Seminar 11

1. Calculati derivatele partiale de ordinul 1, gradientul ∇f si diferentiala $\mathrm{d}f$ pentru functiile

a) $f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x, y, z) = x^2y^3 + y \sin x - 2z$
b) $f : (0, \infty)^2 \to \mathbb{R}, \quad f(x, y) = \operatorname{arctg} \frac{x-y}{x+y}$
c) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x\sqrt{x^2 + y^2}$

2. Aratati ca functia $f(x, y) = y \ln(x^2 - y^2)$ verifica relatia

$$\frac{1}{x}\frac{\partial f}{\partial x} + \frac{1}{y}\frac{\partial f}{\partial y} = \frac{f}{y^2}, \quad \forall x > y > 0$$

Functii omogene. Fie $p \in \mathbb{R}$. O functie $f : (0, \infty)^m \to \mathbb{R}$ se numeste omogena (de grad p) daca $f(tx) = t^p f(x)$, $\forall x \in (0, \infty)^m$ si $\forall t > 0$.

3. Aratati ca functia $f : (0, \infty)^3 \to \mathbb{R}$, $f(x, y, z) = \frac{1}{x+y+z}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$ este omogena (de un anumit grad) si justificati egalitatea

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = -2f, \quad \forall (x, y, z) \in (0, \infty)^3$$

4. Studiati existenta derivatelor partiale in origine si a derivatelor dupa directie in origine pentru

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \begin{cases} \frac{x^2y}{x^4+y^2} & ,(x, y) \neq (0, 0) \\ 0 & ,(x, y) = (0, 0) \end{cases}$$

5. Calculati derivatele partiale ale functiei compuse $g \circ f$ pentru $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x \cos y, x \sin y)$ si $g = g(u, v) : \mathbb{R}^2 \to \mathbb{R}$ o functie oarecare de clasa C^1 pe $\mathbb{R}^2$.

6. Calculati derivatele partiale de ordinul 2 ale functiilor

a) $f : (1, \infty) \times \mathbb{R} \to \mathbb{R}, \quad f(x, y) = \ln(x + y^2 - 1)$
b) $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}, \quad f(x, y) = xy\, \mathrm{e}^{\frac{x}{y}}$

7. Fie $f : \mathbb{R}^m \to \mathbb{R}$, $f(x) = ||x||$. Justificati egalitatea

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \ldots + \frac{\partial^2 f}{\partial x_m^2} = \frac{m-1}{f}, \quad \forall x = (x_1, \ldots, x_m) \neq O_m$$

Exercitii suplimentare

1. Calculati derivatele partiale de ordinul 1, gradientul ∇f si diferentiala $\mathrm{d}f$ pentru functiile

a) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \sin^2(x^3 + y)$

b) $f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x,y,z) = (x + y + z)\mathrm{e}^{x^2+y^2+z^2}$

2. Calculati matricea Jacobi $J(f)$ in punctul $(1,1)$ pentru urmatoarele functii vectoriale

a) $f : \mathbb{R}^2 \to \mathbb{R}^3, \quad f(x,y) = (x^2 - y, 3x - 2y, 2xy + y^2)$

b) $f : (0,\infty)^2 \to \mathbb{R}^2, \quad f(x,y) = (\frac{1}{xy}, \operatorname{arctg} \frac{y}{x})$

3. Aratati ca daca functia $f : (0,\infty)^m \to \mathbb{R}$ este omogena de grad p si de clasa C^1 atunci

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots + x_m \frac{\partial f}{\partial x_m} = pf, \quad \forall (x_1, x_2, \ldots, x_m) \in (0,\infty)^m$$

4. Aratati ca functia de mai jos nu este continua in $(0,0)$, dar admite derivate dupa orice directie in acest punct

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \begin{cases} \frac{xy}{x+y} & , x + y \neq 0 \\ 0 & , x + y = 0 \end{cases}$$

5. Calculati derivatele partiale ale functiei compuse $g \circ f$ pentru $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (x^2 - y, 3x - 2y, 2xy + y^2)$ si $g = g(u,v,w) : \mathbb{R}^3 \to \mathbb{R}$ o functie oarecare de clasa C^1 pe $\mathbb{R}^3$.

6. Aratati ca functia $f(x,y) = (x^2 + y^2)\operatorname{arctg} \frac{y}{x}$ verifica relatia

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 2f, \quad \forall x \in (0,\infty)^2$$