Raziel Moesch

Start with the PDE form of the heat diffusion equation:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \tag{1}$$

Rewrite (expand) the laplacian assuming T is a function of x, y, and z (3D): T(x, y, z)

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{2}$$

For finite difference method we want approximations of the terms we do not know at runtime. Looking at the formula above α is the heat diffusivity constant for the material of the object being calculated. We know this at runtime however our laplacian terms are not known. Let's expand taylor series for this formula starting with the x dimension:

Ascending:

$$T(x + \Delta x,\ y,\ z,\ t) \ \approx T(x,\ y,\ z) \ + \ \Delta x \frac{\partial T}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 T}{\partial x^2} \tag{3}$$

Descending:

$$T(x - \Delta x,\ y,\ z,\ t) \ \approx T(x,\ y,\ z) \ - \ \Delta x \frac{\partial T}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 T}{\partial x^2} \tag{4}$$

Sum the ascending and descending expansions:

$$T(x + \Delta x,\ y,\ z,\ t) + T(x - \Delta x,\ y,\ z,\ t) \ \approx \ 2T(x,\ y,\ z) \ + \ 2\frac{\Delta x^2}{2}\frac{\partial^2 T}{\partial x^2} \tag{5}$$

The 2 and $\frac{1}{2}$ cancel out:

$$T(x + \Delta x,\ y,\ z,\ t) + T(x - \Delta x,\ y,\ z,\ t) \ \approx \ 2T(x,\ y,\ z) \ + \ \Delta x^2\frac{\partial^2 T}{\partial x^2} \tag{6}$$

The unknown terms in our formula (2) are the second partial derivatives in each direction which we can approximate by solving for them in their respective taylor series expansions:

Solve for $\frac{\partial^2 T}{\partial x^2}$

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T(x+\Delta x,\ y,\ z,\ t) + T(x-\Delta x,\ y,\ z,\ t) - 2T(x,\ y,\ z)}{\Delta x^2} \tag{7}$$

Repeat the same process for y and z dimensions:

$$\frac{\partial^2 T}{\partial y^2} \approx \frac{T(x,\ y+\Delta y,\ z,\ t) + T(x,\ y-\Delta y,\ z,\ t) - 2T(x,\ y,\ z)}{\Delta y^2} \tag{8}$$

$$\frac{\partial^2 T}{\partial z^2} \approx \frac{T(x,\ y,\ z+\Delta z,\ t) + T(x,\ y,\ z-\Delta z,\ t) - 2T(x,\ y,\ z)}{\Delta z^2} \tag{9}$$

Plug them into formula (2):

$$\frac{\partial T}{\partial t} \approx \alpha(\frac{T(x+\Delta x,\ y,\ z,\ t) + T(x-\Delta x,\ y,\ z,\ t) - 2T(x,\ y,\ z)}{\Delta x^2} + \frac{T(x,\ y+\Delta y,\ z,\ t) + T(x,\ y-\Delta y,\ z,\ t) - 2T(x,\ y,\ z)}{\Delta y^2} + \frac{T(x,\ y,\ z+\Delta z,\ t) + T(x,\ y,\ z-\Delta z,\ t) - 2T(x,\ y,\ z)}{\Delta z^2}) \tag{10}$$

Finally: (Plug in $\frac{\partial T}{\partial t}$ from formula 10)

$$T(x,\ y,\ z,\ t+\Delta t)\ \approx\ T(x,\ y,\ z,\ t)\ +\ \Delta t\ \frac{\partial T}{\partial t} \tag{11}$$

So for our use cases of computing the change in temperature over time we must know the temperature of the point we are talking about as well as the temperatures of the points surrounding it in each direction as well as the points heat diffusivity constant.