

Seminar 3

Criteriul Stolz-Cesaro. Fie $(a_n)_{n \in \mathbb{N}}$ un sir oarecare de numere reale si $(b_n)_{n \in \mathbb{N}}$ un sir strict monoton si divergent.

$$\text{Daca } \exists \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l \in \overline{\mathbb{R}} \quad \text{atunci} \quad \exists \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l$$

1. Fie $(x_n)_{n \in \mathbb{N}}$ un sir cu termeni strict pozitivi. Daca exista $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = l$ atunci $\lim_{n \rightarrow \infty} \sqrt[n]{x_n} = l$. Reciproca este adevarata?

2. Calculati limita sirurilor

$$\begin{aligned} \text{a) } y_n &= \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{\ln n} \\ \text{b) } y_n &= \sqrt[n]{n!} \\ \text{c) } y_n &= \frac{\sqrt[n]{n!}}{n} \end{aligned}$$

3. Pentru sirurile de mai jos

$$a_n = \sum_{k=1}^n \frac{1 + (-1)^k}{2}, \quad b_n = n, \quad \forall n \in \mathbb{N}^*$$

calculati valoarea limitelor $\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$ si $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$. Contrazice acest lucru criteriul Stolz-Cesaro?

4. Scrieti urmatoarele serii infinite cu ajutorul simbolului suma

$$\begin{aligned} \text{a) } &1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots \\ \text{b) } &1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \\ \text{c) } &1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots \end{aligned}$$

5. Calculati suma urmatoarelor serii

$$\begin{aligned} \text{a) } &\sum_{n=0}^{\infty} \frac{1}{n!} \\ \text{b) } &\sum_{n=1}^{\infty} \frac{1}{5^n} \\ \text{c) } &\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n-1}} \\ \text{d) } &\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \\ \text{e) } &\sum_{n=2}^{\infty} \ln \left(1 - \frac{1}{n^2} \right) \\ \text{f) } &\sum_{n=1}^{\infty} \frac{n 2^n}{(n+2)!} \end{aligned}$$

Exercitii suplimentare

1. Calculati limita sirurilor

a) $y_n = \frac{1+\sqrt{2}+\dots+\sqrt{n}}{n\sqrt{n}}$

b) $y_n = \frac{1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}}{\ln n}$

c) $y_n = \frac{\sqrt[n]{(n+1)(n+2)\dots(n+n)}}{n}$

d) $y_n = \frac{\ln n!}{n \ln n}$

2. Fie $(x_n)_{n \in \mathbb{N}}$ un sir cu termeni strict pozitivi. Daca exista $\lim_{n \rightarrow \infty} x_n = l$ atunci

$$\lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{x_1 x_2 \dots x_n} = l$$

3. Calculati suma urmatoarelor serii

a) $\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n$

b) $\sum_{n=2}^{\infty} \frac{1}{C_n^2}$

c) $\sum_{n=1}^{\infty} \operatorname{arctg} \frac{1}{n^2+n+1}$

d) $\sum_{n=0}^{\infty} \frac{1+a^n}{(1+a)^n}, a > 0$