Counting

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Combinatorics

- Study of arrangement of objects
- Subject was studied in gambling games : 17^{th} century
- Enumeration: Count the number of ways to put things together into various combinations: part of combinatorics

  e.g. If a password is 6-8 letters and/or digits, how many passwords can there be?
- Used to determine
  - the complexity of algorithms
  - Whether there are enough telephone numbers or internet protocol addresses
  - Probabilities of events
  - Sequencing DNA
- Ordered and Unordered Arrangement
- Generating all the arrangements of a specified kind

Basic Counting Rules

Product Rule: Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 choices for the first task and for each of the n_1 choices there are n_2 choices for the second task, then there are n_1n_2 choices to do the procedure.

Set Theoretic Version

- If A is the set of ways to do task 1, and B the set of ways to do task 2, and if A and B are <u>disjoint</u>, then:
- The ways to do both task 1 and 2 can be represented as $A \times B$, and $|A \times B| = |A| \cdot |B|$

Examples

- The chairs of an auditorium are to be labeled with a letter and a positive integer not to exceed 100. What is the largest number of chairs that can be labeled differently? ADDD

- How many different bit strings are there of length seven? BBBBBBB

- How many different license plates are available if each plate contains a sequence of three letters followed by three digits? AAADDD

More Examples

- How many **functions** are there from a set with m elements to a set with n elements ?

- How many **one-to-one** functions are there from a set with m elements to a set with n elements ?

- How many **subsets** are there for a finite set S ?

Example

- What is the value of k after the code is executed ?

$$
\begin{array}{l}
k := 0 \\
\textbf{for } i_1 := 1 \textbf{ to } n_1 \\
\quad \textbf{for } i_2 := 1 \textbf{ to } n_2 \\
\qquad \qquad \vdots \\
\qquad \quad \textbf{for } i_m := 1 \textbf{ to } n_m \\
\qquad \qquad k := k + 1
\end{array}
$$

Sum Rule

- **Sum Rule:** If a task can be done either in one of n_1 ways or in one of n_2 ways where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Set Theoretic Version

- If A is the set of ways to do task 1, and B the set of ways to do task 2, and if A and B are <u>disjoint</u>, then:

"the ways to do either task 1 or 2 are $A \cup B$, and $|A \cup B| = |A| + |B|$"

Example

- Suppose that either a member of the CS faculty or a student who is a CS major can be on a university committee. How many different choices are there if there are 37 CS faculty and 83 CS majors ?
- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects respectively. How many possible projects are there to choose from?

Example

- What is the value of k after the code is executed ?

$$
\begin{array}{l}
k := 0 \\
\textbf{for } i_1 := 1 \textbf{ to } n_1 \\
\qquad k := k + 1 \\
\textbf{for } i_2 := 1 \textbf{ to } n_2 \\
\qquad k := k + 1 \\
\qquad\quad \vdots \\
\textbf{for } i_m := 1 \textbf{ to } n_m \\
\qquad k := k + 1
\end{array}
$$

Example Using Both Rules

- Each user on a computer system has a password, which is six to eight characters long where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
- Pwds = P6 + P7 + P8
- $P6 = 36^6 – 26^6$
- $P7 = 36^7 – 26^7$
- $P8 = 36^8 – 26^8$

IP Address Example (Internet Protocol vers. 4)

- Main computer addresses are in one of 3 types:
  - *Class A:* address contains a 7-bit “netid” ≠ **1**^{7}, and a 24-bit “hostid”
  - *Class B:* address has a 14-bit netid and a 16-bit hostid.
  - *Class C:* address has 21-bit netid and an 8-bit hostid.

Bit Number	0	1	2	3 4 … 8 … 16 … 24 … 31	
Class A	0	netid			hostid
Class B	1	0	netid		hostid
Class C	1	1	0	netid	hostid

  - Hostids that are all 0s or all 1s are **not allowed**.
- How many valid computer addresses are there?

Example Using Both Rules: IP address solution

- (# addrs) = (# class A) + (# class B) + (# class C)

  (by sum rule)
- # class A = (# valid netids)·(# valid hostids)

  (by product rule)
- (# valid class A netids) = $2^7 - 1 = 127$.
- (# valid class A hostids) = $2^{24} - 2 = 16{,}777{,}214$.
- Continuing in this fashion we find the answer is:
  3,737,091,842 (3.7 billion IP addresses)

Subtraction Rule
Inclusion-Exclusion Principle

- **Subtraction Rule:** If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the different ways.

- **Set theory:** If A and B are not disjoint, then $|A \cup B| = |A| + |B| - |A \cap B|$.

Example 1

- How many strings of length eight either start with a 1 bit or end with the two bit string 00?

$$\underline{1}\ \underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}}_{2^7 = 128 \text{ ways}}$$

$$\underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}}_{2^6 = 64 \text{ ways}}\ \underline{0}\ \underline{0}$$

$$\underline{1}\ \underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}}_{2^5 = 32 \text{ ways}}\ \underline{0}\ \underline{0}$$

- 128+64 – 32 = 160

Example 2

- Hypothetical rules for passwords:
  - Passwords must be 2 characters long.
  - Each password must be a letter a-z, a digit 0-9, or one of the 10 punctuation characters !@#$%^&*().
  - Each password must contain at least 1 digit or punctuation character.
- Solution
  - A legal password has a digit or puctuation character in position 1 **or** position 2.
    - These cases overlap, so the principle applies.
  - (# of pwds w. DP sym. in pos. #1) = (10+10)·(10+10+26)
  - (# w. DP sym. in pos. #2): also 20·46
  - (# w. DP sym both places): 20·20
  - Answer: 920+920−400 = 1,440

Example 3

- A class has of 40 students has 20 CS majors and 15 math majors. 5 of these students are dual majors. How many students in the class are neither math nor CS majors?

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316.$$

- 350 – 316 = 34

Division Rule

- **Division Rule:** There are n/d ways to do a task if it can be done by using a procedure that can be carried out in n ways, and for every way *w* exactly *d* of the *n* ways correspond to way *w*.

- **Set theory:** If the finite set *A* is the union of n pairwise disjoint subsets each with d elements, then n = $|A| / d$.

Example 1

- How many different ways are there to seat 4 people around a circular table where 2 seatings are considered the same where each person has the same right neighbor and the same left neighbor?
- 4! Ways to order the four people into the seats. 4 ways to choose seat1. by division rule, 24/4=6 seating arrangements.

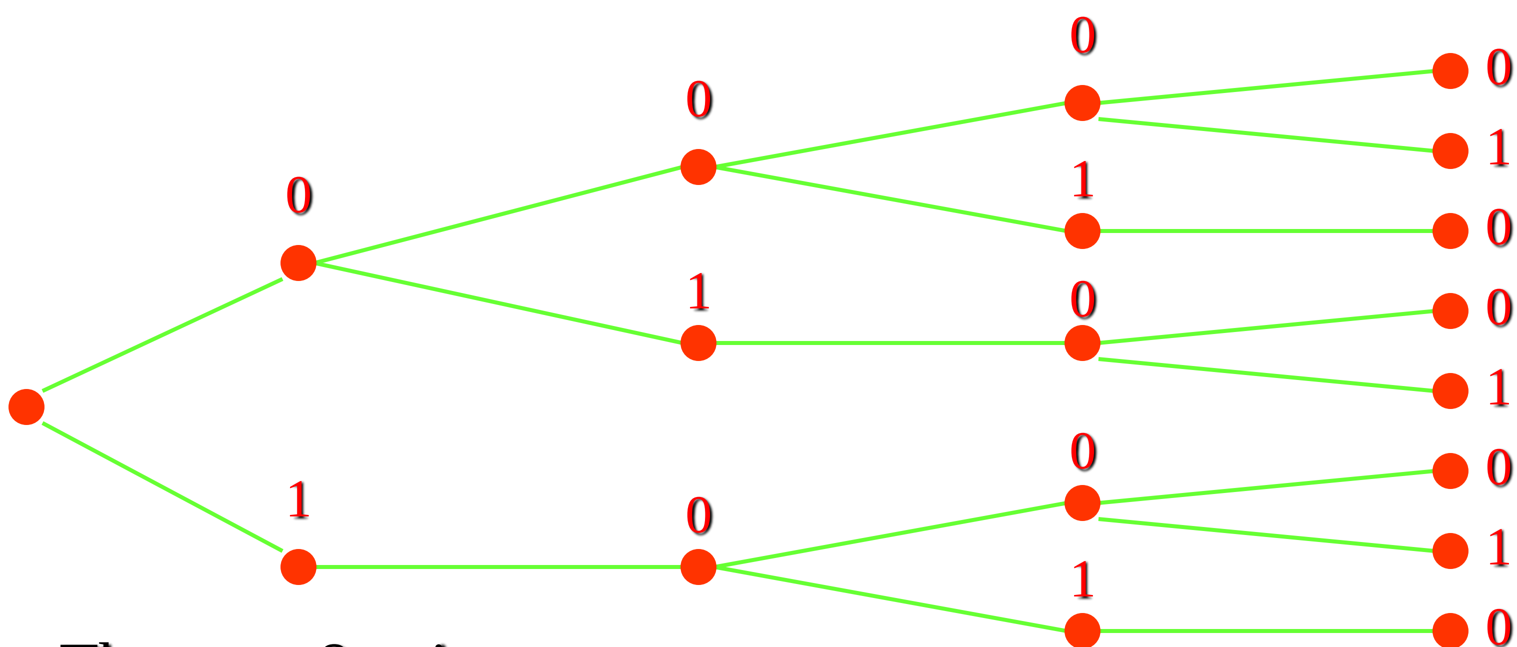
Tree Diagrams

- Counting problems can be solved using tree diagrams
- A branch represents each possible choice
- Leaves represent possible outcomes

Example 1

- How many bit strings of length four do not have two consecutive 1s?

- Task 1 (1st bit) | Task 2 (2nd bit) | Task 3 (3rd bit) | Task 4 (4th bit)



There are 8 strings.

Example 2

- T-Shirts come with 5 different sizes : S, M, L, XL, and XXL. Each size comes with 4 different colors, white red, green and black. XL comes in red, green and black and XXL comes in green and black. How many different shirts does a shop have to stock to have at least one of each available color and size of the T-Shirt?
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