Varianta C - iunie 2014

Extras din lucrare

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Varianta C

Subiectul I (2 puncte)

Se consideră seria de puteri $\sum_{n=1}^{\infty} \frac{1}{5^n}(x-1)^n$ pentru $x \in \mathbb{R}$.

(1 pct.) a) Să se determine mulțimea de convergență a seriei.

(1 pct.) b) Dacă $S(x)$ e suma seriei de puteri iar $f_n : (0,1) \to \mathbb{R}$, e un șir de funcții definit prin

$$f_n(x) = -\frac{n}{nx+1} S(-nx)$$

atunci să se studieze convergența uniformă a șirului $\{f_n(x)\}_{n\geq 1}$ pe $(0,1)$.

Soluție

Notăm $\frac{1}{5^n} = a_n$ $\quad R_y = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{5^n}\cdot\frac{5^{n+1}}{1}\right| = \lim_{n\to\infty}|5| = 5$

$x-1=y$

$\Rightarrow y \in (-5,5)$

pt. $y=-5$ seria devine

$$\sum_{n=1}^{\infty}\frac{1}{5^n}\cdot(-5)^n = \sum_{n=1}^{\infty}\frac{(-1)^n\cdot 5^n}{5^n} = \sum_{n=1}^{\infty}(-1)^n \to \pm\infty \text{ (serie divergentă)}$$

pt. $y=5$ seria devine $\sum_{n=1}^{\infty}\frac{1}{5^n}\cdot 5^n = \sum_{n=1}^{\infty} 1 \to +\infty$ (serie divergentă)

$\Rightarrow y \in (-5,5) \Rightarrow -5 < x-1 < 5 \Rightarrow -4 < x < 6 \Rightarrow x \in (-4,6)$

Mulțimea de convergență a seriei date este $C = (-4,6)$.

b) $S(x) = \lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{1}{5^n}(x-1)^n = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{x-1}{5}\right)^k$

Notăm: $t = \frac{x-1}{5} \Rightarrow S_1(t) = \lim_{n\to\infty}\sum_{k=1}^{n} t^k = \lim_{n\to\infty}(\underbrace{t+t^2+\dots+t^n}_{\text{suma unei progresii geometrice cu rația } q=t})$

$-4 < x < 6$

$-5 < x-1 < 5$

$-1 < \frac{x-1}{5} < 1 \Rightarrow t \in (-1,1)$

$$= \lim_{n\to\infty} t\,\frac{1-t^n}{1-t} = \frac{t}{1-t}$$

$$\Rightarrow S(x) = \frac{\frac{x-1}{5}}{1-\frac{x-1}{5}} = \frac{x-1}{6-x}$$

$f_n(x) = -\frac{n}{nx+1} S(-nx)$

$S(-nx) = \frac{-nx-1}{6-(-nx)} = \frac{-nx-1}{6+nx}$

$\Rightarrow f_n(x) = -\frac{n}{nx+1} \cdot \frac{(-1)(nx+1)}{6+nx} \Rightarrow$

$f_n(x) = -\frac{n\cdot(-1)}{6+nx} = \frac{n}{6+nx}$

$\{f_n(x)\}_{n\geq 1}$ mărginit pe $(0,1)$

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{n}{6+nx} = \frac{1}{x}$ (adică f_n converge punctual la $f(x)=\frac{1}{x}$)

$\frac{1}{x}$ nemărginită pe $(0,1)$ (deoarece $\lim\limits_{x\to 0} \frac{1}{x} = \infty$)

$\xRightarrow{\text{Conform Curs 4}}$ $f_n(x)$ nu converge uniform la $f(x)=\frac{1}{x}$ pe $(0,1)$.

Subiectul II (2,5 puncte)

(1,5 pct.) a) Determinați punctele de extrem local ale funcției $f:\mathbb{R}^3\to\mathbb{R}$

$$f(x,y,z) = \frac{1}{x} + \frac{y}{4} + \frac{x}{z} + \frac{z}{y}$$

(1 pct.) b) Fie $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y) = x\cos y + y\sin x$, calculați $d^2 f\left(\frac{\pi}{4},\frac{\pi}{3}\right)$.

Soluție

C.e.: $x\neq 0,\ y\neq 0,\ z\neq 0$

a) $\begin{cases} f'_x(x,y,z)=0 \\ f'_y(x,y,z)=0 \\ f'_z(x,y,z)=0 \end{cases} \Leftrightarrow \begin{cases} -\frac{1}{x^2}+\frac{1}{z}=0 \\ \frac{1}{4}+z\cdot\frac{(-1)}{y^2}=0 \\ x\frac{(-1)}{z^2}+\frac{1}{y}=0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{z}=\frac{1}{x^2} \\ \frac{1}{4}=\frac{z}{y^2} \\ \frac{1}{y}=\frac{x}{z^2} \end{cases} \Leftrightarrow \begin{cases} x^2=z \\ y^2=4z \\ z^2=xy \end{cases}$

$\Leftrightarrow \begin{cases} z=x^2 \\ z=\frac{y^2}{4} \\ z^2=xy \end{cases} \Leftrightarrow \begin{cases} x=\pm\sqrt{z} \\ y=\pm 2\sqrt{z} \\ z^2=(\pm\sqrt{z})\cdot(\pm 2\sqrt{z})=\pm 2z \end{cases}$ c.e. $z>0$

$z^2=2z \Leftrightarrow z^2-2z=0 \Leftrightarrow z(z-2)=0 \Rightarrow z_1=0$, $z_2=2$

$z^2=-2z \Leftrightarrow z^2+2z=0 \Leftrightarrow z(z+2)=0$, $z_3=0$, $z_4=-2$

$\Rightarrow \begin{cases} z_1=0 \Rightarrow x_1=0,\ y_1=0 \text{ (nu îndeplinește c.e)} \\ z_2=2 \Rightarrow x_2=\sqrt{2},\ y_2=2\sqrt{2} \text{ sau } x_3=-\sqrt{2};\ y_3=-2\sqrt{2} \\ z_3=-2<0 \end{cases}$

$\begin{cases} z_1=2 \\ x_1=\sqrt{2} \\ y_1=2\sqrt{2} \end{cases}$ $\begin{cases} z_2=2 \\ x_2=-\sqrt{2} \\ y_2=-2\sqrt{2} \end{cases}$

$A(\sqrt{2},2\sqrt{2},2)$ și $B(-\sqrt{2},-2\sqrt{2},2)$

Puncte staționare (critice).

Matricea hessiană

$$H_f(x,y,z)=\begin{pmatrix} f''_{x^2} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{y^2} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{z^2} \end{pmatrix}=\begin{pmatrix} \frac{2}{x^3} & 0 & -\frac{1}{z^2} \\ 0 & \frac{2z}{y^3} & -\frac{1}{y^2} \\ -\frac{1}{z^2} & -\frac{1}{y^2} & \frac{2x}{z^3} \end{pmatrix}$$

pentru $A(\sqrt{2}, 2\sqrt{2}, 2)$

$$H_f(\sqrt{2},2\sqrt{2},2)=\begin{pmatrix} \frac{2}{2\sqrt{2}} & 0 & -\frac{1}{4} \\ 0 & \frac{1}{4\sqrt{2}} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{8} & \frac{2\sqrt{2}}{8} \end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{4} \\ 0 & \frac{1}{4\sqrt{2}} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{8} & \frac{\sqrt{2}}{4} \end{pmatrix}$$

$\Delta_1=\frac{1}{\sqrt{2}}>0$

$\Delta_2=\frac{1}{\sqrt{2}}\cdot\frac{1}{4\sqrt{2}}=\frac{1}{8}>0$; $\Delta_3=\det H_f(\sqrt{2},2\sqrt{2},2)=\frac{1}{32}-\frac{1}{32\sqrt{2}}>0$

$\Rightarrow$ A este punct de minim local.

pt. $B(-\sqrt{2},-2\sqrt{2},2)$

$$H_f(-\sqrt{2},-2\sqrt{2},2)=\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4\sqrt{2}} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{8} & \frac{2\sqrt{2}}{8} \end{pmatrix}=\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4\sqrt{2}} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{8} & \frac{\sqrt{2}}{4} \end{pmatrix}$$

$\Delta_1=-\frac{1}{\sqrt{2}}<0$

$\Delta_2=-\frac{1}{\sqrt{2}}\left(-\frac{1}{4\sqrt{2}}\right)>0$ $\Rightarrow$ B este punct de maxim local.

$\Delta_3=\det H_f(B)>0$

b) $f'_x=\cos y+y\cos x$; $f'_y=x\cdot(-\sin y)+\sin x$; $f''_{xy}=-\sin y+\cos x$

$f''_{x^2}=-y\sin x$; $f''_{y^2}=-x\cos y$; $f''_{yx}=-\sin y+\cos x$.

$$d^2f(x,y)=(dx\ dy)\begin{pmatrix} f''_{x^2} & f''_{xy} \\ f''_{yx} & f''_{y^2} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$=\left(-y\sin x\,dx+(\cos x-\sin y)dy \quad (\cos x-\sin y)dx+(-x\cos y)dy\right)\begin{pmatrix} dx \\ dy \end{pmatrix}$$

$dx=x-x_0$

$dy=y-y_0$

$$=(-y\sin x)dx^2+(\cos x-\sin y)dx\,dy+(\cos x-\sin y)dx\,dy+(-x\cos y)dy^2\Rightarrow \boxed{d^2f(x,y)=f''_{x^2}dx^2+2f''_{xy}dx\,dy+f''_{y^2}dy^2}$$

$$d^2f\left(\frac{\pi}{4},\frac{\pi}{3}\right)=-\frac{\pi}{3}\cdot\sin\frac{\pi}{4}\left(x-\frac{\pi}{4}\right)^2+2\left(\cos\frac{\pi}{4}-\sin\frac{\pi}{3}\right)\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{3}\right)-\frac{\pi}{4}\cos\frac{\pi}{3}\left(y-\frac{\pi}{3}\right)^2$$

$$= -\frac{\pi}{3}\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^2 + 2\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\right)\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{3}\right) - \frac{\pi}{4}\cdot\frac{1}{2}\left(y-\frac{\pi}{3}\right)^2$$

$$= -\frac{\pi\sqrt{2}}{6}\left(x-\frac{\pi}{4}\right)^2 + \left(\sqrt{2}-\sqrt{3}\right)\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{3}\right) - \frac{\pi}{8}\left(y-\frac{\pi}{3}\right)^2$$

$$= -\frac{\pi\sqrt{2}}{6}\left(x^2-\frac{2\pi x}{4}+\frac{\pi^2}{16}\right) + \left(\sqrt{2}-\sqrt{3}\right)\left(xy-\frac{\pi x}{3}-\frac{\pi y}{4}+\frac{\pi^2}{12}\right) - \frac{\pi}{8}\left(y^2-\right.$$
$$\left.-\frac{2\pi y}{3}+\frac{\pi^2}{9}\right).$$

Subiectul III (2 puncte)

(0,5 pct.) a) Calculați $\int_0^\infty x^3 e^{-\frac{x^2}{4}}dx$

(1,5 pct.) b) Desenați domeniul $\Delta = \{(x,y)\in\mathbb{R}^2 / 0\le x,\ 0\le y,\ 1\le x^2+y^2\le 9\}$ și calculați

$$\iint_\Delta (x^2+y^2)\,dx\,dy.$$

Soluție

a) $$\int_0^\infty x^3 e^{-\frac{x^2}{4}}dx = \int_0^\infty 8y\sqrt{y}\cdot e^{-y}\cdot\frac{1}{\sqrt{y}}dy = 8\int_0^\infty y^{\frac{3}{2}}y^{-\frac{1}{2}}e^{-y}dy$$
$$= 8\int_0^\infty y\cdot e^{-y}dy = 8\Gamma(2) = 8\cdot 1! = 8$$

notăm $\frac{x^2}{4} = y \Rightarrow x^2 = 4y$

$x = \pm 2\sqrt{y}$

$dx = \pm\frac{1}{\sqrt{y}}dy$ $\qquad \Rightarrow \int_0^\infty x^3 e^{-\frac{x^2}{4}}dx = 8.$

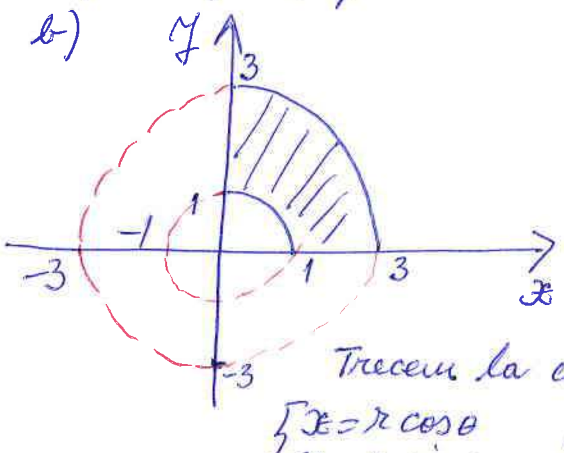
pt. $x=0 \Rightarrow y=0$

pt. $x=\infty \Rightarrow y=\infty$

$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx,\ p>0$

$\Gamma(p+1) = p!\ ,\ p\in\mathbb{N}$

b)



$x\ge 0,\ y\ge 0 \qquad 1\le x^2+y^2$
$\qquad\qquad\qquad\qquad x^2+y^2\le 9$

Ec. cercului $x^2+y^2=r^2,\ r\ge 0$

$x^2+y^2=9 \Rightarrow r=3$

$x^2+y^2=1 \Rightarrow r=1$

Trecem la coordonate polare, din coordonate carteziene

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad r\ge 0 \qquad \theta\in[0,2\pi]$$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2\ (\cos^2\theta+\sin^2\theta = r^2) \Rightarrow$

$1\le x^2+y^2\le 9$

$1\le r^2\le 9$, $r\ge 0$ $\Rightarrow r\in[1,3]$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	1	0	-1	0
$\cos\theta$	1	0	-1	0	1

$$\begin{cases} x \geq 0 \\ y \geq 0 \end{cases} \Leftrightarrow \begin{cases} r\cos\theta \geq 0 \\ r\sin\theta \geq 0 \end{cases}, r \geq 0 \Rightarrow \begin{cases} \cos\theta \geq 0 \\ \sin\theta \geq 0 \end{cases} \Rightarrow \theta \in [0; \tfrac{\pi}{2}]$$

$$\Rightarrow \Delta^* = \{ (r,\theta) \in \mathbb{R}^2 / r \in [1,3], \theta \in [0, \tfrac{\pi}{2}] \}$$

$$dx\,dy = |\det J_T|\, dr\, d\theta$$

$$J_T = \begin{pmatrix} x'_r & x'_\theta \\ y'_r & y'_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det J_T = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r \Rightarrow dx\,dy = r\,dr\,d\theta$$

$$\iint_\Delta (x^2+y^2)\,dx\,dy = \iint_{\Delta^*} (x^2+y^2)\cdot r\,dr\,d\theta = \int_1^3 \left(\int_0^{\pi/2} r^2 \cdot r\,d\theta \right) dr =$$

$$= \int_1^3 \left(r^3 \cdot \theta \Big|_0^{\pi/2} \right) dr = \int_1^3 r^3 \cdot \frac{\pi}{2}\,dr = \frac{\pi}{2} \cdot \frac{r^4}{4} \Big|_1^3 = \frac{\pi}{2}\left(\frac{3^4}{4} - \frac{1}{4} \right) =$$

$$= \frac{\pi}{2} \cdot 20 = 10\pi.$$

Subiectul IV

1) Fie $(X, \mathcal{T})$ spațiu topologic și $A \subset X$. Punctul $x \in X$ se numește punct aderent al mulțimii A dacă pt. orice vecinătate $V \in \mathcal{V}_x(\mathcal{T})$: $V \cap A \neq \emptyset$.

2) $f: \mathbb{R}^2 \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$ se numește punct de minim local dacă există o vecinătate V a lui (a,b) a.î.

$$f(x,y) \geq f(a,b) \quad \forall (x,y) \in V \cap \mathbb{R}^2 = V.$$

3) Fie $\sum_{n=1}^{\infty} (-1)^{n+1} x_n$, $x_n \geq 0$, serie alternată. Dacă

i) $\lim_{n\to\infty} x_n = 0$

ii) $\exists N \in \mathbb{N}$ a.î. $x_{n+1} \leq x_n \quad \forall n \geq N$

atunci seria este convergentă.

BAREM GENERAL

Subiectul I

a) calcul ω: 0,25p
Det. $x \in (-R, R)$: 0,25p
Capete $x = -R$ și $x = R$: 0,25p

b) calcul $S(x)$: 0,5p;
calcul $f_n(x)$ resp. a_n: 0,25p
stud. conv. resp. calcul sumă 0,25p.

Subiect II

a) Calcul deriv. part. ord 1: 0,25p;
rez. sistem 0,5p
Calcul deriv. part. ord 2: 0,25p
matr. hessiană resp. diff. 0,5p.

b) Calc. deriv. part. de ord 1 respectiv scriere modul sau șiruri 0,25p
Calcul deriv. part. ord 2 resp. inegalități 0,25p;
scriere diff. 0,25p;
rez. final 0,25p;

Subiect III

a) Schimb. var. 0,25p; rez. integrală 0,25p;
b) Desen 0,75p; rez. int. 0,75p.

Subiect IV

Se punctează enunțuri parțiale cu 0,25p