Counting 3

N Geetha
AM & CS
PSG Tech

Binomial Coefficients

- Expressions of the form C(n, k) are also called **binomial coefficients**.
- A **binomial expression** is the sum of two terms, such as $(x + y)$.
- Now consider $(x + y)^2 = (x + y)(x + y)$.
- When expanding such expressions, we have to form all possible products of a term in the first factor and a term in the second factor:
- $(x + y)^2 = x \cdot x + x \cdot y + y \cdot x + y \cdot y$
- Then we can sum identical terms:
- $(x + y)^2 = x^2 + 2xy + y^2$

Binomial Coefficients

- For $(x + y)^3 = (x + y)(x + y)(x + y)$ we have
- $(x + y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$
- $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- There is only one term x^3, because there is only one possibility to form it: Choose **x** from all three factors: C(3, 3) = 1.
- There is three times the term x^2y, because there are three possibilities to choose **x** from two out of the three factors: C(3, 2) = 3.
- Similarly, there is three times the term xy^2 (C(3, 1) = 3) and once the term y^3 (C(3, 0) = 1).

Binomial Coefficients

- The binomial theorem gives the coefficients of the expansion of powers of binomial expressions.
- $\binom{n}{r}$ is called a binomial coefficient because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(x+y)^n$

Binomial Theorem

$$
\begin{aligned}
(x+y)^n &= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \\
&= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n
\end{aligned}
$$

Examples

- 1. What is the expansion of $(x+y)^4$?

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$
$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$
$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4.$$

- 2. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?
- From Binomial Theorem, $\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$
- 3. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?
- $(2x+(-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$ = $\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13!\,12!} 2^{12} 3^{13}$

Binomial Coefficient Identities

- Corrolary1: Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

- Proof: x=1, y=1;

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

- Corrolary2: Let n be a positive integer. Then
-

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

- Proof: x=1, y=(-1);

$$0 = 0^n = ((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Binomial Coefficient Identities

- Corrolary3: Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

- Proof: x=1, y=2;

$$(1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Pascal's Triangle

PASCAL'S IDENTITY Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

- Proof : RHS = $\binom{n}{k-1} + \binom{n}{k}$

Pascal's Identity and Triangle

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Pascal's Triangle

- In Pascal's triangle, each number is the sum of the numbers to its upper left and upper right:

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
…	…	…	…	…	…					

Pascal’s Triangle

- Since we have $C(n + 1, k) = C(n, k - 1) + C(n, k)$ and $C(0, 0) = 1$, we can use Pascal’s triangle to simplify the computation of $C(n, k)$:

k

n

$C(0, 0) = 1$

$C(1, 0) = 1 \quad C(1, 1) = 1$

$C(2, 0) = 1 \quad C(2, 1) = 2 \quad C(2, 2) = 1$

$C(3, 0) = 1 \quad C(3, 1) = 3 \quad C(3, 2) = 3 \quad C(3, 3) = 1$

$C(4, 0) = 1 \quad C(4, 1) = 4 \quad C(4, 2) = 6 \quad C(4, 3) = 4 \quad C(4, 4) = 1$

Binomial Coefficients

- This leads us to the following formula:

$$\boxed{(x+y)^n = \sum_{j=0}^{n} C(n,j) \cdot x^{n-j} y^j}$$

(Binomial Theorem)

With the help of Pascal's triangle, this formula can considerably simplify the process of expanding powers of binomial expressions.

For example, the fifth row of Pascal's triangle (1 – 4 – 6 – 4 – 1) helps us to compute $(a + b)^4$:

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Vandermonde's Identity

VANDERMONDE'S IDENTITY Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

- Corollary 4: If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

- Proof: We use Vandermonde's identity with $m = r = n$ to obtain
-

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Example

Suppose that k and n are integers with $1 \leq k < n$. Prove the **hexagon identity**

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1},$$

which relates terms in Pascal's triangle that form a hexagon.

Example

Prove the **hockeystick identity**

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

a) using a combinatorial argument.

b) using Pascal's identity.