Planetary Motion

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The task is to determine the orbits of bound planets and to classify the motions of other small celestial bodies interacting with the sun gravitationally, which can be accomplished to great accuracy without invoking 20th century physics or graduate-level techniques. We also derive Kepler's three laws using a strictly Newtonian approach to the problem.

Quick Introduction to Newton's Law of Gravitation

Gravity is classically understood as a non-contact force that acts along a line between two bodies, known as a *central* force. The gravitational attraction between two objects is proportional to the mass of each body, and *inversely proportional* to the square of their separation. That is,

$$\left|\vec{F}_{12}\right| = G\frac{m_1 m_2}{r^2} .$$

Newton's equation is not at all trivial, particularly because of the r^2 in the denominator. Before the theory of gravitation had been developed by Newton, he was aware (thanks to Kepler) that planetary orbits aren't always circular, but elliptical in general. From this he was able to prove deductively that the magnitude of the gravitational force must go as $1/r^2$. In this study we'll do this proof in reverse – starting with the $1/r^2$ power law and finding ellipses as a consequence.

The term $G = 6.67300 \times 10^{-11} m^3 / kg \cdot s^2$ is called the *Universal Gravitational Constant*. It's magnitude wasn't found until the days of Cavendish, but Newton didn't need to know this number anyway.

Statement of Newton's Laws

Newton's laws should be stated for completeness and to demonstrate the notation used in this document. For example, $\vec{F}_{12}$ should be read: "Force onto particle 1 by particle 2".

Newton's first law:	Law of Inertia
Newton's second law:	$\vec{F} = m\vec{a} = m\ddot{\vec{r}}$
Newton's third law:	$\vec{F}_{12} = -\vec{F}_{21}$

Technically, I've already stated all equations necessary to solve any problem imaginable in the problem space of Newtonian gravitation. ***The whole story is determined by Newton's laws along with the gravitation law.*** A sufficiently talented mathematician could put this paper down and not have missed anything. We, on the other hand, here to acquire such sophistication.

Two-Body Problem, Reduced Mass

To characterize the motions of planets or other celestial objects, a few heuristic arguments made early will ease the mathematics later. First off, assume that the only significant force acting on a body is the gravitational attraction due to its parent star. When looking at a solar system, this means we ignore the attraction between a given planet and any other planet, and concentrate only on the attraction to the sun. This reduces our task to a *two-body problem*. Secondary forces can be introduced as perturbations in a more advanced treatment of the problem.

Let's also agree that the relative separation between celestial bodies is sufficiently greater than their respective diameters, so we may represent each body as a point mass. We will find later that this is really no approximation at all, and it is perfectly valid to represent a voluminous body as a point located at the *center of mass* of the object.

Now picture two particles represented by m_1 and m_2, located with position vectors relative to some origin (see Figure 1).

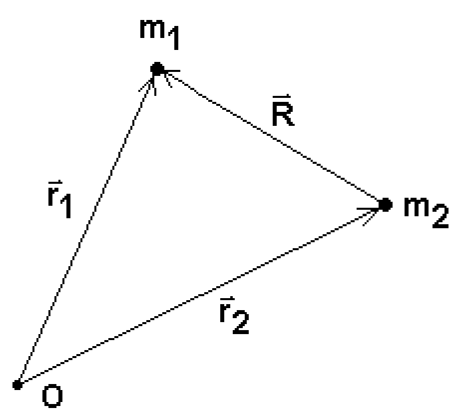

Figure 1. Setup of the two-body problem.

Apply Newton's second law to each particle.

$$\vec{F}_{12} = m_1\vec{a}_1 \qquad\qquad \vec{F}_{21} = m_2\vec{a}_2$$

Multiply through by m_2 and m_1 respectively. In the second equation, $\vec{F}_{21}$ is replaced by $-\vec{F}_{12}$ using Newton's third law.

$$m_2\vec{F}_{12} = m_1 m_2\vec{a}_1 \qquad\qquad -m_1\vec{F}_{12} = m_2 m_1\vec{a}_2$$

Subtract the second equation from the first, giving

$$\vec{F}_{12}(m_1 + m_2) = m_1 m_2 (\vec{a}_1 - \vec{a}_2).$$

Notice from the picture that $\vec{R} = \vec{r}_1 - \vec{r}_2$, and two time derivatives show $\ddot{\vec{R}} = \vec{a}_1 - \vec{a}_2$. Therefore we have $\vec{F}_{12}(m_1 + m_2) = m_1 m_2 \ddot{\vec{R}}$, or,

$$\vec{F}_{12} = \frac{m_1 m_2}{m_1 + m_2} \ddot{\vec{R}}.$$

We've arrived at a curious version of Newton's second law. The effective or *reduced* mass is $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ and the effective acceleration is $\ddot{\vec{R}}$.

Sun-Planet Model

If we now assign m_1 to represent an orbiting body, and m_2 a star (with $m_2 >> m_1$), the problem simplifies further. Examine the reduced mass under this approximation.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = (m_1)\left(\frac{m_2}{m_1 + m_2}\right) \approx (m_1)\left(\frac{m_2}{0 + m_2}\right) = m_1$$

Evidently it suffices to replace μ with simply m_1. Another consequence of $m_2 >> m_1$ is the acceleration of m_2 is negligible compared to that of m_1, so $\ddot{\vec{R}} \approx \ddot{\vec{r}}_1$. It's then natural to locate the origin at the stationary point m_2. After all the dust settles, we are left with

$$\vec{F}_{12} = m_1 \ddot{\vec{r}}_1.$$

In summary, the star lies fixed at the origin, and the orbiting body is fully described by its position vector relative to that origin. To tidy up the notation, let's for now on denote m_2 (the mass of the larger body) as uppercase M, and denote m_1 (mass of the orbiting body) as just m. Also drop the subscript on r_1 and $\ddot{\vec{r}}_1$.

Polar Coordinates

A convenient truth about two-body central force problems is that a given orbit is always confined to a plane (we'll prove this later). To describe the orbital motion then, we choose a *polar coordinate* system, where all locations in 2D space are specified in terms of their distance r from the origin, and angle θ relative to a given axis. (The lesser suitable choice would be insisting on *Cartesian* coordinates, where two numbers x and y map the plane.)

The conversion between Cartesian and polar coordinates is (see Figure 2)

$$\vec{r} = \hat{i}x + \hat{j}y = \hat{i}r\cos\theta + \hat{j}r\sin\theta \ ,$$

where it is readily shown that $r = |\vec{r}| = \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{x^2 + y^2}$.

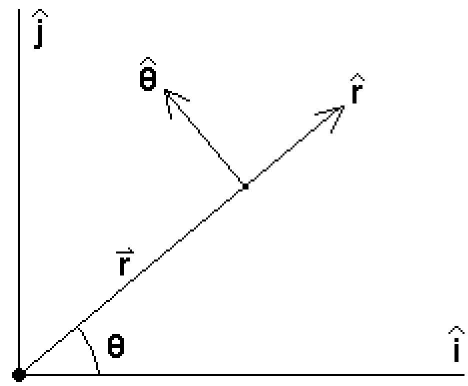

Figure 2. Polar and Cartesian coordinate systems.

Another tool to sharpen is the notion of *unit vectors* in polar coordinates, which are not trivial like their Cartesian counterpart. The equation $\vec{r} = \hat{i}x + \hat{j}y$ implies the two unit vectors are:

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\hat{i}r\cos\theta + \hat{j}r\sin\theta}{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$$

$$\hat{\theta} = \left|\frac{d\vec{r}}{d\theta}\right|^{-1}\frac{d\vec{r}}{d\theta} = \left(\frac{1}{r}\right)\left(-\hat{i}r\sin\theta + \hat{j}r\cos\theta\right) = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

Evidently the unit vectors depend on the coordinates themselves. Observe also that $d\hat{r}/d\theta = \hat{\theta}$ and $d\hat{\theta}/d\theta = -\hat{r}$.

The problem may arise (although not here) where we will want to express $\hat{i}$ and $\hat{j}$ in terms of $\hat{r}$ and $\hat{\theta}$, which amounts to inverting the matrix equation

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\end{bmatrix}.$$

Many methods exist for inverting a 2x2 matrix, my favorite being $AI \rightarrow IA^{-1}$, which amounts to writing $\begin{bmatrix}\cos\theta & \sin\theta & 1 & 0\\ -\sin\theta & \cos\theta & 0 & 1\end{bmatrix}$, and then performing Gauss-Jordan elimination until landing at $\begin{bmatrix} 1 & 0 & \cos\theta & -\sin\theta \\ 0 & 1 & \sin\theta & \cos\theta \end{bmatrix}$, from which the inverse peers back at us. Therefore $\hat{i} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$ and $\hat{j} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$.

Velocity and Acceleration

Another snag involved with polar coordinates is that the unit vectors $\hat{r}$ and $\hat{\theta}$ vary with time, and therefore possess derivatives, making a mess out of the velocity and acceleration vectors.

The velocity is

$$\dot{\vec{r}} = \frac{d}{dt}\left(r\hat{r}\right) = \hat{r}\frac{dr}{dt} + r\frac{d\hat{r}}{dt} = \hat{r}\frac{dr}{dt} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta},$$

and the acceleration is

$$\ddot{\vec{r}} = \frac{d}{dt}\left(\hat{r}\dot{r} + r\dot{\theta}\hat{\theta}\right) = \hat{r}\left(\ddot{r} - r\left(\dot{\theta}\right)^2\right) + \hat{\theta}\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right).$$

These two equations are much nastier than the usual $\dot{\vec{r}} = \dot{x} + \dot{y}$ and $\ddot{\vec{r}} = \ddot{x} + \ddot{y}$, and may be difficult to remember. If you'll tolerate a diversion before moving on, there is a nice trick from complex numbers that helps recover $\dot{\vec{r}}$ and $\ddot{\vec{r}}$ in a pinch.

Start with the complex number $R = re^{i\theta}$ and take one time derivative.

$$\frac{dR}{dt} = \frac{d}{dt}\left(re^{i\theta}\right) = \frac{dr}{dt}e^{i\theta} + ie^{i\theta}r\frac{d\theta}{dt}$$

Since the real numbers are “orthogonal” to the imaginary numbers, it's natural to relate $e^{i\theta}$ to $\hat{r}$ and $ie^{i\theta}$ to $\hat{\theta}$. Therefore $dR/dt = \hat{r}\dot{r} + r\hat{\theta}\dot{\theta} = \dot{\vec{r}}$, as promised. Differentiate again to finish the job, not forgetting $i^2 = -1$.

$$\frac{d^2R}{dt^2} = e^{i\theta}\left(\ddot{r} - r\left(\dot{\theta}\right)^2\right) + ie^{i\theta}\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

Newton's Simplification

With a closer look at Newton's law of gravitation,

$$\vec{F}_G = -G\frac{Mm}{r^2}\hat{r}$$

all we really know is that any two *point* particles obey this law exactly. The next question is, how does this law generalize to sizable objects? Suppose m is a point particle but M is a massive spherical body, and we need to know the mutual attraction between these bodies. This problem involves an integral that fooled Newton for a long time, stalling the publication of his work until it was solved.

The answer that Newton found is very convenient. ***The sphere acts as if all of its mass were concentrated at the center***. There's no need to worry as to how each piece of a star acts on each piece of a planet. To compute the gravitational attraction, all we need are the two masses and the center-to-center distance.

Let's set up the integral and prove Newton's result. The advantage of the 21^{st} century is *spherical coordinates*, something Newton had to do without.

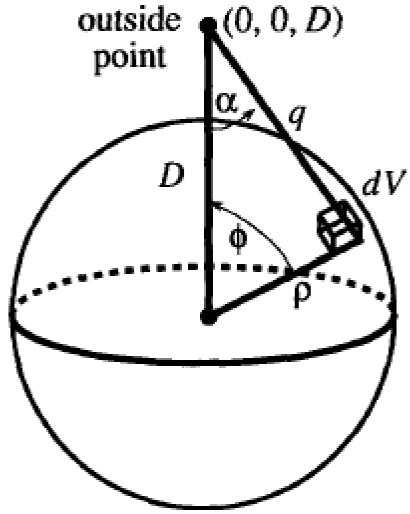

Figure 3. Diagram with Newton's integral.
Image borrowed from Strang.

In Figure 3, we have a point of mass m located at $(0,0,D)$ from the center of the large sphere. From symmetry, it is clear that the total force will be entirely in the $-\hat{z}$ direction. Begin by writing the differential force due to an arbitrary volume element dV in the sphere located distance q from mass m.

$$dF_z = d\vec{F}\cdot\hat{z} = \left|d\vec{F}\right|\left|\hat{z}\right|\cos\alpha = Gm\frac{dM}{q^2}\cos\alpha$$

Assume the sphere to have uniform mass density $\lambda = dM/dV$. The total force is dF_z integrated over the sphere.

$$F = Gm\lambda\int_{Sphere}\frac{\cos\alpha}{q^2}dV$$

Write $\cos\alpha$ using the law of cosines, giving $\cos\alpha = \dfrac{q^2 + D^2 - \rho^2}{2qD}$.

$$F = \frac{Gm\lambda}{2D} \int_{Sphere} \left(\frac{1}{q} + \frac{D^2 - \rho^2}{q^3} \right) \rho^2 d\rho \, \sin\phi \, d\phi \, d\theta$$

The law of cosines used another way gives $q^2 = \rho^2 + D^2 - 2\rho D\cos\phi$. Next let $u = q^2 = \rho^2 + D^2 - 2\rho D\cos\phi$, so $du = 2\rho D\sin\phi \, d\phi$.

$$F = \frac{Gm\lambda}{4D^2} \int_{Sphere} \left(\frac{1}{\sqrt{u}} + \frac{D^2 - \rho^2}{u^{3/2}} \right) \rho \, d\rho \, d\theta \, du$$

Solve the integral in u, noting also that $u(\phi = \pi) = (D + \rho)^2$ and $u(\phi = 0) = (D - \rho)^2$. The rest of the integrals are trivial after a little algebra.

$$F = \frac{Gm\lambda}{4D^2} \int_{Sphere} \left(2\sqrt{u} - 2\frac{D^2 - \rho^2}{\sqrt{u}} \right) \rho \, d\rho \, d\theta$$

$$F = \frac{GMm}{D^2}$$

Just like Newton found, we see the answer looks exactly as if the large sphere were actually a point particle. When setting up the two-body problem, we were safe to represent each object as a point particle after all.

Kepler's Laws and Planetary Motion

Mathematician Kepler painstakingly examined the great tables of astronomical data accumulated by Tycho Brahe, who himself had a 30+ year career as a celestial observer. Before Kepler was finished, he published three remarkable laws that today bear his name:

Law of Ellipses (1609)

The orbit of each planet is an ellipse, with the sun located at a focus.

Law of Equal Areas (1609)

A line drawn between the sun and the planet sweeps out equal areas in equal times as the planet orbits the sun.

Harmonic Law (1618)

The square of the period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

We will use Newton's advantage (calculus) to *prove* Kepler's Laws, something Kepler himself could not do. This is not to downplay Kepler's effort – it's remarkable that his three laws *could* be found in one lifetime without more sophisticated mathematics.

Angular Momentum

I mentioned early that orbital motion is always confined to a plane, which avoids the mess involved with 3D coordinate systems. To finally prove this, write the angular momentum $\vec{l} = \vec{r} \times m\dot{\vec{r}}$ of an orbiting body, and take it's time derivative.

$$\frac{d}{dt}\left(\vec{l}\right) = \frac{d}{dt}\left(\vec{r} \times m\dot{\vec{r}}\right) = m\left(\dot{\vec{r}} \times \dot{\vec{r}}\right) + m\left(\vec{r} \times \ddot{\vec{r}}\right) = 0$$

The first term is zero by definition of the cross product, and the second term is zero because the position and acceleration vectors lie on the same line. Therefore a vector $\vec{l}$ perpendicular to the plane of motion never changes, so that plane must not change either. Since $\vec{r} \times m\dot{\vec{r}}$ is evidently constant, let's find that constant. Calculate $\vec{l} = \vec{r} \times m\dot{\vec{r}}$ using $\dot{\vec{r}} = \hat{r}\dot{r} + r\hat{\theta}\dot{\theta}$.

$$\vec{l} = m\vec{r} \times \left(\hat{r}\dot{r} + r\hat{\theta}\dot{\theta}\right) = mr\left(\dot{r}\left(\hat{r} \times \hat{r}\right) + r\dot{\theta}\left(\hat{r} \times \hat{\theta}\right)\right) = \left(mr^2\dot{\theta}\right)\hat{z}$$

The constant of motion is $l = mr^2\dot{\theta}$, which is a widely general result, as we didn't yet specify any particular force law – gravity hasn't entered the picture. The sole physical assumption has been that the radius and acceleration vectors lie on the same line, a consequence of the two-body problem.

Kepler's Law of Equal Areas

The angular momentum $l = mr^2\dot{\theta}$ is the what's to prove Kepler's law of equal areas. In calculus language, Kepler's claim is $dA/dt = C$, where area A is swept out by the position vector over time and C is a constant. Picture a particle on an arbitrary planar path in two dimensions, as in Figure 4.

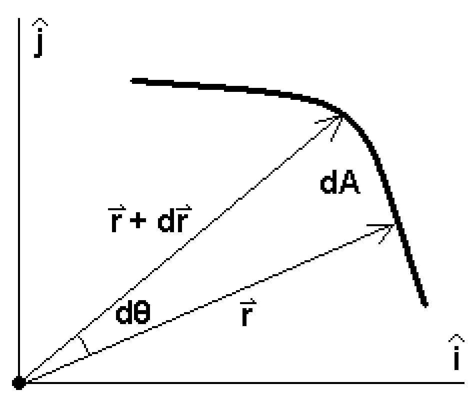

Figure 4: Area swept out by position vector.

In a small time dt, the radius vector changes by $d\vec{r}$ as angle $d\theta$ is swept out. The area dA between $\vec{r}$ and $\vec{r} + d\vec{r}$ is approximately a triangle having area $dA = r^2 d\theta \,/\, 2$. Taking a time derivative of dA, and remembering $l = mr^2\dot{\theta}$, we have

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2m},$$

which proves Kepler's law without doubt. Keep in mind that Kepler found this without using calculus. He was able to notice $dA / dt = C$, but was clueless that $C = l / 2m$.

Another Complicated Velocity Equation

We've been able to coast along by exploiting the benefits of a polar coordinate system, popping out historical results without the historical effort. In that same spirit, let's come up with a strange equation for the velocity of the orbiting body, and squeeze a few answers out of it.

For the first time, combine Newton's second law with his law of gravitation, using all of our achievements from the two-body analysis.

$$m\frac{d\vec{v}}{dt} = -G\frac{Mm}{r^2}\hat{r}$$

Divide by $d\theta \,/\, dt$, and set up an integral to solve for $\vec{v}$, not forgetting $l = mr^2\dot{\theta}$.

$$m\frac{d\vec{v}}{d\theta} = -GM\frac{m}{r^2\dot{\theta}}\hat{r} = -\frac{GMm^2}{l}\hat{r}$$

$$\vec{v} = -\frac{GMm}{l}\int \hat{r}\,d\theta + \vec{C}$$

To evaluate the integral, recall that $d\hat{\theta} / d\theta = -\hat{r}$, so $\int \hat{r}\,d\theta = -\int d\hat{\theta}$, and thus

$$\vec{v} = \frac{GMm}{l}\hat{\theta} + \vec{C}.$$

To interpret the constant $\vec{C}$, imagine the spot in the particle's orbit where $\theta = 0$. At this point, the velocity is purely along $\hat{j}$, so at the very least, $\vec{C} = \hat{j}C$. To proceed we don't actually *need* to know the magnitude $\left|\vec{C}\right| = C$. Let's just be satisfied with

$$\vec{v} = \frac{GMm}{l}\hat{\theta} + \hat{j}C,$$

which holds for *all* points the particle's trajectory, and describes both bound and non-bound motions.

Kepler's Law of Ellipses
With the new velocity equation, calculate the angular momentum $\vec{l} = \vec{r} \times m\vec{v}$, remembering the vector $\vec{l}$ lies solely along $\hat{z}$.

$$\vec{l} = \vec{r} \times m\vec{v} = m\vec{r} \times \left(\frac{GMm}{l}\hat{\theta} + \hat{j}C \right) = \frac{Gm^2M}{l}\left(\vec{r} \times \hat{\theta}\right) + mC\left(\vec{r} \times \hat{j}\right)$$

$$l\hat{z} = \frac{Gm^2M}{l} r\hat{z} + mCr\sin\left(\pi\ / 2 - \theta\right)\hat{z}$$

$$l^2 = Gm^2Mr + lmCr\cos\theta$$

Solve for r. To manage the constants, let $Q = \dfrac{l^2}{Gm^2M}$ and $\varepsilon = \dfrac{Cl}{GmM}$.

$$r(\theta) = \frac{l^2}{Gm^2M + lmC\cos\theta} = \frac{l^2 / Gm^2M}{1 + \dfrac{Cl}{GmM}\cos\theta} = \frac{Q}{1 + \varepsilon\cos\theta}$$

This equation is a loaded statement. Not only does it prove Kepler's law of ellipses, but it also characterizes several *other* kinds of trajectories, particularly the unbound motion of extrasolar objects. We have landed at the equation for the *conic section* in polar coordinates, far beyond what Kepler was looking for.

The elliptical orbit is a special case of this result. Keep in mind that the origin in polar coordinates is <u>not</u> the center of the ellipse, it's located at a focus.

Eccentricity
The dimensionless term ε has a special name called the **eccentricity**, which defines the overall shape of the orbital path, $r(\theta)$. The three main classifications are:

$\varepsilon > 1$	Hyperbola
$\varepsilon = 1$	Parabola
$\varepsilon < 1$	Ellipse

Hyperbolic and parabolic orbits describe objects that enter the solar system only once, bend around the sun once, and leave again. Someone interested in planets, periodic visitors to the solar system, or even the moon's motion will concentrate on the elliptic case.

The Elliptical Orbit

Anything we learn today about ellipses will still be dwarfed by Newton's knowledge of this shape. We've established so far that planets move in elliptical paths with the sun at a focus, and that angular momentum is conserved. What remains is to relate the period of the orbit to the natural scale of the ellipse itself.

$$r(\theta) = \frac{Q}{1+\varepsilon\cos\theta}$$

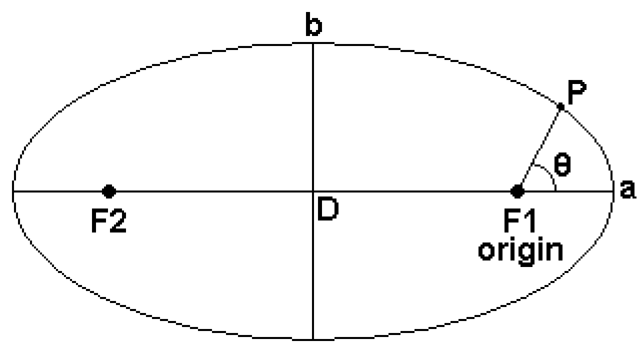

Figure 5: The ellipse.

In Figure 5, length Da, or a for short, is called the *semi-major axis*, and Db is the *semi-minor axis*. The origin is placed at the right-hand focus $F1$.

With the observation $r(0)+r(\pi)=2a$, determine a in terms of Q and ε.

$$a = \frac{1}{2}\left(\frac{Q}{1+\varepsilon}+\frac{Q}{1-\varepsilon}\right) = \frac{Q}{1-\varepsilon^2}$$

Solving for b isn't as easy, but isn't beyond calculus 101. The point b corresponds to a maximum in the $\hat{j}$ component of the radius vector, which is $y = r\sin\theta$. To find the maximum $\tilde{y} = \tilde{r}\sin\tilde{\theta}$, compute $dy/d\theta = 0$.

$$0 = \frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = \frac{d}{d\theta}\left(\frac{Q}{1+\varepsilon\cos\theta}\sin\theta\right) = Q\frac{\cos\theta+\varepsilon}{(1+\varepsilon\cos\theta)^2}$$

Evidently $dy/d\theta = 0$ is satisfied when $0 = \cos\tilde{\theta}+\varepsilon$. This is enough to write $b = \tilde{r}\sin\tilde{\theta}$ all in terms of Q and ε, giving

$$b = \frac{Q}{\sqrt{1-\varepsilon^2}}\ .$$

Finally, eliminate ε between a and b, giving $Q = b^2/a$. Next recall $Q = l^2/Gm^2M$ and write it all out.

$$\frac{b^2}{a} = \frac{l^2}{Gm^2M}$$

The result is written to have geometry on the left and physics on the right: a linkage between physical lengths and constants of motion.

Kepler's Harmonic Law

We're ready to relate the period of a planet's orbit to the size of its orbit. To begin, go back to Kepler's law of equal areas, $dA/dt = l/2m$, and integrate over one full revolution. That is,

$$\int_0^A dA = \frac{l}{2m}\int_0^T dt\ .$$

The left side is just the area of an ellipse, $A = \pi ab$. The right side is a time integral made trivial because anduglar momentum l is constant. Solve for T, expressing everything in terms of $\sqrt{GM}$ and a.

$$T = 2\pi\ \frac{a^{3/2}}{\sqrt{GM}}$$

And that does it! Once again, Newton had outdone Kepler by finding the exact proportionality constants to go along with the $a^{3/2}$ dependence.

Historical Accuracy

Our entire analysis has been backwards in the sense that we knew the $1/r^2$ gravity law all along – a luxury that Kepler and Newton couldn't start with. When Newton tackled the problem, he was already aware of elliptical orbits, but had to deduce the $1/r^2$ law himself, which we'll do now. Our advantage is modern notation and convenient coordinate systems. The proof we do looks very little like the effort Newton made, but the spirit is the same.

The equation of motion for an elliptic orbit is

$$r(\theta) = \frac{1}{C - D\cos\theta},$$

from which we compute $dr/dt = -Dr^2\dot{\theta}\sin\theta$, where Kepler's second law lets us write $dr/dt = -Dh\sin\theta$. We'll need the second derivative also, which comes out to

$$\frac{d^2r}{dt^2} = \frac{h^2}{r^2}\left(\frac{1}{r} - \frac{1}{C}\right).$$

The radial component of the acceleration vector is $a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$.

Substituting d^2r/dt^2 and eliminating $d\theta/dt$ gives

$$a_r = \frac{h^2}{r^2}\left(\frac{1}{r} - \frac{1}{C}\right) - r\frac{h^2}{r^4} = -\frac{h^2/C}{r^2},$$

which finishes the job. The acceleration aims toward the focus, is attractive (hence the negative sign), and follows the inverse-square law. *Now* we may work backwards and retrace Newton's steps. (Back to page 1.)