Recurrence Relations 1

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Outline

- Recurrence Relations
- Modeling Recurrence Relations

Modeling with Recurrence Relations

- **Example 1:**
- Let a_n denote the number of bit strings of length n that do not have two consecutive 0s ("valid strings"). Find a recurrence relation and give initial conditions for the sequence $\{a_n\}$.

- **Solution:**
- Idea: The number of valid strings equals the number of valid strings ending with a 0 plus the number of valid strings ending with a 1.
- Let us assume that $n \geq 3$, so that the string contains at least 3 bits.
- Let us further assume that we know the number a_{n-1} of valid strings of length $(n – 1)$.
- Then how many valid strings of length n are there, if the string ends with a 1?
- There are a_{n-1} such strings, namely the set of valid strings of length $(n – 1)$ with a 1 appended to them.
- **Note:** Whenever we append a 1 to a valid string, that string remains valid.

Sol :

Let a_n be the number of bit strings of length n that do not have two consecutive 0s.

1	2	…	n-3	n-2	n-1	n
			a_{n-1}			1
			a_{n-2}		1	0

$$\therefore a_n = a_{n-1} + a_{n-2},\quad n \geq 3$$

$a_1=2$ (string : 0,1)

$a_2=3$ (string : 01,10,11)

$$\therefore a_3=a_2+a_1=5,\ a_4=8,\ a_5=13$$

Modeling with Recurrence Relations

- What are the **initial conditions**?
- $a_1 = 2$ (0 and 1)
- $a_2 = 3$ (01, 10, and 11)
- $a_3 = a_2 + a_1 = 3 + 2 = 5$
- $a_4 = a_3 + a_2 = 5 + 3 = 8$
- $a_5 = a_4 + a_3 = 8 + 5 = 13$
- …
- This sequence satisfies the same recurrence relation as the **Fibonacci sequence**.
- Since $a_1 = f_2$ and $a_2 = f_3$, we have $a_n = f_{n+1}$.

Modeling with Recurrence Relations

- Example 2: How many valid strings of length n are there, if the string ends with a **0**?
- Valid strings of length n ending with a 0 **must have a 1 as their (n – 1)st bit** (otherwise they would end with 00 and would not be valid).
- The number of valid strings of length (n – 1) that end with a 1?
- There are a_{n-1} strings of length n that end with a 1.
- Therefore, there are a_{n-2} strings of length (n – 1) that end with a 1.
- So there are a_{n-2} valid strings of length n that end with a 0 (all valid strings of length (n – 2) with 10 appended to them).

- The number of valid strings is the number of valid strings ending with a 0 plus the number of valid strings ending with a 1.
- That gives us the following **recurrence relation**:

$$a_n = a_{n-1} + a_{n-2}$$

Introduction

- A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n is terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.
- A recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.

- Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.

- If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

Introduction

•**Example 1:**

Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \text{ for } n = 2, 3, 4, \ldots$$

- Is the sequence $\{a_n\}$ with $a_n=3n$ a solution of this recurrence relation?

- For $n \geq 2$ we see that

$$2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n.$$

•Therefore, $\{a_n\}$ with $a_n=3n$ is a solution of the recurrence relation.

Introduction

- Example 2:
- Is the sequence $\{a_n\}$ with $a_n=5$ a solution of the same recurrence relation?
- For $n \geq 2$ we see that

$$2a_{n-1} - a_{n-2} = 2\cdot5 - 5 = 5 = a_n.$$

- Therefore, $\{a_n\}$ with $a_n=5$ is also a solution of the recurrence relation.

Definition 1:A *recurrence relation* is an infinite sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ in which the formula for the nth term a_n depends on one or more preceding terms, with a finite set of start-up values or **initial conditions**

Examples of recurrence relations

- Example 1 :
  - Initial condition $a_0 = 1$
  - Recursive formula: $a_{n} = 1 + 2a_{n-1}$ for $n \geq 2$
  - First few terms are: 1, 3, 7, 15, 31, 63, …
- Example 2:
  - Initial conditions $a_0 = 1$, $a_1 = 2$
  - Recursive formula: $a_{n} = 3(a_{n-1} + a_{n-2})$ for $n \geq 2$
  - First few terms are: 1, 2, 9, 33, 126, 477, 1809, 6858, 26001,…

Recurrence Relations

- **Example.** Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, \ldots$, and suppose that $a_0 = 3$, and $a_1 = 5$.

Here $a_0 = 3$ and $a_1 = 5$ are the initial conditions.

By the recurrence relation,

$$a_2 = a_1 - a_0 = 2$$
$$a_3 = a_2 - a_1 = -3$$
$$a_4 = a_3 - a_2 = -5$$
$$\vdots$$

Q1: Applications ?

Q2: Are there better ways for computing the terms of $\{a_n\}$?

Applications

- Recurrence relations are very useful in the study and solving of counting problems.
  - Many counting problems can be modeled (or expressed) naturally by recurrence relations.
- Goal of RR:
  - Modeling counting problems in terms of recurrence relations
  - Solving recurrence relations. Namely, find explicit formula f(n) for a_n (i.e., without using previous terms) satisfying the recurrence relation (and initial conditions)

※Modeling with Recurrence Relations

We can use recurrence relations to model (describe) a wide variety of problems.

Example 1. Compound Interest

Suppose that a person deposits Rs10000 in a saving account at a bank yielding 11% per year with interest compounded annually.

How much will be in the account after 30 years ?

Sol : Let P_n denote the amount in the account after n years.

$$P_n = P_{n-1} + 0.11 \times P_{n-1} = 1.11 \times P_{n-1}, \qquad P_0 = 10000$$

$$\therefore P_{30} = 1.11 \times P_{29} = (1.11)^2 \times P_{28} = \ldots = (1.11)^{30} \times P_0$$

$$= 2{,}28{,}922.97$$

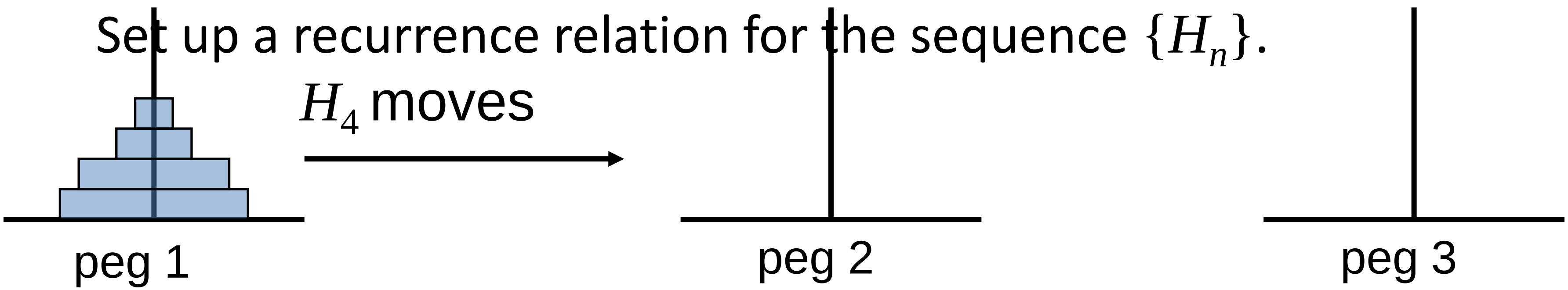
Example 2. (The Tower of Hanoi)

Three pegs numbered 1, 2 and 3 are mounted on a board, n disks of different sizes with holes in their centers are placed in order of increasing size from top to bottom.

The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks.

Object of the game: find the minimum number of moves needed to have all n disks stacked in the same order in peg number 3.

Set up a recurrence relation for the sequence $\{H_n\}$.



Sol : $H_n = 2H_{n-1} + 1$, $H_1 = 1$

(n–1 disks from peg 1→peg 3, n th disk from peg 1→peg 2, n–1 disks from peg 3→peg 2)

$$H_n = 2H_{n-1} + 1, \quad H_1 = 1$$

$$\begin{aligned}
\therefore H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 \\
&= 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 \\
&= 2^3 H_{n-3} + (2^2 + 2 + 1) \\
&\quad \vdots \\
&= 2^{n-1} H_1 + (2^{n-2} + 2^{n-3} + \ldots + 1) \\
&= 2^{n-1} + 2^{n-2} + \ldots + 1 \\
&= \frac{2^n - 1}{2 - 1} = 2^n - 1
\end{aligned}$$

- How big is H_{64} ? Assume one move takes 1 sec => 2^{64} - 1 = 1.8 x 10^{19} = 500 billion years!!

Example 3. (Codeword enumeration)

A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n.

Sol :

$\boxed{1}\ \boxed{2}\ \boxed{3}\ \cdots\ \boxed{n\text{-}1}\ \boxed{n}$

$\underline{\quad a_{n-1} \quad}$ $1\sim9$

$\underline{\quad 10^{n-1} - a_{n-1} \quad}$ 0

$$\therefore a_n = 9a_{n-1} + (10^{n-1} - a_{n-1})$$
$$= 8a_{n-1} + 10^{n-1} \qquad , n \geq 2$$
$$a_1 = 9$$

$a_n:$

$$a_n = 8a_{n-1} + 10^{n-1} = 8(8a_{n-2} + 10^{n-2}) + 10^{n-1}$$

$$= 8^2 a_{n-2} + (8 \cdot 10^{n-2} + 10^{n-1})$$

$$= 8^3 a_{n-3} + (8^2 \cdot 10^{n-3} + 8 \cdot 10^{n-2} + 10^{n-1})$$

$$\vdots$$

$$= 8^{n-1} a_1 + (8^{n-2} \cdot 10 + 8^{n-3} \cdot 10^2 + \ldots + 8 \cdot 10^{n-2} + 10^{n-1})$$

$$= 8^n + \underline{(8^{n-1} + 8^{n-2} \cdot 10 + 8^{n-3} \cdot 10^2 + \ldots + 8 \cdot 10^{n-2} + 8^0 \cdot 10^{n-1})}$$

$$8^{n-1}, \frac{10}{8}, n$$

$$= 8^n + \frac{8^{n-1}((\frac{10}{8})^n - 1)}{(\frac{10}{8} - 1)} = 8^n + 4 \cdot 8^{n-1} \cdot (\frac{10}{8})^n - 4 \cdot 8^{n-1}$$

$$= \frac{1}{2} \cdot 10^n + 4 \cdot 8^{n-1}$$

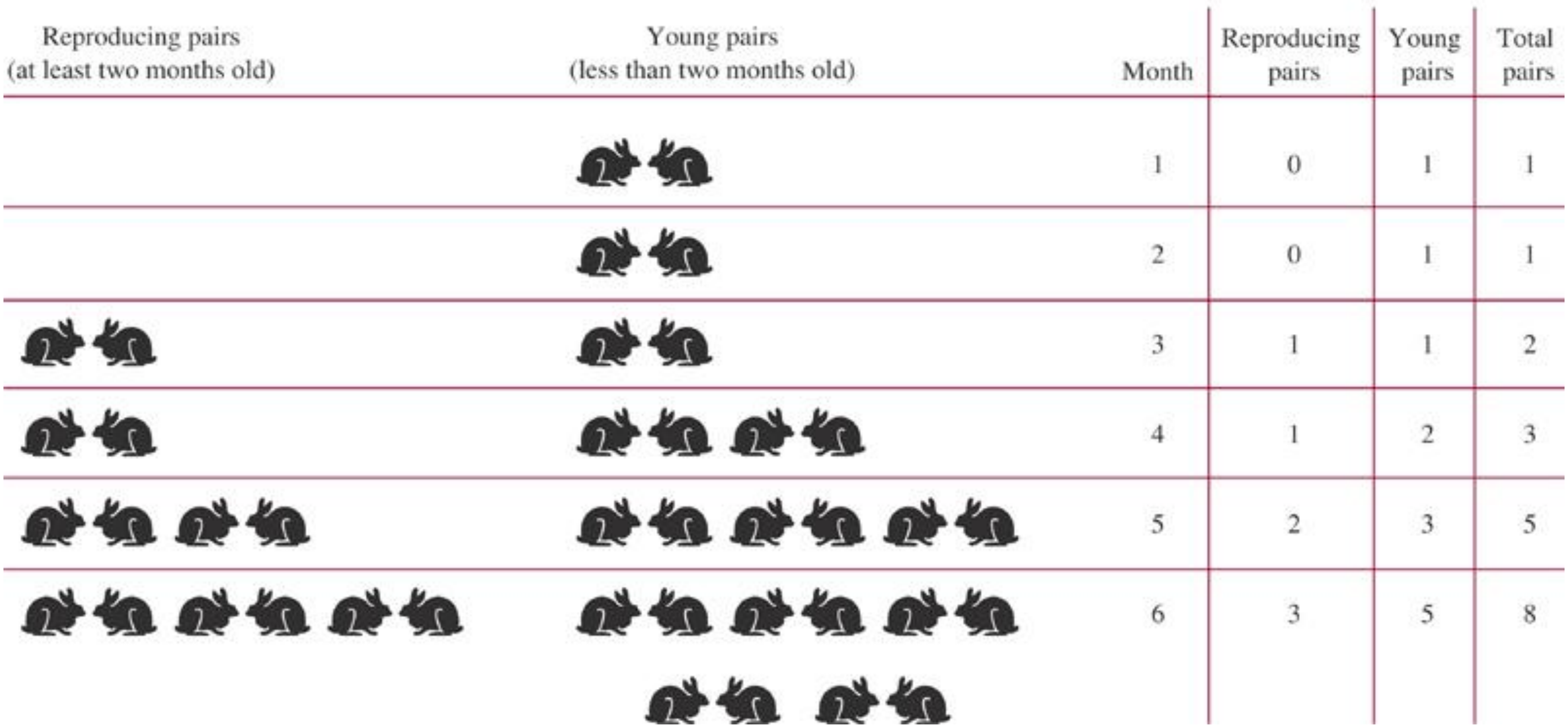
Example 4: Rabbits & Fibonacci

A pair of rabbits produce one pair per month after 2 month old.

Initially there is only one pair.

How many pairs of rabbits are there after n months?

Example 4: Rabbits & Fibonacci sequence



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

FIGURE Rabbits on an Island.

$$f_n = f_{n-1} + f_{n-2} \quad n>=2$$

Example : 5

- The number of bacteria doubles every 2 hours. If a colony begins with 5 bacteria, how many will be present in n hours?
- Let $a_n=2a_{n-1}$ where n is a positive integer with $a_0=5$

Fibonacci sequence

- Initial conditions:
  - $f_1 = 1, f_2 = 2$
- Recursive formula:
  - $f_{n+1} = f_{n-1} + f_n$ for $n \geq 3$
- First few terms:

n	1	2	3	4	5	6	7	8	9	10	11
f_n	1	2	3	5	8	13	21	34	55	89	144

Eugene Catalan

- Belgian mathematician, 1814-1894
- Catalan numbers are generated by the formula:

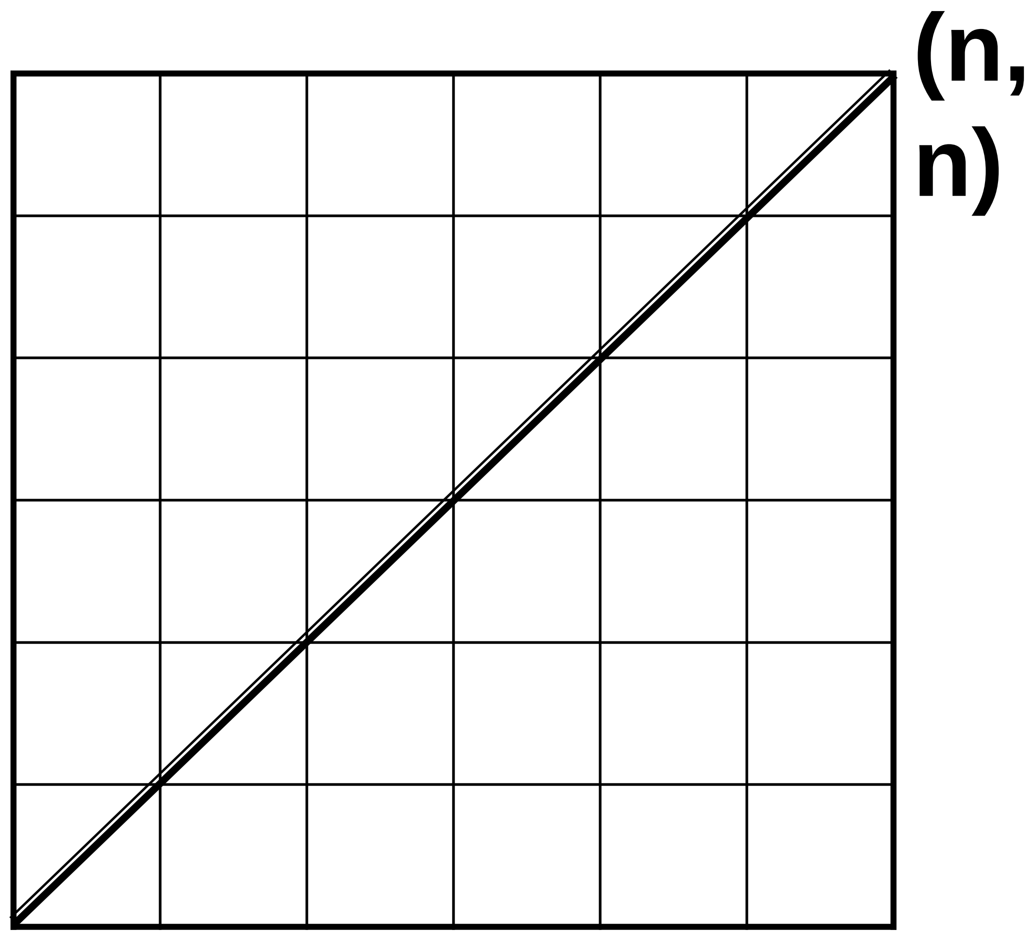
  $C_n = C(2n,n) / (n+1) \quad \text{for } n \geq 0$

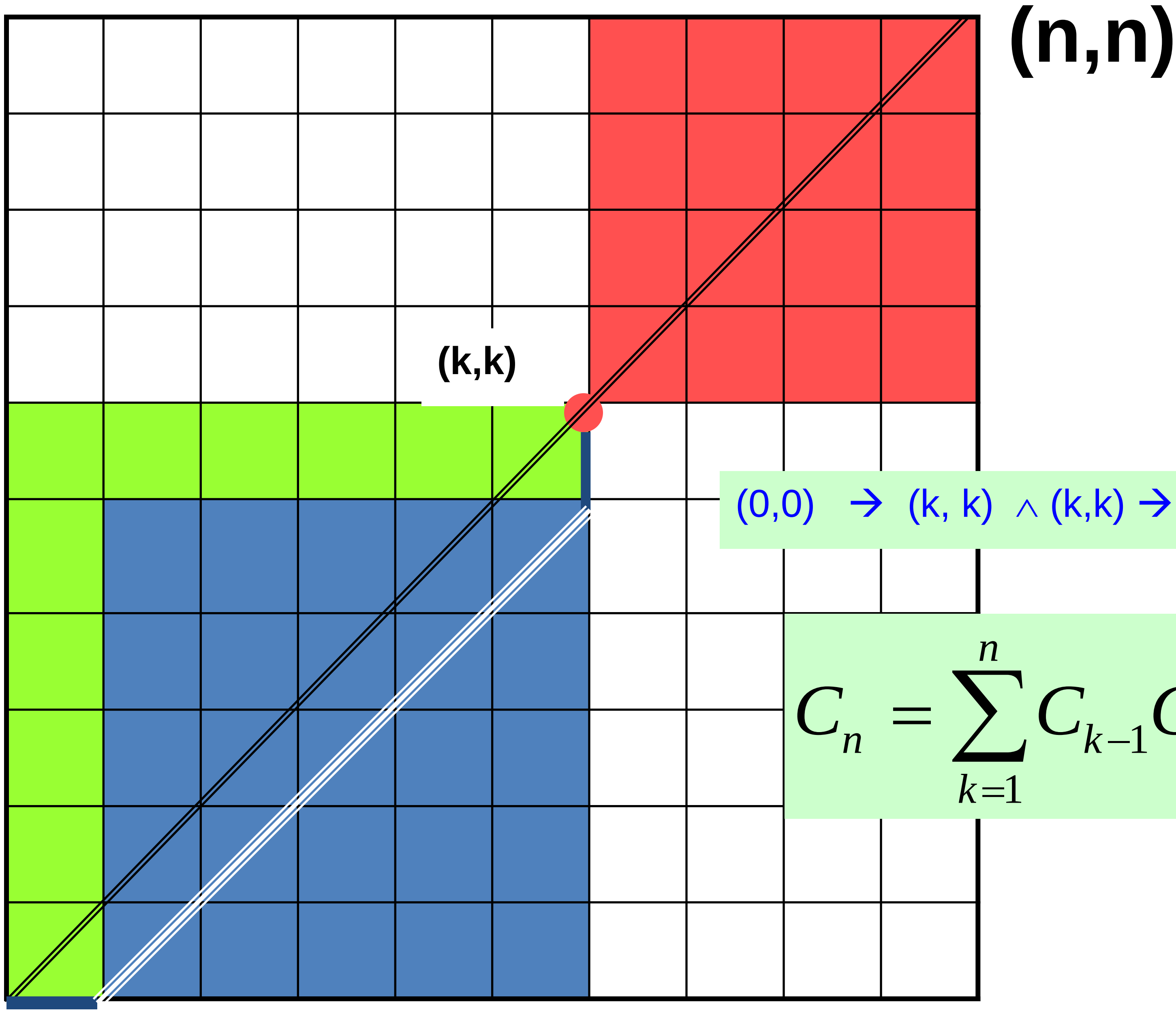
- The first few *Catalan numbers* are:

n	0	1	2	3	4	5	6	7	8	9	10	11
C_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786

Eugene Catalan

- How many routes are there from the lower-left of $n \times n$ square grid to the upper-right corner if we are restricted to traveling only to the right or upward and if we are allowed to touch but not go above a diagonal line from the lower-left to the upper-right corner?





$(0,0) \rightarrow (k, k) \wedge (k,k) \rightarrow (n,n)$

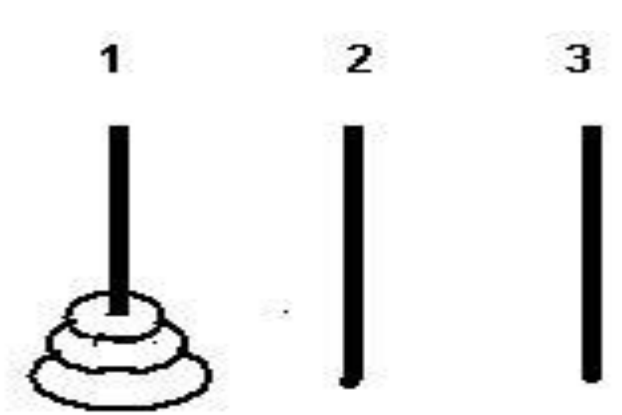
$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$$

End of game: Hanoi towers

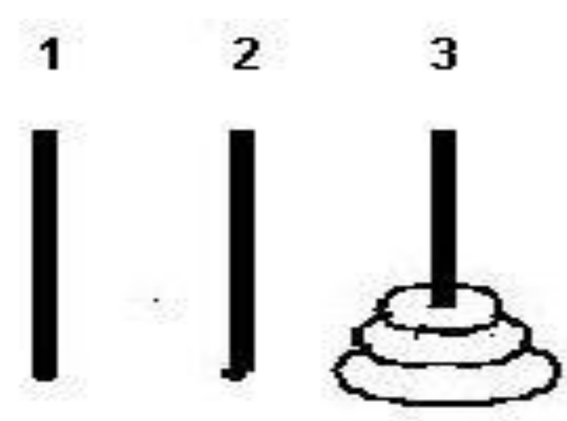
– Game ends when all disks are stacked in peg number 3 in the same order they were stored at the start in peg number 1.

– Verify that the minimum number of moves needed is the Catalan number $C_3 = 7$.

Start

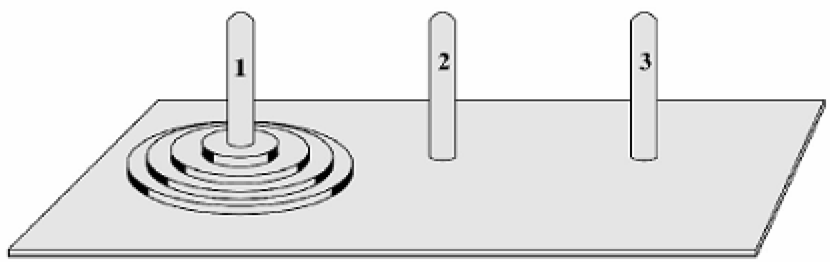


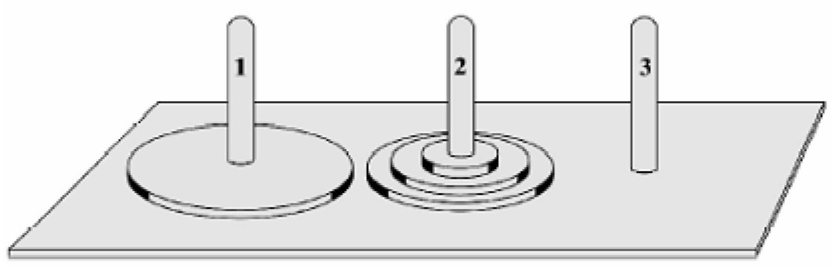
End



Recurrence Relation: Hanoi towers

- $C_n = 2\ C_{n-1} + 1$
- $C_1 = 1$
- We can prove that $C_n = 2^n - 1$





Recurrence relations

- Play an important role in many aspects of algorithms and complexity
- Can be used to
  - analyze the complexity of divide-and-conquer algorithms (e.g., merge sort)
  - Solve dynamic programming problems (e.g., scheduling tasks, shortest-path, hidden Markov model)
  - Fractals