Inverses of Functions

N Geetha
AM & CS
PSG Tech

Progress of Inverses Throughout Math

- Learned Addition and then its inverse operation Subtraction.
- Learned Multiplication and then its inverse operation Division.
- Learning Perfect Squares connects with extracting Square Roots
- **Basically – Inverses are a second operation that reverses the first one!**

Inverse Function

$$f(x) = 9x^2 + 4$$

List the sequence of steps needed to evaluate this function:

1. **Square the Input**
2. **Multiply by 9**
3. **Add 4**

Inverse of a relation

- The inverse of the ordered pairs (**x**, **y**) is the set of all ordered pairs (**y**, **x**).

- The Domain of the function is the range of the inverse and the Range of the function is the Domain of the inverse.

- Symbol: $\boxed{f^{-1}(x)}$

In other words, switch the x's and y's!

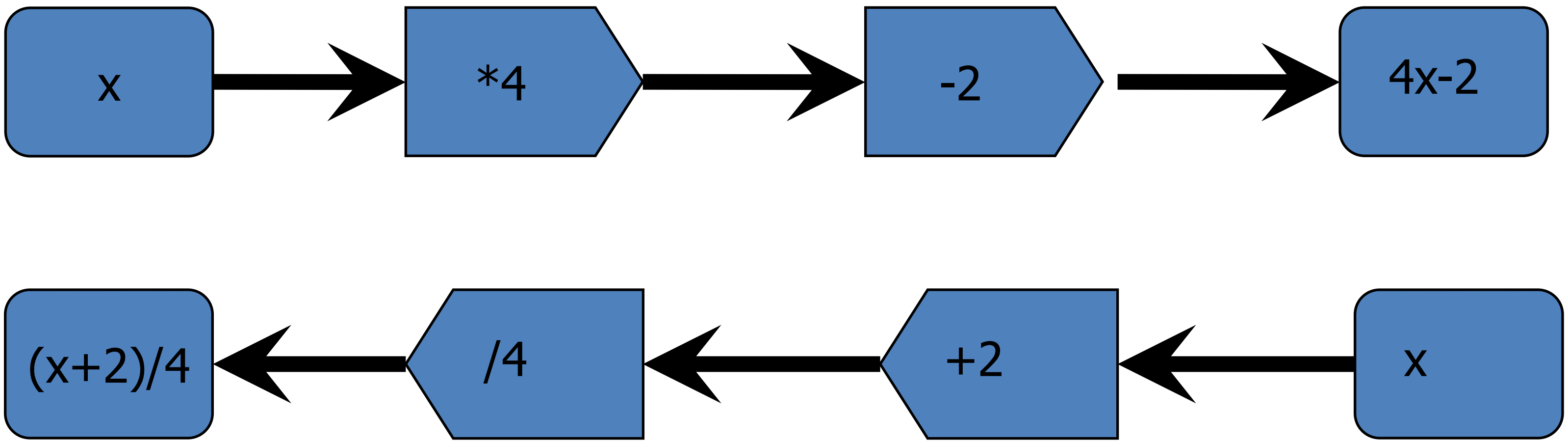
Example: {(1,2), (2, 4), (3, 6), (4, 8)}

Inverse:

$$\{(2,1),(4,2),(6,3),(8,4)\}$$

Function notation? What is really happening when you find the inverse?

Find the inverse of f(x)=4x-2



So

$$f^{-1}(x)=\frac{(x+2)}{4}$$

To find an inverse…

- Replace f(x) as y
- Switch the *x*'s and *y*'s.
- Solve for *y.*
- Change to functional notation.

Find Inverse:

$$\boxed{f(x) = 8x - 1}$$

$$f(x) = 8x - 1$$

$$y = 8x - 1$$

$$x = 8y - 1$$

$$8y = x + 1$$

$$y = \frac{x+1}{8}$$

$$f^{-1}(x) = \frac{x+1}{8}$$

Find Inverse:

$$\boxed{f(x) = 8x - 2}$$

$$f(x) = 8x - 2$$

$$y = 8x - 2$$

$$x = 8y - 2$$

$$8y = x + 2$$

$$y = \frac{x+2}{8}$$

$$f^{-1} = \frac{x+2}{8}$$

Find Inverse:

$$f(x) = \frac{3x+1}{2}$$

$$f(x) = \frac{3x+1}{2}$$

$$y = \frac{3x+1}{2}$$

$$x = \frac{3y+1}{2}$$

$$3y+1 = 2x$$

$$3y = 2x-1$$

$$y = \frac{2x-1}{3}$$

$$f^{-1} = \frac{2x-1}{3}$$

Find Inverse:

$$\boxed{f(x) = x^2 + 4}$$

$$f(x) = x^2 + 4$$

$$y = x^2 + 4$$

$$x = y^2 + 4$$

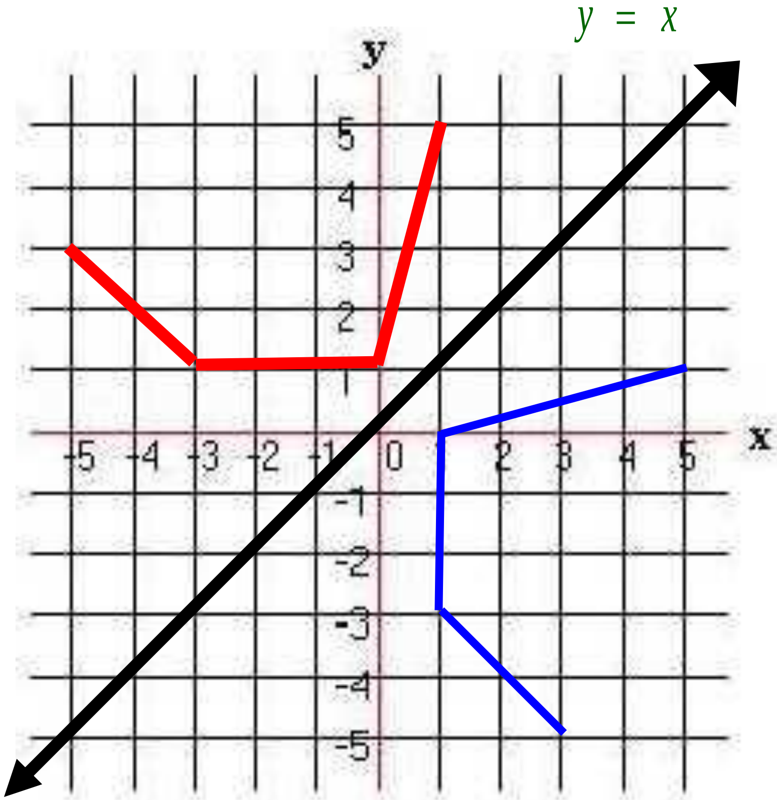
$$y^2 = x - 4$$

$$y = \sqrt{x - 4}$$

$$f^{-1}(x) = \pm\sqrt{x - 4}$$

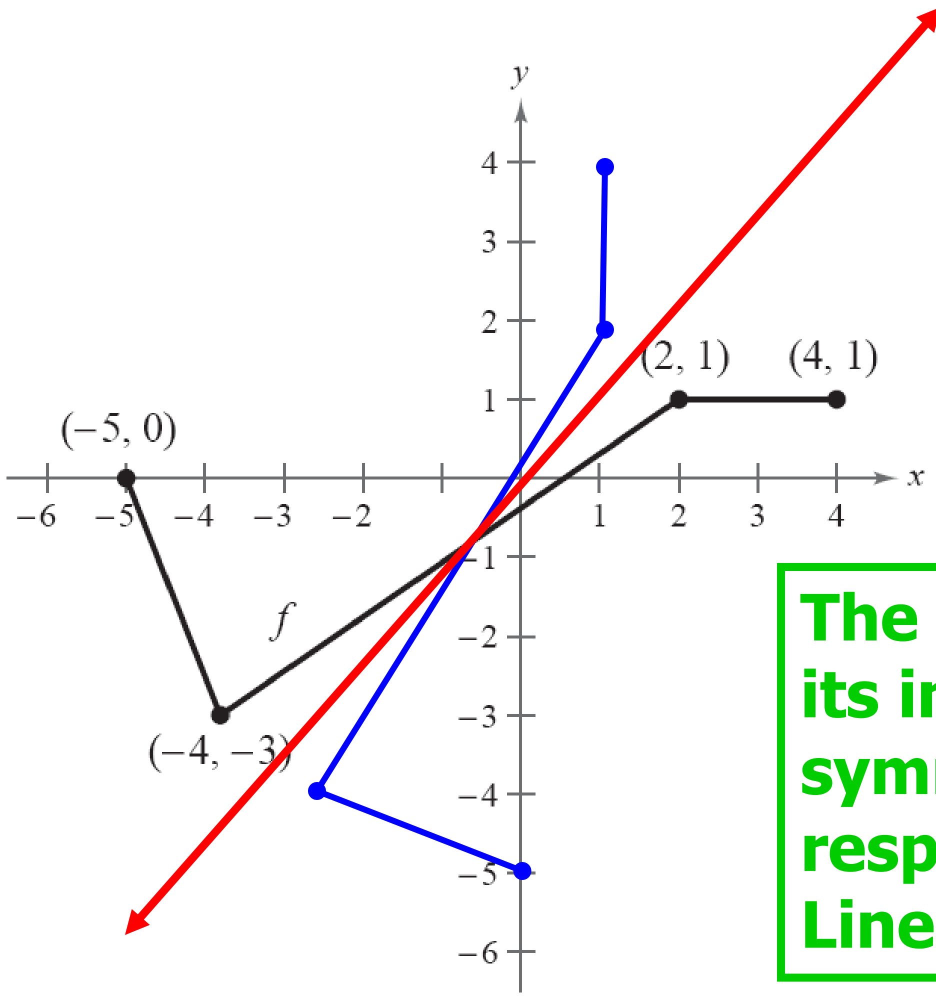
Draw the inverse. Compare to the line y = x. What do you notice?

$(-5,3)$
$(-4,2)$
$(-3,1)$
$(-2,1)$
$(-1,1)$
$(0,1)$
$(1,5)$



$(3,-5)$
$(2,-4)$
$(1,-3)$
$(1,-2)$
$(1,-1)$
$(1,0)$
$(5,1)$

Graph the inverse of the following:



x	y
0	-5
-3	-4
1	2
1	4

The function and its inverse are symmetric with respect to the Line y = x.

How Does an Inverse Exist

- To **have** an inverse function, a function must be **one-to-one**.
- A function is said to be one-to-one if whenever $f(x_1)=f(x_2)$ it implies that $x_1 = x_2$.
- This means that for each y-value there is only one x-value.

More on One-To-One

- Recall that a function is a set of ordered pairs where every first coordinate has exactly one second coordinate.
- A one-to-one function has the added constraint that each 2^{nd} coordinate has exactly one 1^{st} coordinate.

Example

$$\{(1,2),(3,4),(6,10),(8,10)\}$$

Is this relation a function?

Yes – Every 1st coordinate has exactly one 2nd coordinate.

Is this relation one-to-one?

No – Every 2nd coordinate does NOT have exactly one 1st coordinate.

Example

$$\{(1,2),(3,4),(6,10),(8,11)\}$$

Is this relation a function?

Yes – Every 1st coordinate has exactly one 2nd coordinate.

Is this relation one-to-one?

Yes – Every 2nd coordinate has exactly one 1st coordinate.

Example

$$\{(1,2), (1,3), (6,10)\}$$

Is this relation a function?

NO – Every 1st coordinate does NOT have exactly one 2nd coordinate.

Is this relation one-to-one?

NO – Since it is not a function it can NOT be one-to-one.

Things to note..

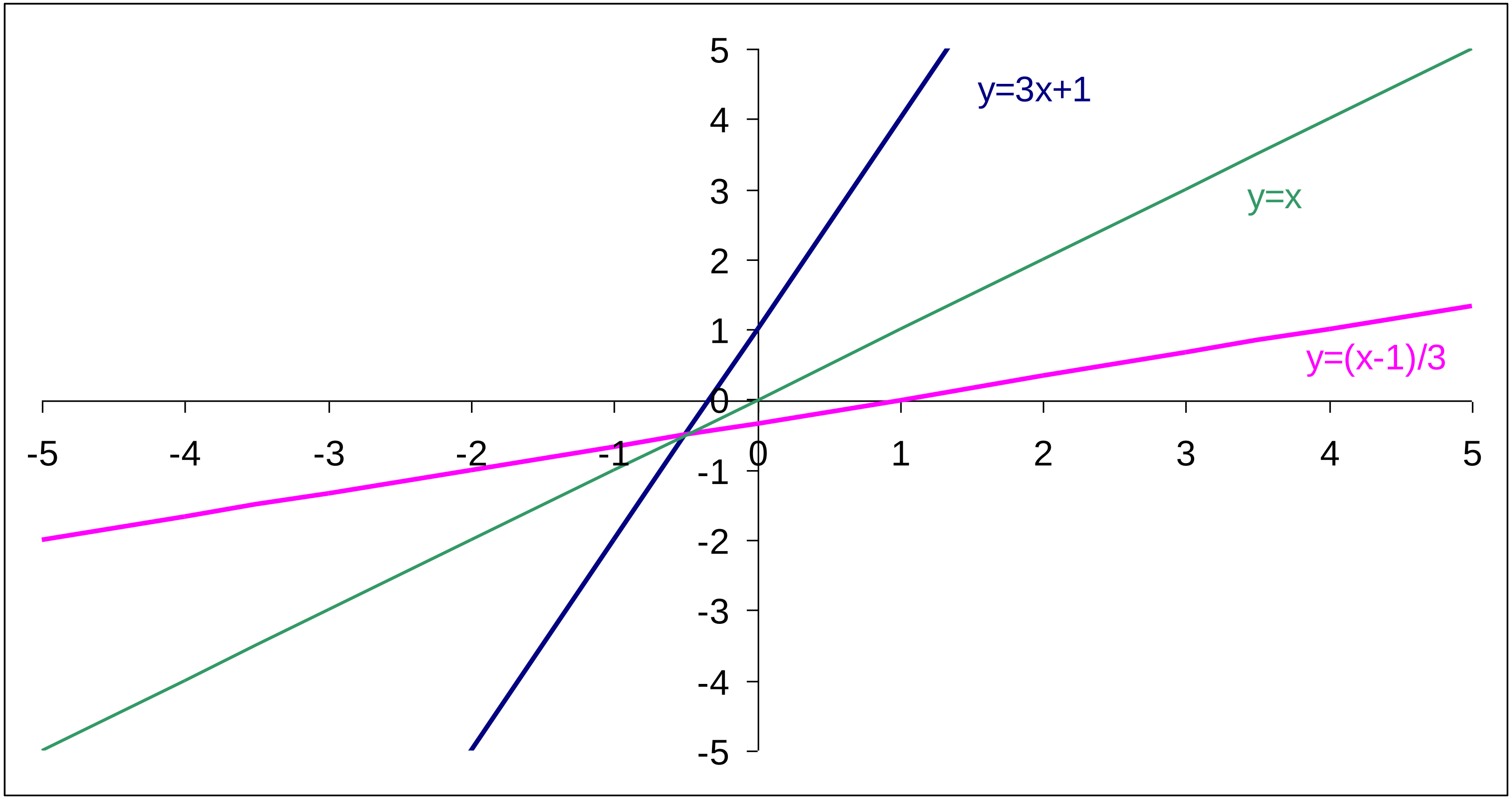
- The domain of $f^{-1}(x)$ is the range of *f(x).*

- The graph of an inverse function can be found by reflecting a function in the line *y=x.*

Check this by plotting *y = 3x + 1* and
$y = \dfrac{x-1}{3}$ on your graphic calculator.

Take a look

Reflecting..



Find the inverse of the function.

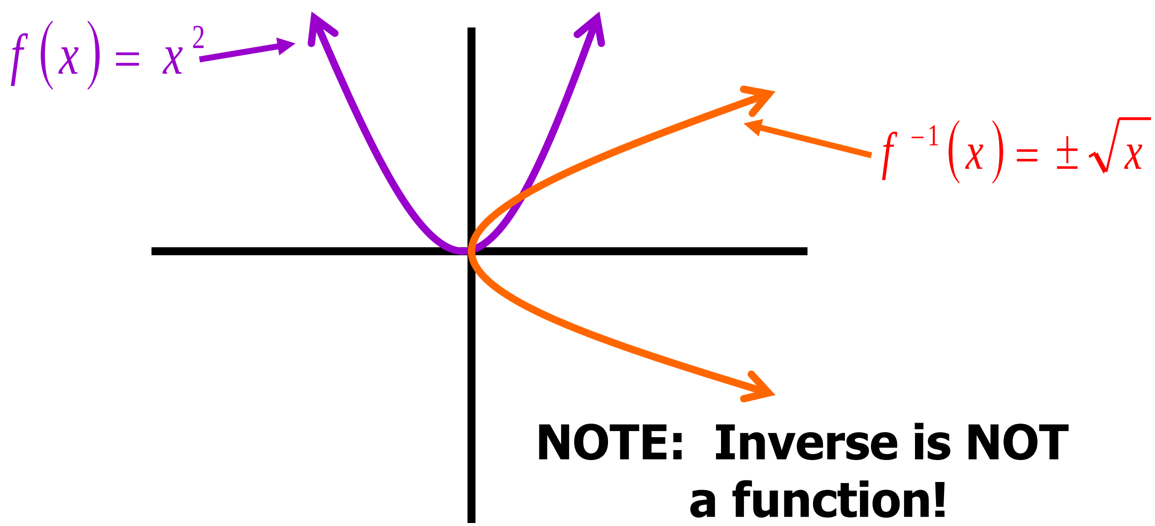
$$f(x) = \sqrt{x}$$

$$x = \sqrt{y}$$
$$x^2 = \left(\sqrt{y}\right)^2$$
$$y = x^2$$

Is the inverse also a function? Let’s look at the graphs.

$$\text{If } f(x) = x^2,$$
$$x = y^2$$
$$\sqrt{x} = \sqrt{y^2}$$
$$y = \pm\sqrt{x}$$

Inverse



NOTE: Inverse is NOT a function!

Horizontal Line Test

- Recall that a function passes the vertical line test.
- The graph of a one-to-one function will pass the horizontal line test. (A horizontal line passes through the function in only one place at a time.)

Is it an Inverse?

A function can only have an inverse if it is one-to-one.

You can use the horizontal line test on graphical representations to see if the function is one-to-one.

Composition and Inverses

- If *f* and *g* are functions and

$$\boxed{(f \circ g)(x) = (g \circ f)(x) = x,}$$

then *f* and *g* are inverses of one another.

Example: Show that the following are inverses of each other.

$$\boxed{f(x) = 7x - 2 \; and \; g(x) = \frac{1}{7}x + \frac{2}{7}}$$

$$(f \circ g)(x) = 7\left(\frac{1}{7}x + \frac{2}{7}\right) - 2$$
$$= x + 2 - 2$$
$$= x$$

$$(g \circ f)(x) = \frac{1}{7}(7x - 2) + \frac{2}{7}$$
$$= x - \frac{2}{7} + \frac{2}{7}$$
$$= x$$

The composition of each both produce a value of x; Therefore, they are inverses of each other.

Are f & g inverses?

$$f(x) = x^3 - 4$$
$$g(x) = \sqrt[3]{x+4}$$

$$(f \circ g)(x) = \left(\sqrt[3]{x+4}\right)^3 - 4$$
$$= x + 4 - 4$$
$$= x$$

$$(g \circ f)(x) = \sqrt[3]{x^3 - 4 + 4}$$
$$= \sqrt[3]{x^3}$$
$$= x$$

YES!

Try….

- Show that

$$\boxed{f(x) = 4x - 3 \; and \;\; g(x) = \frac{1}{4}x + \frac{3}{4}}$$

- are inverses of each other.

$$\boxed{\begin{array}{l} (f \circ g)(x) = (g \circ f)(x) = x \\ \text{Therefore, they ARE} \\ \text{inverses of each other.} \end{array}}$$

Are f & g inverses?

$$f(x) = 3x - 2$$

$$g(x) = \frac{x+2}{3}$$

$$(f \circ g)(x) = 3\left(\frac{x+2}{3}\right) - 2$$

$$= x + 2 - 2$$

$$= x$$

$$(g \circ f)(x) = \frac{3x - 2 + 2}{3}$$

$$= \frac{3x}{3}$$

$$= x$$

$$\boxed{YES!}$$