Recurrence Relations 2

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Outline

- Solving Linear Recurrence Relations

Solving Recurrence Relations

Def 1. A linear homogeneous recurrence relation of degree k (i.e., k terms) with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_i \in \mathbf{R}$ and $c_k \neq 0$

Example.

$f_n = f_{n-1} + f_{n-2}$	(True, deg=2)
$a_n = a_{n-5}$	(True, deg=5)
$a_n = a_{n-1} + a_{n-2}^{2}$	(False, not linear)
$a_n = n a_{n-1}$	(False , not linear)
$H_n = 2H_{n-1} + 1$	(False, not homogeneous)

Linear Homogeneous RR of degree k

- **Linear** : the RHS is the sum of the previous terms
- **Homogeneous**: No terms are not multiples of a_j
- All coefficients are constants rather than functions that depend on n
- **Degree k** : an is expressed in terms of previous k terms of the sequencs

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

Theorem 1.

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ (characteristic equation) has two distinct roots r_1 and $r_{2;}$ Then the sequence $\{a_n\}$ is a solution of the RR $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$,

for $n=0,1,2,\ldots,$ where α_1, α_2 are constants.

Proof: If r_1 and r_2 are roots, then a_n is a solution of the RR.

- Show if $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$ (and r^2-$c_1 r - c_2 = 0$) then $\{a_n\}$ is the solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$
- $c_1 a_{n-1} + c_2 a_{n-2}$=$c_1(\alpha_1 {r_1}^{n-1} + \alpha_2 {r_2}^{n-1}) + c_2(\alpha_1 {r_1}^{n-2} + \alpha_2 {r_2}^{n-2}) =$ $\alpha_1 {r_1}^{n-2}(c_1 r_1 + c_2)$+$\alpha_2 {r_2}^{n-2}$ $(c_1 r_2$+$c_2)$= $\alpha_1 {r_1}^{n-2} {r_1}^2$+ $\alpha_2 {r_2}^{n-2} {r_2}^2$=$\alpha_1 {r_1}^n$+ $\alpha_2 {r_2}^n = a_n$

Proof: if a_n is a solution, then $a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$

- Show if $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ (and r^2-$c_1 r - c_2 = 0$) then $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$ for some α_1 and α_2
- From initial conditions: $a_0 = C_0$=$\alpha_1 + \alpha_2$
  $a_1 = C_1$=$\alpha_1 r_1 + \alpha_2 r_2$
- It follows, $\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$, $\alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2}$ when $r_1 \neq r_2$
- Both $\{a_n\}$ and $\{\alpha_1 {r_1}^n + \alpha_2 {r_2}^n\}$ are solutions
- As there is a unique solution of a linear homogenous recurrence relation of degree 2 with the same initial conditions, these two must be the same

Example 1.

What's the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with a_0=2 and a_1=7 ?

Sol :

The characteristic equation is $r^2 - r - 2 = 0$.

Its two roots are $r_1 = 2$ and $r_2 = -1$.

Hence $a_n = \alpha_1 \times 2^n + \alpha_2 \times (-1)^n$.

$\because a_0 = \alpha_1 + \alpha_2 = 2,\ \ a_1 = 2\alpha_1 - \alpha_2 = 7$

$\therefore \alpha_1 = 3,\ \ \alpha_2 = -1$

$\Rightarrow\ a_n = 3 \times 2^n - (-1)^n$.

$a_2 = a_1 + 2a_0 = 11$
$a_2 = 3 \times 2^2 - 1 = 11$

- **Example 2:** Give an explicit formula for the Fibonacci numbers.
- **Solution:** The Fibonacci numbers satisfy the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.
- The characteristic equation is $r^2 - r - 1 = 0$.
- Its roots are

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

- Therefore, the Fibonacci numbers are given by

$$f_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

- for some constants α_1 and α_2.

- We can determine values for these constants so that the sequence meets the conditions $f_0 = 0$ and $f_1 = 1$:

$$f_0 = \alpha_1 + \alpha_2 = 0 \qquad f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

- The unique solution to this system of two equations and two variables is

$$\alpha_1 = \frac{1}{\sqrt{5}}, \ \alpha_2 = -\frac{1}{\sqrt{5}}$$

- So finally we obtained an explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Theorem 2.

Let c1 and c2 be real numbers with c2 <> 0.

Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0.

A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ **iff**

$$a_n = \alpha_1 \cdot r_0^{\,n} + \alpha_2 \cdot n \cdot r_0^{\,n}$$

for $n=0,1,2,\ldots$, where α_1 and α_2 are constants.

Example 1.

What's the solution of $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0=1$ and $a_1=6$?

Sol :

The root of $r^2 - 6r + 9 = 0$ is $r_0 = 3$.

Hence $a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n$.

$\because a_0 = \alpha_1 = 1$

$a_1 = 3\alpha_1 + 3\alpha_2 = 6$

$\therefore \alpha_1 = 1$ and $\alpha_2 = 1$

$\Rightarrow a_n = 3^n + n \cdot 3^n$

$: a_2 = 6a_1 - 9a_0 = 27$

$a_2 = 3^2 + 2 \times 3^2 = 27$

Theorem 3.

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a recurrence relation with $c_1, c_2, \ldots, c_k \in \mathbf{R}$.

If $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$.

Then the solution of a_n is

$a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n} + \ldots + \alpha_k r_k^{\,n}$, for $n = 0, 1, 2, \ldots$

where $\alpha_1, \alpha_2, \ldots \alpha_k$ are constants.

Example 1: (k = 3)

Find the solution of $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial conditions a_0=2, a_1=5 and a_2=15 .

Sol :

The roots of $r^3 - 6r^2 + 11r - 6 = 0$ are

$$r_1 = 1,\ r_2 = 2, \text{ and } r_3 = 3$$

$$\therefore a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$$

$$\left.\begin{aligned} \because a_0 &= \alpha_1 + \alpha_2 + \alpha_3 = 2 \\ a_1 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \\ a_2 &= \alpha_1 + 4\alpha_2 + 9\alpha_3 = 15 \end{aligned}\right\} \begin{aligned} \alpha_1 &= 1, \\ \alpha_2 &= -1, \\ \alpha_3 &= 2 \end{aligned}$$

$$\therefore a_n = 1 - 2^n + 2 \cdot 3^n$$

$a_3 = 6a_2 - 11a_1 + 6a_0$ =47
a_3= $1 - 2^3 + 2 \cdot 3^3$ =47

Theorem 4.

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a recurrence relation with $c_1, c_2, \ldots, c_k \in \mathbf{R}$.

If $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c^k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively, where $m_i \geq 1, \forall i$, and $m_1 + m_2 + \ldots + m_t = k$,

then

$$
\begin{aligned}
a_n = {} & (\alpha_{1,0} + \alpha_{1,1} \cdot n + \ldots + \alpha_{1,m_1-1} \cdot n^{m_1-1}) r_1^{\,n} \\
& + (\alpha_{2,0} + \alpha_{2,1} \cdot n + \ldots + \alpha_{2,m_2-1} \cdot n^{m_2-1}) \cdot r_2^{\,n} \\
& + \ldots + (\alpha_{t,0} + \alpha_{t,1} \cdot n + \ldots + \alpha_{t,m_t-1} \cdot n^{m_t-1}) \cdot r_t^{\,n}
\end{aligned}
$$

where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Eg.1

Roots are ：1 (2 times),

-2 (3 times),

3 (1 time)

$$a_n = (\alpha_{1,0} + \alpha_{1,1} \cdot n) \cdot 1^n + (\alpha_{2,0} + \alpha_{2,1} \cdot n + \alpha_{2,2} \cdot n^2) \cdot (-2)^n$$
$$+ \alpha_{3,0} \cdot 3^n$$

Example 2. Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Sol :

$r^3 + 3r^2 + 3r + 1 = 0$ has a single root $r_0 = -1$ of multiplicity three.

$$\therefore a_n = (\alpha_1+\alpha_2 n+\alpha_3 n^2)\, r_0^{\,n} = (\alpha_1+\alpha_2 n+\alpha_3 n^2)(-1)^n$$

$$\because a_0 = \alpha_1 = 1$$

$$a_1 = (\alpha_1+\alpha_2+\alpha_3) \cdot (-1) = -2$$

$$a_2 = \alpha_1+2\alpha_2+4\alpha_3 = -1$$

$$\therefore \alpha_1 = 1,\ \alpha_2 = 3,\ \alpha_3 = -2$$

$$\Rightarrow a_n = (1+3n-2n^2) \cdot (-1)^n$$

$$a_3 = -3a_2 - 3a_1 - a_0 = 8$$

$$a_3 = (1+3\cdot 3-2\cdot 3^2)\cdot(-1)^3 = 8$$

Eg.3

What is the general form of the solution of a linear homogeneous RR if its characteristic equation has roots 2,2,2,5,5,9 ?

$$a_n = (\alpha_{1,0} + \alpha_{1,1} \cdot n + \alpha_{1,2} \cdot n^2) \cdot 2^n + (\alpha_{2,0} + \alpha_{2,1} \cdot n) \cdot (5)^n + \alpha_{3,0} \cdot 9^n$$

Eg.4

What is the general form of the solution of a linear homogeneous RR if its characteristic equation has roots 1,1,1,1,-2,-2,-2,3,3,-4 ?

$$a_n = (\alpha_{1,0} + \alpha_{1,1} \cdot n + \alpha_{1,2} \cdot n^2 + \alpha_{1,3} \cdot n^3) \cdot 1^n + (\alpha_{2,0} + \alpha_{2,1} \cdot n + \alpha_{2,2} \cdot n^2) \cdot (-2)^n + (\alpha_{3,0} + \alpha_{3,1} \cdot n) \cdot (3)^n + \alpha_{4,0} \cdot (-4)^n$$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Example: $a_n = 3a_{n-1} + 2n$

A recurrence relation of the form

$$\boldsymbol{a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)},$$

where $c_1, c_2, \ldots, c_k$ are real numbers
and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$\boldsymbol{a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}}$$

is called the associated homogeneous recurrence relation.

Examples:

$a_n = a_{n-1} + 2^n$, associated homogeneous RR

$\Rightarrow \quad a_n = a_{n-1}$

$a_n = a_{n-1} + a_{n-2} + n^2+1$, associated homogeneous RR

$\Rightarrow a_n = a_{n-1} + a_{n-2}$

$a_n = 3a_{n-1} + n3^n$, associated homogeneous RR

$\Rightarrow a_n = 3a_{n-1}$

$a_n = a_{n-1} + a_{n-3} + n!$, associated homogeneous RR

$\Rightarrow a_n = a_{n-1} + a_{n-3}$

Every solution is a sum of a particular solution and a solution of the homogeneous RR.

Theorem 5. If $\{a_n^{(p)}\}$ is a particular solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$, then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

Proof. If $\{a_n^{(p)}\}$ and $\{b_n\}$ are both solutions of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

then $a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \ldots + c_k a_{n-k}^{(p)} + F(n)$,

and $b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots + c_k b_{n-k} + F(n)$.

$$\Rightarrow a_n^{(p)} - b_n = c_1(a_{n-1}^{(p)} - b_{n-1}) + c_2(a_{n-2}^{(p)} - b_{n-2}) + \ldots + c_k(a_{n-k}^{(p)} - b_{n-k})$$

$\Rightarrow \{a_n^{(p)} - b_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$

$\Rightarrow \{a_n^{(p)} - b_n\} = a_n^{(h)} \Rightarrow b_n = a_n^{(p)} + a_n^{(h)}$

Example 1. Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1=3$?

Sol :

{associated homo. RR $a_n = 3a_{n-1}$ }

Characteristic equation: $r - 3 = 0 \Rightarrow r = 3 \Rightarrow a_n^{(h)} = \alpha \times 3^n$.

{particular solution}

$\because F(n) = 2n$ *Guess it is linear in n;* $\therefore$ Let $a_n^{(p)} = cn+d$, where c, $d \in \mathbf{R}$.

If $a_n^{(p)} = cn+d$ is a solution to $a_n = 3a_{n-1} + 2n$,

then $cn+d = 3(c(n-1)+d)+2n = 3cn - 3c + 3d + 2n$

$\Rightarrow 2cn-3c + 2d + 2n = (2c+2)n + (2d-3c) = 0$

$\therefore 2c+2 = 0$, and $2d-3c = 0 \Rightarrow c = -1, d = -3/2$

$\Rightarrow a_n^{(p)} = -n - 3/2 \Rightarrow a_n = a_n^{(h)} + a_n^{(p)} = \alpha \times 3^n - n - 3/2$

If $a_1 = \alpha \times 3 - 1 - 3/2 = 3 \Rightarrow \alpha = 11/6$

$\Rightarrow a_n = (11/6) \times 3^n - n - 3/2$

Example 2. Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Sol :

{associated RR $a_n = 5a_{n-1} - 6a_{n-2}$}

Characteristic equation: $r^2 - 5r + 6 = 0$

$\Rightarrow r_1 = 3, r_2 = 2$

$\Rightarrow a_n^{(h)} = \alpha_1 \times 3^n + \alpha_2 \times 2^n$.

{particular solution}

$\because F(n) = 7^n$ $\therefore$ Let $a_n^{(p)} = c \cdot 7^n$, where $c \in \mathbf{R}$.

If $a_n^{(p)} = c \cdot 7^n$ is a solution to $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$,

then $c \cdot 7^n = 5c \cdot 7^{n-1} - 6c \cdot 7^{n-2} + 7^n$

$\Rightarrow 49c = 35c - 6c + 49$

$\Rightarrow c = 49/20 \Rightarrow a_n^{(p)} = (49/20) \cdot 7^n$

$\Rightarrow a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1 \times 3^n + \alpha_2 \times 2^n + (49/20) \cdot 7^n$

Theorem 6.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

where $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$.

When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n.$$

When s is a root of the characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n.$$

Example 1. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2+1)3^n$.

Sol :

The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$.

characteristic equation: $r^2 - 6r + 9 = 0 \Rightarrow r = 3$ (2times)

$F(n) = 3^n$, and 3 is a root $\Rightarrow a_n^{(p)} = n^2 p_0 3^n$

$F(n) = n3^n$, and 3 is a root $\Rightarrow a_n^{(p)} = n^2(p_1 n + p_0)\, 3^n$

$F(n) = n^2 2^n$, and 2 is not a root $\Rightarrow a_n^{(p)} = (p_2 n^2 + p_1 n + p_0) 2^n$

$F(n) = (n^2+1)3^n$, and 3 is a root

$\Rightarrow a_n^{(p)} = n^2\, (p_2 n^2 + p_1 n + p_0)\, 3^n$

Example 2. Find the solutions of the recurrence relation $a_n = a_{n-1} + n$ with a_1=1.

Sol : The associated linear homogeneous recurrence relation is $a_n = a_{n-1}$.

characteristic eq.: $r - 1 = 0 \Rightarrow r = 1 \Rightarrow a_n^{(h)} = c(1)^n = c$

$F(n) = n = n(1)^n$, and 1 is a root

$$\Rightarrow a_n^{(p)} = n(p_1 n + p_0)1^n = p_1 n^2 + p_0 n$$

$a_n^{(p)}$ $\quad a_n = a_{n-1} + n$

$$\Rightarrow p_1 n^2 + p_0 n = p_1(n-1)^2 + p_0(n-1) + n$$

$$\Rightarrow (2p_1 - 1)n + p_0 - p_1 = 0 \quad \Rightarrow p_1 = ½, \quad p_0 = p_1 = ½$$

$$\Rightarrow a_n^{(p)} = (n^2+n)/2$$

$$\Rightarrow a_n = a_n^{(p)} + a_n^{(h)} = (n^2+n)/2 + c$$

$$a_1 = 1 \Rightarrow c = 0 \Rightarrow a_n = a_n^{(p)} + a_n^{(h)} = (n^2+n)/2$$

ex 3: Solve the simultaneous recurrence relations

$$a_n = 3a_{n-1} + 2b_{n-1}$$
$$b_n = a_{n-1} + 2b_{n-1}$$

with $a_0 = 1$ and $b_0 = 2$.

Sol :

$$a_n - b_n = 2a_{n-1} \quad \Rightarrow \quad b_n = a_n - 2a_{n-1}$$

$$\Rightarrow a_n = 3a_{n-1} + 2b_{n-1} = 3a_{n-1} + 2a_{n-1} - 4a_{n-2}$$

$$\Rightarrow a_n = 5a_{n-1} - 4a_{n-2} \quad \Rightarrow r^2 - 5r + 4 = 0 \quad \Rightarrow r = 1, 4$$

$$\Rightarrow a_n = \alpha_1 + \alpha_2 4^n$$

$$a_0 = \alpha_1 + \alpha_2 = 1$$

$$a_1 = \alpha_1 + 4\alpha_2 = 3a_0 + 2b_0 = 7 \quad \Rightarrow \alpha_1 = -1,\ \alpha_2 = 2$$

$$\Rightarrow a_n = 2\cdot 4^n - 1 \quad \Rightarrow b_n = a_n - 2a_{n-1} = 2\cdot 4^n - 1 - 4^n + 2 = 4^n + 1$$

Modeling example

- A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the Previous two years.
- a) Find a recurrence relation for L_n where L_n is the number of lobsters caught in year n, under the assumption of this model.
- b) Find Ln if 1,00,000 lobsters were caught in year 1 and 2,00,000 lobsters were caught in year 2.

Modeling example Solution

- a) Since Ln is the average of L_{n-1} and L_{n-2}, the recurrence is Ln = 1/2 L_{n-1} + 1/2 L_{n-2}
- b) The characteristic polynomial x2 –x/2-1/2 = 1/2(2x+1)(x-1) has the roots 1 and -1/2. Hence there exist real constants c1 and c2 such that Ln = $\alpha 1 + \alpha 2(-1/2)^n$. The initial conditions imply the linear relations
- $\alpha 1 + \alpha 2 = 100000$ and $\alpha 1 - \alpha 2/2 = 200000$.
- The solution is $\alpha 1 = \frac{500000}{3}\, and\, \alpha 2 = -200000/3$, and we conclude a_n = 500000/3 + 200000/3 . $(-1/2)^n$
- The second term converges to zero. Thus, the steady state scenario is that 1,66,667 lobsters will be caught every year.