Mathematical Foundations of Computing
Propositional Calculus

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Discrete Structures

- Computer Science is not programming. It is the mathematical modeling and study of what computation is—ie, what problems have a computational solution and how efficient that solution can be.
- A strong foundation in mathematics is essential to be a successful computer scientist.
- At the heart of computer science are fundamental, discrete structures which we will study in this course.
- Will learn many of the mathematical definitions, techniques, and ways of thinking that will be useful in Computer Science.

Theory of Computing

- Formal languages and Grammar
- Theoretical models of Computation

  (Automata, Push Down Automata, Turing Machines)
- Power and limitations of the machines

MFOC : Major Sections

- 1. Logic
- 2. Relations & Functions
- 3. Counting
- 4. Recurrence Relations
- 5. Formal Languages, Automata, Push Down Automata and Turing Machine

Logic

- Science of reasoning
- A kind of intelligence
- A formal language for representing knowledge and for making logical inferences
- Helps us to understand how to construct a valid argument
- Defines syntax, meaning, and rules of logical inference
- Eg.
  - All cats are reptiles.
  - Big Bunny is a cat.
  - Therefore Big Bunny is a reptile.

History

- 300 BC Aristotle: founded logic of life
- Late 1600's Leibnitz's goal: mechanization of inference
- 1847 Boole: Mathematical Analysis of Logic
- 1879: Complete Propositional Logic: Frege
- 1965: Resolution Complete (Robinson)
- 1971: Cook: satisfiability NP-complete

Rules of logic are used :

- To provide proofs of theorems in Maths
- To verify the correctness of computer programs
- To draw conclusion from Scientific Experiments
- To design simplified computer circuits
- To solve a multitude of problems

Propositional Calculus

- Algebra of Propositions
- Propositional logic is the reasoning of truth or falsehood of logical expressions
- **Proposition** : A declarative statement which is either true or false but not both.
  - Atomic : no connectives or operators
  - Compound : combining one or more atomic statements using connectives
- **Examples**
  - Paris is in England (F)
  - N Delhi is the capital of India (T)
  - 2 belongs to { 1,2,3,4} (T)
  - 4 + 2 = 6 (T)
  - 42 >= 51 (F)

What is Not a proposition ?

- Exclamation, Question(interrogative), Order(imperative), request, Opinion – Not proposition
- **Examples**:
  - What can we represent in propositional logic?
  - What is the efficiency?
  - What a beautiful scene !
  - Please get me a chalk.
  - Close the door.
  - $X + 2 = 6$
  - Java is the best language

Proposition Logic: Syntax

- TRUE, FALSE are logical constants
- A proposition is represented by alphabets – A, P, X, Q etc; A, P, X, Q are called variables
- **Atomic Sentence**: true, false, variable
- **Complex Sentence**: connectives applied to atomic or complex sentence.
- **Connectives**: not, and, or, conditional, biconditional etc.
- Defined by truth tables.

Connective NOT

- Is not opposite
- Truth table:

The Truth Table for the Negation of a Proposition.	
p	$\neg p$
T	F
F	T

Examples :

P : Radha is good; ~P : Radha is not good

Q : It is cold ~Q: It is not cold

R : Saranya always eats chocolates ;

~R: Sometimes Saranya does not eat chocolates

Logical OR - disjunction (V)

- Inclusive OR : I shall watch the movie on the TV or go to the cinema
- Exclusive OR : There is something wrong with the fan or with the switch

Examples :

1. P : Ravi goes to school; Q : Ravi goes to play

   P V Q : Either Ravi goes to school or to play

   There is something wrong with the switch or with the fan.

   I shall watch the movie on TV or go to the theatre.

2. X : Ravi is tall; Y : Ravi is smart

   X V Y : Ravi is tall or smart

   ~X V ~Y : Ravi is not tall or Ravi is not smart

   ~(~X V ~Y) : It is not true that Ravi is not tall or not smart

The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical AND - conjunction (^)

The Truth Table for the Conjunction of Two Propositions.		
p	*q*	*p* ∧ *q*
T	T	T
T	F	F
F	T	F
F	F	F

Examples :

1. P : Ramu is healthy; Q : Raja is clever

   P ^ Q : Ramu is healthy and Raja is clever

2. S : Ravi is rich

   T : Ravi is happy

   S ^ T : Ravi is rich and happy

   ~S ^ T : Ravi is not rich but happy

   S ^ ~T : Ravi is rich but not happy

   ~S ^ ~ T : Ravi is neither rich nor happy

3. X : Ravi speaks Tamil; Y : Ravi speaks Hindi

   X ^ Y : Ravi speaks Tamil and Hindi

   X ^ ~Y : Ravi speaks Tamil but not Hindi

   ~X ^ ~Y : Ravi speaks neither Tamil nor Hind

   ~(~X) : It is not true that Ravi does not speak Tamil

Logical Implication – conditional (→)

The Truth Table for the Biconditional $p \to q$.		
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- Let p and q be two propositions. The implication $p \rightarrow q$ is the proposition that is false when p is true and q is false and true otherwise
  - – p is called the hypothesis, antecedent, premise
  - – q is called the conclusion, consequence

Logical Connective: Implication

- The implication of $p \to q$ can be also read as
  - If p then q
  - p implies q
  - If p, q
  - p only if q
  - q if p
  - q when p
  - q whenever p
  - q follows from p
  - p is a sufficient condition for q (p is sufficient for q)
  - q is a necessary condition for p (q is necessary for p)
  - q *unless not* p

Logical Connective: Implication

- Examples
  - If you buy you air ticket in advance, it is cheaper.
  - If x is an integer, then $x^2 \geq 0$.
  - If it rains, the grass gets wet.
  - We can succeed in our life only if we work hard.
  - Fuel is necessary for driving the car.
  - Scoring 50% in the exam is sufficient to pass the course
  - The apple trees will bloom if it stays warm for a week.
  - Payment will not be made unless you complete the job.

Exercise: Which of the following implications is true?

- If -1 is a positive number, then 2+2=5

  True. The premise is obviously false, thus no matter what the conclusion is, the implication holds.

- If -1 is a positive number, then 2+2=4

  True. Same as above.

- If $\sin x = 0$, then $x = 0$

False. x can be a multiple of π. If we let $x=2\pi$, then $\sin x=0$ but $x\neq0$. The implication "if $\sin x = 0$, then $x = k\pi$, for some k" is true.

Logical Connective Biconditional (←→)

The Truth Table for the Biconditional $p \leftrightarrow q$.		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." (iff)
- The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.
- Biconditional statements are also called *bi-implications*.
- **Forms**:
- If p then q and conversely
- p is necessary and sufficient for q
- **Eg:** You can vote in the election iff you have the voter id.

Propositional Logic

- Can use connectives to build up complicated compound propositions involving any number of propositional variables, then use truth tables to determine the truth value of these compound propositions.
- Example: Construct the truth table of the compound proposition

$$(p \vee \neg q) \longrightarrow (p \wedge q).$$

The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.					
p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical Operators

- Can use parentheses to specify the order in which logical operators in a compound proposition are to be applied.
- To reduce the number of parentheses, the precedence order is defined for logical operators.

Precedence of Logical Operators.	
Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

E.g. $\neg p \wedge q = (\neg p) \wedge q$

$p \wedge q \vee r = (p \wedge q) \vee r$

$p \vee q \wedge r = p \vee (q \wedge r)$

Symbolize

- English (and every other human language) is often ambiguous. Translating sentences into compound statements removes the ambiguity.
- Example: How can this English sentence be translated into a logical expression?

“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.”

Solution: Let q, r, and s represent “You cannot ride the roller coaster,” “You are under 4 feet tall,”

and “You are older than 16 years old.”

The sentence can be translated into:

$$\neg s \rightarrow (r \rightarrow q).$$

Symbolize

- Example: How can this English sentence be translated into a logical expression?

  "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: Let a, c, and f represent

"You can access the Internet from campus,"

"You are a computer science major," and "You are a freshman."

The sentence can be translated into:

$$a \rightarrow (c \vee \neg f).$$

Logic and bit operations

- Computers represent information using bits.
- A **bit** is a symbol with two possible values, 0 and 1.
- By convention, 1 represents T (true) and 0 represents F (false).
- A variable is called a Boolean variable if its value is either true or false.
- Bit operation – replace true by 1 and false by 0 in logical operations.

Table for the Bit Operators *OR*, *AND*, and *XOR*.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0