Lecture 16

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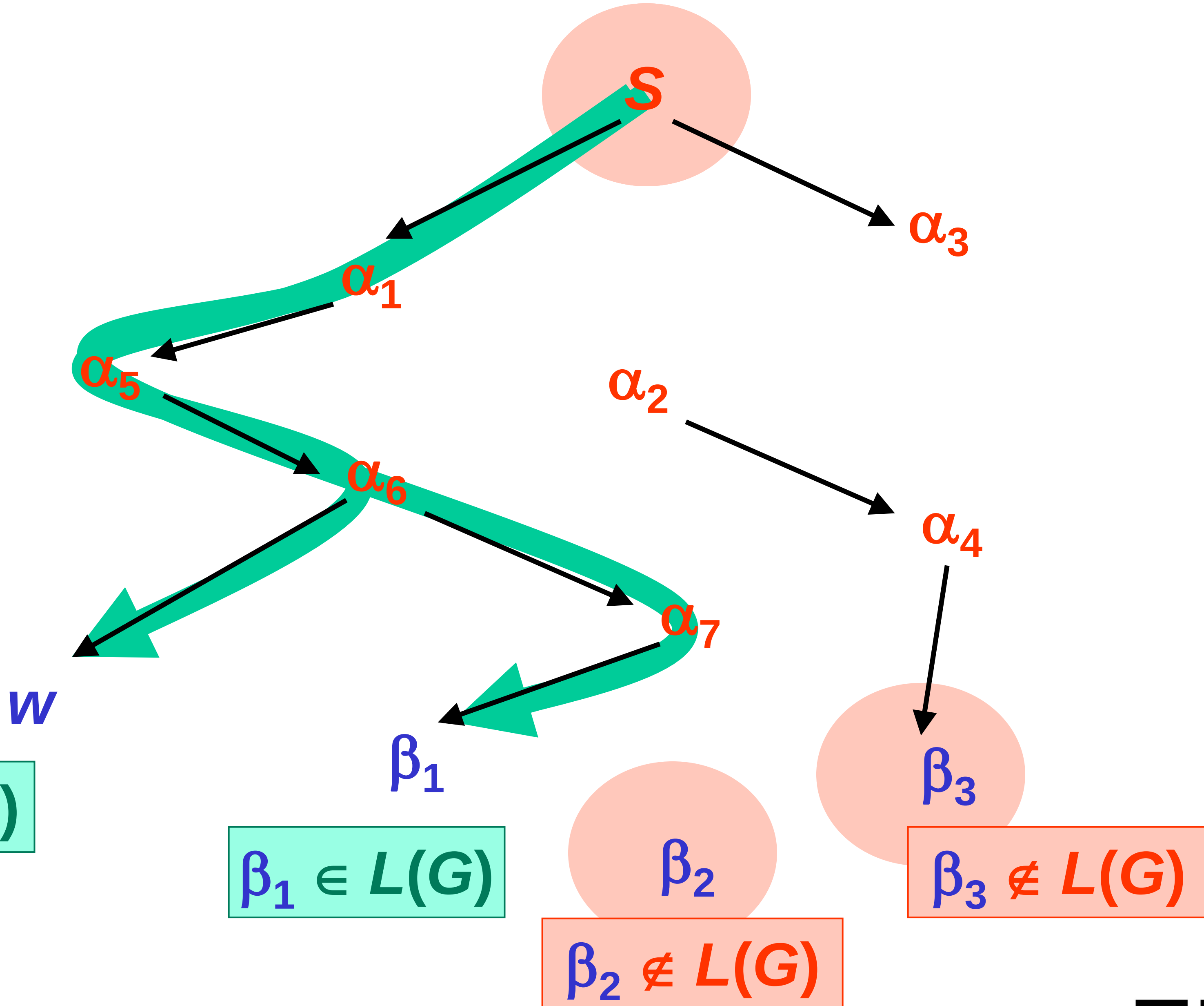
Decidability of context-sensitive languages

$$|\alpha_i| \leq |w|$$

$$\alpha_i \Rightarrow \alpha_j$$



Decidability of context-sensitive languages

- Iterative algorithm that finds a path in a graph
  - build a list K of strings generated by the CSG
    - $|\alpha| \leq |w|$
  - K_i – contents of the list after i-th iteration
  - K_i – strings α
    - $|\alpha| \leq |w|$
    - CSG generates α in at most i steps

Decidability of context-sensitive languages

- **Initialization:**
  - ***K*** **only contains the initial nonterminal** ***S***

- ***i*-th iteration:**
  - $K_i = K_{i-1} \cup \{ \beta \mid \alpha \Rightarrow \beta, \alpha \in K_{i-1} \text{ i } |\beta| \leq |w| \}$

Decidability of context-sensitive languages

1) $S \rightarrow [ACaB]$
2) $[Ca]a \rightarrow aa[Ca]$
$[CaB] \rightarrow a[aCB]$
$[CaB] \rightarrow a[aCB]$
$[ACa]a \rightarrow [Aa]a[Ca]$
$[ACa]a \rightarrow [Aa]a[Ca]$
$[ACa][aB] \rightarrow [Aa]a[CaB]$
$[ACaB] \rightarrow [Aa][aCB]$
3) $[aCB] \rightarrow [aDB]$

4) $[aCB] \rightarrow [aE]$
5) $a[Da] \rightarrow [Da]a$
$[aDB] \rightarrow [DaB]$
$a[DaB] \rightarrow [Da][aB]$
$[Aa][Da] \rightarrow [ADa]a$
$[Aa][DaB] \rightarrow [ADa][aB]$
6) $[ADa] \rightarrow [ACa]$

7) $[aE] \rightarrow [Ea]$
$a[Ea] \rightarrow [Ea]a$
$[Aa][Ea] \rightarrow [AEa]a$
8) $[AEa] \rightarrow a$

$w = a,\ |w| = 1$

$i = 0$: $K_0 = \{ S \}$

$i = 1$: $K_1 = \{ S, [ACaB] \}$

$i = 2$: $K_2 = \{ S, [ACaB] \}$

$\boxed{a \notin L(G)}$

Decidability of context-sensitive languages

1) $S \rightarrow [ACaB]$
2) $[Ca]a \rightarrow aa[Ca]$
$[CaB] \rightarrow a[aCB]$
$[CaB] \rightarrow a[aCB]$
$[ACa]a \rightarrow [Aa]a[Ca]$
$[ACa]a \rightarrow [Aa]a[Ca]$
$[ACa][aB] \rightarrow [Aa]a[CaB]$
$[ACaB] \rightarrow [Aa][aCB]$
3) $[aCB] \rightarrow [aDB]$

4) $[aCB] \rightarrow [aE]$
5) $a[Da] \rightarrow [Da]a$
$[aDB] \rightarrow [DaB]$
$a[DaB] \rightarrow [Da][aB]$
$[Aa][Da] \rightarrow [ADa]a$
$[Aa][DaB] \rightarrow [ADa][aB]$
6) $[ADa] \rightarrow [ACa]$

7) $[aE] \rightarrow [Ea]$
$a[Ea] \rightarrow [Ea]a$
$[Aa][Ea] \rightarrow [AEa]a$
8) $[AEa] \rightarrow a$

$w = aa$, $|w| = 2$

$i = 0$: $K_0 = \{ S \}$

$i = 1$: $K_1 = \{ S, [ACaB] \}$

$i = 2$: $K_2 = \{ S, [ACaB], [Aa][aCB] \}$

$i = 3$: $K_3 = \{ S, [ACaB], [Aa][aCB], [Aa][aDB], [Aa][aE] \}$

Decidability of context-sensitive languages

1) $S \rightarrow [ACaB]$
2) $[Ca]a \rightarrow aa[Ca]$
   $[CaB] \rightarrow a[aCB]$
   $[CaB] \rightarrow a[aCB]$
   $[ACa]a \rightarrow [Aa]a[Ca]$
   $[ACa]a \rightarrow [Aa]a[Ca]$
   $[ACa][aB] \rightarrow [Aa]a[CaB]$
   $[ACaB] \rightarrow [Aa][aCB]$
3) $[aCB] \rightarrow [aDB]$
4) $[aCB] \rightarrow [aE]$
5) $a[Da] \rightarrow [Da]a$
   $[aDB] \rightarrow [DaB]$
   $a[DaB] \rightarrow [Da][aB]$
   $[Aa][Da] \rightarrow [ADa]a$
   $[Aa][DaB] \rightarrow [ADa][aB]$
6) $[ADa] \rightarrow [ACa]$
7) $[aE] \rightarrow [Ea]$
   $a[Ea] \rightarrow [Ea]a$
   $[Aa][Ea] \rightarrow [AEa]a$
8) $[AEa] \rightarrow a$

$i = 4$: $K_4 = \{ S, [ACaB], [Aa][aCB], [Aa][aDB], [Aa][aE],$
$[Aa][DaB], [Aa][Ea] \}$

$i = 5$: $K_5 = \{ S, [ACaB], [Aa][aCB], [Aa][aDB], [Aa][aE],$
$[Aa][DaB], [Aa][Ea], [ADa][aB], [AEa]a,$
$[ACa][aB] \}$

Decidability of context-sensitive languages

1) $S \to [ACaB]$
2) $[Ca]a \to aa[Ca]$
$[CaB] \to a[aCB]$
$[CaB] \to a[aCB]$
$[ACa]a \to [Aa]a[Ca]$
$[ACa]a \to [Aa]a[Ca]$
$[ACa][aB] \to [Aa]a[CaB]$
$[ACaB] \to [Aa][aCB]$
3) $[aCB] \to [aDB]$

4) $[aCB] \to [aE]$
5) $a[Da] \to [Da]a$
$[aDB] \to [DaB]$
$a[DaB] \to [Da][aB]$
$[Aa][Da] \to [ADa]a$
$[Aa][DaB] \to [ADa][aB]$
6) $[ADa] \to [ACa]$

7) $[aE] \to [Ea]$
$a[Ea] \to [Ea]a$
$[Aa][Ea] \to [AEa]a$
8) $[AEa] \to a$

$i = 6$: $K_6 = \{S, [ACaB], [Aa][aCB], [Aa][aDB], [Aa][aE],$
$[Aa][DaB], [Aa][Ea], [ADa][aB], [AEa]a,$
$[ACa][aB], aa\}$

$aa \in L(G)$

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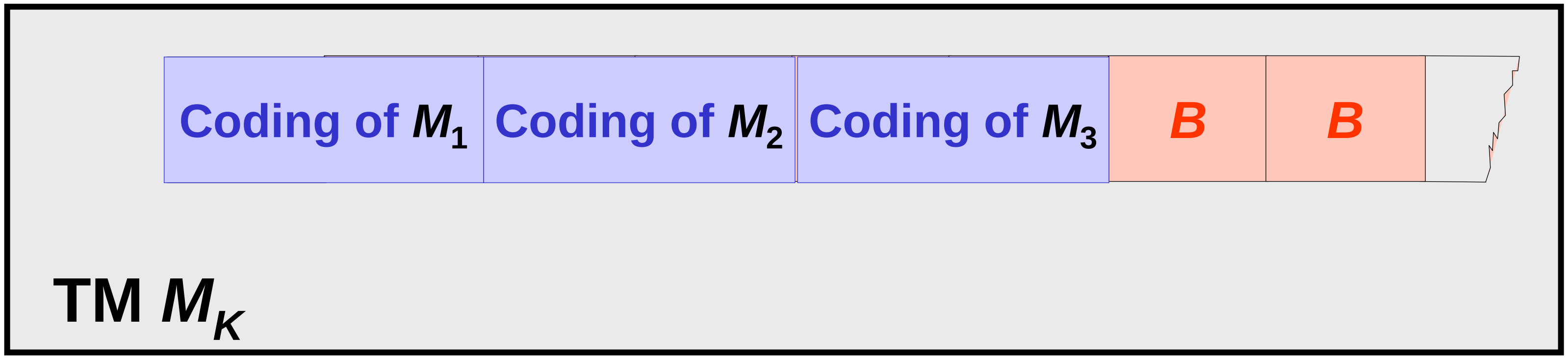
Recursive language that is not context-sensitive

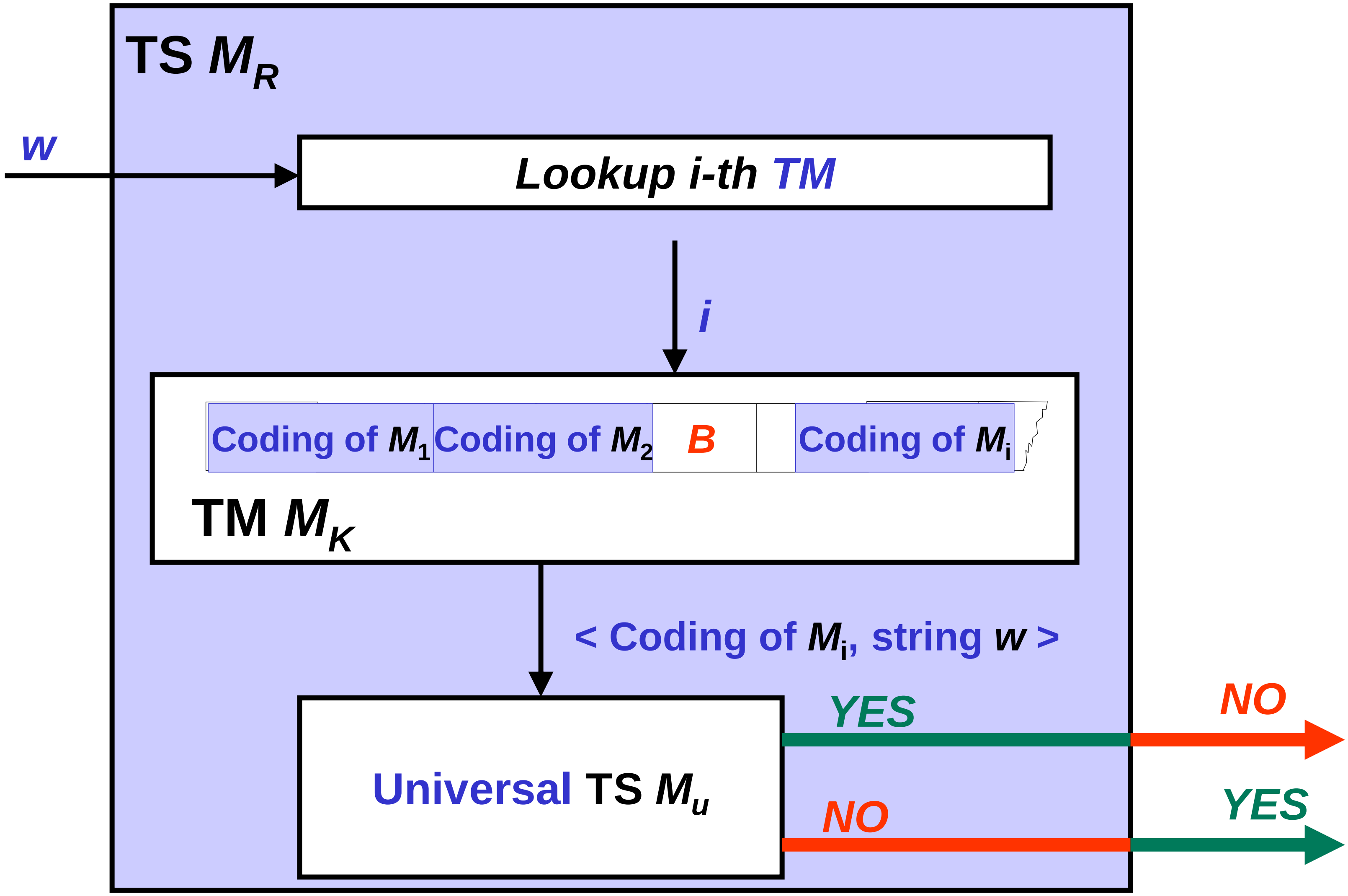
$\{ L_1 , \quad L_2 , \quad L_3 , \ldots \} = K \subset RECL$



- We define language L_R as follows:
  - $w \in L_R$ **if and only if** TM M_i **rejects** w
    - — i is the integer whose binary representation is w
- We will show that language L_R is both:
  - Different from languages or all TMs M_1, M_2, M_3, ...
  - A recursive language.
- *In conclusion,* $L_R \in RECL$ *i* $L_R \notin K \Rightarrow K \subset RECL$

Recursive language that is not context-sensitive

- Language L_R *is recursive* - $L_R \in RECL$



Recursive language that is not context-sensitive

- Language L_R is different from languages of $M_1, M_2, \ldots$
  - Assume language of TM M_j is L_R
  - Let x be the binary representation of j
  - Assumption that M_j accepts L_R leads to contradiction:
    - If $x \in L_R \Rightarrow x \notin L(M_j)$
    - If $x \notin L_R \Rightarrow x \in L(M_j)$
- Therefore, L_R is recursive and different from every $L(M_j)$!

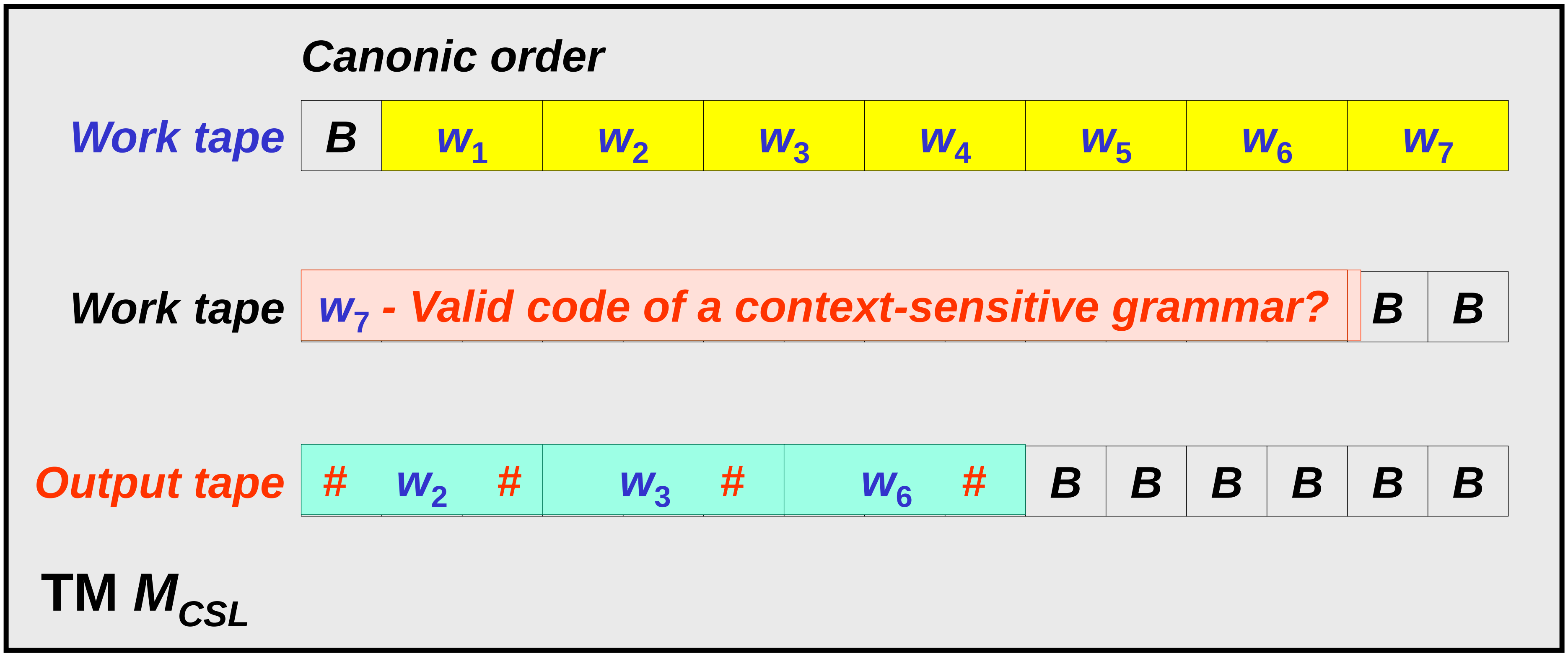
Recursive language that is not context-sensitive

- **Coding grammars using binary code**

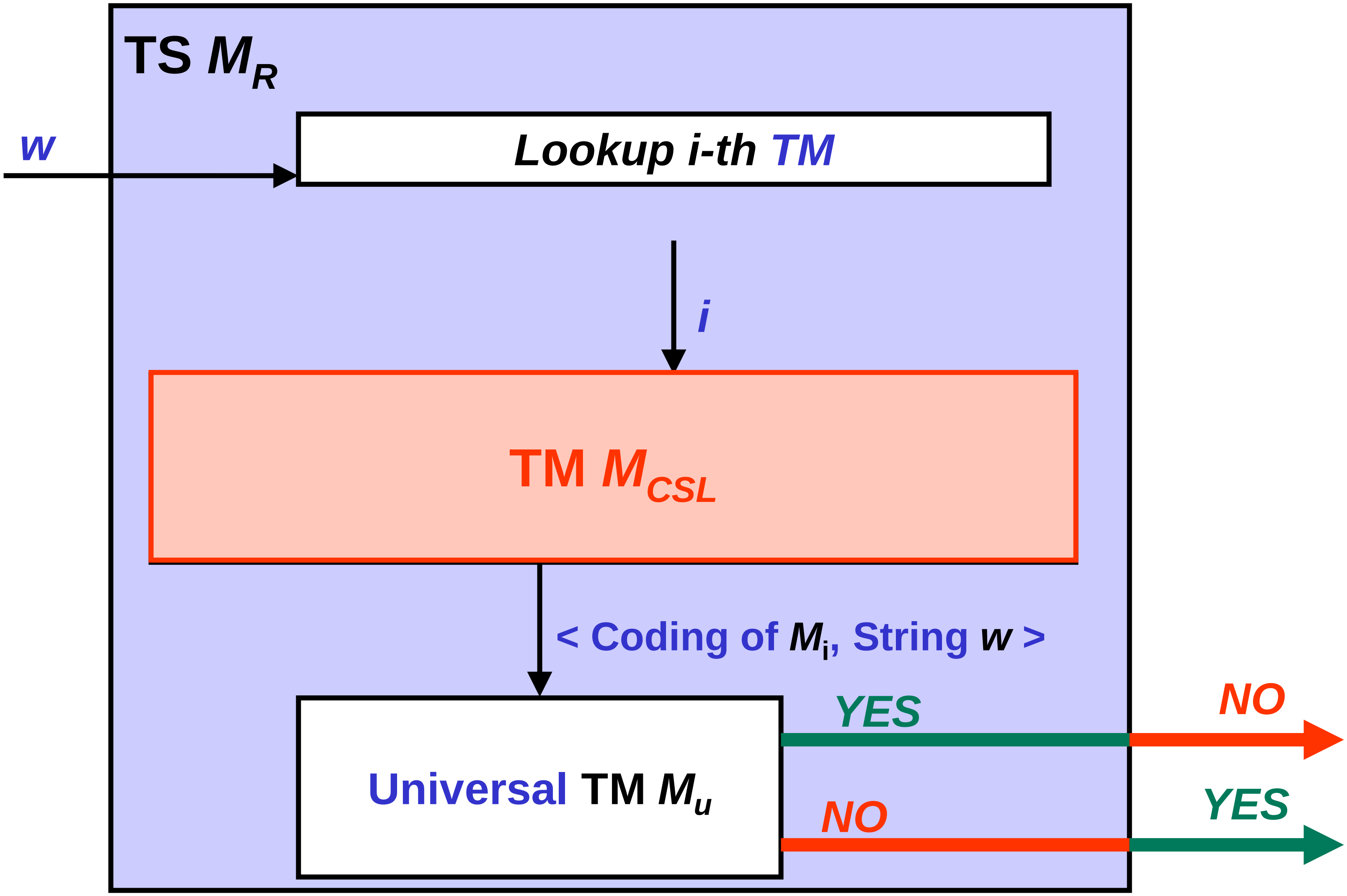
Grammar symbol	*Code*
Terminal 0	10
Terminal 1	100
,	1000
→	10000
{	100000
}	1000000
(	10000000
)	100000000
Nonterminal A_1	10^9
---	---
Nonterminal A_i	10^{i+8}
---	---

Recursive language that is not context-sensitive

- TM M_{CSL}
  - Generates and outputs all valid codes of context-sensitive grammars.



Recursive language that is not context-sensitive



Lecture 16

Chomsky hierarchy

All languages over alphabet: 2^{Σ^*}

Diagonal language $L_d \in 2^{\Sigma^*}$ i $L_d \notin REL$

Recursively enumerable languages: ***REL***

Universal language $L_u \in REL$ i $L_u \notin RECL$

Recursive languages: ***RECL***

Language $L_P \in RECL$ i $L_P \notin CSL$

Context-sensitive languages: ***CSL***

Language L_1: $\{ ww \mid w \in (0+1)^* \text{ i } |w|>1 \}$

$L_1 \in CSL$ i $L_1 \notin NCFL$

Nondeterministic context-free languages: ***NCFL***

Language L_2: $\{ ww^R \mid w \in (0+1)^* \text{ i } |w|>1 \}$

$L_2 \in NCFL$ i $L_2 \notin DCFL$

Deterministic context-free languages: ***DCFL***

Language L_3: $\{ w2w^R \mid w \in (0+1)^* \text{ i } |w|>1 \}$

$L_3 \in DCFL$ i $L_3 \notin REG$

Regular languages: ***RL***

Lecture 16

Hierarchy of grammars and automata

Unrestricted grammar
G_0: $\alpha \rightarrow \beta$

Context-sensitive grammar G_1:
$\alpha \rightarrow \beta, |\alpha| \leq |\beta|$

Context-free grammar G_2:
$A \rightarrow \alpha$

Regular grammar G_3:
$A \rightarrow wB$ i $A \rightarrow w$
$A \rightarrow Bw$ i $A \rightarrow w$

Turing machine:
$M_0 = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$

Linear bounded automaton:
$M_1 = (Q, \Sigma, \Gamma, \delta, q_0, ¢, \$, F)$

Pushdown automaton:
$M_2 = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

Finite automaton:
$M_3 = (Q, \Sigma, \delta, q_0, F)$

Recursively enumerable languages: $L_0=L(G_0)=L(M_0)$

Context-sensitive languages: $L_1=L(G_1)=L(M_1)$

Context-free languages: $L_2=L(G_2)=L(M_2)$

Regular languages: $L_3=L(G_3)=L(M_3)$

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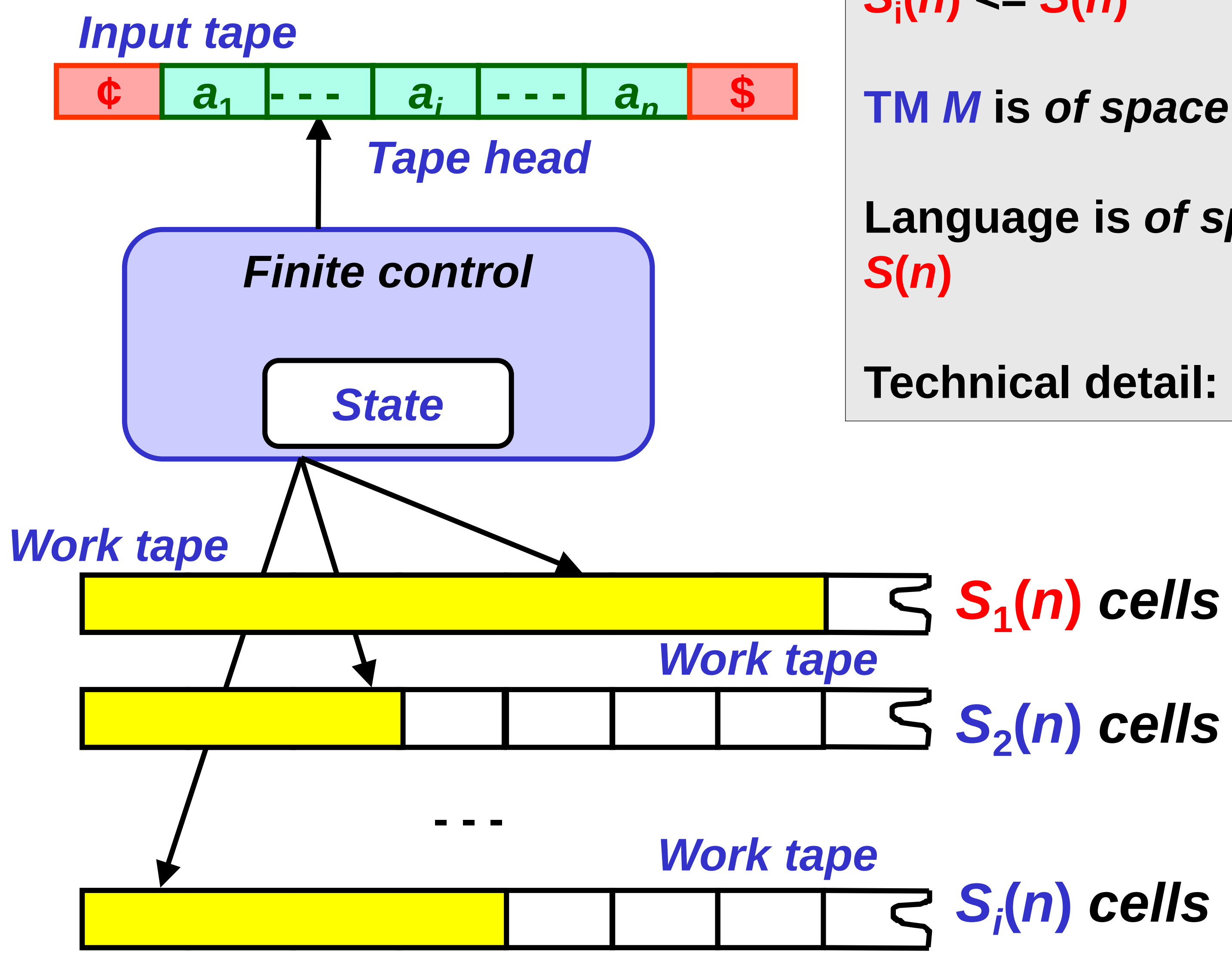
Space complexity



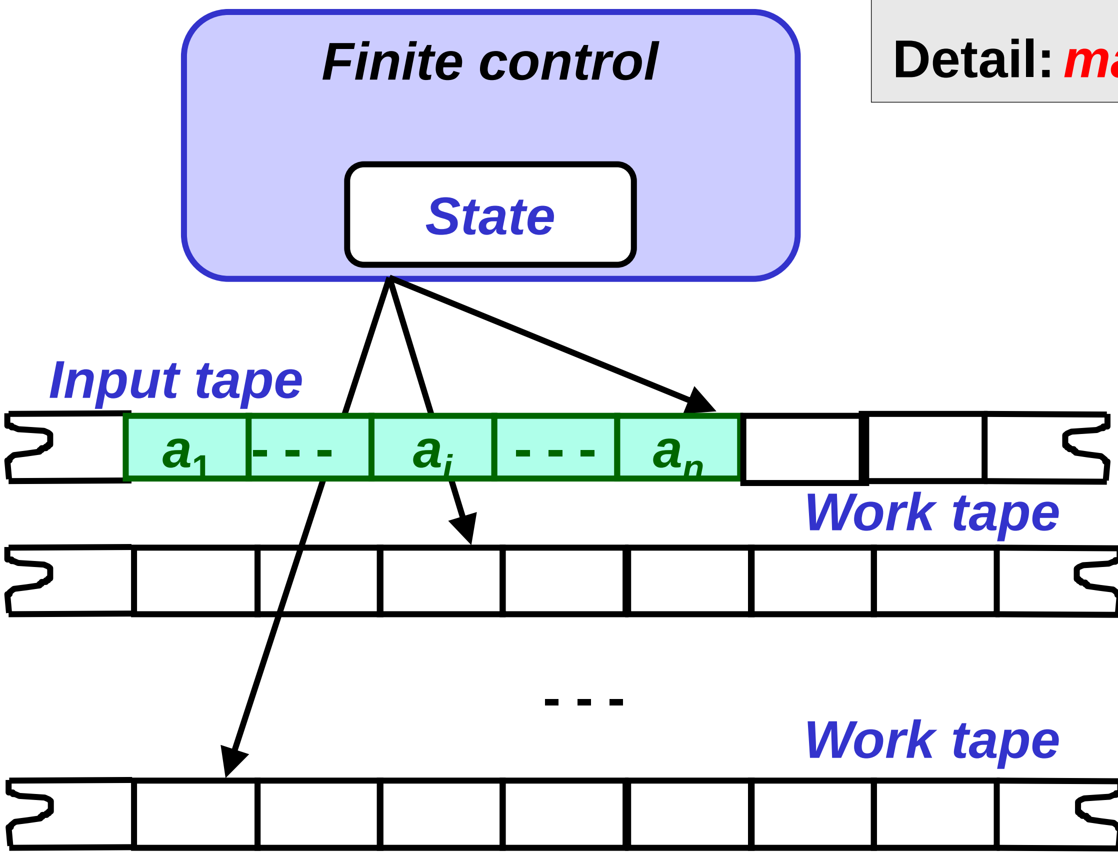
$S_i(n)$ <= $S(n)$

TM M is *of space complexity* $S(n)$

Language is *of space complexity* $S(n)$

Technical detail: $max(1, \lceil S(n) \rceil)$

Time complexity



TS *M* is *of time complexity* $T(n)$

Language is *of time complexty* $T(n)$

Detail: $max(n+1, \lceil T(n) \rceil)$

Example

$L = \{ wcw^R \mid w \in (a+b)^* \}$

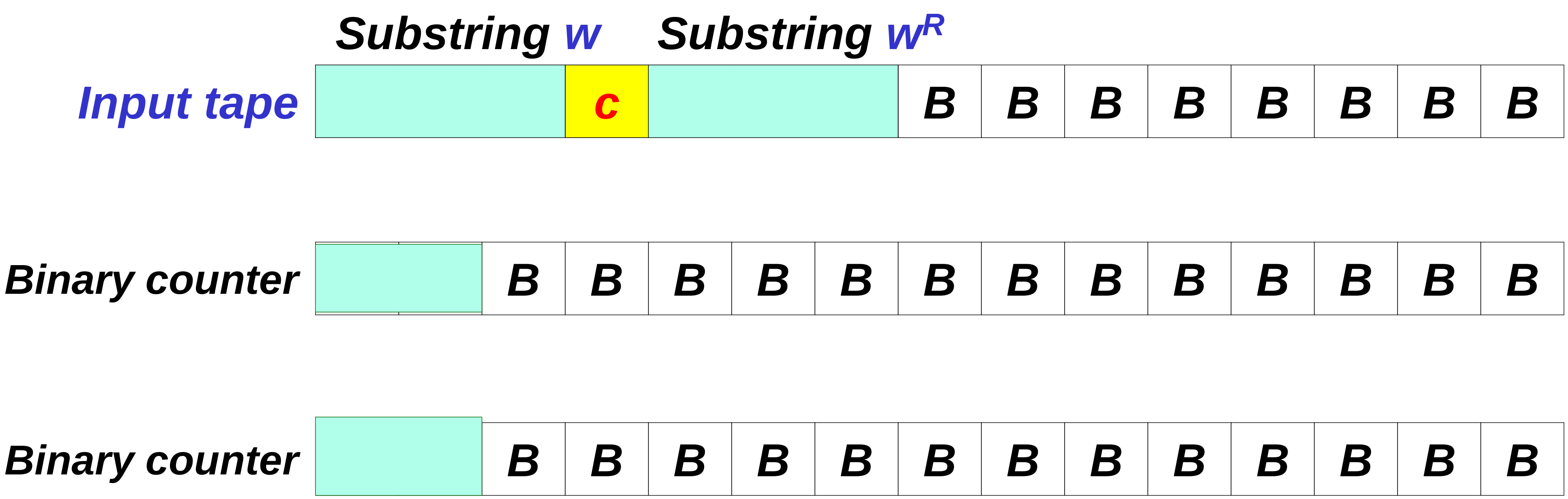
	Substring w				Substring w^R										
Input tape	a_1	a_2	a_3	c	a_3	a_2	a_1	B	B	B	B	B	B	B	B
Work tape	a_1	a_2	a_3	B	B	B	B	B	B	B	B	B	B	B	B

Time complexity

$T(n) = n + 1$

Example

$L = \{\, wcw^R \mid w \in (a+b)^* \,\}$



Example

$L = \{ wcw^R \mid w \in (a+b)^* \}$

Substring w Substring w^R

Input tape	a_1	a_2	a_3	c	a_3	a_2	a_1	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*

Binary counter				*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*

Binary counter				*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*	*B*

Space complexity

$S(n) = \log_2 n$

Lecture 16

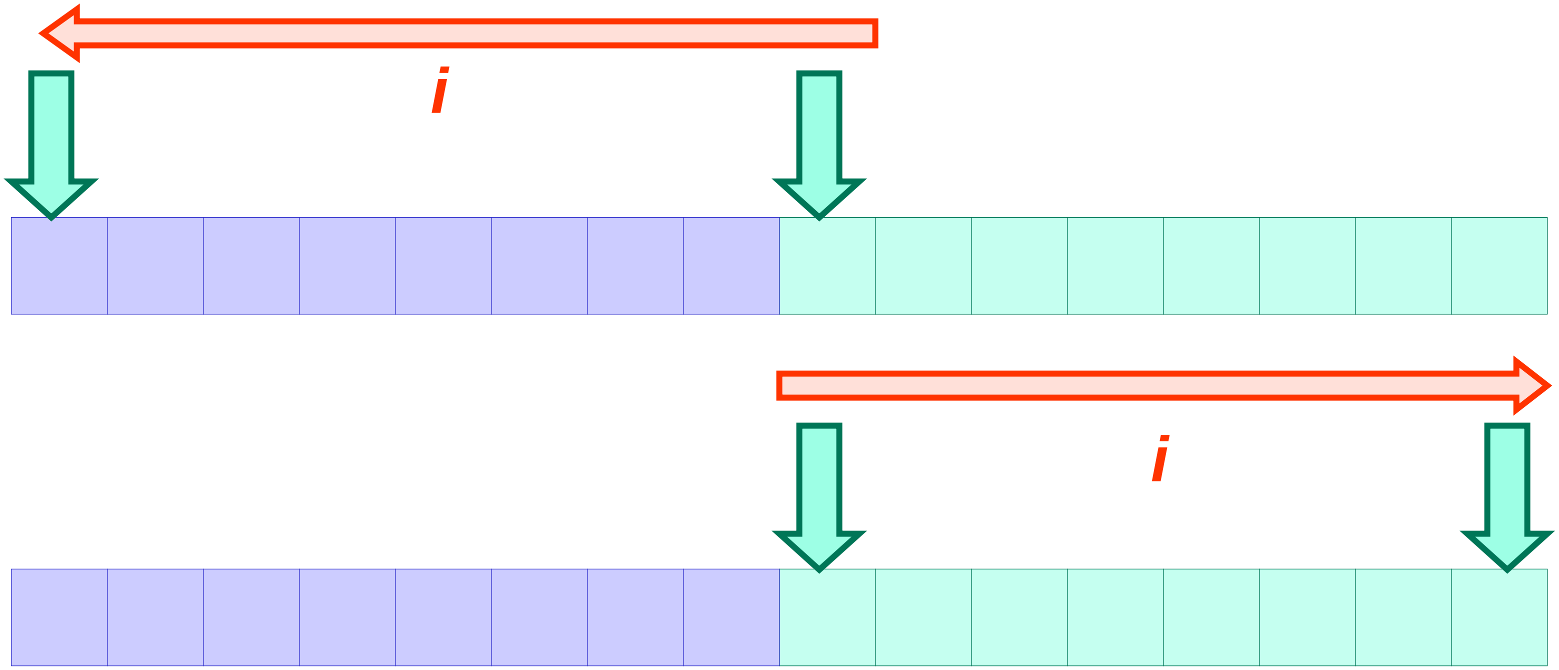
Number of tapes and space complexity

Simulating three tape TM M_1 using six tracks of one tape of TS M_2

Head position 1		X	- - -		- - -	
Contents of tape 1	A_1	A_2	- - -	A_i	- - -	A_m
Head position 2			- - -	X	- - -	
Contents of tape 2	B_1	B_2	- - -	B_i	- - -	B_m
Head position 3	X		- - -		- - -	
Contents of tape 3	C_1	C_2	- - -	C_i	- - -	C_m

[State TS M_1, Counter, Contents of tape 1, ..., Contents of tape k]

Number of tapes and time complexity



Initial step

Distance: 0

***i*-th step**

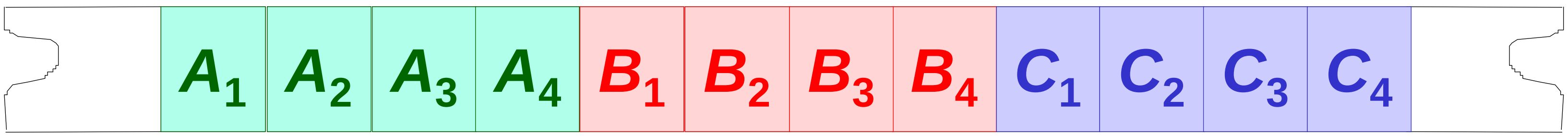
Distance: $2i$

Simulating one transition: at least $4i$ moves

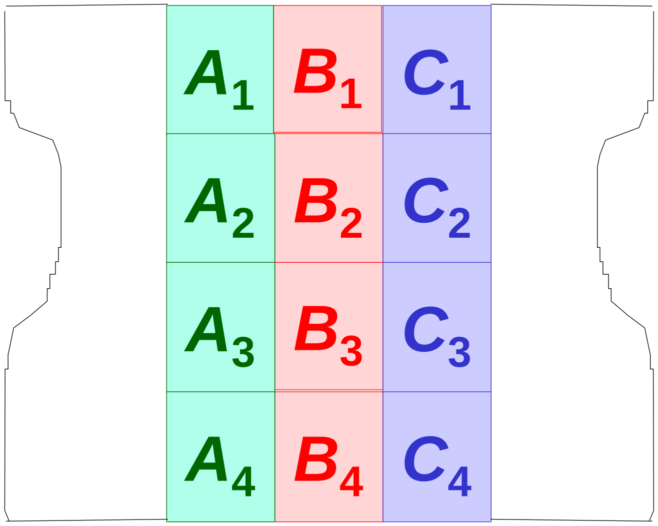
***m* total moves** $$\sum_{i=1}^{m} 4i \approx 2m^2$$

Tape compression

Tape of TM M_1

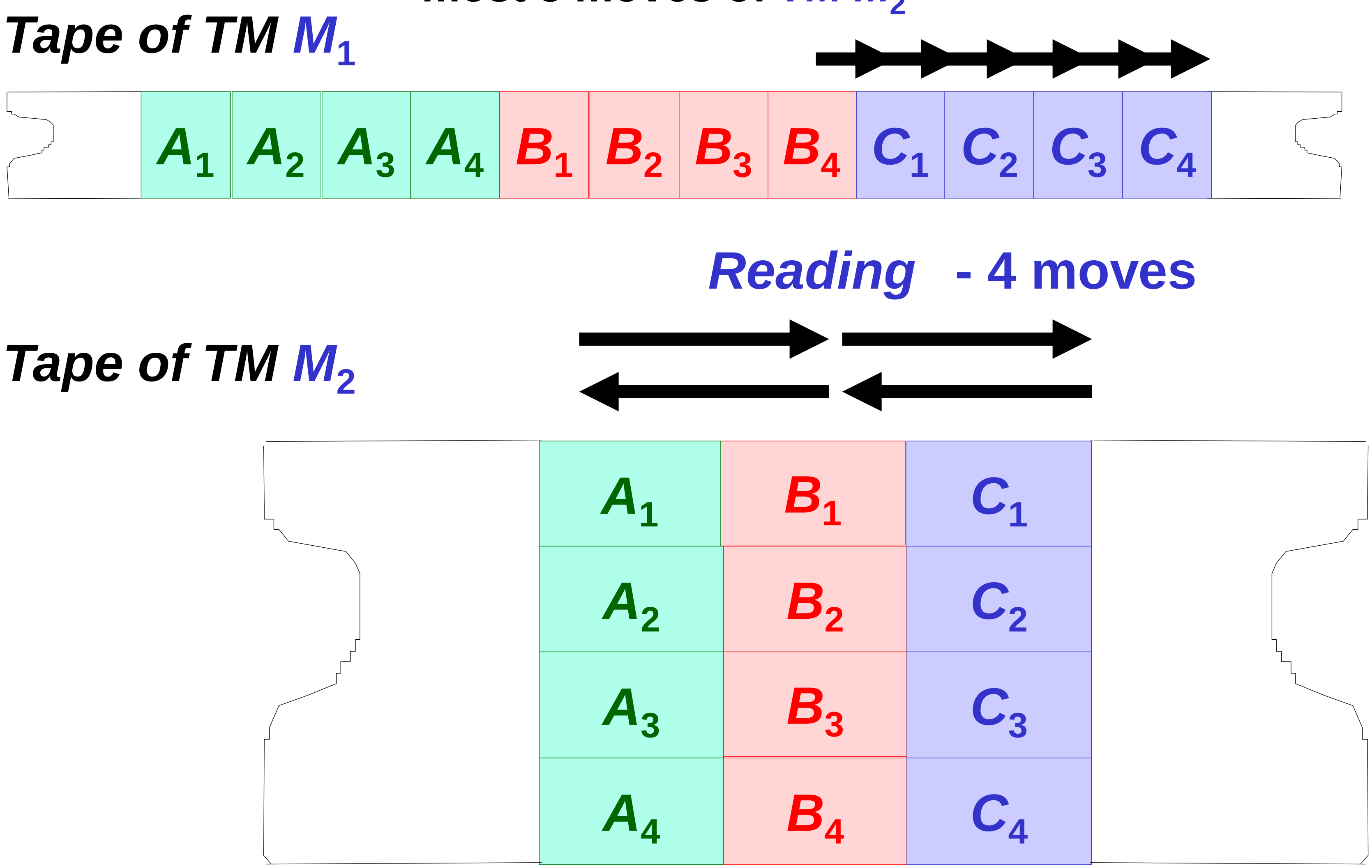


Tape of TM M_2



Linear speed-up

Simulating *m moves of TM M_1* using at most 8 moves of *TM M_2*



Linear speed-up

Simulating *m* *moves of TM M_1* using at most 8 moves of *TM M_2*