Sol. of Prob-S-II
Spring 2022

Sol 1 (a) Let D = Detection
E = Arrive via Email
I = Arrive via Internet

We want $P(D)$.

By Law of Total Probability

$$P(D) = P(D|E)P(E) + P(D|I)P(I)$$

Substituting $P(I) = 70/100$, $P(E) = 30/100$, $P(D|I) = 0.6$ & $P(D|E) = 0.8$ we get

$$P(D) = (0.8)\left(\frac{30}{100}\right) + (0.6)\left(70/100\right)$$

$$\Rightarrow P(D) = 0.66$$

So 66% of times the spyware is detected.

(b)

$$P(A \cup B \cup C) = P(A \cup (B \cup C))$$

$$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$$

$$= P(A) + P(B) + P(C) - P(BC) - P(AB \cup AC)$$

$$= P(A) + P(B) + P(C) - P(BC) - [P(AB) + P(AC) - P(ABAC)]$$

$$\Rightarrow \boxed{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)}$$

proved

Sol 2 (a)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$\Rightarrow f_X(x) = \int_0^{\infty} y\,e^{-y(1+x)}\,dy$$

$$= \left. \frac{y e^{-y(1+x)}}{-(1+x)} \right|_0^{\infty} + \int_0^{\infty} \frac{e^{-y(1+x)}}{1+x}\,dy$$

$$= \frac{-1}{1+x}\left(\frac{y}{e^{y(1+x)}} \right)\Bigg|_0^{\infty} + \frac{1}{1+x} \cdot \left. \frac{e^{-y(1+x)}}{-(1+x)} \right|_0^{\infty}$$

$$= -\frac{1}{1+x}(0-0) - \frac{1}{(1+x)^2}\left(\frac{1}{e^{y(1+x)}} \right)\Bigg|_0^{\infty}$$

$$= -\frac{1}{(1+x)^2}(0-1) = \frac{1}{(1+x)^2}$$

$$\Rightarrow \boxed{f_X(x) = \frac{1}{(1+x)^2}, \; x>0}$$

which is the Required PDF of X.

Similarly

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} y\,e^{-y(1+x)}\,dx = y e^{-y} \int_0^{\infty} e^{-yx}\,dx$$

$$= \cancel{y} e^{-y} \cdot \left. \frac{e^{-yx}}{-\cancel{y}} \right|_0^{\infty} = -e^{-y(1+x)} \Big|_0^{\infty} = -(0 - e^{-y})$$

$$\Rightarrow \boxed{f_Y(y) = e^{-y}, \; y>0}$$

which is the PDF of Y.

(b) $P\{X>2, Y>2\}$

$$= \int_2^\infty \int_2^\infty y e^{-y(1+x)} \, dx \, dy$$

$$= \int_2^\infty \frac{\cancel{y} e^{-y(1+x)}}{-\cancel{y}} \Big|_2^\infty dy = -\int_2^\infty e^{-y(1+x)} \Big|_2^\infty dy$$

$$= -\int_2^\infty (0 - e^{-3y}) dy = \int_2^\infty e^{-3y} dy = \frac{e^{-3y}}{-3} \Big|_2^\infty$$

$$= -\frac{1}{3}\left(\frac{1}{e^{3y}}\right)\Big|_2^\infty = -\frac{1}{3}\left(0 - \frac{1}{e^6}\right) = \frac{1}{3e^6}$$

$$\Rightarrow \boxed{P\{X>2, Y>2\} = \frac{1}{3e^6} \simeq 0.000826}$$ Ans

Let

D_P = Detected Present $\Rightarrow D_P^c$ = Not detected Present

P_R = Present in Range $\Rightarrow P_R^c$ = Not in range

Given that

$$P(D_P \mid P_R) = \frac{98}{100} \Rightarrow P(D_P^c \mid P_R) = \frac{2}{100}$$

$$P(D_P \mid P_R^c) = \frac{5}{100} \Rightarrow P(D_P^c \mid P_R^c) = \frac{95}{100}$$

$$P(P_R) = \frac{7}{100} \Rightarrow P(P_R^c) = 1 - \frac{7}{100} = \frac{93}{100}$$

$$P(P_R^c \mid D_P) = ? \quad \& \quad P(P_R^c \mid D_P^c) = ?$$

By Bayes Rule

$$P(P_R^c \mid D_P) = \frac{P(D_P \mid P_R^c)\,P(P_R^c)}{P(D_P \mid P_R)\,P(P_R) + P(D_P \mid P_R^c)\,P(P_R^c)}$$

$$= \frac{\frac{5}{100} \cdot \frac{93}{100}}{\frac{98}{100}\,\frac{7}{100} + \frac{5}{100} \cdot \frac{93}{100}} = \frac{5 \times 93}{98 \times 7 + 5 \times 93}$$

$$\Rightarrow \boxed{P(P_R^c \mid D_P) \simeq 0.404} \text{ Ans}$$

Similarly

$$P(P_R^c \mid D_P^c) = \frac{P(D_P^c \mid P_R^c)\,P(P_R^c)}{P(D_P^c \mid P_R)\,P(P_R) + P(D_P^c \mid P_R^c)\,P(P_R^c)}$$

$$\Rightarrow P(P_R^c \mid D_P^c) = \frac{95/100 \cdot 93/100}{2/100 \cdot 7/100 + 95/100 \cdot 93/100} = \frac{95 \times 93}{(2 \times 7) + (95 \times 93)} \simeq \boxed{0.998} \text{ Ans}$$

Sol 4

(a)

$$p(x) = \begin{cases} \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{3} & \text{if } x = 2 \\ \frac{1}{6} & \text{if } x = 3 \end{cases}$$

(b)

$$P\{X < 4\} = \int_2^4 \frac{2(1+x)}{27}\,dx$$

$$= \frac{2}{27}\int_2^4 (1+x)\,dx = \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4$$

$$= \frac{2}{27}\left[\left(4 + \frac{4^2}{2}\right) - \left(2 + \frac{2^2}{2}\right)\right]$$

$$= \frac{2}{27}\left[\cancel{4} + 8 - (2 \cancel{+ 2})\right]$$

$$\boxed{P\{X<4\} = \frac{16}{27} \approx 0.5926}$$ Ans

The End