Lab 11

Monday, May 9, 2022 2:50 PM

Cuadraturi Gauss

$$\int_a^b f(x)\cdot w(x)\,dx \simeq ? \qquad \int_a^b f(x)\cdot w(x)\,dx = \underbrace{\sum_{k=1}^{n} A_k\cdot f(x_k)}_{I_n} + R_n(f)$$

Input:
- f = fct. integrată
- a, b = capetele $(-\infty \le a < b \le \infty)$
- w = fct. pondere ($w \geq 0$, w cont. pe (a,b))
- n = nr. de noduri

Output:
- I_n
- noduri: $[x_1, \dots, x_n]$
- coeficienți: $[A_1, \dots, A_n]$

a.î. $R_n(p) = 0, \ \forall p \in \underbrace{\mathbb{P}_{2n-1}}_{\text{sp. pol. de gr.} \le \underbrace{2n-1}_{\text{gr. de exactitate}}}$

Notație: $(f(x), g(x))_w := \int_a^b f(x)\,g(x)\,w(x)\,dx$

- $x_1, \dots, x_n$ sunt rădăcinile pol. monic (coef. dominant $=1$) ortogonal π_n în rap. cu w $\left(\pi_n(x) = (x-x_1)\cdot\ldots\cdot(x-x_n)\right)$

$$A_k = \int_a^b \frac{(x-x_1)\cdot\ldots\cdot(x-x_{k-1})(x-x_{k+1})\cdot\ldots\cdot(x-x_n)}{(x_k-x_1)\cdot\ldots\cdot(x_k-x_{k-1})(x_k-x_{k+1})\cdot\ldots\cdot(x_k-x_n)}\cdot w(x)\,dx = \int_a^b \frac{\pi_n(x)}{(x-x_k)\cdot\pi_n'(x_k)}\,w(x)\,dx,$$

$k = \overline{1,n}$.

- pol. monic ortogonal în rap. w sunt unic determinate de:

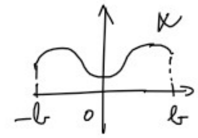
$$\pi_{k+1}(x) = (x-\alpha_k)\,\pi_k(x) - \beta_k\,\pi_{k-1}(x), \quad k = 0, 1, 2, \dots$$

$$\pi_{-1}(x) = 0, \quad \pi_0(x) = 1$$

$$\alpha_k = \frac{(x\pi_k(x), \pi_k(x))_w}{(\pi_k(x), \pi_k(x))_w}, \quad k = 0, 1, 2, \dots$$

$$\beta_k = \frac{(\pi_k(x), \pi_k(x))_w}{(\pi_{k-1}(x), \pi_{k-1}(x))_w}, \quad k = 1, 2, \dots, \qquad \beta_0 := \int_a^b w(x)\,dx.$$

Obs: $a = -b$ și $w(-x) = w(x)$, $x \in (-b, b)$



$$\Rightarrow \begin{cases} \alpha_k = 0 \\ \pi_k(-x) = (-1)^k \pi_k(x) \end{cases}, \quad k = 0, 1, 2, \dots$$

$$\Longrightarrow \begin{cases} x_{n+1-k} = -x_k \\ A_{n+1-k} = A_k \end{cases}, \quad k=\overline{1,n}.$$

Ex.: $w(x) = \dfrac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$ $\longrightarrow$ Cebîșev I

$$x_k = \cos\left(\frac{2k-1}{2n}\cdot\pi\right), \quad A_k = \frac{\pi}{n}, \quad k = \overline{1,n}.$$

Alg. numeric:

$$J_n(w) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ 0 & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}$$

$[V, x] = \text{eig}(J, \text{'vector'})$

- $x \longrightarrow$ nodurile
- $V \longrightarrow$ extrag prima linie
  - $\longrightarrow V(1,:)$
  - $\longrightarrow A = \beta_0 \cdot V(1,:).^{\wedge}2.$

$w(t)$	Support	Name	α_k	β_0	$\beta_k,\ k \geq 1$
1	$[-1,1]$	Legendre	0	2	$1/(4-k^{-2})$
1	$[0,1]$	Shifted Legendre	$\frac{1}{2}$	1	$1/(4(4-k^{-2}))$
$(1-t^2)^{-1/2}$	$[-1,1]$	Chebyshev #1	0	π	$\frac{1}{2}\ (k=1),\ \frac{1}{4}\ (k>1)$
$(1-t^2)^{1/2}$	$[-1,1]$	Chebyshev #2	0	$\frac{1}{2}\pi$	$\frac{1}{4}$
$(1-t)^{-1/2}(1+t)^{1/2}$	$[-1,1]$	Chebyshev #3	$\frac{1}{2}\ (k=0)$ $0\ (k>0)$	π	$\frac{1}{4}$
$(1-t)^{1/2}(1+t)^{-1/2}$	$[-1,1]$	Chebyshev #4	$-\frac{1}{2}\ (k=0)$ $0\ (k>0)$	π	$\frac{1}{4}$
$(1-t^2)^{\lambda-1/2},\ \lambda > -\frac{1}{2}$	$[-1,1]$	Gegenbauer	0	$\sqrt{\pi}\,\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+1)}$	$\frac{k(k+2\lambda-1)}{4(k+\lambda)(k+\lambda-1)}$
$(1-t)^{\alpha}(1+t)^{\beta}$ $\alpha > -1,\ \beta > -1$	$[-1,1]$	Jacobi	α_k^J	β_0^J	β_k^J
e^{-t}	$[0,\infty]$	Laguerre	$2k+1$	1	k^2
$t^{\alpha}\mathrm{e}^{-t},\ \alpha > -1$	$[0,\infty]$	Generalized Laguerre	$2k+\alpha+1$	$\Gamma(1+\alpha)$	$k(k+\alpha)$
e^{-t^2}	$[-\infty,\infty]$	Hermite	0	$\sqrt{\pi}$	$\frac{1}{2}k$
$\lvert t\rvert^{2\mu}\mathrm{e}^{-t^2},\ \mu > -\frac{1}{2}$	$[-\infty,\infty]$	Generalized Hermite	0	$\Gamma(\mu+\frac{1}{2})$	$\frac{1}{2}k\ (k \text{ even})$ $\frac{1}{2}k+\mu\ (k \text{ odd})$
$\frac{1}{2\pi}\mathrm{e}^{(2\phi-\pi)t}\lvert\Gamma(\lambda+it)\rvert^2$ $\lambda > 0,\ 0 < \phi < \pi$	$[-\infty,\infty]$	Meixner–Pollaczek	$-\frac{k+\lambda}{\tan\phi}$	$\frac{\Gamma(2\lambda)}{(2\sin\phi)^{2\lambda}}$	$\frac{k(k+2\lambda-1)}{4\sin^2\phi}$

$$\alpha_k^J = \frac{\beta^2-\alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)}^{*}$$

$$\beta_0^J = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},\quad \beta_k^J = \frac{4k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta)^2(2k+\alpha+\beta+1)(2k+\alpha+\beta-1)}^{\dagger}$$

*If $k=0$, the common factor $\alpha+\beta$ in the numerator and denominator of α_0^J should be (must be, if $\alpha+\beta=0$) cancelled.

†If $k=1$, the last factors in the numerator and denominator of β_1^J should be (must be, if $\alpha+\beta+1=0$) cancelled.

The first few Legendre polynomials are:

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}\left(3x^2-1\right)$
3	$\frac{1}{2}\left(5x^3-3x\right)$
4	$\frac{1}{8}\left(35x^4-30x^2+3\right)$
5	$\frac{1}{8}\left(63x^5-70x^3+15x\right)$
6	$\frac{1}{16}\left(231x^6-315x^4+105x^2-5\right)$
7	$\frac{1}{16}\left(429x^7-693x^5+315x^3-35x\right)$
8	$\frac{1}{128}\left(6435x^8-12012x^6+6930x^4-1260x^2+35\right)$
9	$\frac{1}{128}\left(12155x^9-25740x^7+18018x^5-4620x^3+315x\right)$
10	$\frac{1}{256}\left(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63\right)$

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1

x

$x^2-\frac{1}{3}$

$x^3-\frac{3}{5}x$

$\vdots$

The first few Chebyshev polynomials of the first kind are OEIS: A028297

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2-1 \\
T_3(x) &= 4x^3-3x \\
T_4(x) &= 8x^4-8x^2+1 \\
T_5(x) &= 16x^5-20x^3+5x \\
T_6(x) &= 32x^6-48x^4+18x^2-1 \\
T_7(x) &= 64x^7-112x^5+56x^3-7x \\
T_8(x) &= 128x^8-256x^6+160x^4-32x^2+1 \\
T_9(x) &= 256x^9-576x^7+432x^5-120x^3+9x \\
T_{10}(x) &= 512x^{10}-1280x^8+1120x^6-400x^4+50x^2-1 \\
T_{11}(x) &= 1024x^{11}-2816x^9+2816x^7-1232x^5+220x^3-11x
\end{aligned}
$$

The first few Chebyshev polynomials of the second kind are OEIS: A053117

$$U_0(x) = 1$$
$$U_1(x) = 2x$$
$$U_2(x) = 4x^2 - 1$$
$$U_3(x) = 8x^3 - 4x$$
$$U_4(x) = 16x^4 - 12x^2 + 1$$
$$U_5(x) = 32x^5 - 32x^3 + 6x$$
$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$$
$$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$$
$$U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$$
$$U_9(x) = 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x$$

These are the first few Laguerre polynomials: $\alpha = 0, \ w(x) = e^{-x}$

n	$L_n(x)$
0	1
1	$-x+1$
2	$\frac{1}{2}(x^2-4x+2)$
3	$\frac{1}{6}(-x^3+9x^2-18x+6)$
4	$\frac{1}{24}(x^4-16x^3+72x^2-96x+24)$
5	$\frac{1}{120}(-x^5+25x^4-200x^3+600x^2-600x+120)$
6	$\frac{1}{720}(x^6-36x^5+450x^4-2400x^3+5400x^2-4320x+720)$
n	$\frac{1}{n!}((-x)^n+n^2(-x)^{n-1}+\cdots+n(n!)(-x)+n!)$

The first eleven physicist's Hermite polynomials are:

$$
\begin{aligned}
H_0(x) &= 1,\\
H_1(x) &= 2x,\\
H_2(x) &= 4x^2 - 2,\\
H_3(x) &= 8x^3 - 12x,\\
H_4(x) &= 16x^4 - 48x^2 + 12,\\
H_5(x) &= 32x^5 - 160x^3 + 120x,\\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120,\\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x,\\
H_8(x) &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680,\\
H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x,\\
H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240.
\end{aligned}
$$

Formule pentru resturi în cuadraturi gaussiene

· **Gauss:**

$$\int_a^b f(x)w(x)dx - \sum_{i=1}^{n} w_i \cdot f(x_i) = \frac{f^{(2n)}(\xi)}{(2n)!}\int_a^b \pi_n^2(x) \cdot w(x)dx$$

π_n =polinomul ortogonal (în raport cu ponderea w) de grad n care este monic (coeficientul lui x^n este 1, i.e. $\pi_n(x) = (x - x_1) \cdot \ldots \cdot (x - x_n)$).

· **Gauss-Radau:**

$$\int_a^b f(x)w(x)dx - w_1 \cdot f(a) - \sum_{i=2}^{n} w_i \cdot f(x_i) = \frac{f^{(2n-1)}(\xi)}{(2n-1)!}\int_a^b \pi_{n-1}^2(x) \cdot (x-a) \cdot w(x)dx$$

$$\int_a^b f(x)w(x)dx - \sum_{i=1}^{n-1} w_i \cdot f(x_i) - w_n \cdot f(b) = \frac{f^{(2n-1)}(\xi)}{(2n-1)!}\int_a^b \pi_{n-1}^2(x) \cdot (b-x) \cdot w(x)dx$$

π_{n-1} =polinomul ortogonal (în raport cu ponderea $w_a(x) = (x-a)w(x)$, respectiv $w_b(x) = (b-x)w(x)$) de grad $n-1$ care este monic (coeficientul lui x^{n-1} este 1, i.e. $\pi_{n-1}(x) = (x - x_2) \cdot \ldots \cdot (x - x_n)$, respectiv $\pi_{n-1}(x) = (x - x_1) \cdot \ldots \cdot (x - x_{n-1})$).

· **Gauss-Lobatto:**

$$\int_a^b f(x)w(x)dx - w_1 \cdot f(a) - \sum_{i=2}^{n-1} w_i \cdot f(x_i) - w_n \cdot f(b) = \frac{f^{(2n-2)}(\xi)}{(2n-2)!}\int_a^b \pi_{n-2}^2(x) \cdot (x-a) \cdot (b-x) \cdot w(x)dx$$

π_{n-2} =polinomul ortogonal (în raport cu ponderea $w_{a,b}(x) = (x-a)(b-x)w(x)$) de grad $n-2$ care este monic (coeficientul lui x^{n-2} este 1, i.e. $\pi_{n-2}(x) = (x - x_2) \cdot \ldots \cdot (x - x_{n-1})$).

Problemă:

a) Găsiți o formulă de cuadratură de forma
$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = A_1 f(-1) + A_2 f(x_2) + A_3 f(x_3) + A_4 f(1) + R(f),$$
determinând coeficienții și nodurile necunoscute astfel încât să aibă gradul maxim de exactitate. Determinați formula restului.
(Pentru calculcul simbolic de integrale puteți folosi și https://www.wolframalpha.com/examples/mathematics/calculus-and-analysis
- funcționează fără erori și este mai rapid).

b) Aplicați formula de cuadratură de mai sus pentru a estima integrala
$$\int_{-1}^{1} \frac{e^x}{\sqrt{1-x^2}}\,\mathrm{d}x.$$
Găsiți o margine superioară a erorii de aproximare, folosind formula restului.