Mathematical Foundations of Computing

Predicate Calculus : 1

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Predicate Logic

- *Predicate logic* is an extension of propositional logic that permits concisely reasoning about whole *classes* of entities.

  E.g., “x>1”, “x+y=10”

- Such statements are neither true or false when the values of the variables are not specified.

The Problem

The problem is that the validity of this argument comes from the internal structure of these sentences--which Propositional Logic cannot "see":

1. All men are mortal
2. Socrates is a man

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3. Therefore, Socrates is mortal

Singular statements

Make assertions about persons, places, things or times

Examples:

- Socrates is a man.
- Athens is in Greece.
- Thomas Aquinas preferred Aristotle to Plato

To display the internal structure of such sentences we need two new categories of vocabulary items:

- Individual constants (“names”)
- Predicates

Individual Constants & Predicates

Socrates is a man.

The name of a person—which we'll express by an individual constant

A predicate—assigns a property (being-a-man) to the person named

More Vocabulary

- Add to the vocabulary of Propositional Logic :
- **Individual Constants: tea, Magnesium,**
- **Variables : x,y,z,a,b,c**
- **Predicates: Propositional functions**
- **Functions :** mother-of (bill); maximum-of (7,8)
- **Quantifiers Universal, Existential**

Applications of Predicate Logic

- It is *the* formal notation for writing perfectly clear, concise, and unambiguous mathematical *definitions*, *axioms*, and *theorems* for *any* branch of mathematics.
- Supported by some of the more sophisticated *database query engines*.
- Basis for *automatic theorem provers* and many other Artificial Intelligence systems.

Subjects and Predicates

- The proposition

  "The dog is sleeping"

  has two parts:
  - "the dog" denotes the *subject* - the *object* or *entity* that the sentence is about.
  - "is sleeping" denotes the *predicate*- a property that the subject can have.

Propositional Functions

- A *predicate* is modeled as a *function* $P(\cdot)$ from objects to propositions.
  - $P(x)$ = "x is sleeping" (where x is any object).
- The *result of applying* a predicate P to an object $x=a$ is the *proposition* $P(a)$.
  - *e.g.* if $P(x)$ = "$x > 1$",
    then $P(3)$ is the *proposition* "3 is greater than 1."
- <u>Note:</u> The predicate P **itself** (*e.g.* P="is sleeping") is **not** a proposition (not a complete sentence).

Propositional Functions

- Predicate logic includes propositional functions of **any** number of arguments.

  e.g. let $P(x,y,z)$ = "x gave y the grade z",

  x="Mike", y="Mary", z="A",

  $P(x,y,z)$ = "Mike gave Mary the grade A."

Universe of Discourse(uod)

- The collection of values that a variable x can take is called x's *universe of discourse*.

  e.g., let $P(x)$="$x+1>x$".

  we could define the course of universe as the set of integers.

Quantifier Expressions

- *Quantifiers* allow us to *quantify* (count) *how many* objects in the universe of discourse satisfy a given predicate:
  - "∀" is the FOR ∀LL or *universal* quantifier.
    $\forall x\, P(x)$ means *for all* x in the u.d., P holds.

  - "∃" is the ∃XISTS or *existential* quantifier.
    $\exists x\, P(x)$ means there *exists* an x in the uod. (that is, one or more) such that $P(x)$ is true.

Universal Quantifier $\forall$: Example

- Let $P(x)$ be the *predicate* "x is full."
- Let the uod of x be <u>parking spaces at RSP</u>.
- The *universal quantification of* $P(x)$,

  $\forall x\, P(x)$, is the *proposition:*
  - "All parking spaces at RSP are full." or
  - "Every parking space at RSP is full." or
  - "For each parking space at RSP, that space is full."

The Universal Quantifier $\forall$

- To prove that a statement of the form $\forall x\, P(x)$ is false, it suffices to find a **counterexample** (i.e., one value of x in the universe of discourse such that $P(x)$ is false)
  - e.g., $P(x)$ is the predicate "$x>0$"

Existential Quantifier ∃ Example

- Let $P(x)$ be the *predicate* "x is full."
- Let the uod. of x be <u>parking spaces at RSP</u>.
- The *universal quantification of* $P(x)$,

  $\exists x\, P(x)$, is the *proposition*:
  - "Some parking space at RSP is full." or
  - "There is a parking space at RSP that is full." or
  - "At least one parking space at RSP is full."

Quantifiers

- Definitions of quantifiers: If uod=a,b,c,…
  $\forall x\ P(x) \Leftrightarrow P(\mathrm{a}) \wedge P(\mathrm{b}) \wedge P(\mathrm{c}) \wedge \ldots$
  $\exists x\ P(x) \Leftrightarrow P(\mathrm{a}) \vee P(\mathrm{b}) \vee P(\mathrm{c}) \vee \ldots$
- Every student in III MSc SS has studied calculus
- There is a student in this class who has not studied Scala programming

Scope of Quantifiers

- The part of a logical expression to which a quantifier is applied is called the scope of this quantifier.

$$\text{e.g., } (\forall x\, P(x)) \wedge (\exists y\, \mathrm{Q}(y))$$

$$\text{e.g., } (\forall x\, P(x)) \wedge (\exists x\, \mathrm{Q}(x))$$

Free and Bound Variables

- An expression like $P(x)$ is said to have a *free variable x* (meaning x is undefined).
  - *Eg.* $P(x,y)$ has 2 free variables, x and y.
- A quantifier (either $\forall$ or $\exists$) *operates* on an expression having one or more free variables, and *binds* one or more of those variables, to produce an expression having one or more *bound variables*.
  - $\forall x\, P(x,y)$ has 1 free variable, and one bound variable.
- An expression with <u>zero</u> free variables is an actual proposition.
- An expression with one or more free variables is still only a predicate: $\forall x\, P(x,y)$

More to Know About Binding

- $\forall x\, \exists x\, P(x)$ - x is not a free variable in $\exists x\, P(x)$, therefore the $\forall x$ binding isn't used.
- $(\forall x\, P(x)) \wedge Q(x)$ - The variable x is outside of the *scope* of the $\forall x$ quantifier, and is therefore free. Not a proposition.
- $(\forall x\, P(x)) \wedge (\exists x\, Q(x))$ - Legal because there are 2 <u>different</u> x's!
- Quantifiers bind as loosely as needed: parenthesize $\forall x\, (\; P(x) \wedge Q(x)\,)$

Nested Quantifiers

Exist within the scope of other quantifiers

- Let the uod of x & y be people.
- likes(x,y)="x likes y" (a predicate with 2 f.v.'s)
- $\exists y\ person(y) \wedge person(x) \wedge likes(x,y)$ = "There is someone whom x likes." (a predicate with 1 free variable, x)

- $\forall x\ \exists y\ (person(x) \wedge person(y) \rightarrow likes(x,y))$ = "Everyone has someone whom they like." (A Proposition with 0 free variables.)

Order of Quantifiers Is Important!!

person(x) : x is a person

relies(x,y) : x relies on y

$\forall x \exists y\ (person(x) \wedge person(y) \rightarrow relies(x,y))$ =

Everyone has *someone* to rely on.

$\exists y \forall x$ = There's a poor overworked soul whom *everyone* relies upon (including himself)!

$\exists x \forall y$ = There's some needy person who relies upon *everybody* (including himself).

$\forall y \exists x$ = Everyone has *someone* who relies upon them.

$\forall x \forall y$ = *Everyone* relies upon *everybody*, (including themselves)!

Natural language is ambiguous!

- "Everybody likes somebody."
  - For everybody, there is somebody they like,
    - $\forall x\ \exists y$ person(x) ^ person(y) $\rightarrow$ *likes*(x,y)
  - or, there is somebody (a popular person) whom everyone likes?

    [Probably more likely.]
    - $\exists y\ \forall x$ person(y) ^ person(x) ^ *likes*(x,y)

Symbolize

- Every apple is red
- Every student in this class has studied calculus
- Some parts of this machine are not in good condition
- Poornima has chatted with everyone except Vimala
- Every student has asked questions to Prof John.

Some Number Theory Examples

- Let uod = the *natural numbers* 0, 1, 2, …
- "A number x is *even*, *even*(x), if and only if it is equal to 2 times some other number."
  $\forall x\ \big(\text{natnum}(x) \wedge even(x) \leftrightarrow (\exists y\ natnum(y) \wedge x=2*y)\big)$
- "A number is *prime*, *prime*(x), iff it isn't the product of two non-unity numbers."
  $\forall x\ \Big(\text{natnum}(x) \wedge \text{prime}(x) \leftrightarrow \neg\ (\exists y \exists z\ natnum(y) \wedge natnum(z) \wedge x=y*z \wedge y\neq 1 \wedge z\neq 1)\Big)$

Calculus Example

- Precisely defining the concept of a limit using quantifiers:

$$\left(\lim_{x \to a} f(x) = L\right) \Leftrightarrow$$

$$\left(\begin{array}{l} \forall \varepsilon > 0 : \exists \delta > 0 : \forall x : \\ \left(|x - a| < \delta\right) \rightarrow \left(|f(x) - L| < \varepsilon\right) \end{array}\right)$$

Negation

- Every student in the class has studied calculus
- stud(x), studied(x,calculus)

- Remember:
- $\neg (\forall x\, p(x)) \Leftrightarrow \exists x(\neg p(x))$
- $\neg (\exists x\, p(x)) \Leftrightarrow \forall x(\neg p(x))$

Examples

- There are some cities in Canada which are not clean
- Some components of this machine are either old or of good quality
- There is someone who can be fooled everytime

Function Examples

A person's mother is that person's parent.

$\forall X$ person $(X) \rightarrow$ parent(mother-of(X),X)

There are people who think this class is cool.

$\exists X$ person $(X) \wedge$ Think (X,cool)

Some computers have mouses connected on the USB.

$\exists$ X computer $(X) \wedge$ USB_conn (X, mouse_of(X))

Interpretation

- Not by truth tables
- Domain(x)={1,2}; P(x) is : p(1)=T, p(2)=F
- 1) $\forall x\, p(x)$
- 2) $\exists$x $p(x)$
- Domain(y)={1,2};   p(1,1)=T,p(1,2)=F,
- p(2,1)=F,p(2,2)=T
- q(1)=2,q(2)=1
- 3) $\forall x$ $\exists$y $p(x,y)$
- 4) $\forall x$ $\forall$y $p(x,y)$
- 5) $\forall x$ $\forall$y $p(x,q(y))$

Equivalence Rules

- G :proposition; Q : $\forall$, $\exists$; F(x) : Predicate
- 1. $(Q(x)\ F(x)) \vee G \Leftrightarrow Q(x)\ (F(x) \vee G)$
- 2. $(Q(x)\ F(x)) \wedge G \Leftrightarrow Q(x)\ (F(x) \wedge G)$
- 3. $\neg\ (\forall x\ F(x)) \Leftrightarrow\ \exists x(\neg\ F(x))$
- 4. $\neg\ (\exists x\ F(x)) \Leftrightarrow \forall x(\neg\ F(x))$
- 5. $\forall x\ F(x)\ \wedge \forall x\ H(x) \Leftrightarrow \forall x\ [F(x) \wedge H(x)]$
- 6. $\exists x\ F(x) \vee \exists x\ H(x) \Leftrightarrow \exists x\ [F(x) \vee H(x)]$
- 7. $\forall x\ F(x)\ \vee \forall x\ H(x) \Leftrightarrow \forall x\ F(x) \vee \forall z\ H(z) \Leftrightarrow \forall x\ \forall z[F(x) \vee H(z)]$
- 8. $\exists x\ F(x)\ \wedge \exists x\ H(x) \Leftrightarrow \exists x\ F(x) \wedge \exists z\ H(z) \Leftrightarrow \exists x\ \exists z\ [F(x) \wedge H(z)]$

Rules of Inference & Quantified Statements

Rule of Inference	Name
$\dfrac{\forall x\, P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x\, P(x)}$	Universal generalisation
$\dfrac{\exists x\, P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x\, P(x)}$	Existential generalisation

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all dogs and Fido is a dog.

“All dogs are cuddly.”

“Therefore, Fido is cuddly.”

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

“There is someone who got an A in the course.”
“Let’s call her Linda and say that Linda got an A”

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

“Michelle got an A in the class.”
“Therefore, someone got an A in the class.”

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$\begin{array}{l} \forall x(P(x) \to Q(x)) \\ P(a), \text{where } a \text{ is a particular} \\ \quad \text{element in the domain} \\ \hline \therefore Q(a) \end{array}$$

This rule could be used in the Socrates example.

Revisiting the Socrates Example

- We have the two premises:
  - “All men are mortal.”
  - “Socrates is a man.”
- And the conclusion:
  - “Socrates is mortal.”
- How do we get the conclusion from the premises?

The Argument

- We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\begin{array}{ll} & \forall x(Man(x) \to Mortal(x)) \\ & Man(Socrates) \\ \hline \therefore & Mortal(Socrates) \end{array}$$

Solution for Socrates Example

Valid Argument

Step	**Reason**
1. $\forall x(Man(x) \rightarrow Mortal(x))$	Premise
2. $Man(Socrates) \rightarrow Mortal(Socrates)$	UI from (1)
3. $Man(Socrates)$	Premise
4. $Mortal(Socrates)$	MP from (2) and (3)

Valid Arguments

Inference rules for propositional logic plus additional inference rules to handle variables and quantifiers.

Using Rules of Inference

Example 1: Construct a valid argument to show that

"Someone in this class can get a high paying job"

is a consequence of the premises:

"One student in this class knows how to write programs in Java. Everyone who knows how to write programs in Java can get a high paying job"

Solution: Let $S(x)$: "x is a student" ;

J(x) : " x knows to write programs in Java"

H(x) : x can get a high paying job.

Valid Argument:

Premises : $\exists x\, S(x) \wedge J(x)$

$\forall x\, [J(x) \rightarrow H(x)]$

Therefore, $\exists x\, S(x) \wedge H(x)$

$S(a) \wedge J(a)$	Existential Instantiation
$J(a) \rightarrow H(a)$	Universal Instantiation
$S(a) \wedge J(a) \wedge [J(a) \rightarrow H(a)]$	Modus Ponens
$S(a) \wedge H(a)$	Existential generalization
$\exists x\, S(x) \wedge H(x)$	

Using Rules of Inference

Example 2: Use the rules of inference to construct a valid argument showing that the conclusion

"Someone who passed the first exam has not read the book."

follows from the premises

"A student in this class has not read the book."

"Everyone in this class passed the first exam."

Solution: Let $C(x)$ denote "x is in this class," $B(x)$ denote "x has read the book," and $P(x)$ denote "x passed the first exam."

First we translate the premises and conclusion into symbolic form.

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

Continued on next slide →

Using Rules of Inference

Valid Argument:

Step	**Reason**
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \to P(x))$	Premise
5. $C(a) \to P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	EG from (8)

References

- www.cs.berkeley.edu/%7wilensky/cs188/lectures/index.html
- Kenneth H Rosen, “Discrete Mathematics and its Applications”, Tata McGraw Hill, 2011.