1 Soluţia unei probleme de extrem

Problema 1.1 *Studiaţi punctele de extrem ale funcţiei* $f : \left\{(x,y,z) \mid 3+\frac{x}{2}+5y+z>0\right\} \to \mathbb{R}$,

$$f(x,y,z) = \sqrt{3+\frac{x}{2}+5y+z} - \frac{x^2}{8} + \frac{10y^2}{9} - \frac{2z^2}{25}.$$

Soluţie.

$\sqrt{3+\frac{(1-0.01)}{2}+5\left(-\frac{9}{8}\right)+\left(\frac{25}{8}\right)} - \frac{1}{8}(1-0.01)^2 + \frac{10}{9}\left(-\frac{9}{8}\right)^2 - \frac{2}{25}\left(\frac{25}{8}\right)^2 - 1.5 = -1.5633\times 10^{-5}$

$\sqrt{3+\frac{(1-0.01)}{2}+5\left(-\frac{9}{8}+\frac{9}{8}\cdot 0.01\right)+\left(\frac{25}{8}-\frac{25}{8}\cdot 0.01\right)} - \frac{1}{8}(1-0.01)^2 + \frac{10}{9}\left(-\frac{9}{8}+\frac{9}{8}\cdot 0.01\right)^2$
$-\frac{2}{25}\left(\frac{25}{8}-\frac{25}{8}\cdot 0.01\right)^2 - 1.5 = 4.9384\times 10^{-7}$

Să observăm că punctele critice se obţin din sistemul

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \\ \dfrac{\partial f}{\partial z} = 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{4\sqrt{a}} - \dfrac{x}{4} = 0 \\ \dfrac{5}{2\sqrt{a}} - \dfrac{20y}{9} = 0 \\ \dfrac{1}{2\sqrt{a}} - \dfrac{4z}{25} = 0, \end{cases}$$

unde

$$a = 3 + \frac{x}{2} + 5y + z.$$

De aici rezultă

$$y = -\frac{9x}{8}, \quad z = \frac{25x}{8}, \quad a = 3 - 2x,$$

care introduse în prima ecuaţie ne dau soluţiile $x_1 = 1$, $x_2 = -\dfrac{1}{2}$.

Vom avea deci punctele critice

$$\left(1, -\frac{9}{8}, \frac{25}{8}\right) \quad \text{şi} \quad \left(-\frac{1}{2}, \frac{9}{16}, -\frac{25}{16}\right).$$

Matricea hessiană va fi

$$H_f = \begin{pmatrix} -\dfrac{a^{-3/2}}{16} - \dfrac{1}{4} & -\dfrac{5a^{-3/2}}{8} & -\dfrac{a^{-3/2}}{8} \\ -\dfrac{5a^{-3/2}}{8} & -\dfrac{25a^{-3/2}}{4} - \dfrac{20}{9} & -\dfrac{5a^{-3/2}}{4} \\ -\dfrac{a^{-3/2}}{8} & -\dfrac{5a^{-3/2}}{4} & -\dfrac{a^{-3/2}}{4} - \dfrac{4}{25} \end{pmatrix}.$$

Pentru primul punct critic vom avea

$$H_f\left(1, -\frac{9}{8}, \frac{25}{8}\right) = \begin{pmatrix} -\dfrac{5}{16} & -\dfrac{5}{8} & -\dfrac{1}{8} \\ -\dfrac{5}{8} & -\dfrac{145}{36} & -\dfrac{5}{4} \\ -\dfrac{1}{8} & -\dfrac{5}{4} & -\dfrac{41}{100} \end{pmatrix}.$$

Să observăm că diferenţiala de ordin 2 în acest punct este

$$\begin{aligned} d^2 f\left(1,-\frac{9}{8},\frac{25}{8}\right) &= -\frac{5}{16}\left(h_1^2+\frac{116}{9}h_2^2+\frac{164}{125}h_3^2+4h_1h_2+\frac{4}{5}h_1h_3+8h_2h_3\right) \\ &= -\frac{5}{16}\left[\left(h_1+2h_2+\frac{2h_3}{5}\right)^2+\frac{80}{9}\left(h_2+\frac{9}{25}h_3\right)^2\right], \end{aligned}$$

deci este pozitiv semidefinită, adică nu putem aplica teoria generală pentru a deduce natura punctului critic.

Să observăm însă că, atunci când (x,y,z) variază în jurul lui $\left(1,-\frac{9}{8},\frac{25}{8}\right)$, suma $a^{-3/2} = \left(3+\frac{x}{2}+5y+z\right)^{-3/2}$ va fi de forma $1\pm\varepsilon$, cu $\varepsilon>0$ mic. Observând expresia hessianei, va rezulta

$$H_f = \begin{pmatrix} -\frac{(1\pm\varepsilon)}{16}-\frac{1}{4} & -\frac{5(1\pm\varepsilon)}{8} & -\frac{(1\pm\varepsilon)}{8} \\ -\frac{5(1\pm\varepsilon)}{8} & -\frac{25(1\pm\varepsilon)}{4}-\frac{20}{9} & -\frac{5(1\pm\varepsilon)}{4} \\ -\frac{(1\pm\varepsilon)}{8} & -\frac{5(1\pm\varepsilon)}{4} & -\frac{(1\pm\varepsilon)}{4}-\frac{4}{25} \end{pmatrix}.$$

Atunci, pentru (x,y,z) în jurul lui $\left(1,-\frac{9}{8},\frac{25}{8}\right)$, vom avea că

$$d^2 f(x,y,z) = -\frac{5}{16}\left[\left(h_1+2h_2+\frac{2h_3}{5}\right)^2+\frac{80}{9}\left(h_2+\frac{9}{25}h_3\right)^2\right] \pm \frac{\varepsilon}{16}\left[h_1+10h_2+2h_3\right]^2.$$

Aceasta ne sugerează să alegem

$$f\left(1\pm\varepsilon,-\frac{9}{8}(1\pm\varepsilon),\frac{25}{8}(1\pm\varepsilon)\right).$$

Avem

$$\begin{aligned} f\left(1-\varepsilon,-\frac{9}{8}(1-\varepsilon),\frac{25}{8}(1-\varepsilon)\right) &= \sqrt{3+\frac{1-\varepsilon}{2}-\frac{45}{8}(1-\varepsilon)+\frac{25}{8}(1-\varepsilon)} \\ &\quad -(1-\varepsilon)^2\left[\frac{1}{8}-\frac{10}{9}\frac{81}{64}+\frac{2}{25}\frac{625}{64}\right] \\ &= \sqrt{2\varepsilon+1}+\frac{(1-\varepsilon)^2}{2}. \end{aligned}$$

Atunci

$$f\left(1-\varepsilon,-\frac{9}{8}(1-\varepsilon),\frac{25}{8}(1-\varepsilon)\right) > f\left(1,-\frac{9}{8},\frac{25}{8}\right)$$

dacă şi numai dacă

$$\begin{aligned} \sqrt{2\varepsilon+1}+\frac{(1-\varepsilon)^2}{2} &> \frac{3}{2}, \\ \sqrt{2\varepsilon+1} &> 1+\varepsilon-\frac{\varepsilon^2}{2}, \\ 2\varepsilon+1 &> 1+\varepsilon^2+\frac{\varepsilon^4}{4}+2\varepsilon-\varepsilon^2-\varepsilon^3, \\ \varepsilon^3 &> \frac{\varepsilon^4}{4}, \quad 4>\varepsilon, \end{aligned}$$

ceea ce este adevărat pentru $\varepsilon > 0$ suficient de mic.

Pe de altă parte

$$f\left(1-\varepsilon, -\frac{9}{8}, \frac{25}{8}\right) = \sqrt{3 + \frac{1-\varepsilon}{2} - \frac{45}{8} + \frac{25}{8}} - \frac{(1-\varepsilon)^2}{8} + \frac{10}{9}\frac{81}{64} - \frac{2}{25}\frac{625}{64}$$
$$= \sqrt{1 - \frac{\varepsilon}{2}} + \frac{1}{2} - \frac{\varepsilon^2 - 2\varepsilon}{8}.$$

Atunci

$$f\left(1-\varepsilon, -\frac{9}{8}, \frac{25}{8}\right) < f\left(1, -\frac{9}{8}, \frac{25}{8}\right)$$

dacă şi numai dacă

$$\sqrt{1-\frac{\varepsilon}{2}} + \frac{1}{2} - \frac{\varepsilon^2 - 2\varepsilon}{8} < \frac{3}{2},$$
$$\sqrt{1-\frac{\varepsilon}{2}} < 1 + \frac{\varepsilon^2}{8} - \frac{\varepsilon}{4},$$
$$1 - \frac{\varepsilon}{2} < 1 + \frac{\varepsilon^4}{64} + \frac{\varepsilon^2}{16} - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{4} - \frac{\varepsilon^3}{16},$$
$$\frac{\varepsilon^3}{16} - \frac{\varepsilon^4}{64} < \frac{5\varepsilon^2}{16},$$
$$\varepsilon - \frac{\varepsilon^2}{4} < \frac{5}{16},$$

ceea ce este adevărat pentru $\varepsilon > 0$ suficient de mic.

Obţinem aşadar că punctul $\left(1, -\frac{9}{8}, \frac{25}{8}\right)$ nu este punct de extrem.

Ne ocupăm acum de celălalt punct. Avem

$$a = 3 + \frac{x}{2} + 5y + z\Big|_{\left(-\frac{1}{2}, \frac{9}{16}, -\frac{25}{16}\right)} = 4, \quad a^{-3/2} = \frac{1}{8},$$

deci hessiana va fi

$$H_f = \begin{pmatrix} -\frac{33}{128} & -\frac{5}{64} & -\frac{1}{64} \\ -\frac{5}{64} & -\frac{865}{288} & -\frac{5}{32} \\ -\frac{1}{64} & -\frac{5}{32} & -\frac{153}{800} \end{pmatrix} = -64 \begin{pmatrix} \frac{33}{2} & 5 & 1 \\ 5 & \frac{1730}{9} & 10 \\ 1 & 10 & \frac{306}{25} \end{pmatrix}.$$

Aceasta va fi negativ definită (matrice tare diagonal dominată), deci $\left(-\frac{1}{2}, \frac{9}{16}, -\frac{25}{16}\right)$ este punct de maxim local. $\square$