# Counting 2

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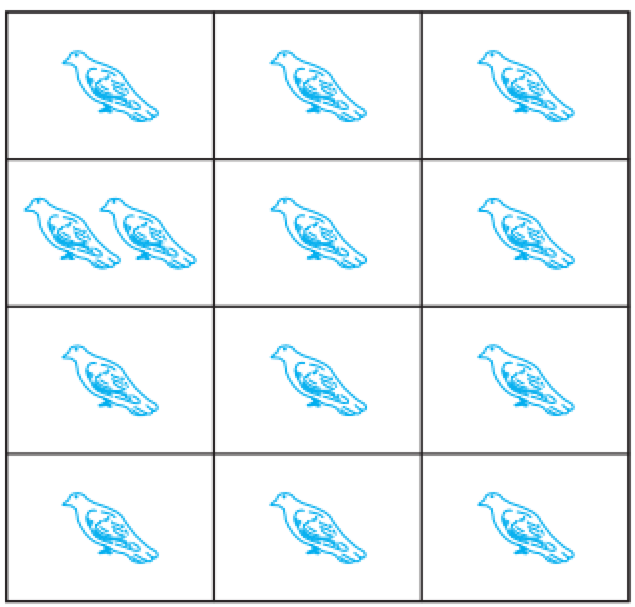
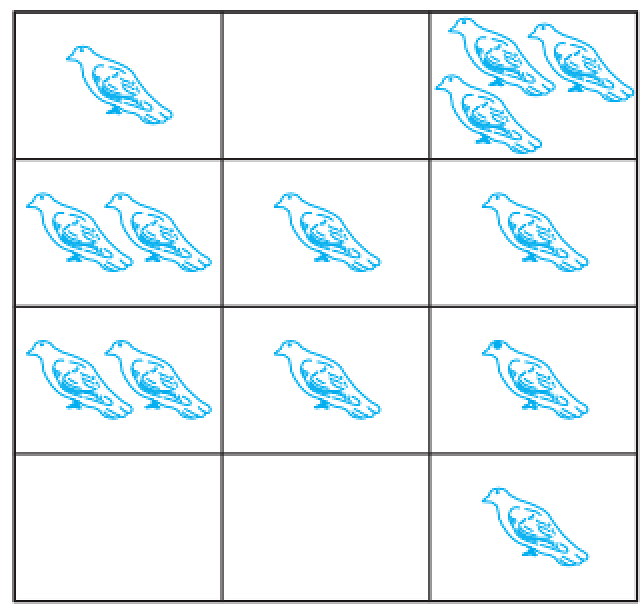
# Pigeonhole Principle

If there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it





# The Pigeonhole Principle

- **The pigeonhole principle:** If (k + 1) or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.
- Proof: By Contraposition.
- Also called Dirichlet Drawer Principle after Lejuene Dirichlet who used this often in his work

- **Example 1:** If there are 11 players in a soccer team that wins 12-0, there must be at least one player in the team who scored at least twice.

- **Example 2:** If you have 6 classes from Monday to Friday, there must be at least one day on which you have at least two classes.

# The Pigeonhole Principle

- **The generalized pigeonhole principle:** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ of the objects.

*Proof:* We will use a proof by contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then, the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < (N/k) + 1$ has been used. This is a contradiction because there are a total of $N$ objects. ◁

- **Example 1:** In our 60-student class, at least 12 students will get the same letter grade (A, B, C, D, or F).

# The Pigeonhole Principle

- **Example 2:** Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

- There are two types of socks, so if you pick at least 3 socks, there must be either at least two brown socks or at least two black socks.

- Generalized pigeonhole principle: $\lceil 3/2 \rceil = 2$.

# The Pigeonhole Principle

- **Example 3:** What is the minimum number of students required in a discrete mathematics class to be sure that at least 6 will receive the same grade ? (5 possible grades : A,B,C,D,F)
- Generalized pigeonhole principle: $\lceil N/5 \rceil = 6$. N = 26
- **Example 4:** a)How many cards must be selected from a standard deck of 52 cards to guarantee that at least 3 cards from the same suite are chosen ? b) How many must be selected to guarantee that at least 3 hearts are selected ?
- **a)** $\lceil N/4 \rceil >= 3$. smallest N = 2.4+1 = 9
- B) No pigeonhole principle; Worst case : 42 cards

# Example

- What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form *NXX-NXX-XXXX, where the first three digits form the area code, N represents a* digit from 2 to 9 inclusive, and *X represents any digit.)*
- *N = 25 million = 25,000,000*
- *Leaving Area Codes, we have 800 x 10,0000= 8 million*
- $\lceil 25000000/8000000 \rceil = \lceil 3.125 \rceil = 4$

# Permutations

- How many ways are there to pick a set of 3 people from a group of 6?
- There are 6 choices for the first person, 5 for the second one, and 4 for the third one, so there are $6 \cdot 5 \cdot 4 = 120$ ways to do this.
- This is not the correct result!
- For example, picking person C, then person A, and then person E leads to the **same group** as first picking E, then C, and then A.
- However, these cases are counted **separately** in the above equation.

# Permutations

- So how can we compute how many different subsets of people can be picked (that is, we want to disregard the order of picking) ?

- To find out about this, we need to look at **permutations**.

- A **permutation** of a set of distinct objects is an ordered arrangement of these objects.
- A *permutation* of a set *S* of objects is an ordered arrangement of the elements of S where each element appears only once:

  e.g., 1 2 3,   2 1 3,   3 1 2

- An ordered arrangement of r elements of a set is called an **r-permutation**.

# Permutations

- **Example:** Let $S = \{1, 2, 3\}$.
- The arrangement 3, 1, 2 is a permutation of S.
- The arrangement 3, 2 is a 2-permutation of S.

- The number of r-permutations of a set with n distinct elements is denoted by **$P(n, r)$.**

- We can calculate $P(n, r)$ with the product rule:
- $P(n, r) = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - r + 1)$.
- (n choices for the first element, $(n - 1)$ for the second one, $(n - 2)$ for the third one…)

# Permutations

- **Example:**
- $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$
- $= (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$

- **General formula:**
- $P(n, r) = n!/(n - r)!$

- Knowing this, we can return to our initial question:
- How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

# Examples

- 1. *How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?*
- 3 elements from a set of 100 elements =
- P(100,3)=100.99.98= 9,70,200
- 2. *Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?*
- 7! = 7 .6.5.4.3.2.1=5040 ways
- 3. How many permutations of the letters *ABCDEFGH contain the string ABC ?*
- ABC is one block and others are separate; 6 objects in any order = 6! = 720 permutations

# Examples

- A terrorist has planted an armed nuclear bomb in your city, and it is your job to disable it by cutting wires to the trigger device.
- There are 10 wires to the device.
- If you cut exactly the right three wires, in exactly the right order, you will disable the bomb, otherwise it will explode!
- If the wires all look the same, what are your chances of survival?

$P(10,3) = 10 \cdot 9 \cdot 8 = 720$,
so there is a 1 in 720 chance that you'll survive!

# Combinations

- An **r-combination** of elements of a set is an unordered selection of r elements from the set.
- Thus, an r-combination is simply a subset of the set with r elements.

- **Example:** Let S = {1, 2, 3, 4}.
- Then {1, 3, 4} is a 3-combination from S.

- The number of r-combinations of a set with n distinct elements is denoted by C(n, r). $\binom{n}{r}$ and is called a binomial coefficient

- **Example:** C(4, 2) = 6, since, for example, the 2-combinations of a set {1, 2, 3, 4} are {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}.

# Combinations

- How can we calculate C(n, r)?
- Consider that we can obtain the r-permutation of a set in the following way:
- **First,** we form all the r-combinations of the set (there are C(n, r) such r-combinations).
- **Then,** we generate all possible orderings in each of these r-combinations (there are P(r, r) such orderings in each case).

- Therefore, we have:
- $P(n, r) = C(n, r) \cdot P(r, r)$

# Combinations

- $C(n, r) = P(n, r)/P(r, r)$
- $= n!/(n-r)!/(r!/(r-r)!)$
- $= n!/(r!(n-r)!)$

- Now we can answer our initial question:
- How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?
- $C(6, 3) = 6!/(3!\cdot3!) = 720/(6\cdot6) = 720/36 = 20$

- There are 20 different ways, that is, 20 different groups to be picked.

# Combinations

- **Corollary:**
- Let n and r be nonnegative integers with $r \leq n$.
- Then $C(n, r) = C(n, n - r)$.

- Note that **"picking a group of r people from a group of n people"** is the same as **"splitting a group of n people into a group of r people and another group of (n – r) people".**

# Combinations

- **Example:**
- A soccer club has 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?

- $C(8, 6) \cdot C(7, 5) = 8!/(6!\cdot 2!) \cdot 7!/(5!\cdot 2!)$
- $= 28\cdot 21$
- $= 588$

# Combinations

- We also saw the following: that C(n,r) = C(n,n-r)

This symmetry is intuitively plausible. For example, let us consider a set containing six elements (n = 6).

**Picking two** elements and **leaving four** is essentially the same as **picking four** elements and **leaving two**.

In either case, our number of choices is the number of possibilities to **divide** the set into one set containing two elements and another set containing four elements.

# Combinations Examples

- 1. How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?
- Since order does not matter,

  C(52,5)=52! /5!47! = 52.51.50.49.48/5.4.3.2.1 = 2598960
- 2. How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?
- C(10,5)= 10! /5!5!= 252
- 3. A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?
- C(30,6) = 30!/6! 24!= 30.29.28.27.26.25/6.5.4.3.2.1 = 593775
- How many bit strings of length 10 *contain exactly 3 1s?*
- *C(10,3)*

# More Examples

- How many distinct 7-card hands can be drawn from a standard 52-card deck?
  - The order of cards in a hand doesn't matter.
- $C(52,7) = P(52,7)/P(7,7)$
- How many ways are there to select a committee to develop a discrete mathematics course if the committee is to consist of 3 faculty members from the Math department and 4 from the CS department, if there are 9 faculty members from Math and 11 from CS?
- C(9,3) . C(11,4) = 9! /3! 6! * 11!/4!7! = 84.330 = 27,720

# Permutations vs. Combinations

- How many ways are there of selecting 1st, 2nd, and 3rd place from a group of 10 sprinters?
- How many ways are there of selecting the top three finishers from a group of 10 sprinters?
- How many binary strings of length 10 with 3 0's ?
- How many binary strings of length 10 with 7 1's?
- How many different ways of assigning 38 students to the 5 seats in the front of the class?
- How many different ways of assigning 38 students to a table that seats 5 students?

# Prove C(n, r) = C(n, n-r) [Proof 1]

- Proof by formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$$

# Prove C(n, r) = C(n, n-r)  [Proof 2]

- Combinatorial proof
  - Set S with n elements
  - Every subset A of S with r elements corresponds to a subset of S with n – r elements (the complement of A)

# Combinations

- Imagine a set S containing n elements and a set T containing (n + 1) elements, namely all elements in S plus a new element **a**.
- Calculating C(n + 1, k) is equivalent to answering the question: How many subsets of T containing k items are there?
- Case I: The subset contains (k – 1) elements of S plus the element **a**: C(n, k – 1) choices.
- Case II: The subset contains k elements of S and does not contain **a**: C(n, k) choices.
- **Sum Rule:** C(n + 1, k) = C(n, k – 1) + C(n, k).

# Combinations vs Permutations

- Essentially <u>unordered permutations</u> …

$$P(n,r) = C(n,r)P(r,r)$$

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!(n-r)!}$$

- Note that $C(n,r) = C(n, n-r)$