Lab 4

Rezolvarea sistemelor lineare

Sisteme liniare triunghiulare → *backward/forward substitution*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots \quad \cdots \quad \cdots + a_{1n}x_n = b_1 \\ \qquad a_{22}x_2 + a_{23}x_3 + \cdots \quad \cdots \quad \cdots + a_{2n}x_n = b_2 \\ \qquad\quad \ddots \\ \qquad\qquad a_{ii}x_i + a_{ii+1}x_{i+1} + \cdots \quad \cdots + a_{in}x_n = b_i \\ \qquad\qquad\quad \ddots \\ \qquad\qquad\qquad a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n = b_{n-1} \\ \qquad\qquad\qquad\qquad a_{nn}x_n = b_n \end{cases}$$

Sisteme lineare compatibile determinate

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \Leftrightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

$$A \text{ inversabilă } (\det A \neq 0),\ A \cdot x = b \ \Rightarrow \ x = A^{-1} \cdot b.$$

Eliminarea gaussiană naivă

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} & \dots & a_{1j} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2k} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & a_{k2} & \dots & a_{kk} & \dots & a_{kj} & \dots & a_{kn} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & a_{i2} & \dots & a_{ik} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & a_{n2} & \dots & a_{nk} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} & \dots & a_{1j} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2k} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & a_{kk} & \dots & a_{kj} & \dots & a_{kn} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \dots & a_{kk} & \dots & a_{kj} & \dots & a_{kn} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & a_{ik} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & a_{nk} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \vdots \\ x_k \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vdots \\ b_k \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} & \dots & a_{1j} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2k} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & a_{kk} & \dots & a_{kj} & \dots & a_{kn} \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & \dots & a_{jj} & \dots & a_{jn} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

Exemplu (în care eliminarea gaussiană naivă nu funcționează):

$$\begin{cases} \varepsilon x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases} \Rightarrow \begin{cases} \varepsilon x_1 + x_2 = \ 1 \\ (1-\varepsilon^{-1})x_2 = 2-\varepsilon^{-1} \end{cases} \Rightarrow x_2 = \frac{2-\varepsilon^{-1}}{1-\varepsilon^{-1}} \approx 1,\ x_1 = \frac{1-x_2}{\varepsilon} \approx 0$$

$$\begin{cases} x_1 + x_2 = 2 \\ \varepsilon x_1 + x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 + \ x_2 = 2 \\ (1-\varepsilon)x_2 = 1-2\varepsilon \end{cases} \Rightarrow x_2 = \frac{1-2\varepsilon}{1-\varepsilon} \approx 1, \quad x_1 = 2 - x_2 \approx 1$$

Descompunerea L,U,P și alte descompuneri

$$\boldsymbol{L} = \begin{bmatrix} 1 & & & & \\ \ell_{21} & 1 & & \mathbf{0} & \\ \ell_{31} & \ell_{32} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & 1 \end{bmatrix} \qquad \boldsymbol{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ & u_{22} & u_{23} & \cdots & u_{2n} \\ & & u_{33} & \cdots & u_{3n} \\ & \mathbf{0} & & \ddots & \vdots \\ & & & & u_{nn} \end{bmatrix}$$

$\mathrm{P} = I_n$ cu liniile permutate

Descompunerea L,U,P: $P \cdot A = L \cdot U$

Rezolvarea sistemului $A \cdot x = b$ se reduce la rez. a două sisteme triunghiulare:

$$A \cdot x = b \Leftrightarrow (P \cdot A) \cdot x = P \cdot b \Leftrightarrow L \cdot U \cdot x = P \cdot b \Leftrightarrow L \cdot y = P \cdot b, U \cdot x = y$$

! Descompunerea *L,U,P* a matricei A se poate folosi pentru b-uri diferite în rezolvarea sistemului $A \cdot x = b \Longrightarrow$ economie de memorie și timp de execuție.

★ Dacă A este **simetrică pozitiv definită**, atunci putem obține descompunerea Cholesky ($L = R^*, U = R, P = I_n$):

$A = R^* \cdot R$, unde R este triunghiulară superior.