1.) Find the Fourier Cosine Transformation of the function

$$f(x) = \begin{cases} x & , 0<x<1 \\ 2-x & ; 1<x<2 \\ 0 & , x>2 \end{cases}$$

$$F_c\{f(t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st . dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 t . \cos st \, dt + \int_1^2 (2-t) \cos st . dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\frac{t \sin st}{s} - \frac{\cos st}{s^2} \right]_0^1 + \left[(2-t) \frac{\sin st}{s} + \frac{\cos st}{s^2} \right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\cancel{\frac{\sin s}{s}} - \frac{\cos s}{s^2} + \frac{1}{s^2} \right] + \left[\frac{\cos 2s}{s^2} - \cancel{\frac{\sin s}{s}} - \frac{\cos s}{s^2} \right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1 + \cos 2s - 2\cos s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1 + (2\cos^2 s - 1) - 2\cos s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\cos s(1-\cos s)}{s^2} \right]$$

2) Find the Fourier Cousine Transformation of

$$f(x) = \begin{cases} 1 & , \text{for } 0 < x < a \\ 0 & , \text{otherwise} \end{cases}$$

$$F_c\{f(t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos st \, dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin st}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} . \frac{\sin as}{s}$$

3) Find Fourier Sine Transform of $f(x) = \dfrac{x}{x^2+1}$

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \sin x}{x^2+1}\, dx$$

$$I(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x . \sin sx}{x^2+1}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{x} \times \frac{x . \sin sx}{x^2+1}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x^2 \sin sx}{x(x^2+1)}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty \frac{(x^2+1) \mp 1}{x(x^2+1)} \sin sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty \frac{\sin sx}{x} dx - \int_0^\infty \frac{\sin sx}{x(x^2+1)} dx \right]$$

$t = sx \qquad dt = s\,dx$

$dx = dt/s$ $\qquad \left| \; \int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2} \right.$

$$I(s) = \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} - \int_0^\infty \frac{\sin sx}{x(x^2+1)}\, dx \right]$$

$$I'(s) = \sqrt{\frac{2}{\pi}} \left[\int_0^\infty -\frac{x . \cos sx}{x(x^2+1)} dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty \frac{\cos sx}{(x^2+1)}\, dx \right]$$

$$I''(s) = \sqrt{\frac{2}{\pi}} \left[\int_0^\infty \frac{x \sin sx}{x^2+1} . dx \right]$$

so here,

$$I(s) = I''(s)$$

$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$I(s) = c_1 e^{\lambda_1 s} + c_2 e^{\lambda_2 s}$

$I(s) = c_1 e^{s} + c_2 e^{-s}$ $\qquad I(0) = c_1 + c_2$

$I'(s) = c_1 e^{s} - c_2$ $\qquad I'(0) = c_1 - c_2$

$I(0) = \sqrt{\frac{\pi}{2}}$

$I'(0) = -\sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{1}{x^2+1} \, dx = -\sqrt{\frac{2}{\pi}} \left[\tan^{-1} x\right]_0^\infty = -\sqrt{\frac{\pi}{2}}$

4) Find the Fourier Transform of $f(x) = \begin{cases} 1-|x|, & \text{for } |x| < x \\ 0 & \text{otherwise} \end{cases}$.

Hence P.T. $\int_0^\infty \frac{\sin^4 x}{x^4} \, dx = \frac{\pi}{3}$

$1 - |x|$ is even func so $F\{f(t)\} = F_c\{f(t)\}$

$F_c\{f(t)\} = \sqrt{\frac{2}{\pi}} \int_0^1 [1-(x)] \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}} \left[(1-x) \frac{\sin sx}{s} - \frac{(-1)(-\cos sx)}{s^2} \right]_0^1$

$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos s}{s^2} - \left[\frac{\sin 0}{s} - \frac{\cos 0}{s^2} \right] \right]$

$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos s}{s^2} + \frac{1}{s^2} \right]$

$F_c = \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos s}{s^2} \right]$

$\cos s = 1 - 2 \sin^2 s/2$

$\Rightarrow 2 \sin^2 s/2 = 1 - \cos s$

$$F_c = \sqrt{\frac{2}{\pi}}\left[\frac{(2\sin^2 s/2)}{2^2 \cancel{\times} (s/2)^2}\right]$$

$$= \frac{\cancel{2}}{4}\frac{1}{\cancel{2}}\sqrt{\frac{2}{\pi}}\left[\frac{\sin^2 s/2}{(s/2)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin^2 s/2}{(s/2)^2}\right]$$

W.K.T.

$$\int_0^1 (1-x)^2 dx = \int_0^\infty (F_c(x))^2 dx = \frac{1}{2\pi}\left[\frac{\sin^2 s/2}{(s/2)^2}\right]^2 ds$$

$s/2 = x \quad \Rightarrow ds = 2dx$

$$= \frac{1}{2\pi}\left[\frac{\sin^2 x}{(x)^2}\right]^2 2.dx$$

$$\int_0^1 (1-x)^2 dx = \frac{1}{\pi}\left[\frac{\sin^4 x}{x^4}\right]dx$$

$$\left| \int_0^1 (1-x)^2 dx = \frac{1}{3}\right.$$

$$\Rightarrow \boxed{\frac{\pi}{3} = \frac{\sin^4 x}{x^4} dx}$$

5) Find the Inverse Fourier Transform of $F(s)$, where $F(s) = \begin{cases} a-|s| & \text{for } |s|<a \\ 0 & \text{for } |s|>0 \end{cases}$ and eval $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$

$$F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ist} F(s)\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} e^{ist}(a-|s|)\, ds\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} (\cos st + i\sin st)(a-|s|)\, ds\right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\int_0^a (\cos st)(a - |s|)\, ds) \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\int_0^a (\cos st)(a)\, ds - \int_0^a (s) \cos st\, ds \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left[\frac{a(\sin st)}{t} \right]_0^a - \left[\frac{s(\sin st)}{t} + \frac{\cos st}{t^2} \right]_0^a \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{(a)\sin at}{t} - \left[\frac{(a)\sin at}{t} + \frac{\cos at}{t^2} - \frac{1}{t^2} \right] \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{1 - \cos at}{t^2} \right]$$

$$f(t) = \sqrt{\frac{2}{\pi}} \left[\frac{-(\cos at - 1)}{t^2} \right]$$

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} \sqrt{\frac{2}{\pi}} \frac{-(\cos at - 1)}{t^2}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-ist} \frac{-(\cos at - 1)}{t^2}$$

| $\frac{\cos at - 1}{t^2}$ is Even func

$$= \frac{1}{\pi} 2 \int_0^{\infty} \cos st \frac{-(\cos at - 1)}{t^2}$$

| $e^{-ist} = \cos st - i \sin st$

$$F(s) = \frac{2}{\pi} \int_0^{\infty} (\cos st)(2) \left(\frac{\sin^2 at/2}{t^2}\right) dt$$

$s = 0$; $a = 2$;

$$2 - |0| = \frac{2}{\pi}(2) \int_0^{\infty} \frac{\sin^2 t}{t^2} dt$$

$$\Rightarrow \boxed{\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}}$$