TUTORING
1

I. 2017 VARIANTA C

1) Să se studieze convergența simplă și uniformă a șirului de funcții : $(f_n)_{n\in\mathbb{N}}$, $f_n:\mathbb{R}\to\mathbb{R}$, $f_n(x) = \frac{\sin(nx)}{2n^3+1}$

<u>Convergența simplă</u> : ($\exists$ și e finit)

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin(nx)}{2n^3+1}$$

$$-1 \le \sin(nx) \le 1 \quad \Big| \cdot \frac{1}{2n^3+1} > 0$$

$$\underbrace{-\frac{1}{2n^3+1}}_{\downarrow\, 0} \le \underbrace{\frac{\sin(nx)}{2n^3+1}}_{\downarrow\, 0} \le \underbrace{\frac{1}{2n^3+1}}_{\downarrow\, 0}$$

$$\lim_{n\to\infty} \frac{\sin(nx)}{2n^3+1} = 0,\ \forall x\in\mathbb{R} \Rightarrow f_n \xrightarrow[\text{simplu pe } \mathbb{R}]{\text{converge}} 0$$

<u>Convergența uniformă</u>

Criteriul lui Weierstrass pt șirul de funcții

Fie f_n șir de funcții $f_n : A\to\mathbb{R}$

Dacă $\exists (a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ a.î. $|f_n(x) - f(x)| \le a_n$, $\forall x\in C$ și

$\lim_{n\to\infty} a_n = 0$

$f_n \xrightarrow[C]{cu} f$ unde $f: A\to\mathbb{R}$

$$|f_n(x) - 0| = \left|\frac{\sin(nx)}{2n^3+1}\right| \le \frac{1}{2n^3+1} = a_n, \forall x \in \mathbb{R} \quad \Big\} \Rightarrow$$

$\lim\limits_{n\to\infty} a_n = 0$

$\overset{\text{Crit}}{\underset{W}{\Rightarrow}} f_n \xrightarrow[\mathbb{R}]{C.U} 0$

Pentru serii:

Conv simplă și uniformă a seriei de funcții $\sum\limits_{n=1}^{\infty} \frac{\sin(nx)}{2n^3+1}$

Criteriul lui Weierstrass pt serii:

$(f_n)_{n\in\mathbb{N}}$, $f_n : A \to \mathbb{R}$ și $\sum\limits_{n=1}^{\infty} f_n(x)$.

Dacă $\exists \sum\limits_{n=1}^{\infty} a_n$ cu termeni pozitivi $\forall n \in \mathbb{N}^*$ a.î. $|f_n(x)| \le a_n$, $\forall x \in C$ și $\sum\limits_{n=1}^{\infty} a_n$ convergentă $\Rightarrow \sum\limits_{n=1}^{\infty} f_n \xrightarrow[C]{cu}$

$\frac{\sin(nx)}{2n^3+1} \le \frac{1}{2n^3+1}$, $\forall n \in \mathbb{N}^*$ (1)

$\sum\limits_{n=1}^{\infty} \frac{1}{2n^3+1}$ (Se compară cu $\sum \frac{1}{n^3}$)

Fie $a_n = \frac{1}{2n^3+1}$; $b_n = \frac{1}{n^3}$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{n^3}{2n^3+1} = \frac{1}{2} \in (0, \infty) \overset{\text{Crit de}}{\underset{\text{comp}}{\Rightarrow}}$ seriile au aceeași natură

$\sum\limits_{n=1}^{\infty} \frac{1}{n^\alpha}$ — converg pt $\alpha > 1$; diverg pt $\alpha \le 1$ (Serie armonică gen)

$\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ serie armonică gen, $\alpha = 3 > 1 \Rightarrow$

$\Rightarrow \sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ este convergentă $\Rightarrow \sum\limits_{n=1}^{\infty} \frac{1}{2n^3+1}$ este convergentă (2)

Din (1) și (2) $\overset{\text{Crit}}{\underset{W}{\Longrightarrow}} \sum\limits_{n=1}^{\infty} \frac{\sin(nx)}{2n^3+1}$ este unif converg pe $\mathbb{R} \Rightarrow$

$\Rightarrow \sum\limits_{n=1}^{\infty} \frac{\sin(nx)}{2n^3+1}$ simplu converg pe $\mathbb{R}$.

2) Se consideră seria de puteri:

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+1}} \cdot (4x-1)^n, \quad x \in \mathbb{R}$$

Aflați mulțimea de convergență și calculați suma seriei pt valorile x din mulțimea de convergență.

Not $y = 4x-1$

$a_n = \frac{(-2)^n}{3^{n+1}}$

Forma generală: $\sum_{n=1}^{\infty} a_n \cdot y^n$

$$\omega = \overline{\lim_{n\to\infty}} \frac{|a_{n+1}|}{|a_n|} = \overline{\lim_{n\to\infty}} \left| \frac{(-2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{(-2)^n} \right| = \overline{\lim_{n\to\infty}} \left| \frac{\cancel{(-2)^n} \cdot (-2)}{\cancel{3^n} \cdot 3^2} \cdot \frac{\cancel{3^n} \cdot 3}{\cancel{(-2)^n}} \right| =$$

$$= \overline{\lim_{n\to\infty}} \left| \frac{-2}{3} \right| = \frac{2}{3}$$

$\omega = \frac{2}{3} \Rightarrow R = \frac{3}{2}$

Conform T. lui Abel avem:

$$\begin{cases} \text{pt } y \in \left(-\frac{3}{2}; \frac{3}{2}\right) \Rightarrow \text{seria este absolut convergentă} \Rightarrow \text{serie convergentă} \\ \text{pt } y \in \left(-\infty; -\frac{3}{2}\right) \cup \left(\frac{3}{2}; \infty\right) \Rightarrow \text{seria este divergentă} \end{cases}$$

$$\text{Pt. } y = -\frac{3}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+1}} \cdot \left(-\frac{3}{2}\right)^n = \sum_{n=1}^{\infty} \frac{\cancel{(-2)^n}}{\cancel{3^n} \cdot 3} \cdot \frac{\cancel{(3)^n}}{\cancel{(-2)^n}} = \sum_{n=1}^{\infty} \frac{1}{3}$$

$$\lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} \neq 0 \underset{\text{nul de div}}{\overset{\text{crit}}{\Rightarrow}} \sum_{n=1}^{\infty} \frac{1}{3} \text{ e divergentă}$$

$$\text{Pt } y = \frac{3}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+1}} \cdot \left(\frac{3}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{3}$$

Not $c_n = (-1)^n \cdot \frac{1}{3}$

$$\lim_{n\to\infty} c_{2n} = \frac{1}{3} \neq 0 \Rightarrow \lim_{\substack{n\to\infty \\ c_n \to 0}} c_n \neq 0 \underset{\text{nul de div}}{\overset{\text{crit}}{\Rightarrow}} \text{seria e divergentă}$$

Seria de puteri e conv pt $y \in \left(-\frac{3}{2}; \frac{3}{2}\right)$

$-\frac{1}{8} \leq x \leq \frac{5}{8} \Rightarrow$ multimea de convergentă

$C = \left(-\frac{1}{8}; \frac{5}{8}\right)$

$-\frac{3}{2} \leq y \leq \frac{3}{2} \Leftrightarrow -\frac{3}{2} \leq 4x-1 \leq \frac{3}{2} \quad |+1$

$-\frac{3}{2}+1 \leq 4x \leq \frac{3}{2}+1 \Leftrightarrow -\frac{1}{2} \leq 4x \leq \frac{5}{2} \quad |:4$

$-\frac{1}{8} \leq x \leq \frac{5}{8}$

Pentru $x \in \left(-\frac{1}{8}; \frac{5}{8}\right)$ notăm $S(x) = \sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+1}} (4x-1)^n$

$$\boxed{\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \ q \in (-1,1) \ \text{serie geometrică}}$$

$$\sum_{n=1}^{\infty} q^n = \frac{q}{1-q} \ ; \ \sum_{n=2}^{\infty} \frac{q^2}{1-q}$$

$$S(x) = \sum_{n=1}^{\infty} \frac{(-2)^n}{3^n \cdot 3} \cdot (4x-1)^n$$

$$\frac{1}{3}\left(\sum_{n=1}^{\infty} \left(\frac{2-8x}{3}\right)^n = \frac{1}{3} \frac{\frac{2-8x}{3}}{1-\frac{2-8x}{3}}\right.$$

$$S(x) = \frac{2-8x}{3(1+8x)}, \ \forall x \in \left(-\frac{1}{8}; \frac{5}{8}\right)$$

3) Determinati punctele de extrem conditionat ale funcției $f: \mathbb{R}^2 \to \mathbb{R}$, utilizând metoda multiplicatorului Lagrange

$$f(x,y) = y^2 - 2y + x^2, \text{ cu restricția } y^2 - x^2 = 1$$

Etapa 1: Notăm cu $g: \mathbb{R}^2 \to \mathbb{R}$

$$g(x,y) = y^2 - x^2 - 1$$

$$L(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y)$$

$$L(x,y,\lambda) = y^2 - 2y + x^2 + \lambda(y^2 - x^2 - 1)$$

Etapa 2: $\begin{cases} L'_x(x,y,\lambda) = 2x + (-2x\lambda) \\ L'_y(x,y,\lambda) = 2y - 2 - 2y\lambda \\ L'_\lambda(x,y,\lambda) = y^2 - x^2 - 1 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} 2x - 2x\lambda = 0 \\ 2y - 2 + 2y\lambda = 0 \\ y^2 - x^2 - 1 = 0 \end{cases}$

$2x - 2x\lambda = 0 \Leftrightarrow 2x(1-\lambda) = 0 \Rightarrow$

I. $2x = 0$
$x = 0$

$\begin{cases} 2(y - 1 + y\lambda) = 0 \\ y^2 - 1 = 0 \Rightarrow y = \pm 1 \end{cases}$

Pt $y = 1 \Rightarrow \lambda = 0 \Rightarrow P_1(0,1); 0\lambda$

Pt $y = -1 \Rightarrow \lambda = -2 \Rightarrow P_2(0,-1); -2\lambda$

II. $1 - \lambda = 0$
$\lambda = 1$

$4y - 2 = 0 \Rightarrow y = \frac{1}{2}$

$\frac{1}{4} - x^2 - 1 = 0 \Rightarrow x^2 = -\frac{3}{4} \Rightarrow x \notin \mathbb{R}$

Etapa 3: Pt $P_1(0,1)$ și $\lambda = 0$ calculăm dif de ord 2.

$d^2L(x,y;0,1) \Leftrightarrow d^2L(P_1) =$

$= L''_{x^2}(P_1)dx^2 + 2L''_{xy}(P_1)dx\,dy + L''_{y^2}(P_1)dy$

$L''_{x^2} = 2 - 2\lambda \Rightarrow L''_{x^2}(P_1) = 2$

$L''_{y^2} = 2 + 2\lambda \Rightarrow L''_{y^2}(P_1) = 2$

$L''_{xy} = 0 \Rightarrow L''_{xy} = 0$

$d^2L(P_1) = 2dx^2 + 2dy^2 > 0$; $d^2L(P_1)$ poz def $\Rightarrow$ pct de min local condiționat

Pt $P_2(0,-1)$ și $\lambda = -2$

$L^{u}_{x^2}(P_2) = 6$

$L^{u}_{y^2}(P_2) = -2$

$L^{u}_{xy} = 0$

$d^2L(P_2) = 6dx^2 - 2dy^2$

Se diferențiază legătura:

$d^1g(P_2) = 0 \Leftrightarrow g'_x(P_2)dx + g'_y(P_2)dy = 0$

$f(x,y) = y^2 - x^2 - 1$

$\left\{\begin{array}{l} g'_x(x,y) = -2x \\ g'_y(x,y) = 2y \end{array}\right\} \Rightarrow \begin{array}{l} g'_x(P_2) = 0 \\ g'_y(P_2) = -2 \end{array} \Big\} \Rightarrow$

$\Rightarrow d^1g(P_2) = -2dy = 0 \Rightarrow dy = 0$

$d^2L(P_2) = 6dx^2 \Rightarrow$ pozitiv def $\Rightarrow$ pct de min local condiționat

4) În 4 ani consecutivi s-a obținut producția: 1, 4, 5, 6.
Să se afle prognoza pt. anul următor ajustând datele după o d

$f(x) = ax + b \to$ funcția de ajustare

$\left\{\begin{array}{l} a\sum_{i=1}^{4} x_i^2 + b\sum_{i=1}^{4} x_i = \sum_{i=1}^{4} x_i y_i \\ a\sum_{i=1}^{4} x_i + b\sum_{i=1}^{4} 1 = \sum_{i=1}^{4} y_i \end{array}\right.$

	x_i	y_i	x_i^2	$x_i y_i$
	1	1	1	1
	2	4	4	8
	3	5	9	15
	4	6	16	24
Σ	10	16	30	48

$\left\{\begin{array}{l} 30a + 10b = 48 \\ 10a + 4b = 16 \quad | \cdot (-3) \end{array}\right.$

$/ \quad -2b = 0 \Rightarrow b = 0$

$30a = 48 \Rightarrow a = \frac{48}{30} \Rightarrow a = 1{,}6$ $\Big\} \Rightarrow f(x) = 1{,}6x$

$f(5) = 1{,}6 \cdot 5 \Rightarrow f(5) = 8$ (prognoza)

5) Rezolvați ecuația diferențială:

$$x^3 y' = 3x^2 y + x^4 \cdot e^{-\frac{3y}{x^3}}, \quad x<0$$

Cu schimbarea de variabilă $z = \frac{y}{x^3}$

$$z = \frac{y}{x^3} \Rightarrow y = z \cdot x^3 \Rightarrow y' = z' \cdot x^3 + 3x^2 \cdot z$$

$$x^3(z' \cdot x^3 + 3x^2 \cdot z) = 3x^2 z x^3 + x^4 e^{-3z}$$

$$x^6 z' + \cancel{3x^5 z} = \cancel{3x^5 z} + x^4 e^{-3z}$$

$$x^6 z' = x^4 e^{-3z} \quad \Big| \cdot \frac{1}{x^6}, \ x<0$$

$$z' = \frac{1}{x^2} \cdot e^{-3z} \qquad f(x) \cdot g(z) = z'$$

ec cu variabile separabile

$$\frac{dz}{dx} = \frac{1}{x^2} \cdot e^{-3z} \Leftrightarrow \frac{dz}{e^{-3z}} = \frac{1}{x^2} dx$$

$$e^{3z} dz = \frac{1}{x^2} dx$$

$$\int e^{3z} dz = \int \frac{1}{x^2} dx \Leftrightarrow \frac{1}{3} e^{3z} = -\frac{1}{x} + \mathcal{C} \quad \Big| \cdot 3$$

$$e^{3z} = -\frac{3}{x} + \mathcal{C}, \ \mathcal{C} \in \mathbb{R}$$

$$e^{3\frac{y}{x^3}} = -\frac{3}{x} + \mathcal{C}, \ \mathcal{C} \in \mathbb{R}$$

Soluția gen. a ec. în formă implicită.

- Să se afle sol particulară a ec ce trece prin $(-1,0)$

$$\left.\begin{matrix} x=-1 \\ y=0 \end{matrix}\right\} \Rightarrow e^0 = 3 + \mathcal{C} \Rightarrow \mathcal{C} = -2$$

Sol particulară ce trece prin $(-1,0)$ este $e^{\frac{3y}{x^3}} = -\frac{3}{x} - 2$

- Să se afle sol generală sub formă explicită

$$e^{\frac{3y}{x^3}} = -\frac{3}{x} + \mathscr{C}, \; x<0, \; \mathscr{C} \in \mathbb{R}$$

$$\ln\left(e^{\frac{3y}{x^3}}\right) = \ln\left(\mathscr{C} - \frac{3}{x}\right), \quad \mathscr{C} \in \mathbb{R} \text{ a.î. } \mathscr{C} - \frac{3}{x} > 0$$

$$\frac{3y}{x^3} = \ln\left(\mathscr{C} - \frac{3}{x}\right) \Rightarrow y = \frac{x^3}{3}\ln\left(\mathscr{C} - \frac{3}{x}\right), \quad x<0, \; \mathscr{C} \in \mathbb{R}$$

6) Calculați:

$$I = \int_{-2}^{\infty} e^{-x^2-4x+1}\, dx$$

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x}\, dx, \; a>0$$

Prop:

1) $\Gamma(1) = 1$

2) $\Gamma(n) = (n-1)!, \; \forall n \in \mathbb{N}^*$

3) $\Gamma(a) = (a-1) \cdot \Gamma(a-1), \; \forall a>1$

4) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

5) $\Gamma(a)$ convergentă $\forall a>0$

Schimbare de variabilă

$$y = x+2 \Rightarrow x = y-2$$
$$dx = dy$$

$$x = -2 \Rightarrow y = 0$$
$$x \to \infty \Rightarrow y \to \infty$$

$$I = \int_0^{\infty} e^{-(y-2)^2 - 4(y-2)+1}\, dy = \int_0^{\infty} e^{-y^2+5}\, dy =$$

$$= \int_0^{\infty} e^{-y^2} \cdot e^5\, dy = e^5 \int_0^{\infty} e^{-y^2}\, dy = \frac{e^5 \cdot \sqrt{\pi}}{2}$$

$$\boxed{\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}}$$

7) Calculați

$$\int_0^\infty e^{-2x}(1-e^{-x})^7\,dx$$

$$\beta(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\,dx\ ;a,b>0$$

$$\beta(a,b)=\int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}\,dx\ ;\ a,b>0$$

Prop:

1) $\beta(a,b)=\dfrac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}$

2) $\beta(a,b)=\beta(b,a)$

3) $\beta(a,b)=\dfrac{\pi}{\sin(a\pi)}$, $a,b>0,\ a+b=1$

4) $\beta(a,b)$ convergen $\forall\, a,b>0$

Schimbare de variabilă:

$$e^{-x}=y \Rightarrow -e^{-x}dx=dy$$

$x=0 \Rightarrow y=1$

$x\to\infty \Rightarrow y=0$

$$I=-\int_0^\infty e^{-x}(1-e^{-x})^7\underbrace{(-e^{-x})dx}_{dy}$$

$$I=-\int_0^\infty y(1-y)^7dy=\int_0^1 y^1(1-y)^7dy$$

$\left.\begin{cases}a-1=1 \Rightarrow a=2\\ b-1=7 \Rightarrow b=8\end{cases}\right\} \Rightarrow I=\beta(2,8)$

$$\beta(2,8)=\frac{\Gamma(2)\cdot\Gamma(8)}{\Gamma(10)}=\frac{1!\cdot7!}{9!}=\frac{1}{72}$$

8) Calculați:

$$J=\int_0^\infty \frac{x\sqrt{x}}{(1+2x)^6}dx$$

Schimbare de variabilă

$$2x=y \Rightarrow x=\frac{1}{2}y$$
$$dx=\frac{1}{2}dy$$

$$x=0 \Rightarrow y=0$$
$$x\to\infty \Rightarrow y\to\infty$$

$$J=\int_0^\infty \frac{(\frac{1}{2}y)^{\frac{3}{2}}}{(1+y)^6}\cdot\frac{1}{2}dy=\frac{1}{4\sqrt{2}}\int_0^\infty \frac{y^{\frac{3}{2}}}{(1+y)^6}dy$$

$$\begin{cases} a-1=\frac{3}{2} \\ a+b=6 \end{cases} \Rightarrow \begin{matrix} a=\frac{5}{2} \\ b=\frac{7}{2} \end{matrix} \Rightarrow J=\frac{1}{4\sqrt{2}}\beta\left(\frac{5}{2};\frac{7}{2}\right)$$

$$J=\frac{1}{4\sqrt{2}}\cdot\frac{9\sqrt{\pi}}{768}$$

$$\beta\left(\frac{5}{2};\frac{7}{2}\right)=\frac{\Gamma(\frac{5}{2})\Gamma(\frac{7}{2})}{\Gamma(\frac{5}{2}+\frac{7}{2})}=\frac{\frac{3}{4}\sqrt{\pi}\cdot\frac{15}{8}\sqrt{\pi}}{120}=\frac{\cancel{45}^{9}\pi}{32}\cdot\frac{1}{\cancel{120}_{24}}$$

$$\Gamma\left(\frac{5}{2}\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{3}{4}\sqrt{\pi}$$
$$\Gamma\left(\frac{7}{2}\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}\cdot\frac{3}{4}\cdot\Gamma\left(\frac{1}{2}\right)=\frac{15}{8}\sqrt{\pi}$$

$$\Gamma(6)=5!=120$$

9) Desenați domeniul $D = \{(x,y) \in \mathbb{R}^2 \mid -y \leq x \leq y,\ x^2+y^2 \leq 9\}$ și calculați $\iint_D dx\,dy$.

$C_2(a,b) = (x-a)^2 + (y-b)^2 = r^2$

$x^2+y^2 = 3^2 \to C_3(0,0)$

$x^2+y^2 \leq 9 \to$ interiorul cercului

$-y = x \Rightarrow y = -x$ a doua bisectoare

$y = x$ I bis

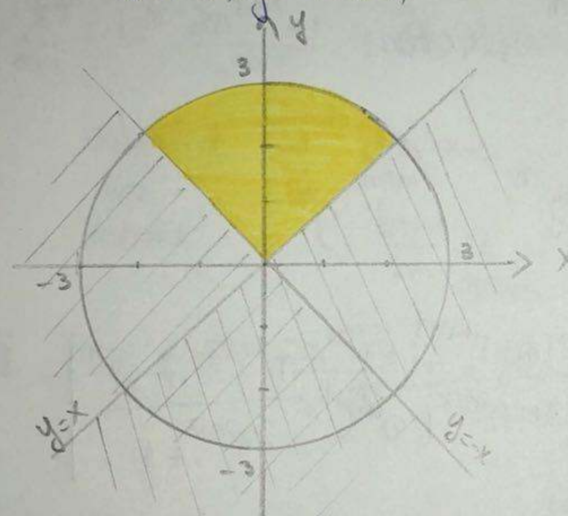
$x = 0 \Rightarrow y = 0\ (0,0)$

$x = 1 \Rightarrow y = -1\ (1,-1)$

$-y \leq x$

$(3,0) \Rightarrow 0 \leq 3\ (A)$

$(3,0) \Rightarrow 3 \leq 0\ (F)$



Schimbarea de variabilă:

$$\begin{cases} x = \rho \sin\theta \\ y = \rho \cos\theta \end{cases}$$

$$D \to D^* = \left\{(\rho,\theta) \in \mathbb{R}^2 \mid 0 \leq \rho \leq 3,\ \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}\right\}$$

$$D^* = [0,3] \times \left[\frac{\pi}{4}, \frac{3\pi}{4}\right.$$

$$\iint_D f(x,y)\,dx\,dy = \iint_D f(\rho\cos\theta, \rho\sin\theta)\,d\rho\,d\theta$$

$$I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\int_0^3 \rho\, d\rho \right) d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\rho^2}{2} \Big|_0^3 d\theta$$

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{9}{2}\, d\theta = \frac{9}{2}\,\theta \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{9\pi}{4}$$

10) Aflați mulțimea de convergență

$$\sum_{n=1}^{\infty} \frac{2n^{2013}}{n^{2015}(-4)^n}(1-2x)^n \quad , x \in \mathbb{R}$$

$$\left.\begin{array}{l} a_n = \dfrac{2n^{2013}}{n^{2015}(-4)^n} \\ y = 1-2x \end{array}\right\} \Rightarrow \sum_{n=1}^{\infty} a_n y^n$$

$$\omega = \overline{\lim_{n\to\infty}} \frac{|a_{n+1}|}{|a_n|} = \overline{\lim_{n\to\infty}} \left| \frac{2(n+1)^{2013}+1}{(n+1)^{2015}\cancel{(-4)}^{n+1}} \cdot \frac{n^{2015}\cdot\cancel{(-4)^n}}{2n^{2013}+1} \right|$$

$$= \frac{1}{4} \lim_{n\to\infty} \frac{2(n+1)^{2013}+1}{2n^{2013}+1} \cdot \frac{n^{2015}}{(n+1)^{2015}} = \frac{1}{4}$$

$\omega = \frac{1}{4} \Rightarrow R = 4$

Conform T. lui Abel avem:

$$\begin{cases} \text{pt } y \in (-4,4) \Rightarrow \text{seria este absolut conv} \Rightarrow \text{serie convergentă} \\ \text{pt } y \in (-\infty,-4) \cup (4,\infty) \Rightarrow \text{seria este divergentă} \end{cases}$$

Pt $y = -4 \Rightarrow$ seria devine

$$\sum_{n=1}^{\infty} \frac{2n^{2013}+1}{n^{2015}\cancel{(-4)^n}} \cdot \cancel{(-4)^n} = \sum_{n=1}^{\infty} \frac{2n^{2013}+1}{n^{2015}} \quad ; \quad n = \frac{n^{2013}}{n^{2015}} = \frac{1}{n^2}$$

Se compară cu $\sum \frac{1}{n^2}$

Not $b_n = \frac{2n^{2013}+1}{n^{2015}}$; $c_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{2n^{2013}+1}{n^{2015}} \cdot n^2 = 2 \in (0,\infty) \Rightarrow$ seriile au aceeași natură

$\sum c_n$ este seria armonică cu $\alpha = 2 > 1 \Rightarrow \sum c_n$ este converg

$\sum c_n$ converg $\Rightarrow \sum b_n$ converg

Pt $y = 4 \Rightarrow \sum_{n=1}^{\infty} \frac{2n^{2013}+1}{n^{2015}(-4)^n} \cdot 4^n = \sum_{n=1}^{\infty} \frac{1^n}{(-1)^n} \cdot \frac{2n^{2013}+1}{n^{2015}}$

$\sum_{n=1}^{\infty} (-1)^n \frac{2n^{2013}+1}{n^{2015}}$ (serie alternată)

Criteriul lui Leibniz $\begin{cases} \lim_{n\to\infty} = 0 \\ \frac{b_{n+1}}{b_n} < 1 \text{ descresc} \end{cases} \Rightarrow$ serie convergentă

MII. $\sum \left| (-1)^n \cdot \frac{2n^{2013}+1}{n^{2015}} \right| = \underbrace{\sum \frac{2n^{2013}+1}{n^{2015}}}_{\text{converg}}$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot \frac{2n^{2013}+1}{n^{2015}}$ absolut converg

$\Rightarrow \sum (-1)^n \frac{2n^{2013}+1}{n^{2015}}$ converg

Concluzie: Seria de puteri e conv pt $y \in [-4, 4]$

$-4 \le y \le 4$

$-4 \le 1-2x \le 4 \quad | -1$

$-5 \le -2x \le 3 \quad | :(-2)$

$+\frac{5}{2} \ge x \ge -\frac{3}{2} \Rightarrow$ mulțimea de converg este $\left[-\frac{3}{2}; \frac{5}{2}\right]$

$$\sum_{n=1}^{\infty} 2 \cdot \frac{2016^n}{(2017)^{n+1} \cdot n!}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a, \forall a \in \mathbb{R}}$$

$$\frac{2}{2017} \sum_{n=1}^{\infty} \frac{\left(\frac{2016}{2017}\right)^n}{n!} = \frac{2}{2017}\left(e^{\frac{2016}{2017}} - 1\right)$$

$$\sum_{n=1}^{\infty} 2 \cdot \frac{2016}{2017^{n+1}} = \frac{2}{2017} \sum_{n=1}^{\infty} \left(\frac{2017}{2016}\right)^n = \frac{2}{2017} \cdot \frac{\frac{2017}{2016}}{1 - \frac{2017}{2016}} =$$

$$= \frac{2}{2017} \cdot 2016 = \frac{4032}{2017}$$

11) Det pct. de extrem condiționat ale funcției

$$f : (\mathbb{R}^*)^3 \to \mathbb{R},\ f(x,y,z) = y + \frac{1}{4x} + \frac{z}{y} + \frac{x}{z}$$

Etapa 1:
$$\begin{cases} f'_x = -\frac{1}{4x^2} + \frac{1}{z} \\ f'_y = 1 - \frac{z}{y^2} \\ f'_z = \frac{1}{y} - \frac{x}{z^2} \end{cases} \Rightarrow \begin{cases} -\frac{1}{4x^2} + \frac{1}{z} = 0 \\ 1 - \frac{z}{y^2} = 0 \\ \frac{1}{y} - \frac{x}{z^2} = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} -z + 4x^2 = 0 \\ y^2 - z = 0 \\ z^2 - xy = 0 \end{cases} \Leftrightarrow \begin{cases} z = 4x^2 \\ y^2 - 4x^2 = 0 \Rightarrow y = \pm 2x \\ 16x^4 - xy = 0 \end{cases}$$

$y = 2x$

$$16x^4 - 2x^2 = 0 \Leftrightarrow 2x^2(8x^2 - 1) = 0$$

$x = 0 \notin D_f$

$x_{1;2} = \pm \frac{1}{2\sqrt{2}}$

$$x = \frac{1}{2\sqrt{2}} \Rightarrow y = \frac{1}{\sqrt{2}} \Rightarrow z = \frac{1}{2} \Rightarrow P\left(\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$

$$x = -\frac{1}{2\sqrt{2}} \Rightarrow y = -\frac{1}{\sqrt{2}} \Rightarrow z = \frac{1}{2} \Rightarrow P\left(-\frac{1}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$

$y = -2x \Rightarrow 16x^4 + 2x^2 = 0$

Etapa 2: $H(x,y,z) = \begin{pmatrix} f''_{x^2}() & f''_{xy}() & f''_{xz}() \\ f''_{yx}() & f''_{y^2}() & f''_{yz}() \\ f''_{zx}() & f''_{zy}() & f''_{z^2}() \end{pmatrix}$

$f''_{x^2} = (f'_x)'_x = \left[-\frac{1}{4x^2} + \frac{1}{z}\right]'_x = \frac{1}{2x^3}$

$f''_{xy} = 0 = f''_{yx}$

$f''_{xz} = -\frac{1}{z^2} = f''_{zx}$

$f''_{y^2} = \frac{2z}{y^3}$;

$f''_{yz} = -\frac{1}{y^2} = f''_{zy}()$

$f''_{z^2} = \frac{2x}{z^3}$

$H(x,y,z) = \begin{pmatrix} \frac{1}{2x^3} & 0 & -\frac{1}{z^2} \\ 0 & \frac{2z}{y^3} & -\frac{1}{y^2} \\ -\frac{1}{z^2} & -\frac{1}{y^2} & \frac{2x}{z^3} \end{pmatrix}$

$H(P_1) = \begin{pmatrix} 8\sqrt{2} & 0 & -4 \\ 0 & 2\sqrt{2} & -2 \\ -4 & -2 & 4\sqrt{2} \end{pmatrix}$ $\left.\begin{matrix} \Delta_1 = 8\sqrt{2} > 0 \\ \Delta_2 = 32 > 0 \\ \Delta_3 = \quad > 0 \end{matrix}\right\} \Rightarrow$

$\Rightarrow P_1\left(\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ este pct de minim

12) Rez. ec. dif $xy' - y = 2x^2 e^{2x}$, $x > 0$ $\quad \left| \cdot \frac{1}{x} \right., x \neq 0.$

$$y' - \frac{1}{x} \cdot y - 2x e^{2x} = 0$$

$$y + P(x) \cdot y + Q(x) = 0$$

Pasul 1: Rezolvăm ec. omogenă

$$y' - \frac{1}{x} \cdot y = 0 \Rightarrow \text{ec. cu variabile separabile}$$

$$y' = \frac{1}{x} \cdot y$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot y$$

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx$$

$$\ln|y| = \ln|x| + \mathscr{C}, \ \mathscr{C} \in \mathbb{R}$$

$$|y| = |x| \cdot e^{\mathscr{C}} \Rightarrow y = \pm e^{\mathscr{C}} \cdot x \Rightarrow y = kx, \ k \in \mathbb{R}^*$$

Pasul 2: Se caută o sol a ec (1) de forma

$$\left.\begin{array}{l} y = k(x) \cdot x \\ y' = k'(x) \cdot x + k(x) \end{array}\right\} \Rightarrow \xrightarrow[\text{ec (1)}]{\text{înlocuim în}}$$

$$\Rightarrow k'(x) \cdot x + \cancel{k(x)} - \frac{1}{\cancel{x}} \cdot \cancel{k(x)} \cdot \cancel{x} - 2x e^{2x} = 0$$

$$k'(x) \cdot x - 2x e^{2x} = 0 \quad \left| \cdot \frac{1}{x} \right.$$

$$k'(x) = 2e^{2x}$$

$$k(x) = \int 2e^{2x} dx$$

$$k(x) = e^{2x} + \mathscr{C}, \ \mathscr{C} \in \mathbb{R}$$

$$y = (e^{2x} + \mathscr{C}) \cdot x, \ \mathscr{C} \in \mathbb{R}$$

13) $2xyy' - y^2 + x = 0 \mid \cdot \frac{1}{2xy}$, $x \neq 0$, $y \neq 0$

$y' - \frac{y}{2x} + \frac{1}{2y} = 0$ (1) Bernoulli

$y' + P(x)\cdot y + Q(x)\cdot y^{\alpha} = 0$, $\alpha \in \mathbb{R} \setminus \{0,1\}$

Ec. Bernoulli:

1) Se numește ec 1 cu $y^{-\alpha}$

$y' - \frac{y}{2x} + \frac{1}{2y} = 0 \mid \cdot y' \Rightarrow$

$y' \cdot y - \frac{y^2}{2x} + \frac{1}{2} = 0$ (2)

2) Schimbare de variabilă $y = y(x)$

$y^{1-\alpha} = z \Rightarrow y^2 = z$

$2y \cdot y' = z'$

Înlocuim în (2)

$y'y - \frac{y^2}{2x} + \frac{1}{2} = 0 \mid \cdot 2$

$2y'y - \frac{y^2}{x} + 1 = 0$

14) $z' - \frac{1}{x}\cdot z + 1 \Rightarrow$ ec lin de ord 1

Pas 2.1. Rez. ec. omogenă atașată

$z' - \frac{1}{x}\cdot z = 0 \Leftarrow$ ec. cu var separabile

$\frac{dz}{dx} = \frac{1}{x}\cdot z \Rightarrow \frac{dz}{z} = \frac{1}{x}dx$

$\int \frac{1}{z}dz = \int \frac{1}{x}dx \Rightarrow \ln|z| = \ln|x| + C$

$e^{\ln|z|} = e^{\ln|x| + C}$

$|z| = |x| \cdot e^{C}$

$z = \pm x \cdot e^{\mathscr{C}} \cdot k$

$z = k \cdot x$

Pas 2.2. Det o sol a ec 3 de forma

$$\left.\begin{aligned} z &= k(x) \cdot x \\ z' &= k'(x) \cdot x + k(x) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow k'(x) \cdot \cancel{x} + k(x) - \frac{1}{\cancel{x}} \cdot k(x) \cdot \cancel{x} + 1 = 0 \quad \Big| \cdot \frac{1}{x}$$

$k'(x) = -\frac{1}{x}$

$k(x) = -\ln|x| + \mathscr{C}$

$z = x(\mathscr{C} - \ln|x|), \mathscr{C} \in \mathbb{R}$

$y^2 = z \Rightarrow y = \pm\sqrt{x(\mathscr{C} - \ln|x|)}$

$x \in \mathbb{R}^*, c \in \mathbb{R}$ a.î.

$x(c - \ln|x|) \geq 0$