Modele de constructie a multimii numerelor reale

Multimea numerelor reale reprezinta fundamentul analizei matematice.

Cum ne putem imagina aceasta multime?

- colectia tuturor punctelor de pe axa reala
- colectia tuturor fractiilor zecimale

$$a_0, a_1 a_2 \ldots a_n \ldots, \quad a_0 \in \mathbb{Z}, a_n \in \{0, 1, \ldots, 9\}, \forall n \in \mathbb{N}^\star$$

Enumeram mai jos cateva *modele de constructie* a multimii numerelor reale.

a) cu ajutorul sirurilor de numere rationale (Cantor)

$$\mathbb{R} = \left\{ x \middle| \, \exists (q_n)_{n \in \mathbb{N}} \subseteq \mathbb{Q} \text{ sir convergent} : \lim_{n \to \infty} q_n = x \right\}$$

$q_0 = 1$, $q_1 = 1.4$, $q_2 = 1.41$, $q_3 = 1.414$, $q_4 = 1.4142$, ...

$$\lim_{n \to \infty} q_n = \sqrt{2}$$

$$q_{n+1} = \frac{q_n}{2} + \frac{1}{q_n}, \quad \forall n \in \mathbb{N},\ q_0 = 1.$$

b) cu ajutorul submultimilor de numere rationale (Dedekind)

O submultime $S \subseteq \mathbb{Q}$ se numeste *sectiune (taietura) Dedekind* daca

1. $\emptyset \neq S \neq \mathbb{Q}$

2. $\forall s \in S,\ t \in \mathbb{Q} \setminus S : s < t$

3. $\forall s_1 \in S,\ \exists s_2 \in S : s_1 < s_2$ (S nu admite un cel mai mare element).

Oricarei sectiuni S i se poate asocia un numar real unic x cu proprietatea

$$s < x \leq t, \quad \forall s \in S,\ t \in \mathbb{Q} \setminus S.$$

$S = \{x \in \mathbb{Q} | \, x < 0\} \cup \{x \in \mathbb{Q} | \, x^2 < 2\}$

$$S = (-\infty, \sqrt{2}) \cap \mathbb{Q}, \quad \mathbb{Q} \setminus S = [\sqrt{2}, \infty) \cap \mathbb{Q}$$

$$\mathbb{R} = \left\{ S \subseteq \mathbb{Q} \middle| \, S \text{ sectiune} \right\}$$

c) cu ajutorul numerelor suprareale (Conway)

Fie L si R doua submultimi de numere deja construite (initial ambele vor fi multimea vida). Construim forma algebrica $\{L|R\}$ care, in anumite conditii, va reprezenta un *numar*.

Dorim sa definim o relatie de ordine care sa implice, pentru orice numar $x = \{X_L|X_R\}$, ca

$$\boxed{x_L < x < x_R, \quad \forall x_L \in X_L, \forall x_R \in X_R} \tag{1}$$

Relatia de ordine. Fie $x = \{X_L|X_R\}$ si $y = \{Y_L|Y_R\}$ doua forme algebrice. Definim

$$x \leq y \Leftrightarrow \begin{cases} \nexists x_L \in X_L : y \leq x_L \\ \\ \nexists y_R \in Y_R : y_R \leq x \end{cases}$$

$$x = y \Leftrightarrow (x \leq y) \quad \text{si} \quad (y \leq x)$$

$$x < y \Leftrightarrow (x \leq y) \quad \text{si} \quad \text{not}(y \leq x)$$

Relatia de ordine $\leq$ este definita in mod natural (si minimal) din perspectiva conditiei (1).

Numar (suprareal). Forma algebrica $\{L|R\}$ defineste un numar (suprareal) daca

$$\nexists l \in L, \nexists r \in R : r \leq l.$$

"Tot ceea ce nu este interzis, este permis." (von Schiller)

$L = R = \emptyset \Rightarrow \{\,|\,\} \overset{\text{not}}{=} 0$ este un numar. Mai mult, $0 \leq 0$ si $0 = 0$

$L = \{0\}, R = \emptyset \Rightarrow \{0|\,\} \overset{\text{not}}{=} 1$ este un numar. Mai mult, $0 < 1$ caci

$$0 \leq 1 \Leftrightarrow \{\,|\,\} \leq \{0|\,\} \Leftrightarrow \nexists x_L \in \emptyset : 1 \leq x_L, \quad \nexists y_R \in \emptyset : y_R \leq 0$$

$$\text{not}(1 \le 0) \Leftrightarrow \text{not}(\{0|\ \} \le \{\ |\ \}) \Leftrightarrow \exists x_L \in \{0\} : 0 \le x_L \quad \text{sau} \quad \dots$$

$L = \{0\}, R = \{0\} \Rightarrow \{0|0\}$ nu este un numar!

Numar negativ. Fie $x = \{X_L|X_R\}$ un numar. Definim

$$-x = \{-X_R| - X_L\},$$

unde s-a notat $-A = \{-a|\, a \in A\}$.

Aceasta definitie este sugerata de implicatia $x_L < x < x_R \Rightarrow -x_R < -x < -x_L$.

Evident $-0 = 0$.

$L = \emptyset, R = \{0\} \Rightarrow \{\ |0\} = -1$. Mai mult, $-1 < 0$ si $-1 < 1$ (tema).

Putem forma alte numere $\{-1, 0|\ \}$, $\{-1|1\}$, $\{1|\}$, $\{0, 1|\}$ etc.

Proprietati. Pentru orice numere $x = \{X_L|X_R\}$, $y = \{Y_L|Y_R\}$ si $z = \{Z_L|Z_R\}$ se pot demonstra succesiv urmatoarele proprietati

1) $x \le x$ (reflexivitatea)
2) Daca $x \le y$ si $y \le z \Rightarrow x \le z$ (tranzitivitatea)
3) $x \le y$ sau $y \le x$ (total ordonare)
4) $x < y \Leftrightarrow \text{not}(y \le x)$
5) $x_L < x < x_R, \quad \forall x_L \in X_L, \forall x_R \in X_R$ (conditia 1)
6) Daca $y < x \Rightarrow \{y, X_L|X_R\} = x$
7) Daca $x < y \Rightarrow \{X_L|y, X_R\} = x$

Atfel rezulta $1 = \{0|\ \} = \{-1, 0|\ \}$ si $0 = \{\ |\ \} = \{-1|\ \} = \{-1|1\}$.

Sa notam $x = \{1|\ \}$, atunci

$$0 < x \Leftrightarrow \text{not}(x \le 0) \Leftrightarrow \text{not}(\{1|\ \} \le \{\ |\ \}) \Leftrightarrow \exists x_L \in \{1\} : 0 \le x_L \quad \text{sau} \quad \dots$$

$$1 < x \Leftrightarrow \text{not}(x \le 1) \Leftrightarrow \text{not}(\{1|\ \} \le \{0|\ \}) \Leftrightarrow \exists x_L \in \{1\} : 1 \le x_L \quad \text{sau} \quad \dots$$

Ce valoare sa atribuim lui x?

Adunarea numerelor. Fie $x = \{X_L|X_R\}$, $y = \{Y_L|Y_R\}$ doua numere. Definim

$$x + y = \{X_L + y, x + Y_L|X_R + y, x + Y_R\},$$

unde s-a notat $a + B = \{a + b|\, b \in B\}$ si $A + b = \{a + b|\, a \in A\}$.

Aceasta definitie este sugerata de implicatiile

$$x_L < x < x_R \Rightarrow x_L + y < x + y < x_R + y$$

$$y_L < y < y_R \Rightarrow x + y_L < x + y < x + y_R$$

Se justifica imediat ca $x + 0 = 0 + x = x$ (elementul neutru).

Revenim la numarul $x = \{1|\,\}$.

$$1 + 1 = \{0|\,\} + \{0|\,\} = \{0 + 1, 1 + 0|\} = \{1|\,\} \overset{\text{not}}{=} 2.$$

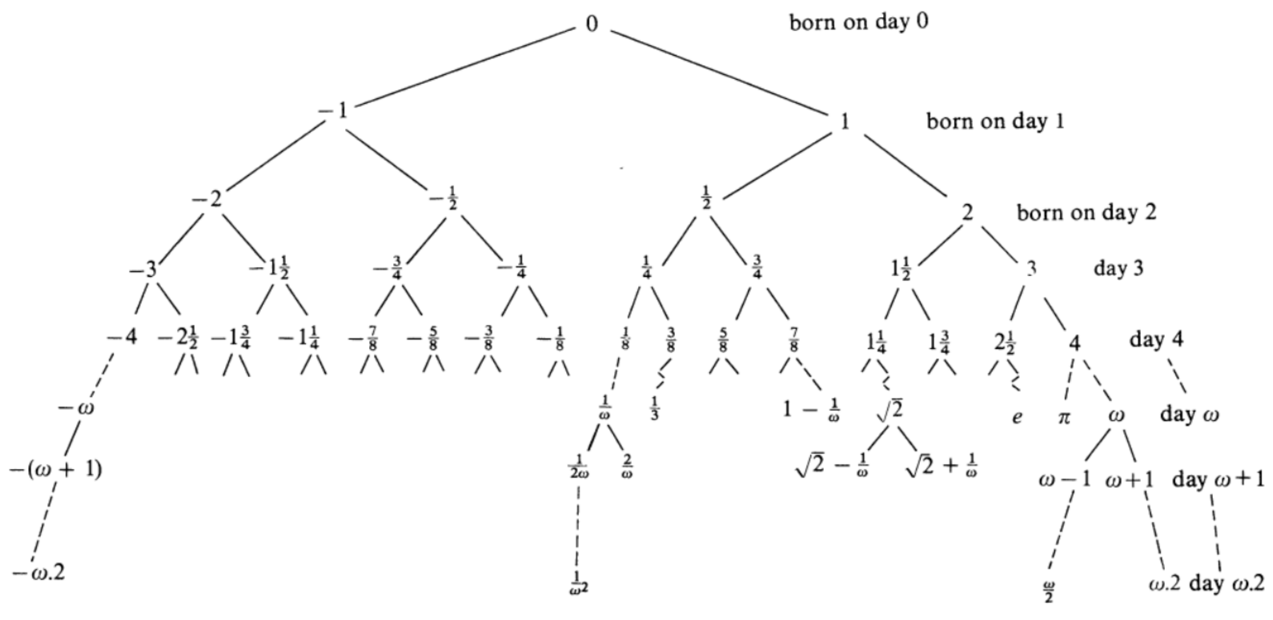
In general, $\{n - 1|\,\} = n$ si $\{\,| - (n - 1)\} = -n$, $\forall n \in \mathbb{N}^\star$.

Fie acum $x = \{0|1\}$. Justificati (tema) ca $0 < x < 1$ si $x + 1 > 1$.

$$x + x = \{0|1\} + \{0|1\} = \{0 + x, x + 0|1 + x, x + 1\} = \{x|x + 1\}$$
$$1 = \{0|\,\} = \{0|x + 1\} = \{0, x|x + 1\} = \{x|x + 1\}$$

Deci $x = \{0|1\} \overset{\text{not}}{=} \frac{1}{2}$.

Succesiv se formeaza toate *numerele diadice*, adica cele de forma $\pm\frac{m}{2^n}$, $\forall n, m \in \mathbb{N}$.



Varsta unui numar. Este *numarul minim de iteratii* $v(x)$ necesar construirii numarului x, plecand de la elementul nul $0 = \{\,|\,\}$.

$v(0) = 0$, $v(-1) = v(1) = 1$, $v(\frac{1}{2}) = v(2) = 2$, $v(\frac{1}{4}) = v(3) = 3, \ldots$

Restul numerelor se obtin aplicand acest procedeu de un numar *transfinit* de ori.

$$\{0, 1, 2, 3, \ldots |\,\} \overset{\text{not}}{=} \omega$$

$$\{0, 1, 2, 3, \ldots | \omega\} = \omega - 1$$

Fie $a_n = \frac{1}{4} + \frac{1}{4^2} + \ldots + \frac{1}{4^n}$ si $b_n = \frac{1}{2} - \frac{1}{2}a_n$, $\forall n \geq 1$. Se arata usor (tema) ca $(a_n)_{n\geq 1}$ si $(b_n)_{n\geq 1}$ sunt siruri de numere diadice care satisfac $a_n < \frac{1}{3} < b_n$, $\forall n \geq 1$ si $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \frac{1}{3}$. Asfel

$$\frac{1}{3} = \{a_1, a_2, \ldots | b_1, b_2, \ldots\}.$$

Analog se pot construi toate numerele *rationale* si cele *irationale*. Spre exemplu

$$\sqrt{2} = \{1, \frac{5}{4}, \frac{11}{8}, \frac{45}{32}, \frac{181}{128} \ldots | \ldots, \frac{363}{256}, \frac{91}{64}, \frac{23}{16}, \frac{3}{2}, 2\}.$$

Numerele diadice din membrul drept se obtin succesiv prin *metoda injumatatirii intervalului*.

Deci $v(\frac{1}{3}) = v(\sqrt{2}) = v(\pi) = v(\omega) = \omega$

Procedeul de constructie poate continua.

$$\{0, 1, 2, 3, \ldots | \omega, \omega - 1, \omega - 2, \ldots\} \stackrel{\text{not}}{=} \frac{\omega}{2}$$
$$\{0, 1, 2, 3, \ldots | \omega, \frac{\omega}{2}, \frac{\omega}{4}, \frac{\omega}{8}, \ldots\} \stackrel{\text{not}}{=} \sqrt{\omega}$$

Inmultirea numerelor. Fie $x = \{X_L | X_R\}$, $y = \{Y_L | Y_R\}$ doua numere. Definim

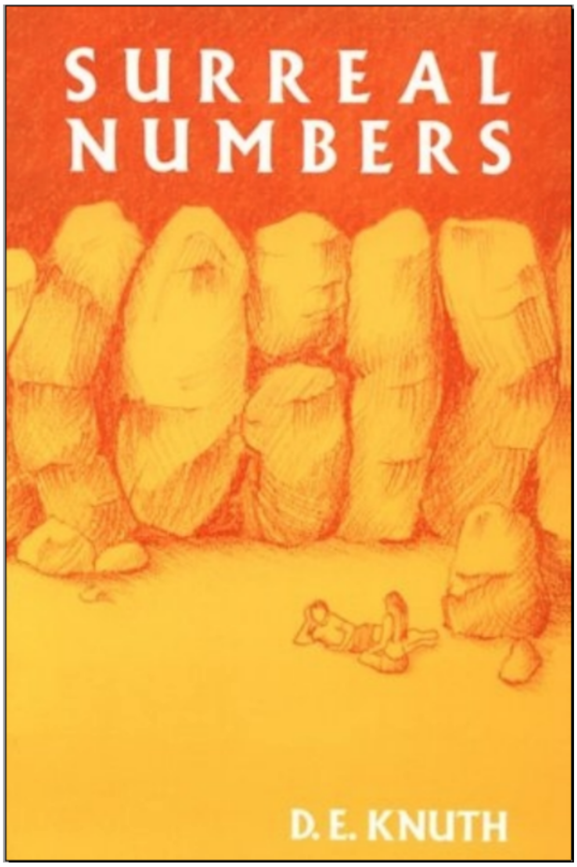
$$xy = \{X_L y + xY_L - X_L Y_L, xY_R + X_R y - X_R Y_R | X_L y + xY_R - X_L Y_R, xY_L + X_R y - X_R Y_L\},$$

unde s-a notat $aB = \{ab |\, b \in B\}$, $Ab = \{ab |\, a \in A\}$, respectiv $AB = \{ab |\, a \in A,\, b \in B\}$. Aceasta definitie este sugerata de implicatiile

$$x_L < x < x_R,\ y_L < y < y_R \Rightarrow \begin{cases} (x_L - x)(y - y_L) < 0 \Rightarrow x_L y + x y_L - x_L y_L < xy \\ \\ (x - x_R)(y_R - y) < 0 \Rightarrow x y_R + x_R y - x_R y_R < xy \\ \\ (x_L - x)(y - y_R) > 0 \Rightarrow x_L y + x y_R - x_L y_R > xy \\ \\ (x - x_R)(y_L - y) > 0 \Rightarrow x y_L + x_R y - x_R y_L > xy \end{cases}$$

Operatiile de adunare si inmultire definite mai sus vor avea toate proprietatile uzuale.

"Surreal numbers" - a mathematical novelette by D.E.Knuth



https://math.ubbcluj.ro/~sberinde/info/Surreal_Numbers.pdf