Mathematical Foundations of Computing

Propositional Calculus : 5

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Theory of Inference

- Deriving a logical conclusion by combining many propositions and using formal logic: hence, determining the truth of arguments.
- An argument is a sequence of statements in which the conjunction of the initial statements (called the premises/hypotheses) is said to imply the final statement (called the conclusion).
- An argument can be presented symbolically as

$$(H_1 \land H_2 \land \ldots \land H_n) \Longrightarrow C$$

where $H_1, H_2, \ldots, H_n$ represent the hypotheses and C represents the conclusion.

What is an argument?

- A sequence of statements the ends with a conclusion.
- (Not the common language usage of a debate or dispute.)

Structure of an argument

Statement 1 (H_1)
Statement 2 (H_2)
…
Statement n (H_n)

Therefore, Statement C (conclusion)

premises or antecedents

Formal Proof

- If A and B are statement formulas, $A \Rightarrow B$ iff $A \rightarrow B$ is a tautology.
- Eg1. $P \Rightarrow P \vee Q$ ie. $P \rightarrow (P \vee Q)$ is a tautology
- Eg2:
  - H_1: Neil Armstrong was the first human to step on the moon.
  - H_2 : Mars is a red planet
  - C: No human has ever been to Mars.
  - This wff $H_1 \wedge H_2 \rightarrow C$ is not a tautology
- An argument is valid if whenever the hypotheses are all true, the conclusion must also be true. A valid argument is intrinsically true ie. $(H_1 \wedge H_2 \wedge \ldots \wedge H_n) \rightarrow C$ is a tautology.
- Use a proof sequence to arrive at a valid argument.
- A proof sequence is a sequence of wffs in which each wff is either a hypothesis or the result of applying one of the formal system's derivation rules to earlier wffs in the sequence.

Formal Proof

- Proofs in mathematics are *valid arguments*
- An *argument* is a sequence of statements that end in a conclusion
- By *valid* we mean the conclusion must follow from the truth of the preceding statements or premises.
- How do we show that an argument is valid?
  - We can use a truth table, or
  - We can show that $(H_1 \wedge H_2 \wedge \ldots \wedge H_n \rightarrow C)$ is a tautology using some rules of inference.
- Eg.
- Is this a valid argument?
- If you listen, you will hear what I'm saying
- You are listening. Therefore, you hear what I am saying.
- Let P represent "you listen"
- Let Q represent "you hear what I am saying"
- $(P \rightarrow Q) \wedge P \Rightarrow Q$. It is a valid argument.

1. Truth Table Technique

Theory of Inference

1. Truth Table Technique

$H_1, H_2 \ldots H_n$ are premises, C is Conclusion

If $(H_1 \wedge H_2 \ldots \wedge H_n) \Rightarrow C$, then C is a valid conclusion.

eg $H_1 : \neg P$ $H_2 : P \vee Q$, $C : Q$.

P	Q	$\neg P$	$P \vee Q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

$H_1 \wedge H_2$ is T only in 3rd row, In that row, C is also T

$\therefore$ C is a valid conclusion.

1. Truth Table Technique

Problems:

1) Find if C follows from H_1, H_2 & H_3 using TT.

$H_1 : \neg P$, $H_2 : P \vee Q$,

		H_1	H_2	C
P	Q	$\neg P$	$P \vee Q$	$P \wedge Q$
T	T	F	T	T
T	F	F	T	F
→ F	T	T	T	F
F	F	T	F	F

H_1 and H_2 are T in 3rd row,

But C is false.

Hence C is invalid.

1. Truth Table Technique

- Eg3.:
- $P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$

- Issues
  - Constructing a truth table is time consuming!
  - If we have *n* propositions, what is the size of the truth table? 2^n, which means that the table doubles in size with every proposition.

2. Rules for Propositional Logic

- Derivation rules for propositional logic

Equivalence Rules	Inference Rules
Allows individual wffs to be rewritten	Allows new wffs to be derived
Truth preserving rules	Work only in one direction

Inference Rules

From	Can Derive	Rule
$P \land Q$	P, Q	Conjunctive Simplification
P	$P \lor Q$	Disjunctive Amplification
$P, P \to Q$	Q	Modus Ponens-***modus ponens*** (Latin) translates to "***mode that affirms***" Aka ***law of detachment***
$P \to Q, \sim Q$	$\sim P$	Modus Tollens **"method of denying"**
$(P \to Q) \land (Q \to R)$	$P \to R$	Chain Rule or Hypothetical Syllogism
$(P \lor Q) \land (\sim P \lor R)$	$(Q \lor R)$	Resolution

- Note: Inference rules ***do not*** work in both directions, unlike equivalence rules.

Rules of Inference for Propositional Logic

modus ponens

$$p \to q$$
$$p$$
$$\therefore q$$

$$\sqrt{2} > \frac{3}{2} \to (\sqrt{2})^2 > \left(\frac{3}{2}\right)^2$$
$$\sqrt{2} > \frac{3}{2}$$
$$\therefore 2 > \frac{9}{4}$$

A valid argument can lead to an incorrect conclusion if one of its premises is wrong/false!

$$p: \ \sqrt{2} > \frac{3}{2}$$

$$q: \ 2 > \left(\frac{3}{2}\right)^2$$

$$p \to q$$

The argument is valid as it is constructed using modus ponens
But one of the premises is false (p is false)
So, we cannot derive the conclusion

Examples

- Example for using Equivalence rule in a proof sequence:
  - Simplify (~A V ~B) V C

1. (~A V ~B) V C
2. ~(A Λ B) V C — De Morgan
3. (A Λ B) → C — Conditional

- Example of using Inference Rule
  - If it is bright and sunny today, then I will wear my sunglasses.

<u>Modus Ponens</u>

It is bright and sunny today. Therefore, I will wear my sunglasses.

<u>Modus Tollens</u>

I will not wear my sunglasses. Therefore, it is not bright and sunny today.

Using the resolution rule (an example)

1. Anna is skiing or it is not snowing.
2. It is snowing or Bart is playing hockey.
3. Consequently Anna is skiing or Bart is playing hockey.

We want to show that (3) follows from (1) and (2)

Types of Inferences

- Deduction (what we use in propositions)
  - Inference based on logical certainty
  - Starts from a general principle and infers something about specific cases
  - Valid deduction guarantees the truth of the conclusion
  - Eg: Grapes are poisonous to all dogs. Your dog too
- Induction
  - Inference based on probability
  - Starts from specific information and infers the general principle
  - Truth of the conclusion is not guaranteed
  - Eg. For the last two years Amanda has woken up at 6 am everyday. Tomorrow too.
- Abduction
  - Inference based on the best explanation
  - given a theory relating faults with their effects and a set of observed effects, abduction can be used to derive sets of faults that are likely to be the cause of the problem

Deduction Proof

- Eg1. :Prove the argument

  $A \land (B \rightarrow C) \land [(A \land B) \rightarrow (D \lor \sim C)] \land B \Longrightarrow D$

  Use the inference and equivalence rules to get at the conclusion D.

1. $A \land (B \rightarrow C) \land [(A \land B) \rightarrow (D \lor \sim C)] \land B$ Associative
2. $A \land B \land (B \rightarrow C) \land [(A \land B) \rightarrow (D \lor \sim C)]$ Modus Ponens
3. $A \land C \land [(A \land B) \rightarrow (D \lor \sim C)]$ Conditional
5. $A \land C \land [\sim(A \land B) \lor (D \lor \sim C)]$ DeMorgan
6. $A \land C \land [\sim A \lor \sim B \lor (D \lor \sim C)]$ Associative
7. $A \land C \land [\sim A \lor \sim C \lor (D \lor \sim B)]$ DeMorgan
8. $A \land C \land [\sim(A \land C) \lor (D \lor \sim B)]$ Conditional
9. $A \land C \land [(A \land C) \rightarrow (D \lor \sim B)]$ Modus Ponens
10. $(D \lor \sim B) \land B$ Comm,Conditio
11. $B \land (B \rightarrow D)$ Modus Ponens
12. D Valid conclusion

Eg2: More Proofs

- Prove that $\sim(A \wedge B) \wedge \sim(\sim C \wedge A) \wedge \sim(C \wedge \sim B) \Longrightarrow \sim A$ is valid

1. $\sim A \vee \sim B \wedge \sim(\sim C \wedge A) \wedge \sim(C \wedge \sim B)$ De Morgan
2. $\sim B \vee \sim A \wedge \sim(\sim C \wedge A) \wedge \sim(C \wedge \sim B)$ Comm
3. $(B \rightarrow \sim A) \wedge \sim(\sim C \wedge A) \wedge \sim(C \wedge \sim B)$ Condnl
4. $(B \rightarrow \sim A) \wedge (C \vee \sim A) \wedge \sim C \vee B$ De Morgan (2)
5. $(B \rightarrow \sim A) \wedge (\sim C \rightarrow \sim A) \wedge (C \rightarrow B)$ Condnl (2)
6. $(B \rightarrow \sim A) \wedge (\sim C \rightarrow \sim A) \wedge (C \rightarrow B)$ Assoc, Chain
7. $(C \rightarrow \sim A) \wedge (\sim C \rightarrow \sim A)$ Condnl
8. $(\sim C \vee \sim A) \wedge (C \vee \sim A)$ Distrib
9. $(\sim C \wedge C) \vee \sim A$ Complement
10. $(F) \vee \sim A$ Identity
11. $\sim A$ A valid conclusion

Eg3:Proving Verbal Arguments

- Russia was a superior power, and either France was not strong or Napoleon made an error. Napoleon did not make an error, but if the army did not fail, then France was strong. Hence the army failed and Russia was a superior power.
- Converting it to a propositional form using letters R, S, N and A

  R: Russia was a superior power

  S: France was strong

  N: Napoleon made an error

  A: The army failed
- Combining, the statements using logic

$(R \wedge (\sim S \vee N))$	hypothesis
$\sim N$	hypothesis
$(\sim A \rightarrow S)$	hypothesis
$(A \wedge R)$	conclusion

- Combining them, the propositional form is

  $(R \wedge (\sim S \vee N)) \wedge \sim N \wedge (\sim A \rightarrow S) \Longrightarrow (A \wedge R)$

Verbal Argument Proof

- Prove $(R \land (\sim S \lor N)) \land \sim N \land (\sim A \to S) \Rightarrow (A \land R)$

- Proof sequence
  1. $(R \land (\sim S \lor N)) \land \sim N \land (\sim A \to S)$ condnl
  2. $R \land (\sim N \to \sim S) \land \sim N \land (\sim A \to S)$ Modus Ponens
  3. $R \land \sim S \land (\sim A \to S)$ Modus Tollens
  4. $R \land A$

Class Exercise

- 1. Prove the following arguments
  - a) $(\sim A \rightarrow \sim B) \wedge (A \rightarrow C) \Longrightarrow (B \rightarrow C)$
  - b) $(Y \rightarrow \sim Z) \wedge (\sim X \rightarrow Y) \wedge [Y \rightarrow (X \rightarrow W)] \wedge (Y \rightarrow Z) \Longrightarrow (Y \rightarrow W)$

- 2. If the program is efficient, it executes quickly. Either the program is efficient, or it has a bug. However, the program does not execute quickly. Therefore it has a bug. (use letters E, Q, B)

- 3. The crop is good, but there is not enough water. If there is a lot of rain or not a lot of sun, then there is enough water. Therefore the crop is good and there is a lot of sun. (use letters C, W, R, S)

Class Exercise

- 4. Is the argument valid ?

- If my client is guilty, then the knife was in the drawer. Either the knife was not in the drawer or Jason Pritchard saw the knife. If the knife was not there on October 10, it follows that Jason Pritchard didn’t see the knife. Furthermore, if the knife was there on October 10, then the knife was in the drawer and also the hammer was in the barn. But we all know that the hammer was not in the barn. Therefore, ladies and gentlemen of the jury, my client is innocent. (C,K,J,O,H)

  (Hint : Take innocent as not guilty)

Inference rules

- **Logical inference** creates new sentences that logically follow from a set of sentences (KB)
- An inference rule is **sound** if every sentence X it produces when operating on a KB logically follows from the KB
  - i.e., inference rule creates no contradictions
- An inference rule is **complete** if it can produce every expression that logically follows from (is entailed by) the KB.
  - All valid statements can be proved

Application of Inference

- 1. To check the validity of an argument
- 2. Direct Proof (same methodology)
- 3. To check a set of given specification is consistent or not
- 4. Indirect Proof or Proof by Contradiction

3. Inconsistent Specification

- A set of specification S1, S2, … Sn is said to be inconsistent if their conjunction implies a contradiction
- $S1 \wedge S2 \wedge \ldots Sn \Leftrightarrow F$
- A system of specification S1,S2, ..Sn is said to be consistent if it is not inconsistent
- $S1 \wedge S2 \wedge \ldots Sn$ is atleast satisfiable
- (Use rules to see satisfiability)

3. Check for consistency

- Given Specification:
- $(P \rightarrow Q) \wedge (Q \rightarrow R) \wedge (S \rightarrow \sim R) \wedge P \wedge S$

1. $(P \rightarrow Q) \wedge (Q \rightarrow R) \wedge (S \rightarrow \sim R) \wedge P \wedge S$ chain rule
2. $(P \rightarrow R) \wedge (S \rightarrow \sim R) \wedge P \wedge S$ Assoc, Modus Ponens
3. $R \wedge \sim R$ Complement
4. F

- Hence the given spec is inconsistent

4. Indirect Proof

- Proof by contradiction or Reduction and Absurdum
- A set of premises H1, H2, … Hn and a conclusion C is given. We assume C is false and ~C to be true and add it with (H1^H2…^Hn) as (H1^H2…^Hn ^~C)
- If (H1^H2…^Hn ^~C) leads to a contradiction, then our assumption is false. Hence C is true.

4. Indirect Proof Example

- Proof by contradiction that
- $(A \rightarrow B) \wedge \sim(B \vee C) \Rightarrow \sim A$

1.	$(A \rightarrow B) \wedge \sim(B \vee C) \wedge \sim(\sim A)$	add ~C as a premise
2.	$(A \rightarrow B) \wedge \sim(B \vee C) \wedge \sim(\sim A)$	Involution
3.	$(A \rightarrow B) \wedge \sim(B \vee C) \wedge A$	Modus Ponens
4.	$B \wedge \sim(B \vee C)$	distributive
5.	$B \wedge \sim B \wedge \sim C$	complement
6.	F	

- It is a contradiction, so our assumption is false and so C is true.

Propositional logic

- Advantages
  - Simple Knowledge Representation language sufficient for some problems
  - Lays the foundation for higher order logic (e.g., FOL- first order logic)
  - Reasoning is decidable, though NP complete, and efficient techniques exist for many problems
- Disadvantages
  - Not expressive enough for most problems
  - Even when it is, it can be very “un-concise”

PL is a weak KR language

- Hard to identify “individuals” (e.g., Mary, 3)
- Can’t directly talk about properties of individuals or relations between individuals (e.g., “Bill is tall”)
- Generalizations, patterns, regularities can’t easily be represented (e.g., “all triangles have 3 sides”)
- First-Order Logic (FOL) is expressive enough to represent this kind of information using relations, variables and quantifiers, e.g.,
  - *Every elephant is gray:* ∀ x (elephant(x) → gray(x))
  - *There is a white alligator:* ∃ x (alligator(X) ^ white(X))

PL Example

- Consider the problem :
  - Every person is mortal. Confucius is a person. Confucius is mortal.
- How can these sentences be represented to infer the third ?
- In PL we have to create propositional symbols to stand for all or part of each sentence, e.g.: P = “person”; Q = “mortal”; R = “Confucius”
- The above 3 sentences are represented as:

  $P \rightarrow Q$; $R \rightarrow P$; $R \rightarrow Q$
- 3rd sentence is entailed by the first two, but we need an explicit symbol, R, to represent an individual, Confucius, who is a member of the classes *person* and *mortal*
- Representing other individuals requires introducing separate symbols for each, with some way to represent the fact that all individuals who are “people” are also “mortal”

Propositional logic summary

- Inference is the process of deriving new sentences from old
  - **Sound** inference derives true conclusions given true premises; i.e., inference rule creates no contradictions
  - **Complete** inference derives all true conclusions from a set of premises
- A **valid sentence** is true in all worlds under all interpretations
- **Propositional logic** commits only to the existence of facts that may or may not be the case in the world being represented
  - Simple syntax and semantics suffices to illustrate the process of inference
  - Propositional logic can become impractical, even for very small worlds

Propositional logic: summary of typical problems

1. A proof of a tautology / contradiction
2. Prove that a set of formulas is not satisfiable
3. (Direct / indirect) proof of validity of an argument
4. Inferring consequences of assumptions

- We can solve by:
  - truth-table technique
  - Equivalence and inference rules
  - Normal forms
  - Resolution method

References

- Formal Logic, W.H. Freeman & Co
- Kenneth H Rosen, “Discrete Mathematics and its Applications”, Tata McGraw Hill, 2011.