Metoda celor mai mici pătrate (m.c.m.m.p.)

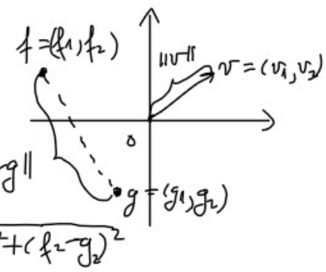
$\langle f, g\rangle$ = prod. scalar dintre f și g (vectori/funcții)

$\|f-g\|$ = distanța dintre f și g = norma dif. $f-g$

$= \sqrt{\langle f-g, f-g\rangle}$

Ex. (pt. azi): $\langle v, w\rangle = v_1 w_1 + \ldots + v_n w_n$

$\|v\| = \sqrt{\langle v, v\rangle} = \sqrt{v_1^2 + \ldots + v_n^2}$



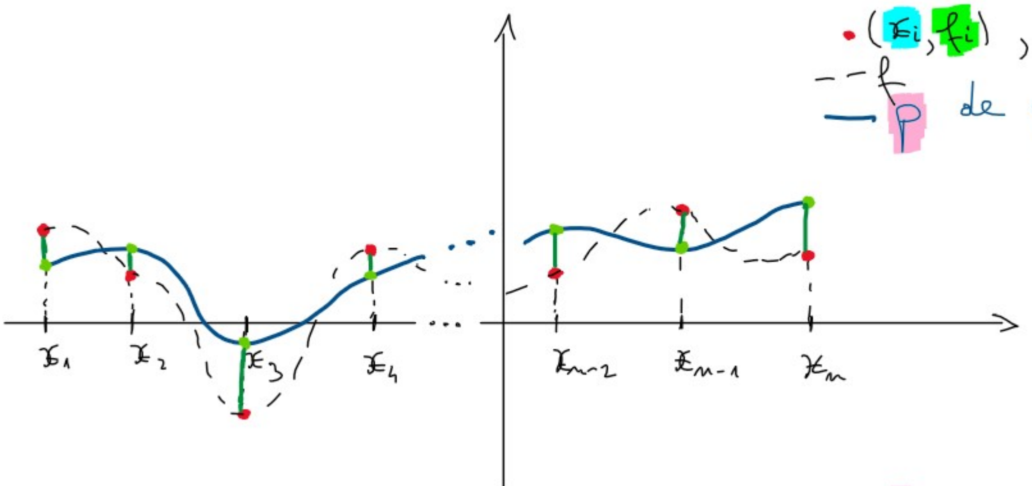
Ex. (pt. alt lab.): $\langle f, g\rangle = \int_a^b f(x) g(x)\, dx$

M.c.m.m.p. discretă $\left(\|f-g\| = \sqrt{(f_1-g_1)^2 + \ldots + (f_n-g_n)^2}\right)$

• (x_i, f_i), $i=\overline{1,n}$

-- f

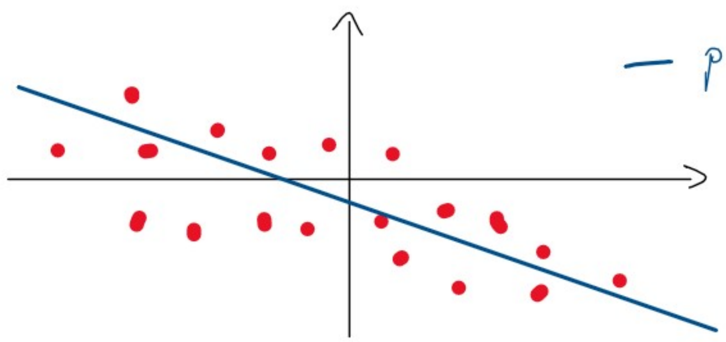
— p de gr. $\le k < n$



(?) Căutăm $\hat{p}$ pol. de gr. $\le k$ a.î. $\|\hat{p} - f\| = \min_{p \in \mathbb{P}_k} \|p - f\|$

↳ spațiul pol. de gr. $\le k$

— p de gr. ≤ 1



$$p(X) = c_k X^k + c_{k-1} X^{k-1} + \dots + c_1 X + c_0 \cdot 1$$

$$\left\{\begin{array}{l} p(x_1) = c_k x_1^k + c_{k-1} x_1^{k-1} + \dots + c_1 x_1 + c_0 \cdot 1 \simeq f_1 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ p(x_n) = c_k x_n^k + c_{k-1} x_n^{k-1} + \dots + c_1 x_n + c_0 \cdot 1 \simeq f_n \end{array}\right| \begin{array}{l} k+1 \text{ necunosc.} \\ \wedge \\ n \text{ ecuații} \end{array}$$

$$\underbrace{\begin{bmatrix} x_1^k & x_1^{k-1} & \dots & x_1 & 1 \\ \vdots & & & & \\ x_n^k & x_n^{k-1} & \dots & x_n & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} c_k \\ \vdots \\ c_0 \end{bmatrix}}_{c} \simeq \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}}_{f}$$

Sist. supradeterminat: $A \cdot c \simeq f$, $A \in \mathcal{M}_{n,m}(\mathbb{R})$, $n \geq m \, (=k+1)$

Căutăm $\hat{c} \in \mathbb{R}^m$ a.î. $\|A\hat{c} - f\| = \min_{c \in \mathbb{R}^m} \|Ac - f\|$.

Descompunerea QR: $\forall A \in \mathcal{M}_{n,m}(\mathbb{R})$, $n \geq m$:
$\exists Q \in \mathcal{M}_{n,n}(\mathbb{R})$, $Q^{-1} = Q^T$, și $\exists R \in \mathcal{M}_{n,m}(\mathbb{R})$,
↳ triunghiulară superior

$$R = \underbrace{\begin{bmatrix} 0 \diagdown \\ 0 \end{bmatrix}}_{m} \begin{matrix} \}m \\ \}n-m \end{matrix}, \quad \text{a.î.} \quad A = Q \cdot R.$$

- $Q^{-1} = Q^T \Rightarrow \|Qv\| = \|v\|, \ \forall v \in \mathbb{R}^n$.
- $\|A \cdot c - f\| = \|Q(Rc) - f\| = \|Q(\underbrace{Rc - Q^T f}_{v})\|$

$$= \|\underbrace{R \cdot c}_{y} - \underbrace{Q^T f}_{b}\| = \left\| \begin{bmatrix} \hat{y} \\ 0 \end{bmatrix} - \begin{bmatrix} \hat{b} \\ \tilde{b} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \hat{y} - \hat{b} \\ -\tilde{b} \end{bmatrix} \right\|$$

Sol. sist. supradet. $A \cdot c \simeq f$:

$$= \| \underbrace{R\cdot c}_{y} - \underbrace{Q^T f}_{b} \|$$

$$\underbrace{\begin{bmatrix} \begin{matrix} \diagdown \\ 0 \end{matrix} \\ 0 \end{bmatrix}}_{R} \cdot \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \}\hat{y} \\ \\ \}0 \end{matrix} \qquad \begin{bmatrix} b_1 \\ \vdots \\ b_m \\ b_{m+1} \\ \vdots \\ b_n \end{bmatrix} \begin{matrix} \}\hat{b} \\ \\ \}\tilde{b} \end{matrix}$$

Sol. sist. supradet. $A\cdot c \simeq f$:

pt. $\hat{y} = \hat{b} \Rightarrow \|A\cdot\hat{c} - f\| = \underbrace{\min_{c\in\mathbb{R}^m} \|A\cdot c - f\|}_{\|\tilde{b}\|}$

$$\Updownarrow$$

$$R(1{:}m,:)\cdot\hat{c} = \hat{b}$$

$$\boxed{\hat{c} = R(1{:}m,:)\setminus\hat{b}\,.}$$