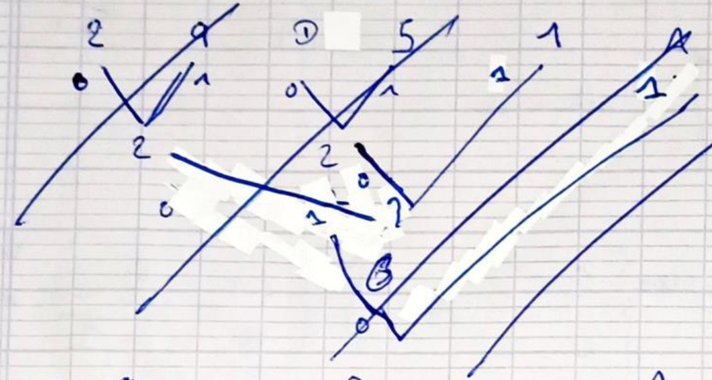
Multimedia systems: 1st Homework

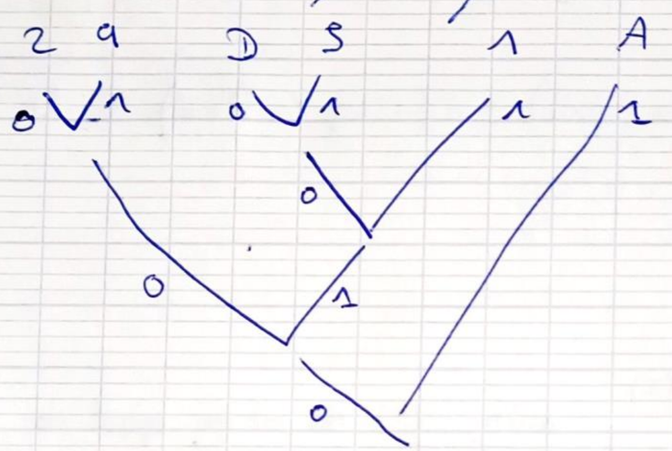
1. $N_{(16)} = 9A15AD2A1AA$

A	1	5	D	9	2
5	2	1	1	1	1



~~A : 1~~
~~1 : 011~~
~~5 : 0101~~
~~D : 0100~~
~~9 : 0001~~
~~2 : 0000~~

a)



A : 1
1 : 011
5 : 0101
D : 0100
9 : 001
2 : 000

b) $H(1) = \sum p_i \log_2\left(\frac{1}{p_i}\right) = \frac{1}{11} \times 4 \times \log_2(11) + \frac{2}{11} \times 1 \times \log_2\left(\frac{11}{2}\right)$
$+ \frac{5}{11} \times 1 \times \log_2\left(\frac{11}{5}\right)$
$= 2.222$

Average code length $= \frac{5}{11} \times 1 + \frac{3}{11} \times 4 + \frac{4}{11} \times 2 = 2,27$

c) $\frac{2,27}{2.22} = 1,02$ The ratio of average length and entropy is equal to 1.02.

d) $\frac{4}{2.27} \approx 1.76$ The compression ratio is equal to ~ 1.76.

2.

$$f_x(x) = \begin{cases} 2x^{-2} & x \in [2/3, 1] \\ 0 & \text{otherwise} \end{cases}$$

a. $$\Phi_x(x) = \begin{cases} \int 2x^{-2} dx = 2 \dfrac{x^{-2+1}}{-2+1} = 2 \dfrac{x^{-1}}{-1} = -2x^{-1} & x \in [2/3, 1] \\ 0 & \text{otherwise} \end{cases}$$

because $\frac{x}{0}$ do not exist and $x^{-1} = 1/x$ and $\int \frac{1}{x} dx = \ln x$

b. $$E(X) = \int x \times f(x) dx = \begin{cases} 2x^{-1} \\ 0 \end{cases} = \int \begin{cases} 2x^{-1} dx \\ 0 \end{cases} = \begin{cases} 2 \ln x & x \in [2/3, 1] \\ 0 \end{cases}$$

$\log a^b$
$= b \log a$

$$h(x) = E(-\log(f(x))) = E(+2\log(x) - \log(2)) = 2 \ln(2 \log(x) - \log(2))$$

$\log a \cdot b$
$= \log a + \log b$

c. $\Delta = 2^{h(x) - H(I)}$ with $H(I)$ the entropy equal to 5.

$$\Delta = 2^{2\ln(2\log(x) - \log(2)) - 5}$$

d. $$D = \frac{\Delta^2}{12} = \frac{\left(2^{2\ln(2\log(x) - \log(2)) - 5}\right)^2}{12}$$

and $\sigma_e^2 = E((e - E(e))^2)$

$$= 2\ln\left(\left(\frac{(2^{2\ln(2\log(x) - \log(2)) - 5})^2}{12}\right)^2 - 2\ln\left(\frac{(2^{2\ln(2\log(x) - \log(2)) - 5})^2}{12}\right)\right)$$

3. Time limited discrete signal $x[n]$ is given:

$[0]=1 \quad x[1]=0 \quad x[2]=2 \quad x[3]=-1 \quad x[4]=2$

a) autocorrelation $x[n]$ for shifts $j=0,1,2$

$$R(j)=\sum_{m=0}^{N-1-j} x[m].\,x[m+j]$$

$$R(0)=\sum_{m=0}^{5-1-0} = \sum_{m=0}^{4} [m]\times[m] = 1\times1+0\times0+2\times2+(-1)\times(-1)+2\times2$$
$$= 1+0+4+1+4=10$$

$$R(1)=\sum_{m=0}^{5-1-1} = \sum_{m=0}^{3} [m]\times[m+1] = 1\times0+0\times2+2\times(-1)+(-1)\times2$$
$$=0+0+(-2)+(-2) = -4$$

$$R(2)=\sum_{m=0}^{5-1-2} = \sum_{m=0}^{2} [m]\times[m+2] = 1\times2+0\times(-1)+2\times2$$
$$=2+0+4=6$$

b) $$\begin{pmatrix}10 & -4\\ -4 & 10\end{pmatrix}\times\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}=\begin{pmatrix}-4\\ 6\end{pmatrix}$$ with and $p=2$ $$\begin{bmatrix}R(0) & R(1) & \dots & R(p-1)\\ R(1) & \dots & \dots & R(p-2)\\ \vdots & \dots & \dots & \vdots\\ R(p-1) & R(p-2) & \dots & R(0)\end{bmatrix}\times\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_p\end{bmatrix}$$

$$\begin{cases}10\alpha_1-4\alpha_2=-4\\ -4\alpha_1+10\alpha_2=6\end{cases} \qquad \begin{cases}10\alpha_1-4\alpha_2=-4\\ -10\alpha_1+25\alpha_2=15\end{cases}$$

$$\begin{cases}10\alpha_1-4\alpha_2=10 \Leftrightarrow 21\alpha_2=11\\ \alpha_2=\frac{11}{21}\end{cases} \qquad \begin{cases}\alpha_1=-\frac{4}{21}\\ \alpha_2=\frac{11}{21}\end{cases}$$

$10\alpha_1-\frac{44}{21}=-4$

$\frac{84-44}{21}=-10\alpha_1$

$\frac{40}{21}=-10\alpha_1$

$-\frac{4}{21}=\alpha_1$

$P(2)=-\frac{4}{21}\times0+\frac{11}{21}\times1=\frac{11}{21}$

$$P(z)=\sum_{k=1}^{p}\alpha_k z^{-k} = \alpha_1\cdot z^{-1}+\alpha_2\cdot z^{-2}$$

$P(3)=-\frac{4}{21}\times2+\frac{11}{21}\times0=-\frac{8}{21}$

c) $$e[n]=x[n]-\sum_{k=1}^{p}\alpha_k\, x[n-k]$$

$P(2)=\frac{11}{21}$

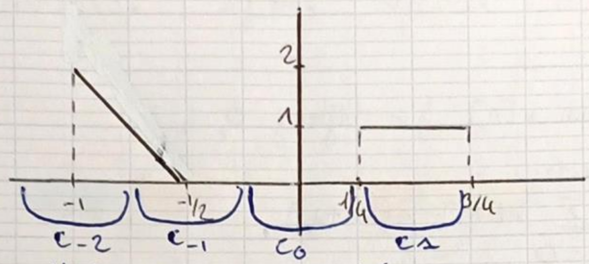
$P(3)=-\frac{8}{21}$

$$e[2]=2-\sum_{k=1}^{2}\alpha_k\, x[2-k] = 2-(-\tfrac{4}{21}\times \underset{=0}{x[1]}+\tfrac{11}{21}\times \underset{=1}{x[0]})$$
$$=2-\frac{11}{21}$$
$$=\frac{31}{21}$$

$$e[3]=-1-\sum_{k=1}^{2}\alpha_k\, x[3-k] = -1-(-\tfrac{4}{21}\times2+\tfrac{11}{21}\times0)$$
$$=-1+\frac{8}{21} = -\frac{13}{21}$$

4. $E(x) = -1/6 \qquad \hat{x} = \text{round}(2x)/2$

a.



We know that $\hat{x} = \text{round}(2x)/2$

so $\Delta = 1/2$

Our function is defined on $[-1, -1/2] \cup [1/4, 3/4]$

$C_i = [\Delta \cdot i - \Delta/2 \,,\, \Delta \cdot i + \Delta/2]$

$C_0 = [1/2 \cdot 0 - 1/4 \,,\, 1/2 \cdot i + 1/4] = [-1/4 \,,\, 1/4]$

So we get 3 defined symbols C_{-2} ; C_{-1} and C_1

$P(1) = 1/2 \qquad P(-1) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} \qquad P(-2) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

$P(C_0) = 0 \qquad P(C_1) = 1/2 \qquad P(C_{-1}) = 1/8 \qquad P(C_2) = 3/8$

b. $H(I) = -\sum p_i \log_2 p_i = -\left[\frac{1}{2}\log_2\left(\frac{1}{2}\right) + \frac{1}{8}\log_2\left(\frac{1}{8}\right) + \frac{3}{8}\log_2\left(\frac{3}{8}\right)\right]$

$= 1.72$ bit

c. $D = \sum_i D_i = \sum_i \int_{x \in C_i} f_x(x) \cdot (x - x_{q_i})^2 dx$ but also $D = E[(x - x_q)^2]$

and we know that $E(x) = -1/6$, so $D = -1/6$

$f_x = \begin{cases} -4x & x \in [-1, -1/2] \\ x & x \in [1/4, 3/4] \\ 0 & \text{otherwise} \end{cases}$

d. $\sigma_x^2 = \int (x - E(x))^2 f_x dx \qquad \sigma_x^2 = \int \left(x + \frac{1}{6}\right)^2 f_x \, dx$

$\int \begin{cases} (x + 1/6)^2 \times (-4x) \\ (x + 1/6)^2 \times (x) \\ 0 \text{ otherwise} \end{cases}$

d) $\int \begin{cases} (x^2 + 2/6\, x + 1/36) \times -4x \, dx \\ (x^2 + 2/6\, x + 1/36) \times x \, dx \\ 0 \text{ otherwise} \end{cases} = \int \begin{cases} -4x^3 - 8/6\, x^2 - 4/36\, x \, dx \\ x^2 + 2/6\, x + 1/36 \, dx \\ 0 \text{ otherwise} \end{cases}$

$= \frac{-4x^4}{4} - \frac{8}{6}\frac{x^3}{3} - \frac{4}{36} \cdot \frac{x^2}{2}$

$= -x^4 - \frac{8x^3}{18} - \frac{4x^2}{72} = -x^4 - \frac{8x^3}{18} - \frac{1x^2}{18}$

$\frac{x^3}{3} + \frac{2}{6} \times \frac{x^2}{2} + \frac{1}{36} \times \frac{x}{1}$

$= \frac{x^3}{3} + \frac{x^2}{6} + \frac{x}{36}$

$\begin{cases} \left[-x^4 - \frac{8x^3}{18} - \frac{x^2}{18}\right]_{-1}^{-1/2} & x \in [-1, -1/2] \\ \left[\frac{x^3}{3} + \frac{x^2}{6} + \frac{x}{36}\right]_{1/4}^{3/4} & x \in [1/4, 3/4] \\ 0 \text{ otherwise} \end{cases} =$

$SQNR = 10 \log_{10}\left(\frac{\sigma^2_x}{D}\right) = 10 \log_{10}\left(\frac{0,406}{-1/6}\right)$

$\begin{cases} 0,570 \\ 0,232 \end{cases} = 0,570 + 0,232 = 0,806$

But since D is < 0, we can not compute SQNR.

c) $D = E[(x - x_q)^2] = -1/6$

$D(-2) = \int_{-1}^{-3/4} -4x(x+1)\, dx = \int_{-1}^{-3/4} -4x(x^2 + 2x + 1)\, dx$

$= \left[-\frac{4x^4}{4} - \frac{8x^3}{3} - \frac{4x^2}{2}\right]_{-1}^{-3/4}$

$= \left[-x^4 - \frac{8x^3}{3} - 2x^2\right]_{-1}^{-3/4} = 0,017$

$D(0) = 0$

Same as the others with

$D(-1) = 0,014$

$D(1) = -0.208$

$\sum D_i = -0,208 + 0,014 + 0,017 \approx -1/6$