Relations

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Cartesian Product

- If we want to describe a relationship between elements of two sets A and B, we use a relation.
- For Relations we require a cartesian product
- **Definition:** The Cartesian product of A and B is denoted by $A \times B$ and equals $\{(a, b): a \in A \text{ and } b \in B\}$. The elements of $A \times B$ are ordered pairs. The elements of $A_1 \times A_2 \times \ldots \times A_n$ are ordered n-tuples.
- Eg.1: A= {1,2,3} B={ a, b} AxB = ?, B x B, B^3
- Eg.2 : A = Real Set B = Real Set; AxB=
- Eg.3: $A = B = \{x \in R: -1 \leq x \leq 1\}$
- Eg.4: For any set A, $A \times \phi = \phi$. Likewise, $\phi \times A = \phi$.
- $|A \times B| = |A| \times |B|$ or #(AxB) = #(A) x #(B)

Relations

- If we want to describe a relationship between elements of two sets A and B, we can use **ordered pairs** with their first element taken from A and their second element taken from B.
- Since this is a relation between **two sets**, it is called a **binary relation**.
- **Definition:** Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.
- In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and a(notR)b to denote that $(a, b) \notin R$.

Relations

- When (a, b) belongs to R, a is said to be **related** to b by R.
- **Example:** Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).
- P = {Carl, Suzanne, Peter, Carla},
- C = {Mercedes, BMW, tricycle}
- D = {(Carl, Mercedes), (Suzanne, Mercedes), (Suzanne, BMW), (Peter, tricycle)}
- This means that Carl drives a Mercedes,
- Suzanne drives a Mercedes and a BMW,
- Peter drives a tricycle, and
- Carla does not drive any of these vehicles.

Relations on a Set

- **Definition:** A relation on the set A is a relation from A to A.
- In other words, a relation on the set A is a subset of $A \times A$.
- Eg1.: $A = \{2,4,6,8\}$ $R = \{(x,y) : x \text{ divides } y \text{ and } x,y \in A\}$?

- Eg2:$A=N$; $R=\{ (x,y) : x \text{ and } y \text{ have the same remainder when divided by } y\} = ?$

- **Eg3:** Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a < b\}$?

Domain, Range, Field of a Relation

- Dom (R) = { x : (x,y) $\in$ R }
- Ran (R) = { y : (x,y) $\in$ R }
- Fld (R) = Dom (R) $\cup$ Ran (R)

- Eg1.
- R1 = {(-1,3), (2,0), (2,5), (-3,2)}
- Dom R1 = {-1, 2, -3}
- Ran R1 = { 3, 0, 5, 2}
- Fld R1 = { -3, -1, 0, 2, 3, 5}

Representation of Relations

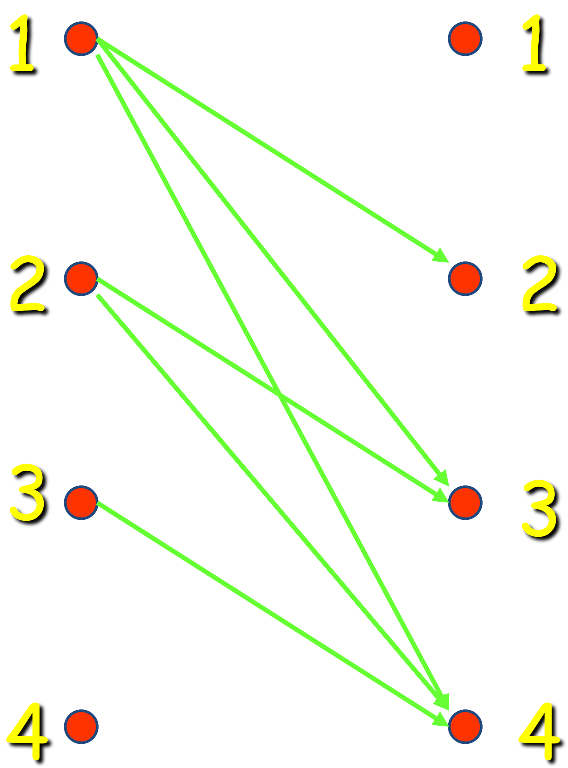
- Let A = {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a < b}
- 1) Set of Ordered Pairs:
-     R = {(1,2), (1,3), (1,4), (2,3), (2,4), (3,4) }
- 2) Arrow Diagram :
- 3) Table :
- 4) Matrix :
- 5) Directed Graph :
- 6) Algebraic Equation:

  $y = x^2$; R = {(x,y) : $y=x^2$ for all x,y ϵ $\mathbb{R}$
- 7) Verbal Sentence:

A = $\mathbb{R}$ B= $\mathbb{R}$ x ϵ A , y ϵ B, x is related to y such that R consists of all points on the parabola $y = x^2$

Relations on a Set: Ordered Pairs, Arrow Diagram, Table

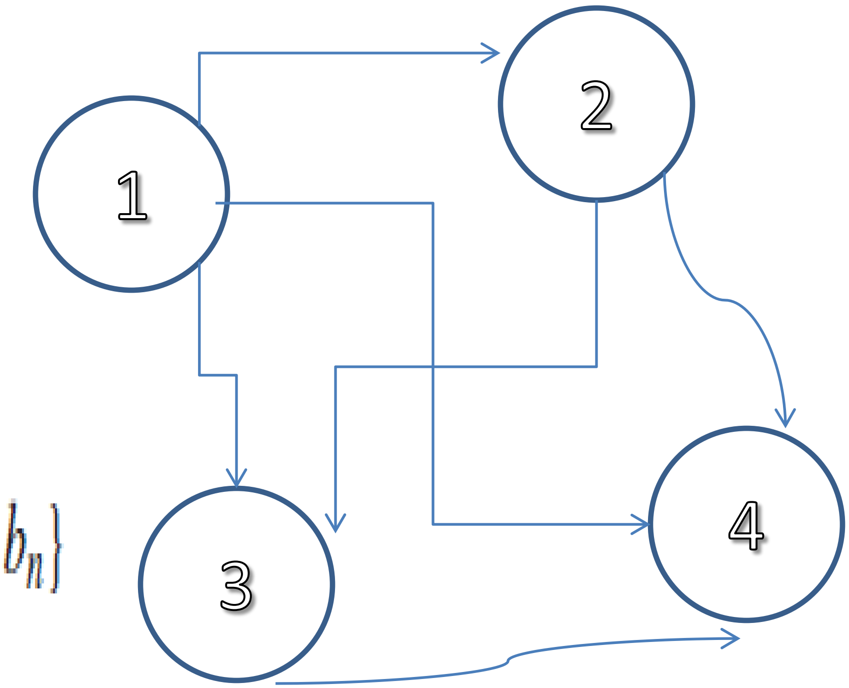
- **Solution:** R = { (1, 2),(1, 3),(1, 4),(2, 3),(2, 4),(3, 4)}



R	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

Matrix, Directed Graph

$$M_R = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



$$A = \{a_1, a_2, \ldots, a_m\} \text{ to } B = \{b_1, b_2, \ldots, b_n\}$$

$$\mathbf{M}_R = [m_{ij}]$$

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

Relations on a Set

- **How many different relations can we define on a set A with n elements?**

- A relation on a set A is a subset of $A \times A$.
- How many elements are in $A \times A$?

- There are n^2 elements in $A \times A$, so how many subsets (= relations on A) does $A \times A$ have?

- The number of subsets that we can form out of a set with m elements is 2^m. Therefore, 2^{n^2} subsets can be formed out of $A \times A$.

- **Answer:** We can define 2^{n^2} different relations on A.

Properties of Relations : REFLEXIVE

- We will now look at some useful ways to classify relations.
- **Definition:** A relation R on a set A is called **reflexive** if (a, a)$\in$R for every element a$\in$A.
- Symbolically, $\forall a \in A, ((a, a) \in R)$
- Are the following relations on {1, 2, 3, 4} reflexive?

R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)} No.

R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)} Yes.

R = {(1, 1), (2, 2), (3, 3)} No.

Definition: A relation on a set A is called **irreflexive** if (a, a)$\notin$R for every element a$\in$A.

Properties of Relations : SYMMETRIC

- **Definitions:**
- A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- $\forall a,b \in A, ((a, b) \in R \rightarrow (b, a) \in R)$
- A relation R on a set A is called **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$.
- $\forall a,b \in A, ((a, b) \in R \land (b, a) \in R \rightarrow (a=b))$
- A relation R on a set A is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$.
- $\forall a,b \in A, ((a, b) \in R \rightarrow (b, a) \notin R)$

Properties of Relations

- Are the following relations on {1, 2, 3, 4} symmetric, antisymmetric, or asymmetric?

$R = \{(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)\}$	symmetric
$R = \{(1, 1)\}$	sym. and antisym.
$R = \{(1, 3), (3, 2), (2, 1)\}$	antisym. and asym.
$R = \{(4, 4), (3, 3), (1, 4)\}$	antisym.

Properties of Relations : TRANSITIVE

- **Definition:** A relation R on a set A is called **transitive** if whenever $(a, b)\in R$ and $(b, c)\in R$, then $(a, c)\in R$ for $a, b, c\in A$.
- $\forall\ a,b,c \in A, ((a, b)\in R \wedge (b, c)\in R \rightarrow (a, c) \in R\ )$
- Are the following relations on $\{1, 2, 3, 4\}$ transitive?

$R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$ Yes.

$R = \{(1, 3), (3, 2), (2, 1)\}$ No.

$R = \{(2, 4), (4, 3), (2, 3), (4, 1)\}$ No.

Relations

- **Inverse of a Relation** : Let R be a relation from a set A to a set B. The inverse relation from B to A, denoted by R^{-1}, is the set of ordered pairs $\{ (b,a) : (a,b) \in R \}$
- **Complimentary Relation** : R^C is the set of ordered pairs $\{ (a,b) : (a,b) \notin R\}$
- Eg: A = {1,2,3}   B = {1,2,3,4}
- R1 = {(1,1), (2,2), (3,3) }
- R2 = { (1,1), (1,2), (1,3), (1,4) }
- $R1^{-1}$ = { (1,1), (2,2), (3,3) }
- R1(BAR) = {(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2),(3,4) }

Counting Relations

- **Example:** How many different reflexive relations can be defined on a set A containing n elements?

- **Solution:** Relations on R are subsets of $A \times A$, which contains n^2 elements.
- Therefore, different relations on A can be generated by choosing different subsets out of these n^2 elements, so there are 2^{n^2} relations.
- A **reflexive** relation, however, **must** contain the n elements (a, a) for every $a \in A$.
- Consequently, we can only choose among $n^2 - n = n(n - 1)$ elements to generate reflexive relations, so there are $2^{n(n-1)}$ of them.

Combining Relations: Operations on Relations

- Relations are sets, and therefore, we can apply the usual **set operations** to them.

- If we have two relations R_1 and R_2, and both of them are from a set A to a set B, then we can combine them to have (union)$R_1 \cup R_2$, (Intersection)$R_1 \cap R_2$, (Difference) $R_1 - R_2$, or (Symmetric Difference)$R_1 \Delta R_2$.
- $R_1 - R_2 = R_1 \cap R_2^C$
- $R_1 \Delta R_2 = (R_1 \cup R_2) - (R_1 \cap R_2)$
- In each case, the result will be **another relation from A to B**.

Combining Relations : Composition

- … and there is another important way to combine relations.

- **Definition:** Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and
$(b, c) \in S$. We denote the composite of R and S by
$S \circ R$.

- In other words, if relation R contains a pair (a, b) and relation S contains a pair (b, c), then $S \circ R$ contains a pair (a, c).

Combining Relations

- **Example:** Let D and S be relations on A = {1, 2, 3, 4}.
- $D = \{(a, b) \mid b = 5 - a\}$ "b equals (5 – a)"
- $S = \{(a, b) \mid a < b\}$ "a is smaller than b"

- D = {(1, 4), (2, 3), (3, 2), (4, 1)}
- S = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}
- S∘D = { (2,4), (3,3), (3,4), (4,2), (4,3), (4,4) }

D maps an element a to the element (5 – a), and afterwards S maps (5 – a) to all elements larger than (5 – a), resulting in **$S \circ D = \{(a,b) \mid b > 5 - a\}$** or **$S \circ D = \{(a,b) \mid a + b > 5\}$.**

Combining Relations

- **Definition:** Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined inductively by
- $R^1 = R$
- $R^{n+1} = R^n \circ R$

- In other words:
- $R^n = R \circ R \circ \ldots \circ R$ (n times the letter R)

Combining Relations

- **Theorem:** The relation R on a set A is transitive if and only if $R^n \subseteq R$ for all positive integers n.
- Remember the definition of transitivity:
- **Definition:** A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.
- The composite of R with itself contains exactly these pairs (a, c).
- Therefore, for a transitive relation R, $R \circ R$ does not contain any pairs that are not in R, so $R \circ R \subseteq R$.
- Since $R \circ R$ does not introduce any pairs that are not already in R, it must also be true that $(R \circ R) \circ R \subseteq R$, and so on, so that $R^n \subseteq R$.