Solutions of Equations in One Variable

Newton Method

Newton's (or the Newton-Raphson) method is one of the most powerful and well-known numerical methods for solving a root-finding problem.

Consider the first Taylor polynomial for $f(x)$ expanded about p_0 nd evaluated at $x = p$.

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p))$$

there $\xi(p)$ lies between p and p_0. Since $f(p) = 0$, this equation gives

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p))$$

Newton's method is derived by assuming that since $|p - p_0|$ is small, the term involving $(p - p_0)^2$ is much smaller, so

$$0 \approx f(p_0) + (p - p_0)f'(p_0)$$

Solving for p gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1$$

This sets the stage for Newton's method, which starts with an initial approximation p_0 nd generates the sequence $\{p_n\}_{n=0}^{\infty}$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1$$

Algorithm:

To find a solution to $f(x) = 0$ given an initial approximation p_0 :

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3-6.

Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. (Compute p_i.)

Step 4 If $|p - p_0| < TOL$ then

OUTPUT (p); (The procedure was successful.)

STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (Update p_0.)

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =' N_0$);

(The procedure was unsuccessful.)

STOP.

Example:

Consider the function $f(x) = \cos\ x - x = 0$. Approximate a root of f using Newton's Method

Solution:

To apply Newton's method to this problem we need $f'(x) = -\sin\ x - 1$. Starting again with $p_0 = \pi/4$, we generate the sequence defined, for $n \geq 1$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f(p'_{n-1})} = p_{n-1} - \frac{\cos\ p_{n-1} - p_{n-1}}{-\sin\ p_{n-1} - 1}$$

This gives the approximations in Table 2.4. An excellent approximation is obtained with $n = 3$. Because of the agreement of p_3 and p_4 we could reasonably expect this result to be accurate to the places listed.

Table: Newton's Method

n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

EXERCISES

1. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.
2. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used?
3. Use Newton's method to find solutions accurate to within 10^{-4} for the following problems.
   a. $x^3 - 2x^2 - 5 = 0,\ [1,4]$
   b. $x^3 + 3x^2 - 1 = 0,\ [-3,-2]$
   c. $x - \cos x = 0,\ [0,\pi/2]$
   d. $x - 0.8 - 0.2\sin x = 0,\ [0,\pi/2]$