Q.3) Find the maximum likelihood estimation (MLE) for the following distribution

(i) Binomial

Let x takes n no. of observation (i.i.d) from binomial distribution

$X \sim binom(n, p)$

where,

$n \longrightarrow$ known

$p \longrightarrow$ unknown (estimate)

Likelihood function is,

$$L(x;\theta) = \prod_{i=1}^{N} f(x;\theta)$$

$$= \prod_{i=1}^{N} {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$$

$$= \prod_{i=1}^{N} \frac{n!}{n-x!\, x!} p^{x} (1-p)^{n-x}$$

Log likelihood,

$$\log(L(x;\theta)) = k + \sum_{i=1}^{n} x_i \log P + \sum_{i=1}^{n} (n-x_i)\log(1-p) \quad — (1)$$

k — constant (does not have parameter p)

For maximizing likelihood,

differentiate eqn (1) w.r.t P,

set $\frac{\partial \log(L)}{\partial p} = 0$

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$$0 = 0 + \sum_{i=1}^{N} \frac{x_i}{p} + \sum_{i=1}^{N} \frac{(n-x_i)}{(1-p)}(-1)$$

$$\sum \frac{x_i}{p} = \sum \frac{(n-x_i)}{1-p}$$

$$(1-p)\sum x_i = p\sum(n-x_i)$$

$$\sum x_i - p\sum x_i = pn - p\sum x_i$$

$$\boxed{p_{ML} = \frac{\sum_{i=1}^{N} x_i}{n}}$$

(ii) Poisson

Lets take $x = \{x_1, x_2, \dots, x_n\}$ are i.i.d observation from poisson distribution

unknown $\longrightarrow \lambda$.

$$L[x:\lambda] = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$= \frac{\lambda^{\sum x_i} e^{-n\lambda}}{x_1!\, x_2! \dots x_n!}$$

log likelihood,

$$\log[L(x;\lambda)] = \sum_{i=1}^{n} x_i \log\lambda - n\lambda - k \quad \text{(2)}$$

$$K = x_1!\, x_2!\, x_3! \dots x_n!$$

differentiate eqn (2) w.r.t λ,

$$\frac{\partial \log(L)}{\partial\lambda} = 0$$

$$0 = \frac{1}{\lambda}\sum_{i=1}^{n} x_i - n$$

$$\frac{1}{\lambda}\sum_{i=1}^{n} x_i = n$$

$$\boxed{\lambda_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

(iii) Exponential

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & , \text{ if } x \geq 0 \\ 0 & , \text{ otherwise} \end{cases}$$

likelihood function is,

$$L[\lambda, x_1, \ldots x_n] = \prod_{i=1}^{n} f(x_i, \lambda)$$

$$= \prod_{i=1}^{n} \lambda e^{-\lambda x}$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

The log likelihood,

$$\text{Log}\left[L(\lambda; x_i)\right] = n \log\lambda - \lambda\sum_{i=1}^{n} x_i \quad \text{--- (3)}$$

differentiate eqn (3) w.r.t λ,

$$\frac{\partial \text{Log}\left[L(\lambda; x_i)\right]}{\partial\lambda} = 0$$

$$0 = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

$$\frac{n}{\lambda} = \sum_{i=1}^{n} x_i$$

$$\boxed{\lambda_{ML} = \frac{n}{\sum_{i=1}^{n} x_i}}$$

(iv) Gaussian Distribution

unknown parameters $\longrightarrow \mu, \sigma^2$.

$$f(x;\theta) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

likelihood function,

$$L[x;\theta] = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

log likelihood function,

$$\log[L(x;\theta)] = n\log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2} \quad (4)$$

differentiate eqn (4) to maximize μ,

$$\frac{\partial \log[L(x;\theta)]}{\partial \mu} = 0$$

$$0 = -2\sum_{i=1}^{n} \frac{(x_i-\mu)}{2\sigma^2}(-1)$$

$$\boxed{\mu_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

differentiate w.r.t. σ to maximize variance,

$$\frac{\partial \log[L(x;\theta)]}{\partial \sigma} = 0$$

$$0 = -\frac{n}{\sigma} + 2\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^3} = 0$$

$$\frac{n}{\sigma} = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^3}$$

$$\boxed{\sigma^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$$

(v) <u>Laplacian</u>

$$f_x(x;\theta) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$

unknown $\longrightarrow$ b, μ

likelihood function,

$$L(x;\theta) = \prod_{i=1}^{n} \frac{1}{2b} \exp\left(-\frac{|x_i-\mu|}{b}\right)$$

$$= \left(\frac{1}{2b}\right)^n \prod_{i=1}^{n} \exp\left(-\frac{|x_i-\mu|}{b}\right)$$

log likelihood,

$$\log[L(x;\mu,b)] = -n\log(2b) - \frac{1}{b}\sum_{i=1}^{n}|x_i-\mu| \quad \text{⑤}$$

differentiate eqn ⑤ w.r.t. b,

$$\frac{\partial \log[L(x;\mu,b)]}{\partial b} = 0$$

$$0 = -\frac{n}{b} + \frac{1}{b^2}\sum_{i=1}^{n}|x_i-\mu|$$

$$\frac{n}{b} = \frac{1}{b^2}\sum_{i=1}^{n}|x_i-\mu|$$

$$\boxed{b_{MLE} = \frac{1}{n}\sum_{i=1}^{n}|x_i-\mu|}$$

differentiate ~~w.r.t~~ eqn ⑤ w.r.t μ,

$$\frac{\partial \log[L(x;\mu,b)]}{\partial \mu} = 0$$

$$0 = -\frac{1}{b}\sum_{i=1}^{n}\frac{\partial|x_i-\mu|}{\partial\mu}$$

using identity,

$$\frac{\partial|x|}{\partial x} = \frac{\partial\sqrt{x^2}}{\partial x} = x(x^2)^{-1/2} = \frac{x}{|x|} = \mathrm{sgn}(x)$$

$$0 = \frac{1}{b} \sum_{i=1}^{n} sgn(x_i - \mu)$$

$$\sum_{i=1}^{n} sgn(x_i - \mu) = 0$$

We have two cases for N

if N is <u>odd</u>,

we choose $\mu_{MLE} = \text{median}\{x_1, \dots x_n\}$.

$\therefore$ there are $\frac{N-1}{2}$ cases where $x_i < \mu$,

and $\frac{N-1}{2}$ cases where $x_i > \mu$

if N is <u>even</u>,

we choose μ to be between $x_{\frac{n}{2}}$ & $x_{\frac{n}{2}+1}$.

In general, $\boxed{\hat{\mu}_{MLE} = \text{median}\{x_1, \dots, x_n\}}$