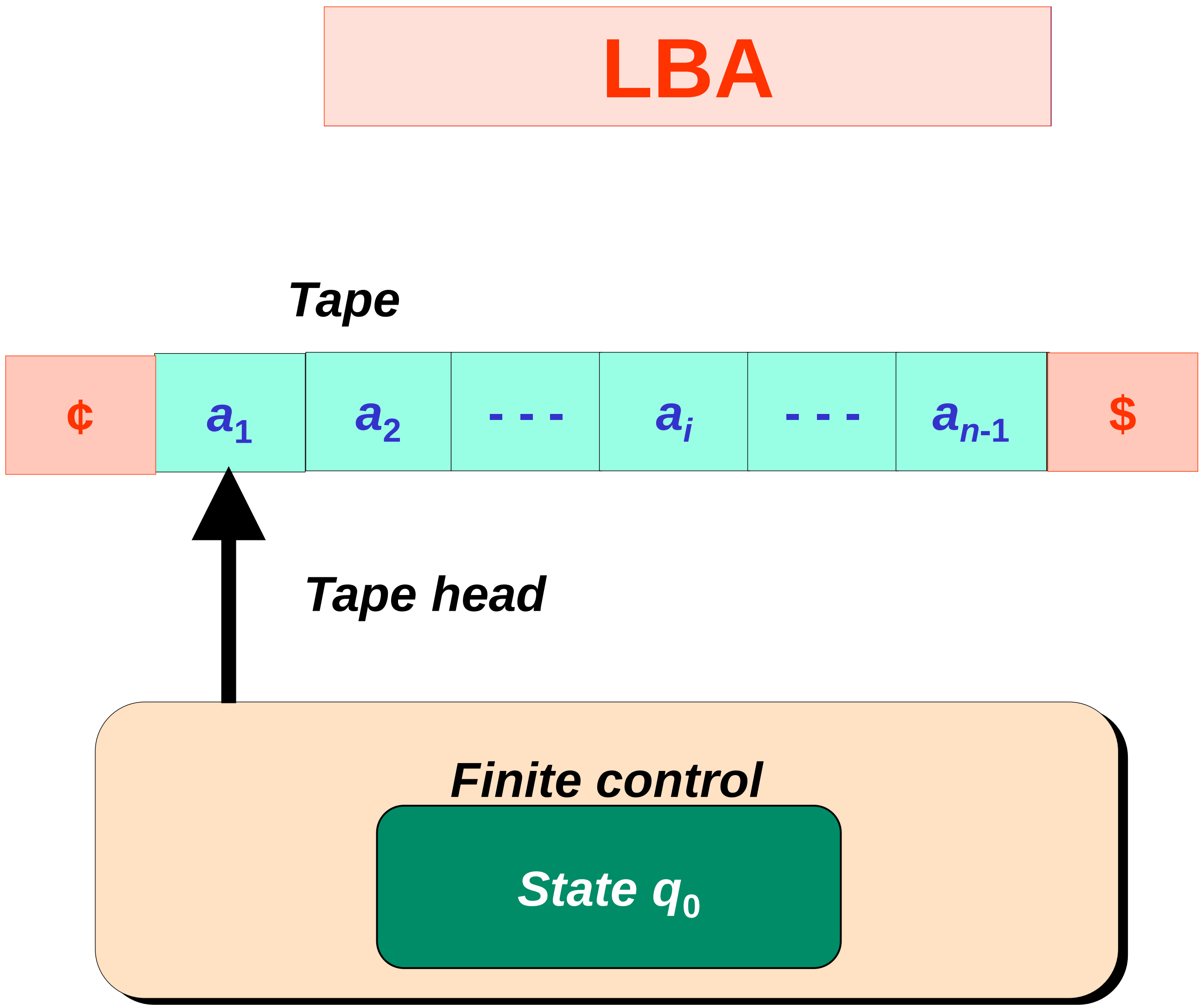
8) $[AEa] \to a$

$S \Rightarrow [ACaB] \Rightarrow [Aa]\,[aCB] \Rightarrow [Aa]\,[aE] \Rightarrow [Aa]\,[Ea] \Rightarrow [AEa]\,a \Rightarrow a\,a$

$S \Rightarrow \underline{ACaB} \Rightarrow Aa\underline{aCB} \Rightarrow A\underline{aaE} \Rightarrow A\underline{aEa} \Rightarrow \underline{AEaa} \Rightarrow a\,a$

Lecture 15

Linear bounded automata (LBA)



Linear bounded automata (LBA)

$$LBA = (Q, \Sigma, \Gamma, \delta, q_0, ¢, \$, F)$$

Q	- finite set of states
Γ	- finite set of tape symbols
$\Sigma \subseteq \Gamma$	- finite set of input symbols
δ	- transition function
	$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$
$q_0 \in Q$	- initial (start) state
$F \subseteq Q$	- set of accepting states
¢, \$	- left and right endmarkers

Linear bounded automata (LBA)

LBA $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ **accepts language:**

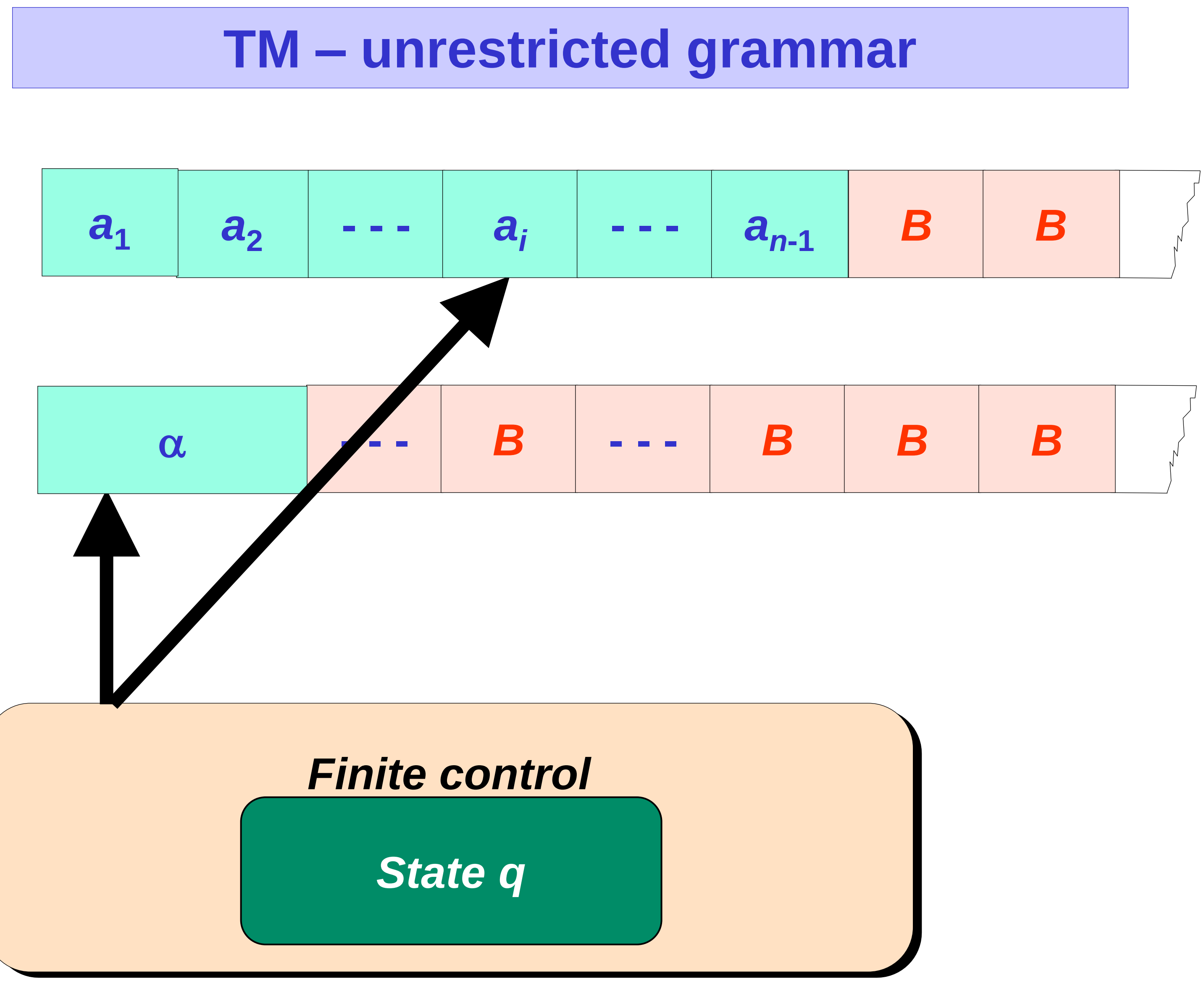
$$L(M) = \{\ w \ \mid \ w \in (\Sigma \setminus \{¢, \$\})^*$$

$$\text{i}\ \ q_0 ¢ w \$ \ \overset{*}{\vdash}\ \alpha\, q\, \beta,$$

$$q \in F\ \}$$

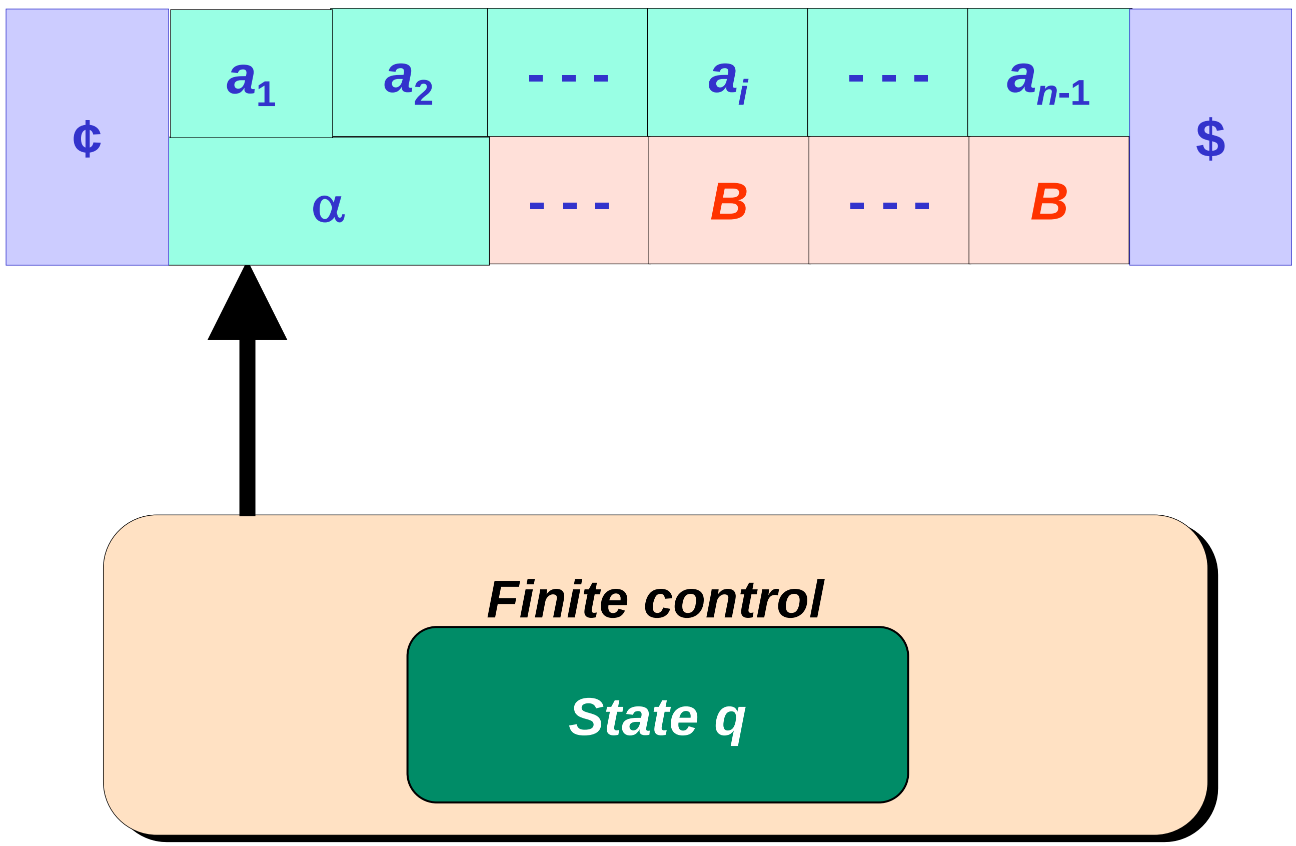
Lecture 15

Construction of an equivalent LBA for a given context-sensitive grammar



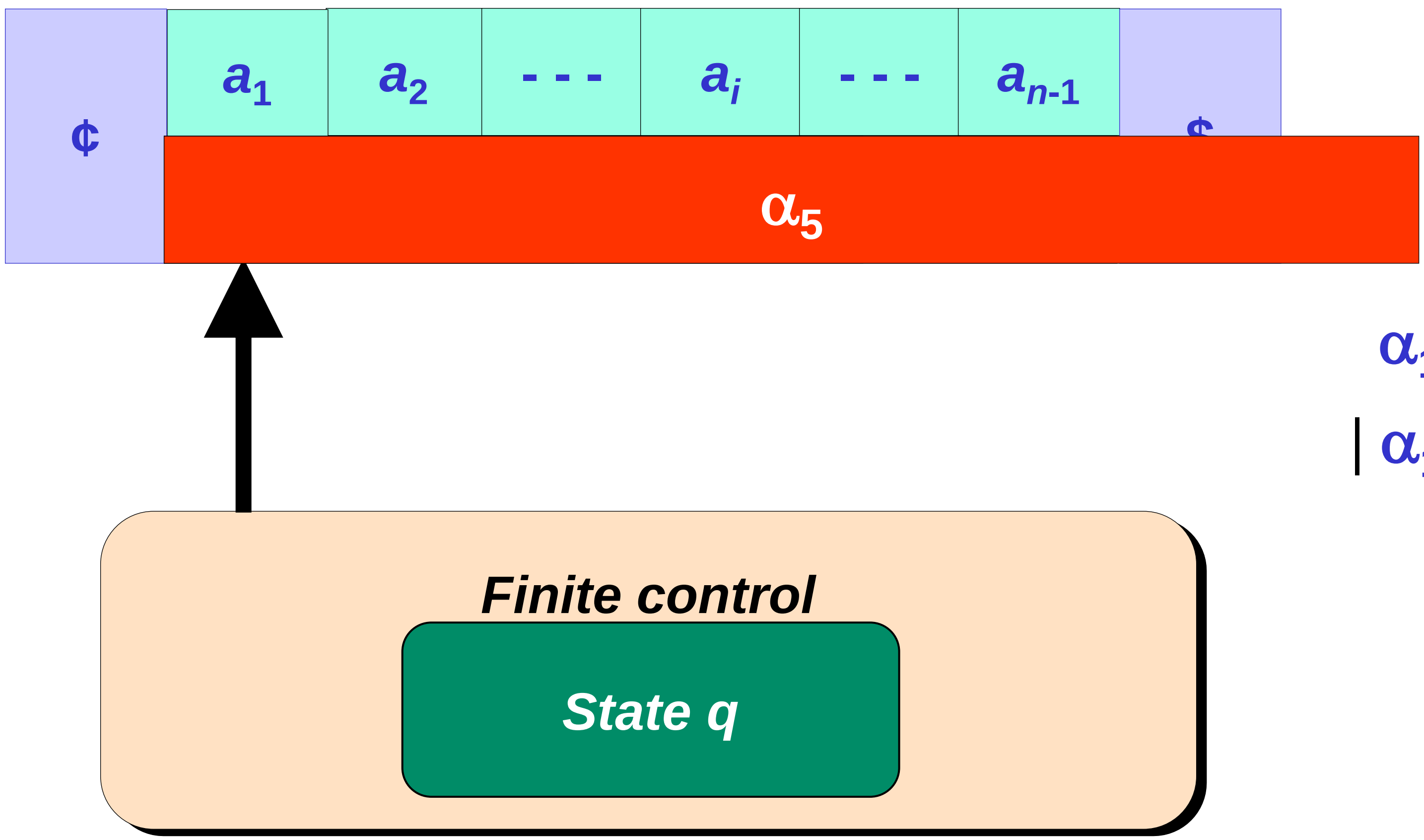
Construction of an equivalent LBA for a given context-sensitive grammar

LBA – context-sensitive grammar



Construction of an equivalent LBA for a given context-sensitive grammar

LBA – context-sensitive grammar



$\alpha_1 \Rightarrow \alpha_2$

$|\alpha_1| \leq |\alpha_2|$

Lecture 15

Construction of an equivalent context-sensitive grammar for a given LBA

Initial configuration LBA: q_0 ¢ $a_1\ a_2$ --- a_n \$

String derived in grammar: $[a_1,\ q_0$ ¢ $a_1]\ [a_2,\ a_2]$ --- $[a_n,\ a_n$ \$]

Grammar productions:

1)	$A_1 \rightarrow [a,\ q_0$ ¢ a \$]
2)	$A_1 \rightarrow [a,\ q_0$ ¢ $a]\ A_2$
3)	$A_2 \rightarrow [a,\ a]\ A_2$
4)	$A_2 \rightarrow [a,\ a$ \$]

Construction of an equivalent context-sensitive grammar for a given LBA

Transition of the TM M : $\delta(q, X) = (p, Y, R)$

Grammar productions:

5)	$[b, q\,X]\,[a, Z]$	$\rightarrow$	$[b, Y]\,[a, p\,Z]$
	$[b, ¢\,q\,X]\,[a, Z]$	$\rightarrow$	$[b, ¢\,Y]\,[a, p\,Z]$
	$[b, q\,X]\,[a, Z\,\$]$	$\rightarrow$	$[b, Y]\,[a, p\,Z\,\$]$
	$[b, q\,X\,\$]$	$\rightarrow$	$[b, Y\,p\,\$]$
	$[a, ¢\,q\,X\,\$]$	$\rightarrow$	$[a, ¢\,Y\,p\,\$]$

Construction of an equivalent context-sensitive grammar for a given LBA

Transition of the TM M: $\delta(q, X) = (p, Y, L)$

Grammar productions:

6)	$[b, Z]\,[a, q\,X]$	$\rightarrow$	$[b, p\,Z]\,[a, Y]$
	$[b, Z]\,[a, q\,X\,\$]$	$\rightarrow$	$[b, p\,Z]\,[a, Y\,\$]$
	$[b, ¢\,Z]\,[a, q\,X]$	$\rightarrow$	$[b, ¢\,p\,Z]\,[a, Y]$
	$[b, ¢\,q\,X]$	$\rightarrow$	$[b, p\,¢\,Y]$
	$[a, ¢\,q\,X\,\$]$	$\rightarrow$	$[a, p\,¢\,Y\,\$]$

Construction of an equivalent context-sensitive grammar for a given LBA

Grammar productions:

7) $[b,\ q\ ¢\ X] \rightarrow [b,\ ¢\ p\ X]$

$[b,\ X\ q\ \$] \rightarrow [b,\ p\ X\ \$]$

$[a,\ q\ ¢\ X\ \$] \rightarrow [a,\ ¢\ p\ X\ \$]$

$[a,\ ¢\ X\ q\ \$] \rightarrow [a,\ ¢\ p\ X\ \$]$

Construction of an equivalent context-sensitive grammar for a given LBA

For all accepting states $q \in F$

For all symbols $a \in \Sigma \setminus \{ ¢, \$ \}$

Grammar productions:

8) $[a, \alpha\, q\, \beta] \rightarrow a$

9) $[a, \alpha]\, b \rightarrow a\, b$

10) $b\, [a, \alpha] \rightarrow b\, a$

Lecture 15

Properties of context-sensitive languages

- **Union**
  - $G_1 = (V_1, T_1, P_1, S_1)$
  - $G_2 = (V_2, T_2, P_2, S_2)$

- $L(G_3) = L(G_1) \cup L(G_2)$
  - $V_1 \cap V_2 = \emptyset$

---

1) $V_3 = V_1 \cup V_2 \cup \{S_3\}$, $S_3 \notin V_1$ i $S_3 \notin V_2$

2) $T_3 = T_1 \cup T_2$

3) $P_3 = P_1 \cup P_2 \cup \{ S_3 \to S_1 \mid S_2 \}$

Properties of context-sensitive languages

- Concatenation
  - $G_1 = (V_1, T_1, P_1, S_1)$
  - $G_2 = (V_2, T_2, P_2, S_2)$

- $L(G_4) = L(G_1)\, L(G_2)$
  - $V_1 \cap V_2 = \emptyset$

---

1) $V_4 = V_1 \cup V_2 \cup \{S_4\}$, $S_4 \notin V_1$ i $S_4 \notin V_2$

2) $T_4 = T_1 \cup T_2$

3) $P_4 = P_1 \cup P_2 \cup \{\, S_4 \rightarrow S_1\, S_2 \,\}$

Properties of context-sensitive languages

$V_1 \cap V_2 = \varnothing$ - not a sufficient condition!

$\alpha_1 A \alpha_2 \rightarrow \alpha_1 \beta \alpha_2$ – productions of grammar G_2

$$S_1 \underset{G_1}{\overset{*}{\Rightarrow}} \boxed{\gamma' \alpha_1} \qquad S_2 \underset{G_2}{\overset{*}{\Rightarrow}} \boxed{A \alpha_2 \delta'}$$

$$S_4 \underset{G_4}{\Rightarrow} S_1 S_2 \underset{G_1}{\overset{*}{\Rightarrow}} \boxed{\gamma' \alpha_1} S_2 \underset{G_2}{\overset{*}{\Rightarrow}} \boxed{\gamma'}\boxed{\alpha_1 A \alpha_2}\boxed{\delta'}$$

$$\underset{G_2}{\overset{*}{\Rightarrow}} \boxed{\gamma' \alpha_1 \beta \alpha_2 \delta'} \overset{*}{\Rightarrow} w \notin L(G_4) = L(G_1)\, L(G_2)$$

Properties of context-sensitive languages

$a \in T$

$A_a \in V'$

$A_a \to a$

For each production $\alpha \to \beta$

In strings α and β replace each terminal a with a fresh nonterminal A_a:

$$\alpha' \to \beta'$$

Properties of context-sensitive languages

- Positive closure L^+
  - $G = (V, T, P, S)$
- $L(G_5) = L(G)^+$

Build an auxiliary grammar $G' = (V', T, P', S')$, $V \cap V' = \emptyset$

Use G and G' to define grammar G_5

1) $V_5 = V \cup V' \cup \{S_5, S_5'\}$

$S_5', S_5 \notin V$

$S_5', S_5 \notin V'$

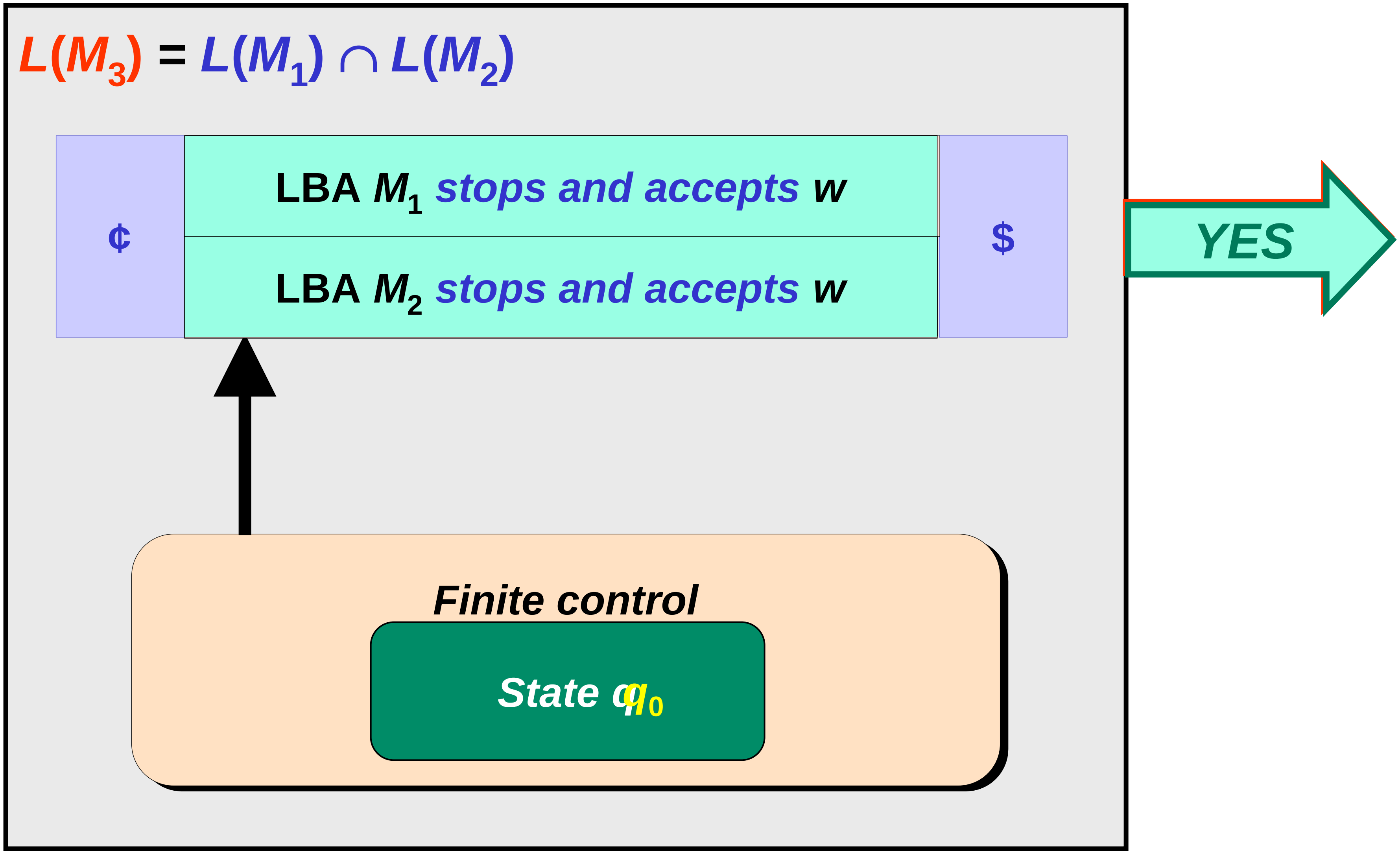
2) $T_5 = T$

3) $S_5 \rightarrow S\, S_5' \mid S$

$S_5' \rightarrow S'\, S_5 \mid S'$

Lecture 15

Properties of context-sensitive languages



Properties of context-sensitive languages

- Complment of deterministic context-sensitive language
- Given a DLBA M

  s - number of states of M

  n - length of the input *w*

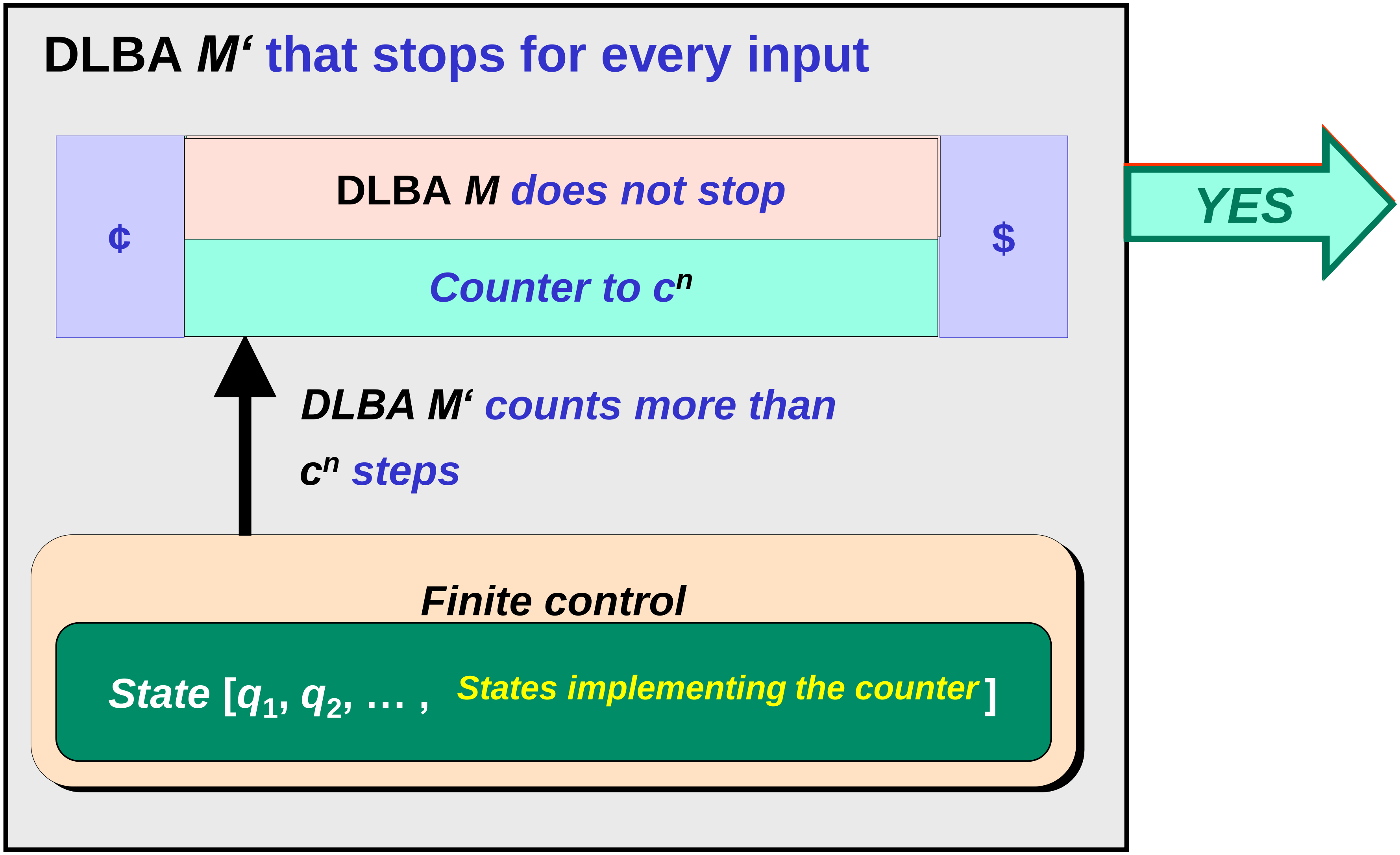
  t - number of tape symbols of M

Number of different configurations of M with input w

$$s\,(n+2)\,t^n < c^n \quad \textit{where } c \textit{ is a constant}$$

- Transition of DLBA from configuration to configuration
  - $K_x \rightarrow K_y$

Properties of context-sensitive languages



Properties of context-sensitive languages

- **„First LBA problem”: Are deterministic LBAs equivalent to nondeterministic LBAs?**
  - Still an open problem.
- **„Second LBA problem”: Is a complement of a (nondeterministic) context-sensitive language also a (nondeterministic) context-sensitive language?**
  - Yes! Immerman–Szelepcsényi theorem (1987.)
  - Gödel prize 1995.