Potion Bonding Curve Generation for Fat-Tailed Models Using the Kelly Criterion

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Abstract

One of the main ideas behind options contracts in the Potion Protocol is that they allow the Liquidity Provider (LP, the entity writing or selling the contract) to adjust the amount of premium that the LP would be charging the buyer of the option contract as a function of the LP's capital utilization (bonding curve). This ability of the Potion contracts allows the user to save on gas costs (and therefore transaction costs) when offering a quote to a hypothetical market. The ability also raises the question as to what the shape of this function should be. How can an LP intelligently specify this function, and how does the function shape relate to the LP's investing risk? Presented here is an application of the Kelly Criterion which demonstrates an optimal solution for the LP according to a specified probability model. While the model presented here is a simplified version of the dynamics of a market, the method can be used with any probability model to generate bonding curves and optimal quotes for Potion Protocol option contracts without a loss of generality.

1 Problem Statement

When an options contract is created there are two parties, a buyer who pays an insurance premium to become the owner of the contract, and a seller who collects the premium and underwrites the payout of the contract if it is exercised by the buyer. In the Potion Protocol, this seller is known as the Liquidity Provider or LP because they would be providing their capital to underwrite these contracts for buyers.

The issue of immediate concern for the LP is not 'What is the price of this option according to some pricing formula like Black-Scholes?' The issue of concern is 'How can I avoid ruin and ensure my capital is growing at an average rate?' To address this question, this paper will examine the use of the Kelly Criterion and demonstrate how an LP could use it to give themselves an average advantage in their bets. This is similar to how a casino has a 'house edge' in gambling games or an insurance company has an average profit over all of their customers.

In Section 2 the mathematical background of the problem is presented. This will give the reader a high-level overview of topics such as return distributions, random walks, convolutions, options contract payoffs, and fair betting odds. In Section 3 results and example bonding curves of different assets, strikes, and expirations are shown. Finally, in Section 4 future work and conclusions are discussed.

The format of the Liquidity Provider bonding curve is as follows. On the X-axis is the betting fraction of the LP. This ranges from 0 to 1 where 0 the LP is betting 0% of their capital and at 1 the LP would be betting 100% of their capital. On the Y-axis is the optimal premium to charge the buyer at each specified betting fraction. An example curve can be seen in Figure 1.

2 Background

An overview is presented of the mathematical background required to use the Kelly Criterion with option payouts. This overview begins with a review of the return distribution and its calculation. It then proceeds to discuss random walks and propagating the return distribution forward in time using convolution. Next, the overview presents a commonly used statistical distribution for modeling market processes. Afterward, the overview presents the payoff functions of option contracts and how to represent them in terms of betting odds. Finally, the overview is complete and the reader is shown the Kelly Criterion directly and how to use it to generate the bonding curve.

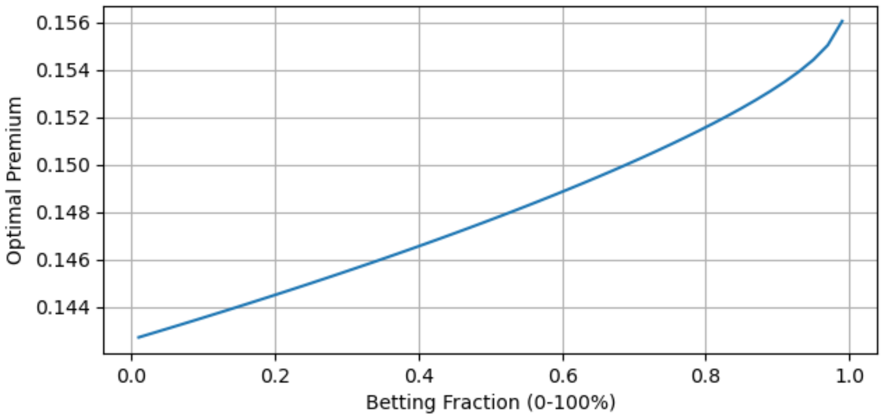


Figure 1: An example bonding curve

2.1 Return Distribution

The distribution of financial returns is the building block of the probability model presented here. First, some time step must be picked like 1 day, 1 hour, or 5 minutes. The return distribution will be calculated by examining changes in the price of the asset at each time step. After the return data has been calculated, a technique called Maximum Likelihood Estimation (MLE) is used to fit a probability distribution to the historical data.

2.1.1 Simple vs. Log Returns

There are many methods of representing the return of an investment. Two of the most common are Simple Returns and Log Returns. Each has its advantages and disadvantages. The Simple Return is calculated in Equation 1

$$r_s = \frac{P_i - P_{i-1}}{P_{i-1}}, \tag{1}$$

where r_s is the Simple Return, P_i is the price of the asset on day i, and P_{i-1} is the price of the asset on day $i-1$. The Log Return formula is derived from the compound interest rate formula and is calculated in Equation 2

$$r = \ln\left(\frac{P_i}{P_{i-1}}\right), \tag{2}$$

where r is the Log Return.

While it is more intuitive to understand quantities expressed in terms of Simple Returns, it is more intuitive to do math calculations using Log Returns. For example, a Simple Return of 1.0 where the asset doubles in price is undone by a Simple Return of -0.5 where the asset falls to the same price. For Log Returns, the same price path is a 0.693 increase and a -0.693 decrease. If one were to average these two values, for Simple Returns the incorrect value of 0.25 would be produced as the average return, even though the price did not change. For Log Returns, the average of the two returns is 0. This is due to the addition property of the logarithm. To calculate the Log Return over 30 days, one simply needs to add up the daily Log Return for each of the 30 days. In addition, over small price changes, the Log Return is still approximately equal to the percent return.

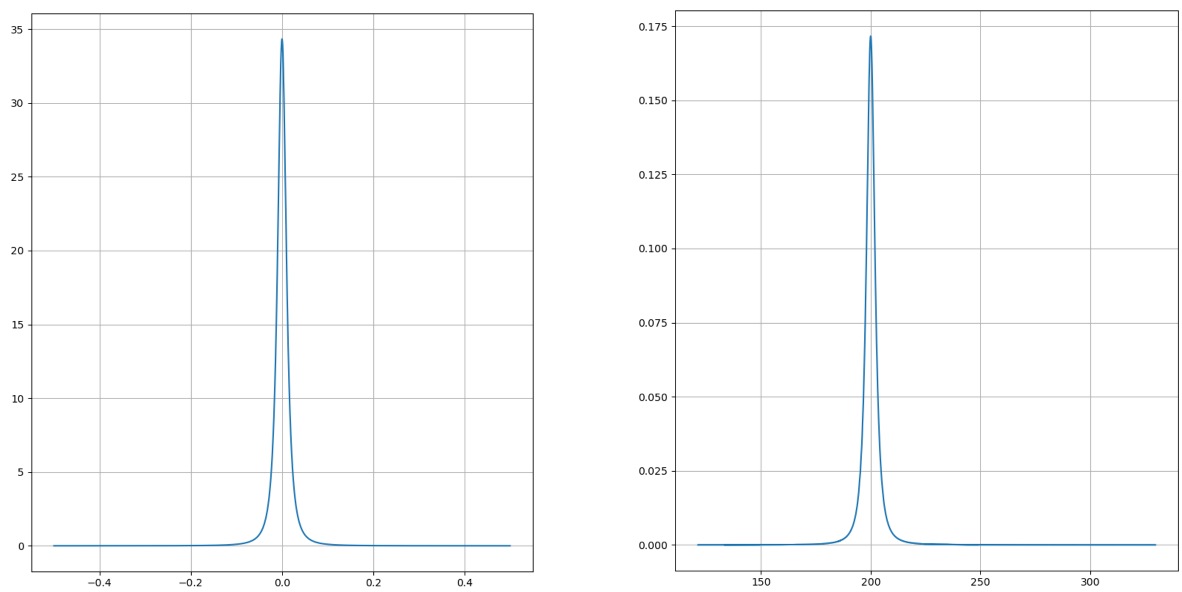
One final helpful property of the Log Return occurs when it is used to represent the return of assets that cannot drop below 0 in price. Ordinarily, if the distribution of Simple Returns were used special consideration would need to be made to add a boundary on the left tail of the distribution to represent this limitation around 0. When the same return distribution is represented using Log Returns, this 0 value occurs at $-\infty$ and no bounds on the tails need to be considered.

2.1.2 Transformations Between Domains

One additional useful tool is needed to compare the two inputs to the Kelly formula. The option payoff is defined over possible prices, while the probability distribution is defined over possible returns. To transform the probability density function (PDF) in one domain like Log Returns to another domain like probability density over possible prices two steps must be taken. First, the sample points of the function in one domain need to be transformed into sample points in the other domain. This can be accomplished directly using the Log Return formula in Equation 2 by specifying a current price around which the distribution will be centered. The second step is to scale the height of the discrete bin of density. The amount of probability present in the bin must be preserved as constant during the transformation, so by using some numerical integration rule like the Trapezoidal Rule the function can be transformed. Equation 3 demonstrates this transformation using the Trapezoidal Rule by

$$y_{t_2} = \frac{(x_2 - x_1)(y_1 + y_2)}{x_{t_2} - x_{t_1}} - y_{t_1}, \tag{3}$$

where x_1 and x_2 are the X values of the two bin edges point 1 and point 2 in the starting domain (e.g. possible log returns). Values y_1 and y_2 are the Y values of point 1 and point 2 in the starting domain. The values x_{t_1} and x_{t_2} are the X values in the transformed domain (e.g. possible prices), and y_{t_1} and y_{t_2} are the Y values of the density function in the transformed domain. The full density function can be calculated by iterating over all of the sample points or using a numerical optimizer.



(a) (b)

Figure 2: The same probability density function in both the log return domain (a) and over possible prices (b) with a current price of 200. The Y-axis is a measurement of probability density.

2.1.3 Maximum Likelihood Estimation

Maximum Likelihood Estimation is a technique for fitting a parameterized probability distribution to a set of data. First, for a given starting set of parameters, the distribution function is calculated. Next, using the likelihood function (or log-likelihood) a score is calculated representing how likely it is that the distribution with the chosen set of parameters generated the random data that was observed. Finally, the parameter set is changed and the process is repeated using a numerical optimization algorithm. This optimization is repeated until the likelihood is maximized (or negative likelihood minimized). This process produces the parameter set which was most likely to have generated the observed data. Some convenient properties of this method are that it works even when the data fits a distribution that has an infinite variance, and that the method has been proven to give parameter estimates that are accurate in the limit of large sample sizes[1].

An example of fitting a probability distribution to a series of log return samples can be seen in Figure 3. The histogram of observed data can be seen in blue and the fit statistical distribution can be seen in orange. Other techniques for fitting empirical data like least-squares fitting can give inaccurate estimates for distributions that are fat-tailed. Since financial data is often modeled using these distributions, MLE is

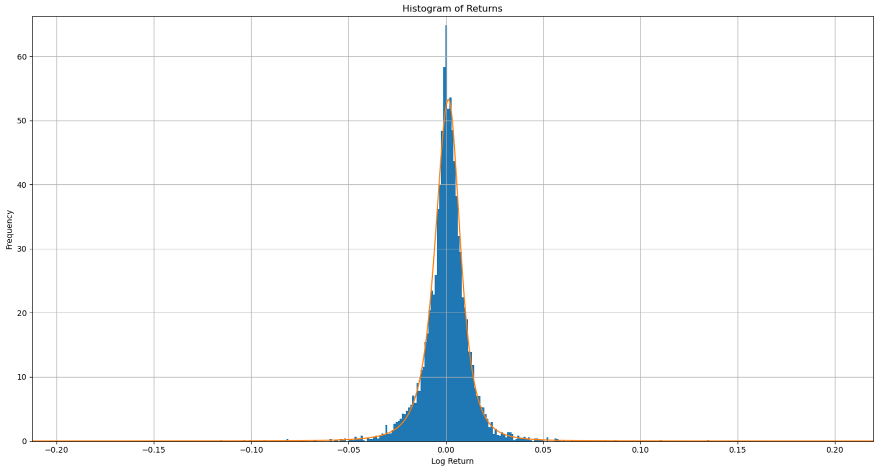


Figure 3: Fitting a PDF function to log returns using Maximum Likelihood Estimation

a useful technique for the purposes presented here. Though it is necessary for fitting these distributions to empirical data, it is not sufficient. When rigorously fitting empirical data, MLE should be supplemented with goodness-of-fit and statistical tests using alternative candidate distributions. If a candidate distribution is rejected by the test it is inappropriate to use it when modeling a set of observed data[1]. Since these rigorous techniques are not necessary to illustrate the technique for bonding curve generation they are not discussed here further.

2.2 Random Walks

The concept of a random walk will be introduced briefly here, but it can be used for Monte Carlo simulation and Backtesting to empirically verify the analytical curves generated with the Kelly method covered later in this paper.

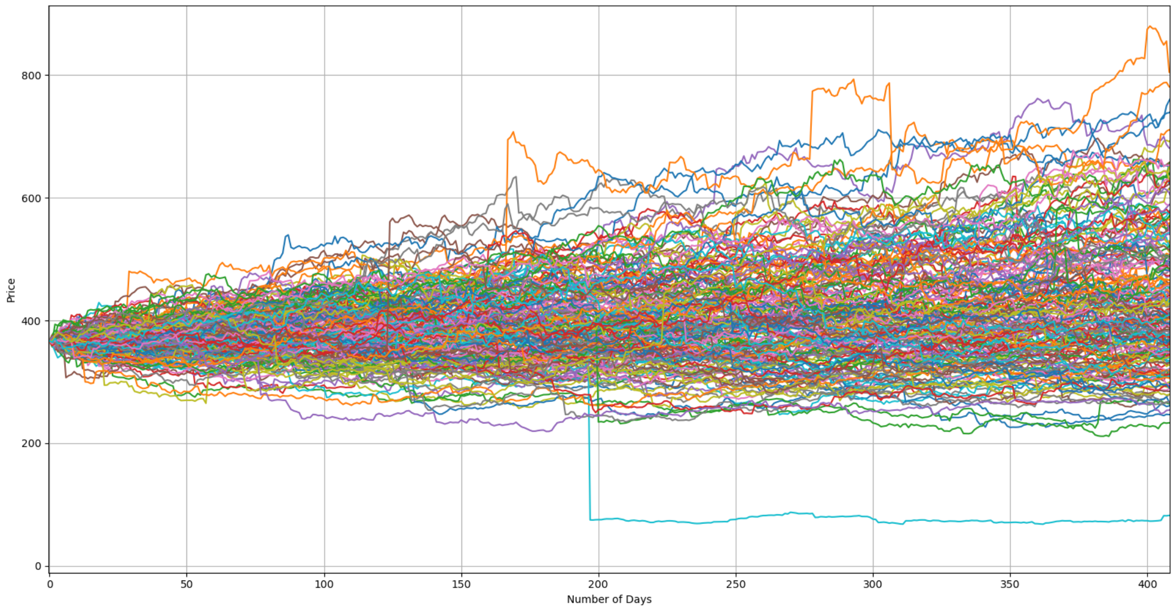


Figure 4: Possible random walks generated from a return distribution

To simulate possible future price movements for the asset for which the return distribution was fit, one technique is to generate a path using a random walk. In brief, for each future time step, a random sample is drawn from the fit return distribution. These samples are converted into price movements and used to build the path starting from the current price of the asset. This path produced is one possible integral with respect to time, and the technique can be repeated as many times as desired to produce a set of possible future paths of the asset. An example of possible future paths can be seen in Figure 4.

This method can be computationally intensive, so rather than use it for the curve generation technique it is used in a companion paper to verify the results presented here. For an extensive discussion of random walks and stochastic calculus, see Shreve[2]

2.3 Convolution

Suppose that one wanted to create a probability distribution for returns 30 days from now. One could examine the historical data and calculate the return for each 30-day period and add it to a histogram. Unfortunately, this would only yield around 12 samples per year of data for the histogram. Instead, one can use the 1-day distribution and propagate it forward in time using a technique called Convolution. This technique assumes that returns from one day to the next are independent of each other. Convolution is often used in fields like signal processing and acoustics. A convolution is an operation on two mathematical functions which produces a third function as output. This output function describes how one function is modified by the other. In this case, the probability density function of the return distribution is convolved with itself. This produces as output the return distribution for the next period. For example, taking the 1-day return distribution and performing the convolution with itself produces the 2-day return distribution assuming independent returns.[3]

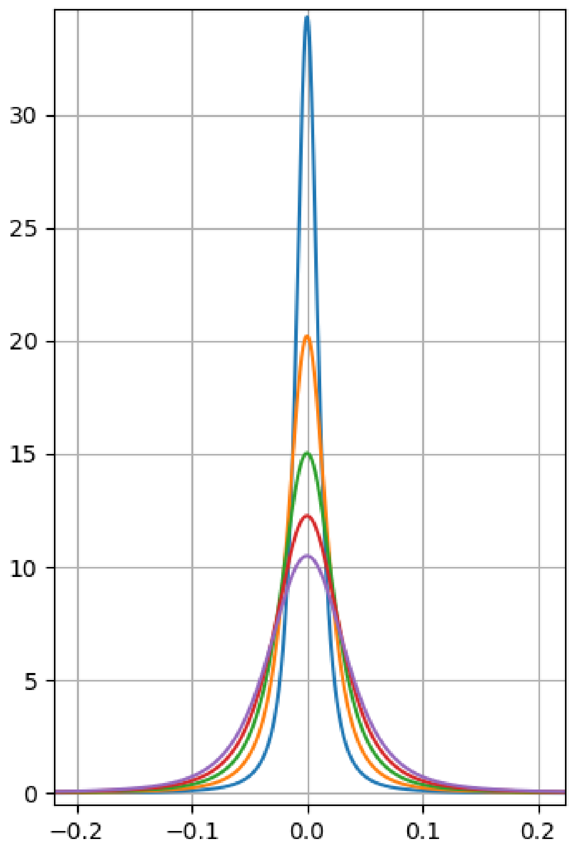


Figure 5: A log return distribution under convolution with itself 4 times. X axis represents log return and Y axis probability density.

This process causes the peak of the probability distribution to decrease in amplitude and the 'shoulders' of the distribution to get thicker. This process can be seen in Figure 5. The return distribution with the highest peak is the 1-day return distribution. Each successive distribution is the 2-day, 3-day, 4-day, and 5-day return distribution. Each day the uncertainty of the outcome increases and the probability density is 'spread out'.

Mathematically, this is expressed as follows. The convolution $C(z)$ is defined as

$$C(z) = \sum_{x=-\infty}^{\infty} f(x)g(z-x), \tag{4}$$

where f and g are any two functions. In this case, both f and g are the probability density function of the returns. Supposing that X and Y were independent random variables like the return on the first day and the return of the asset on the second day. The random variable $Z = X + Y$ has the distribution which is the convolution of the distribution $f(x)$ for X and $g(y)$ for Y.[3]

This technique is convenient because it requires much less computational effort to propagate the return distribution forward in time than the Random Walk technique presented in Section 2.2. The computations can be performed quickly utilizing the Fourier transform and multiplication of the transformed densities.

2.4 Skewed Student's T

The Student's T distribution is commonly used in science and engineering applications. The distribution is also a well-studied distribution for modeling financial returns. It has a parameter ν, called the Degrees of Freedom which controls the tail behavior and the number of statistical moments defined for the distribution. For $0 < \nu \leq 1$ the mean is not defined. For $1 < \nu \leq 2$ The variance of the distribution is not defined. For $2 < \nu \leq 3$ the third moment is not defined, etc. When $\nu \to \infty$ the Student's T approaches the Normal distribution.

Student's T is used as a two-tailed power-law distribution to model bell-shaped unimodal distributions. These distributions have tails that asymptotically approach zero and have a continuous and smooth density function [4].

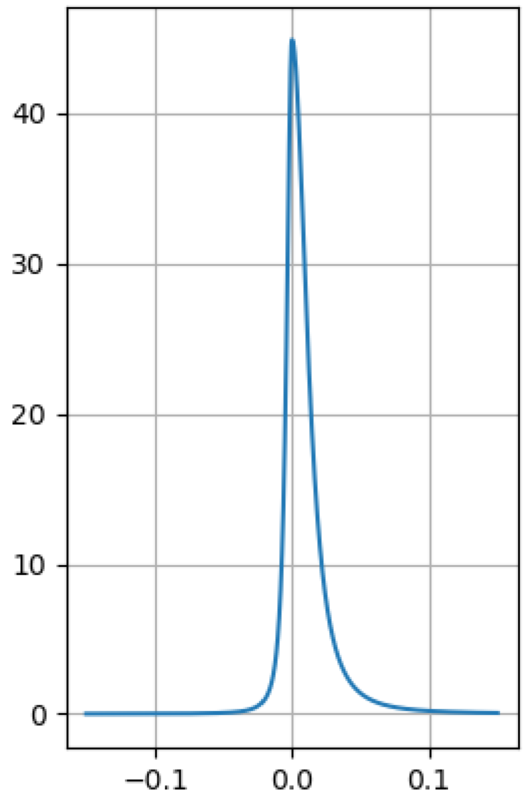


Figure 6: A Skewed Student's T distribution representing an asset that rises more often than it falls in price. X axis represents log return and Y axis probability density.

To capture skew in the probability distribution of an asset, a process for introducing skew to symmetrical distributions was used. This skewness procedure outlined in Fernandez[5] allows the introduction of skew to a symmetric Student's T distribution without affecting the tail behavior of the distribution. This allows skew to be controlled independently through an added skew parameter which is estimated during MLE of the distribution's parameters. The procedure is as follows: Assume a unimodal, univariate, and symmetric PDF function f. The skewed distribution is generated using a scalar parameter $\gamma \in (0, \infty)$ such that

$$p(x) = \frac{2}{\gamma + \frac{1}{\gamma}} \left[f\left(\frac{x}{\gamma}\right) I_+(x) + f(\gamma x) I_-(x) \right] \tag{5}$$

where $p(x)$ is the PDF of the Skewed T distribution, I_+ is the indicator function for values $[0, \infty)$, and I_- is the indicator function for values $(-\infty, 0)$. Using this method the amount of probability mass on each side of the symmetry point 0 can be controlled using γ. An example of this can be seen in Figure 6, however, the skew is magnified far above the normal level for an asset for illustration here.

It is worth noting that the skew parameter presented here is related but is not the same as the third moment of the distribution, which is often called the Skew of the distribution. An asset with an asymmetrical return distribution is known as a biased asset. These assets tend to have decreased volatility while rallying and increased volatility during sell-offs. This is the origin of the adage 'up the escalator and down the chute'. For a thorough discussion of biased assets, skew, and the relationship to an asset's volatility, see Chapter 15 of Taleb's Dynamic Hedging[6]

2.5 Fair Betting Odds and House Edge

The betting odds of a game are the payout that a player receives when the game has different outcomes. There are many conventions for expressing the odds in a game. The convention expressed here is called Decimal Odds. If the odds are 3, the outcome pays out a multiple of 3 times the amount wagered, including the amount bet. Otherwise, the player loses the full amount wagered. An example that illustrates the concept is rolling a fair die. For the bet where the player rolls a 6, the payout odds are 5. There are six possible outcomes, where on outcomes 1-5 the player loses the total bet ($-1 * a$ where a is the amount wagered), and on outcome 6, the player earns $5 * a$. If the die is fair and each outcome is equally likely the average payout of this game is $\$0$, i.e. $-1 - 1 - 1 - 1 - 1 + 5 = 0$. The odds for the opposite side of this bet (the player rolls any number except 6) are the inverse, i.e. $0.2 + 0.2 + 0.2 + 0.2 + 0.2 - 1 = 0$. These odds are always scaled to be in terms of the maximum loss of the bet.

This concept of a 'fair' bet is related to the expected value over the possible outcomes. The expected value is defined as

$$E[X] = \sum_{i=0}^{n} p_i b_i, \tag{6}$$

where E is the expectation operator, X is the random variable with n possible outcomes, and p_i represents the probability of outcome i, and b_i is the payout odds for outcome i. The expected value is the average over all outcomes[7]. Normally, when calculating an average the calculation involves dividing by the total number of outcomes after performing the sum. Conveniently, probability values are already normalized to add up to 1, so the extra division is unnecessary. Bets which have their expected value as a positive number are called positive expectation bets, and a successful investor or gambler is usually aiming to find and make these bets. Bets which have their expected value as a negative number are called negative expectation bets and are encountered frequently. For example, casino games have a 'house edge' and are games with a negative expected value. An alternative way of thinking about fair payout odds is that odds are fair when the payout odds are equal to the reciprocal of the probability value for each outcome.

2.6 Option Payoffs and Spreads

For options contracts, the payouts can also be expressed in terms of their payout odds. This is easiest in the case of buying an option contract, in which case the max loss is simply the premium paid for the contract. It is also possible when writing a put since the maximum loss exists when the price of the underlying drops to zero. It is not possible however when writing a call or writing a put on assets where the price of the underlying can be negative. The reason for this is if the payout is being scaled by the worst-case loss, i.e. $b_i = {}^{y}/_{m}$ where y is the premium collected and m is the max loss. As $m \to \infty$, the value $b_i \to 0$. In the Potion Protocol, it is not possible to place these types of bets because every position must be fully collateralized. A position where loss is unlimited is not possible. As a result, these cases need not be considered here.

Spread positions are also possible. For example, to create a vertical spread the LP only needs to turn around and act as the buyer on a different strike than the strike which they are writing contracts. This gives a different shape for the payout function. Examples of different spreads can be seen in Figure 7. Green areas represent outcomes where a profit is made. Red areas represent outcomes where the investor loses money. The black horizontal line occurs at -1 the outcome with maximum loss. In general, there is an inverse relationship between the payout of a position and the probability of the bet having a favorable outcome. Highly probable outcomes will have low payouts and highly improbable outcomes will have high payouts.

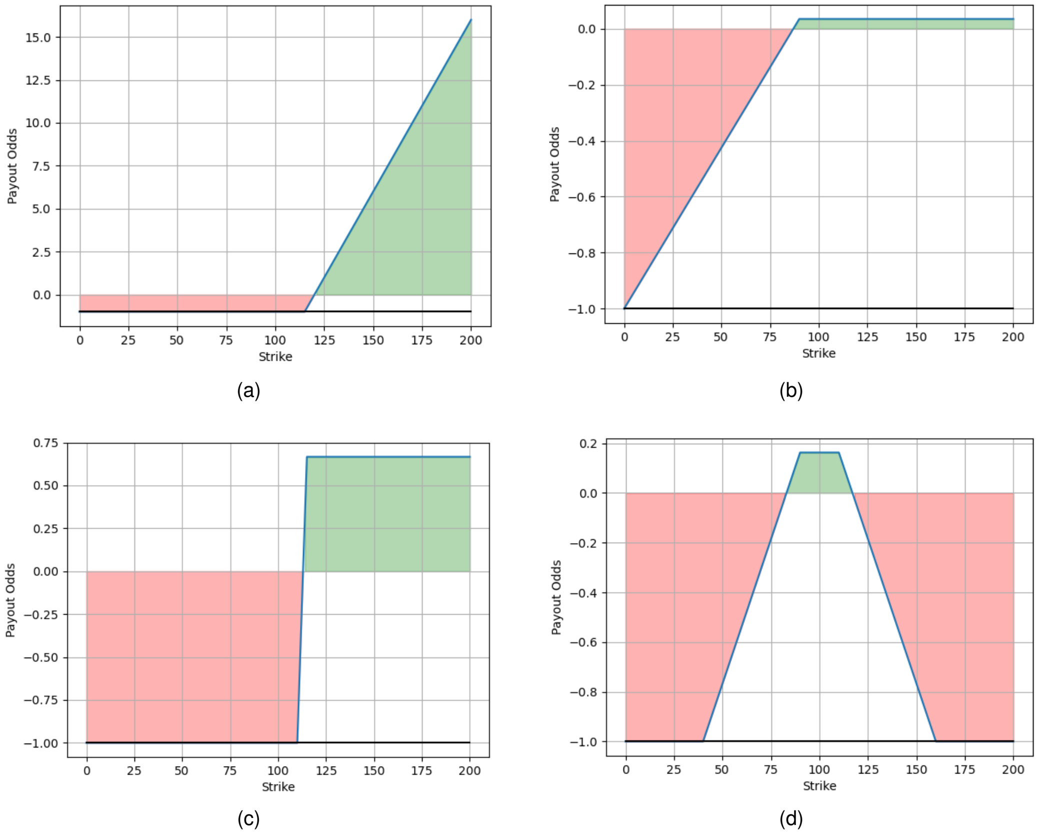


Figure 7: Example payout odds functions for different options and spreads. (a) Payout odds for a long call (b) Payout odds for a short put (c) Payout odds for a vertical spread (d) Payout odds for an iron condor.

2.7 Kelly Formula

The Kelly Formula first appeared in its modern form in Kelly's 1956 paper[8]. In this paper, Kelly uses a thought experiment to illustrate to the reader the intuitive meaning behind the mathematical formula. Kelly proposes a gambler is betting on a baseball game between two evenly matched teams. Since the teams are evenly matched, the payout odds are even. This gambler has a private wire or communication channel where a friend with advanced knowledge of the game's outcome is transmitting to the gambler 0 or 1 depending on the outcome of the game. This allows the gambler to place bets at even odds. With the gambler's advanced knowledge, what is the optimal betting strategy?

On first inspection, the answer appears to be that the gambler should bet 100% of their capital for a maximized final value of 2^N times their original capital after N bets. Due to the Law of Large Numbers, this strategy leads to the ruin of the gambler with probability one[8]. The cause of this unfortunate outcome is due to this side channel of information and the formula's relationship to Information Theory. It does not matter what encoding scheme or redundancy mechanisms are used to reduce the noise present in the side channel to the gambler's friend. The probability of the data being corrupted from a 0 to a 1 or a 1 to a 0 is never exactly zero, it is just small. As a result, due to the Law of Large Numbers, the corruption of the data will eventually occur and if the gambler is betting 100% of their capital they will be ruined. Since with modern communication equipment, the error rate in communication can be made very low, all that the gambler needed to do to avoid ruin with this setup was to adopt a less greedy betting strategy. The link has now been intuitively established that the optimal betting strategy relates in some way to the fraction of the gambler's capital at risk.

This link now opens the opportunity for closer inspection of the optimal strategy. This side channel of information to the gambler's friend has a non-zero entropy which leads to uncertainty about the possible outcomes. On the transmission side of the wire, symbols representing perfect information about the game's outcome were transmitted. On the receiving side of the wire, there is uncertainty about what specific perfect future information has now been corrupted.

In a gambling game where there are n possible outcomes, each outcome has an associated payout odds b_i where if outcome i happens the gambler multiplies the original bet by b_i and also is returned their original bet. In other words, on an outcome where the gambler loses everything they bet $b_i = -1$, on an outcome where the gambler breaks even $b_i = 0$, and an outcome where the gambler doubles what they bet $b_i = 1$. Each outcome i also has an associated probability p_i which is the probability that the outcome happens. For a starting capital of X dollars and the gambler bets fX dollars, where f is a value between 0 and 1 representing a fraction of the total. On the next discrete time step or 'turn' of the game, the expected value of the natural log of the total capital is given by

$$E\left[\ln\left(X_{t+1}\right)\right] = \ln\left(X_t\right) + \sum_{i=0}^{n} p_i \ln\left(1 + b_i f\right), \tag{7}$$

where X_t and X_{t+1} is the total capital at time step t and $t+1$. From this equation, it can be seen that the value $\ln\left(1 + b_i f\right)$ is the log return on the capital the gambler bet, should outcome i occur. It can be seen that the sum term of the equation is the average or expected growth rate of the gambler's capital because of log addition rules. It also becomes clear that the reason the game's payout was scaled to be in terms of the payout odds b_i is because it is input to the logarithm, and since f is between 0 and 1 it means b_i cannot be below -1 or it is undefined.

The value which must be maximized for the optimal betting strategy is the sum term that is the expected growth rate. To maximize this function given by

$$k\left(f\right) = \sum_{i=0}^{n} p_i \ln\left(1 + b_i f\right), \tag{8}$$

take the derivative dk/df and set it equal to zero:

$$\frac{dk}{df} = \frac{d}{df} \sum_{i=0}^{n} p_i \ln\left(1 + b_i f\right) = 0, \tag{9}$$

which yields

$$\frac{dk}{df} = \sum_{i=0}^{n} \frac{p_i b_i}{1 + b_i f} = 0, \tag{10}$$

by the chain rule. By solving Equation 10 for f, the optimum value f_* is obtained, where the growth rate is maximized for the gambler's betting fraction[9]. By taking Equation 7 and undoing the natural logarithm,

$$E\left[X_{t+1}\right] = X_t e^{k(f_*)}, \tag{11}$$

the compound interest formula is obtained and the optimal average value of the gambler's capital at the next time step. By taking the growth per bet g, and making an assumption m about the number of times per year this bet will occur, the growth can be annualized into an equivalent CAGR percentage

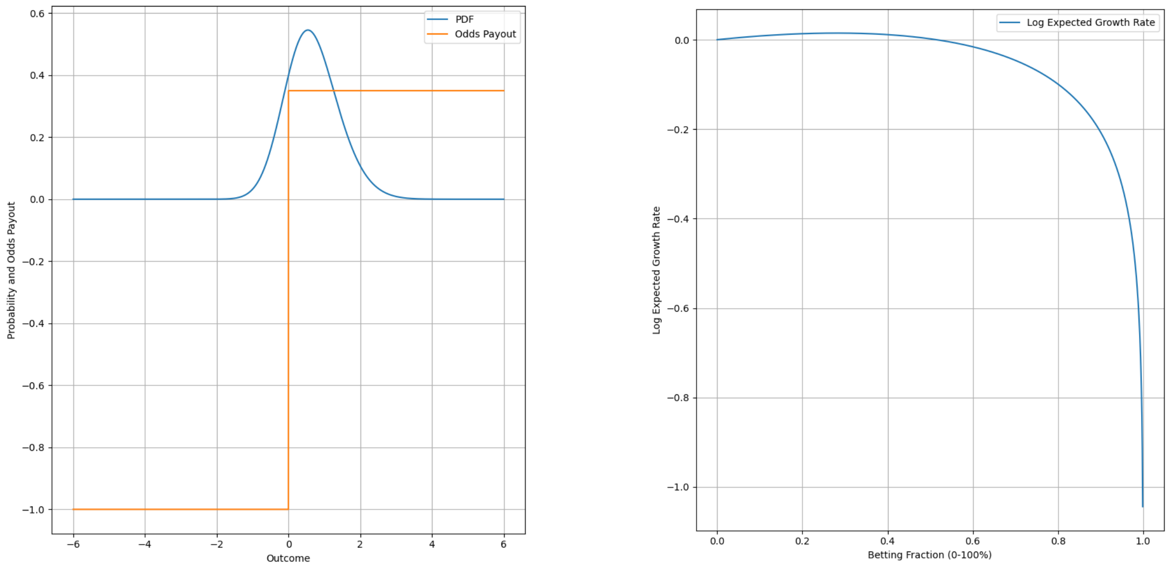
$$g = e^{k(f_*)}, \tag{12}$$

$$CAGR = \left(g^m - 1\right) * 100. \tag{13}$$

2.7.1 Coin Flip

Here, a concrete application of the formula is examined so that the reader can obtain a better intuitive understanding of its use. Consider a biased coin flip. The probability distribution can be seen in the blue line in Figure 8a. This Normal Distribution is located slightly off-center from 0 so that most of the probability mass is above zero. In this biased coin flip game, a random sample is generated from this probability distribution. If the number is positive, it is considered Heads. If the number is negative, it is considered Tails.

The payout function for this game will also be changed from a normal coin flip. Rather than an even payout to the amount the gambler bets, the game will pay out 35% of what the gambler bets if the outcome is Heads, and the gambler will lose everything they bet if the outcome is Tails.



(a) (b)

Figure 8: (a) Payout odds for a game with two outcomes paired with the PDF of a Normal Distribution (b) The output of Equation 8: $k(f)$ using the odds and PDF from (a)

One would like to know whether this bet has a positive expected value and is going to make money on average. If it is, one would also like to know the optimal amount to bet to ensure that the gambler's capital is growing at the fastest rate that is possible. Equation 8 is applied by iterating over the possible bet fractions from 0 to 1 and producing the plot of $k(f)$ in 8b. If the entire function $k(f)$ lines below zero, the bet is never profitable on average. Here, there is a bet fraction region that is profitable. The optimal bet fraction is found by finding where Equation 10 crosses zero. This value f_* is located at 0.281 and gives a value $k(f_*) = 0.01481$. By applying Equation 12 to calculate the average growth per bet as 1.0149 or about an average of 1.5%. Then, Equation 13 can be applied assuming some value like $m = 5$ for an equivalent average annual CAGR of 7.686%.

This behavior as the gambler's betting fraction approaches 100% is a characteristic of the logarithm. It illustrates why no matter how low the entropy of the gambler's side channel is in Kelly's thought experiment, the gambler will be ruined with a probability of 1 if the entropy is not 0. As the bet fraction approaches 100% the log of the average growth rate approaches $-\infty$ in the limit. It can also be seen that the consequences of over-betting are high. Bet fractions where the line is below zero will mean that the gambler would lose capital over time, even if the probabilities appear to be in the gambler's favor.

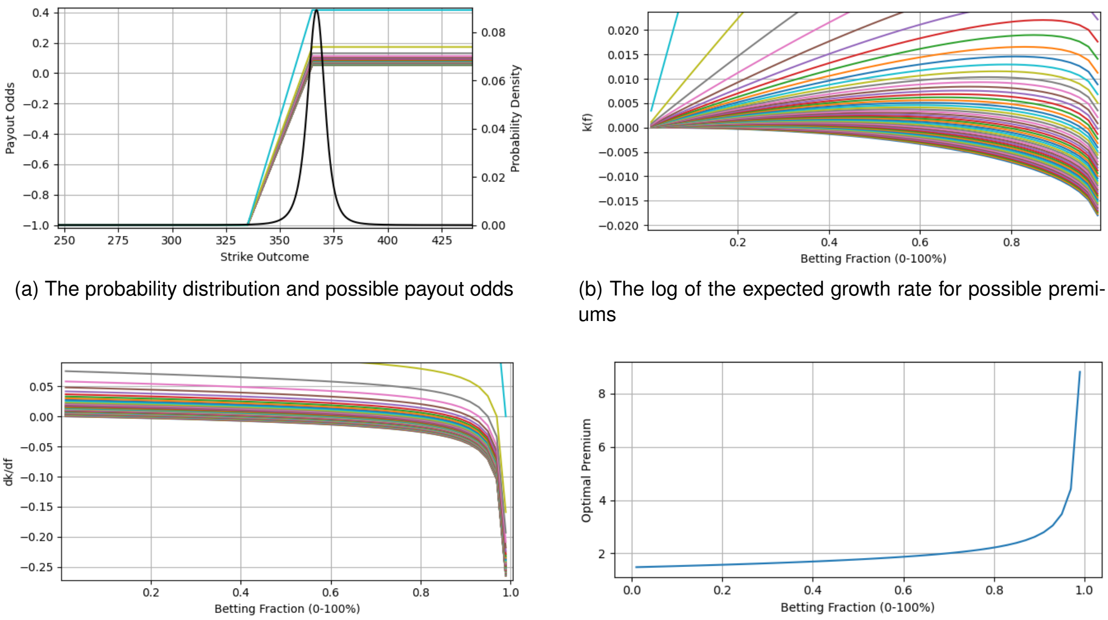
2.7.2 Contract Writing

This section moves to the problem of generating the bonding curves for Potion Protocol options contracts. This problem considered is slightly inverted from the Coin Flip. Rather than consider the problem from the point of view of a speculator, it is considered from the perspective of the 'casino', i.e. "How much premium should the LP charge the speculator to give themselves 'edge' in the game?" One should note that the LP is not the one sensitive to time in this trade, it is the speculator who must buy the option 'now' to act on their trade idea. As a result, the LP has the luxury of being able to wait for a good deal that will give them an edge in the payout of the game.

The problem follows the same process and can be used regardless of whether it is a single contract or multiple on the same asset as part of a spread. First, the LP will consider the maximum possible loss of the position. Next, this value will be used with the payout function to calculate the payout in terms of the betting odds b_i. This is possible for positions that are not writing a 'naked' call. In the Potion Protocol, a requirement is that all contracts must be fully collateralized, so it would not be possible to take such a position anyway. The call writing would need to either be part of a spread or a covered call to meet the collateral requirement, which would have a different payout odds function than one with an infinite max loss.

Next, the distribution fit from the log-returns needs to undergo convolution with itself the number of time steps between now and the option's expiration. For example, the daily log return distribution would need convolution with itself 30 times to produce the 30-day distribution. This process assumes independent returns. With both the distribution and payout odds function, the plot in Figure 9a can be calculated. For visualization in this paper, a bull put spread was chosen because it gave more convex curves with a more exaggerated payout to illustrate the concept than a 'naked' put.

To generate the plots and the bonding curve, the LP iterates over each possible bet fraction and varies the amount of premium collected for the position. This generates the multicolored lines in Figure 9. One line exists for each bet fraction. Figure 9b is the log expected growth rate Equation 8 and similar to Figure 8b in the coin flip, only one line for each bet fraction. Figure 9c is Equation 10 at each bet fraction. Finally Figure 9d, the bonding curve is generated by varying the premium values until each point the lines in Figure 9c cross zero, which is the maximum values of the curves in Figure 9b. As a result, the curve of optimal premiums at each bet fraction is generated.



(a) The probability distribution and possible payout odds

(b) The log of the expected growth rate for possible premiums

(c) The derivative of the log expected growth rate

(d) The curve of optimal premiums for each betting fraction

Figure 9: Analysis of a hypothetical OTM bull put spread on SPY 2 days until expiration. The short leg is 365 and the long leg is 335 with the underlying at 366.28. Multiplier 1. The premium of the bull put spread is varied at each betting fraction until the maximum value of (b) occurs at that fraction. This happens when each line in (c) crosses zero. These optimal premiums are assembled into curve (d) which is the optimal bonding curve of the Potion Protocol option. This optimality is according to the assumptions of the estimated probability values in the distribution in (a).

Normally, with a spread like the one in Figure 9 the person making the bet must also consider the fact that the contract has a discrete value and minimum bet. For example, on an Equity contract like SPY with a multiplier of 100 the max loss of the minimum bet of 1 contract would be $\$2,948.00$. This might be a prohibitively high bet fraction for small accounts. Due to the high divisibility of cryptocurrency assets, it is possible to bet fractional amounts of contracts. The multiplier for Potion Protocol options is 1 making the equivalent bet $\$29.48$ and a bet could be placed for a fraction of a token 0.1 for $\$2.948$ or 0.01 for $\$0.2948$ etc. The analysis of a minimum bet size on the LP will not be considered further here.

For any probability model, the LP would be wise to consider the premium curve generated as a 'minimum profitable bet' and add a personal factor of safety to move the curve in Figure 9d upward. The probabilities in trading are not fixed like a gambling game and will change with time. The optimal premium curve is only optimal under the assumptions made by the probability model and no mathematical model is ever perfect. It could be that while the estimated probability distribution says the LP has an edge, according to the unknown 'true' probability distribution the LP does not. This factor of safety could be based on how much the LP trusts the model. There is no harm to the LP to wait for an extra 'edge' in the game and a better deal, it only means that the LP's positions will be traded less frequently because the higher factor of safety, the more likely it is another LP will offer a more competitive price.

The method by which the LP models the probability distribution is the 'side channel' from Kelly's thought experiment. Here, the distribution of historical returns was used which assumes that future price movements would be similar to past ones. The similarity of the two distributions affects how much edge the LP has. Side channels could take many forms. It could be the historical outcomes of a game, a fancy financial indicator, the financial statements of a company, the investor's grandmother fortune telling the outcome from coffee grounds, or an expert card player reading people's faces during a game. Depending on how much each of these channels corrupts the information they deliver about the future outcome decides how useful the information is for the gambler or investor's betting. Low entropy channels are the most profitable. It has been shown that the maximum financial value of the information side channel is equal to the mutual information between the game outcome and the information delivered from the side channel[7].

The same statement can be expressed in terms of Relative Entropy. The Relative Entropy is a measure of how different one probability distribution is from another and how 'surprised' the user is. If the estimated distribution said an event was very common, but in the true distribution it was rare (or vice versa) the degree of surprise i.e. the Relative Entropy would be a high value. In contrast, if the estimated distribution was fairly close to the true distribution the Relative Entropy and surprise would be a low value. The rate at which the gambler's capital grows is proportional to the difference in the Relative Entropy of the Casino's estimate of the true probability distribution and the Relative Entropy of the gambler's estimate of the true probability distribution. In other words, whichever player in the game has a more accurate estimate of the odds will increase their capital over time.[7] The Casino's estimate is the probability estimate implied by the game's payout odds. In the case of the options market, this is the probability estimate implied by the market prices. Relative Entropy can also be thought of as information gain, i.e. 'How much information is gained by using the true probability distribution instead of the estimated distribution?'

Suppose an investor were to use the historical return distribution as an estimate of the true probability. Also suppose that the true probability is the same distribution, but has its skew parameter change over time. In addition, the investor bets according to the Kelly Formula as was presented here. The investor would have a positive average growth rate of their capital during the periods where the historical distribution better represented the 'true' probability. During the periods where the distribution implied by the market prices was a better estimate of the true probability the investor would lose money. When there are fixed known events in the future, the implied distribution can be a better estimate of the true probability since the market prices will reflect this information. Some examples of these events include earnings announcements for a stock, an election, or perhaps a contentious upgrade or fork of a cryptocurrency. At other times, the market's belief could be wrong and the historical distribution might be a better estimate of the true distribution.

2.7.3 Curve Fit

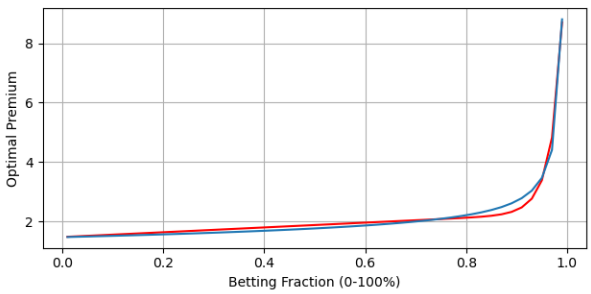


Figure 10: A parameterized curve fit according to Equation 14

To save on gas i.e. transaction costs, it is necessary to store the optimal premium curve as a continuous function. Parametrization requires only a few values to store. Storing all curve data points is many more values and therefore is much more costly. To help reduce this burden, the curves are stored parameterized to some fit. With this fit, it is important to capture the behavior of the formula as the bet fraction approaches one.

The fit function used has 4 fit parameters, A, B, C, and D. The function is of the form

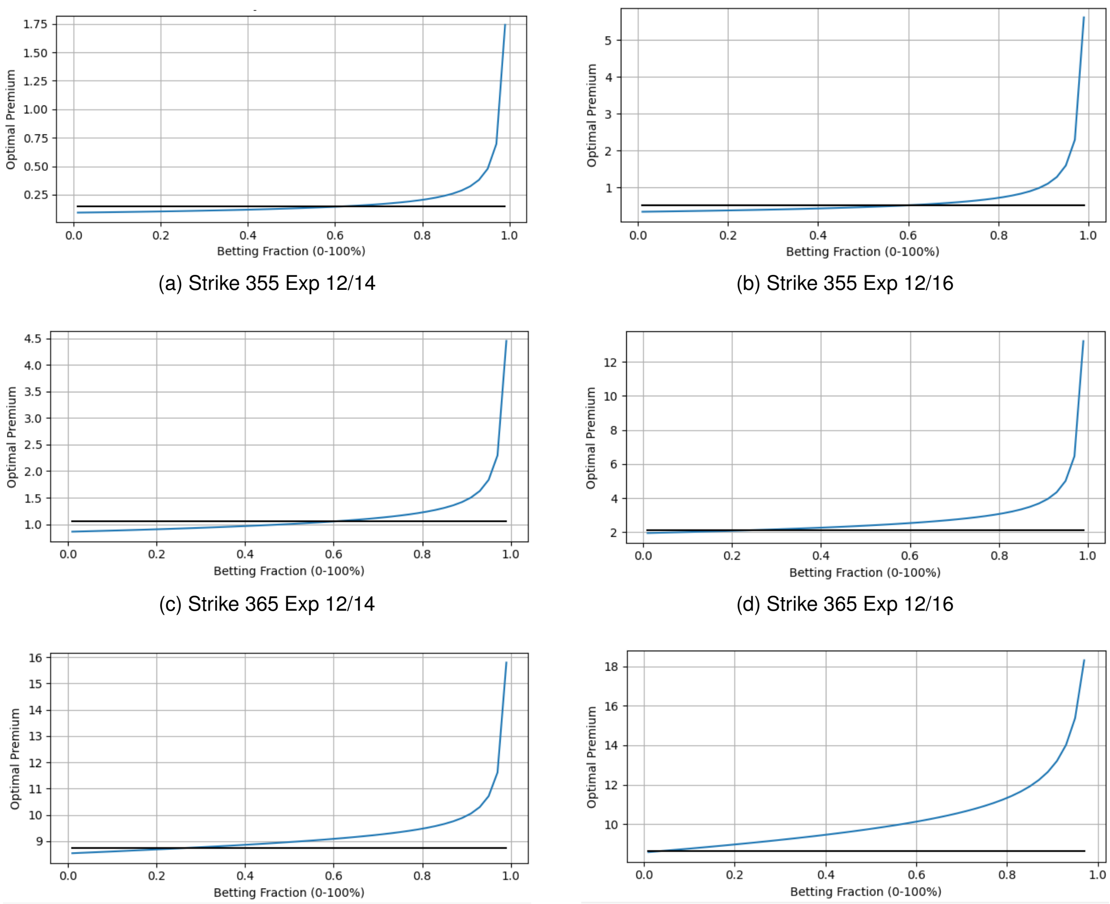
$$f(t) = At * cosh(Bt^C) + D \tag{14}$$

where t is the bet fraction and $f(t)$ is the premium calculated from the fit parameters. An example fit can be seen in Figure 10. By adjusting the fit parameters the LP can ensure the parameterized fit is an overestimate of the optimal premium curve.

3 Results

As a demonstration of the method's ability, result curves are generated here for SPY and compared to the actual market prices for those bull put spreads. This is presented in Figure 11. These 3 bull put spreads are compared against the market closing prices across two expirations. The closing prices of the SPY contracts were divided by 100 to match the multiplier of 1 for the Potion Protocol option.

Each of the market prices for the spread falls within the range of the curve, suggesting that at lower betting fractions the LP would have a trading edge according to the historical probability model. There appears to be a greater edge for the strikes which are more OTM. It could be that since the bull put spread is a position that makes money from the passage of time, the spreads farther OTM have more edge. These options have a larger portion of their value from their extrinsic value. However, no relationship between the two can be established with this quick visual comparison.



(a) Strike 355 Exp 12/14 (b) Strike 355 Exp 12/16

(c) Strike 365 Exp 12/14 (d) Strike 365 Exp 12/16

(e) Strike 375 Exp 12/14 (f) Strike 375 Exp 12/16

Figure 11: Optimal premium curves for bull put spreads on SPY after the market close on 12/11/2020. Blue lines represent the premium curves. Black lines represent the closing midpoint price of that spread on 12/11 scaled to a contract multiplier of 1 (the multiplier of Potion Protocol options). Each spread has the short leg at the strike specified and a width of 30.

These results are quite encouraging. However, as the results were propagated forward in time they would begin to diverge from the market prices. This is primarily because the simplified model presented here assumes that the historical return distribution is fixed and unchanging as it moves through time. It is not capturing the rich set of information present in the current market prices of the options and the implied volatility surface, so this model would miss future expected changes in the market and discrete events like the earnings release of a stock or an election. For further discussion of the volatility surface, see Gatheral[10].

4 Conclusion

This paper demonstrated using the Kelly Criterion to calculate a hypothetical market quote bonding curve for an LP in the Potion Protocol. By assuming that an asset's returns follow a supplied probability distribution, this tool allows an LP to avoid ruin and give their investment bets a probabilistic edge. The Kelly formula gives the LP the ability to calculate which of their premium quotes is expected to make money on average or lose money on average. Armed with this knowledge, an LP can better protect its capital while providing liquidity for Potion Protocol option buyers.

For further reading on this topic, please see the Potion Protocol documentation[11][12].

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