Exerciții suplimentare

1. Det. constanta $a \in \mathbb{R}$ pt. care fc. $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 4x^2 + 4xy + ay^2$ are pct. de extrem local și determinați aceste puncte.

$$\nabla f(x,y) = (0,0) \Rightarrow \begin{cases} 8x + 4y = 0 \\ 4x + 2ay = 0 \mid \cdot 2 \end{cases} \quad \text{"-"} \quad \begin{aligned} 4y - 4ay &= 0 \\ 4y(1-a) &= 0 \end{aligned}$$

* Pt. $a \neq 1$:

$y = 0 \Rightarrow x = 0.$

$$H(f)(x,y) = \begin{pmatrix} 8 & 4 \\ 4 & 2a \end{pmatrix} = H(f)(0,0)$$

$\Delta_1 = 8 > 0$

$\Delta_2 = 16a - 16$

Dacă $\Delta_2 > 0$, atunci $(0,0)$ e pct. de minim local.

$16a - 16 > 0 \Rightarrow 16a > 16 \Rightarrow a > 1.$

Dacă $a < 1$, atunci

$d^2 f(0,0)(u_1, u_2) = 8u_1^2 + 8u_1u_2 + 2a\,u_2^2.$

$d^2 f(0,0)(1,-2) = 8 - \cancel{16} + \cancel{8}a = 8a - 8 < 0$

$d^2 f(0,0)(1,0) = 8 > 0$

$\Rightarrow (0,0)$ pct. șa

* $a = 1.$

$f(x,y) = 4x^2 + 4xy + y^2 = (2x+y)^2 \geq 0$

~~Deci (0,0) pct. de minim local (~~

$2x + y = 0 \Rightarrow y = -2x \Rightarrow$ pct. $(d, -2d)$ minim local

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③ Det. pct. critice și pct. de extrem local (specificând tipul acestora) pt. următoarele funcții:

a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + y^3 - 3xy$

I $\nabla f(x,y) = (0,0) \Rightarrow \begin{cases} 3x^2 - 3y = 0 & \Rightarrow x^2 = y \\ 3y^2 - 3x = 0 & \Rightarrow x = y^2 \end{cases}$

$\Rightarrow y^4 = y \Rightarrow y = 0$ sau $y = 1$

$x = 0$ sau $x = 1$

Pct. critice: $P_1(0,0)$, $P_2(1,1)$

II. $H(f)(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$

* $H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ $\Delta_1 = 0 \Rightarrow$ nu putem aplica Sylvester.

$d^2 f(0,0)(u_1,u_2) = -6u_1u_2$

$d^2 f(0,0)(1,1) = -6 < 0$

$d^2 f(0,0)(1,-1) = 6 > 0$ $\Bigg| \Rightarrow P_1(0,0)$ pct. șa

* $H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ $\Delta_1 = 6 > 0$, $\Delta_2 = 27 > 0$ $\Bigg| \Rightarrow P_2(1,1)$ pct. minim local

b) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^3 - x + y^2 + z^2$

I. $\nabla f(x,y,z) = (0,0,0) = \begin{cases} 3x^2 - 1 = 0 \Rightarrow x^2 = 1/3 \Rightarrow x = \pm\sqrt{3}/3 \\ 2y = 0 \Rightarrow y = 0 \\ 2z = 0 \Rightarrow y = 0 \end{cases}$

Pct. critice: $P_1(\sqrt{3}/3, 0, 0)$ și $P_2(-\sqrt{3}/3, 0, 0)$

II. $H(f)(x,y,z) = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$H(f)\left(\sqrt{3}/3, 0, 0\right) = \begin{pmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad \Delta_1 = 2\sqrt{3} > 0$, $\Delta_2 = 4\sqrt{3} > 0$, $\Delta_3 = 8\sqrt{3} > 0$ $\Rightarrow$ P_1 pct. de minim local

$H(f)\left(-\sqrt{3}/3, 0, 0\right) = \begin{pmatrix} -2\sqrt{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad \Delta_1 = -2\sqrt{3} < 0$, $\Delta_2 = -4\sqrt{3} < 0$, $\Delta_3 = -8\sqrt{3} < 0$

$\Rightarrow d^2 f\left(-\sqrt{3}/3, 0, 0\right)(u_1, u_2, u_3) = -2\sqrt{3}\,u_1^2 + 2u_2^2 + 2u_3^2$

$d^2 f\left(-\sqrt{3}/3, 0, 0\right)(1, 0, 0) = -2\sqrt{3} < 0$

$d^2 f\left(-\sqrt{3}/3, 0, 0\right)(0, 1, 1) = 4 > 0$ $\quad\Big|\Rightarrow$ P_2 pct. şa

c) $f: (0,\infty) \times \mathbb{R} \to \mathbb{R}$, $f(x,y) = x(y^2 + \ln^2 x)$

I. $\nabla f(x,y) = (0,0) \Rightarrow \begin{cases} y^2 + \ln^2 x + \cancel{x} \cdot 2\ln x \cdot \frac{1}{\cancel{x}} = 0 \\ 2yx = 0 \Rightarrow y = 0 \end{cases}$

($x=0$ nu se poate, deoarece $x \in (0,\infty)$)

$\overset{(1)}{\Rightarrow} \ln^2 x + 2\ln x = 0$

$\ln x(\ln x + 2) = 0$

$\Rightarrow \ln x = 0 \Rightarrow x = 1$

$\ln x = -2 \Rightarrow x = e^{-2}$

Pct. critice: $P_1(1,0)$ şi $P_2(e^{-2}, 0)$

II. $H(f)(x,y) = \begin{pmatrix} \frac{2\ln x}{x} + \frac{2}{x} & 2y \\ 2y & 2x \end{pmatrix}$

$* H(f)(1,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\Delta_1 = 2 > 0$, $\Delta_2 = 4 > 0$ $\Rightarrow$ P_1 minim local

$* H(f)(e^{-2},0) = \begin{pmatrix} \frac{-2}{e^{-2}} & 0 \\ 0 & 2e^{-2} \end{pmatrix}$ $\Delta_1 = \frac{-2}{e^{-2}} = -2e^2 < 0$, $\Delta_2 = -4 < 0$

$d^2f(e^{-2},0)(u_1,u_2) = -2e^2 u_1^2 + 2e^{-2} u_2^2$

$\left.\begin{array}{l} d^2f(e^{-2},0)(1,0) = -2e^2 < 0 \\ d^2f(e^{-2},0)(0,1) = 2e^{-2} > 0 \end{array}\right\} \Rightarrow d^2f$ indef. $\Rightarrow$ P_2 pct. şa

d) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = z^2(1+xy) + xy$

I. $\nabla f(x,y,z) = O_3 \Rightarrow \begin{cases} yz^2 + y = 0 \Rightarrow y(z^2+1) = 0 \Rightarrow y = 0 \\ xz^2 + x = 0 \Rightarrow x = 0 \\ (1+xy)\cdot 2z = 0 \Rightarrow z = 0 \end{cases}$

Pct. critic: $P(0,0,0)$

$H(f)(x,y,z) = \begin{pmatrix} 0 & z^2+1 & 2yz \\ z^2+1 & 0 & 2xz \\ 2yz & 2xz & 2(1+xy) \end{pmatrix}$

$H(f)(0,0,0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\Delta_1 = 0$, $\Delta_2 = -1$, $\Delta_3 = -2$ $\Rightarrow$ nu putem det. astfel.

$d^2f(0,0,0)(u_1,u_2,u_3) = 2u_1u_2 + 2u_3^2$

$d^2f(0,0,0)(1,1,0) = 2 > 0$ $\quad d^2f(0,0,0)(1,-1,0) = -2 < 0$ $\Rightarrow$ indef $\Rightarrow$ P pct. şa

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e) $f:(0,\infty)^2\to\mathbb{R},\ f(x,y)=xy+\frac{8}{x}+\frac{8}{y}$

I. $\nabla f(x,y)=0_2 \Rightarrow \begin{cases} y-\frac{8}{x^2}=0 \Rightarrow y=\frac{8}{x^2} \\ x-\frac{8}{y^2}=0 \Rightarrow x=\frac{8}{y^2}\end{cases}$

$$y=\frac{8}{\left(\frac{8}{y^2}\right)^2}=\frac{8}{\frac{64}{y^4}}=\frac{8y^4}{64}=\frac{y^4}{8} \Rightarrow$$

$8y=y^4 \Rightarrow y^4-8y=0 \Rightarrow y(y^3-8)=0 \Rightarrow y=2$

$y=0$, nu se poate ($(x,y)\in(0,\infty)^2$)

$\Rightarrow x=2$

Pct. critic: $P(2,2)$

$$H(f)(x,y)=\begin{pmatrix}\frac{16}{x^3} & 1\\ 1 & \frac{16}{y^3}\end{pmatrix} \qquad H(f)(2,2)=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

$\left.\begin{array}{l}\Delta_1=2>0\\ \Delta_2=3>0\end{array}\right\} \Rightarrow P$ pct. de minim local

f) $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=(1+e^x)\cos y-x\cdot e^x$

I. $\frac{\partial f}{\partial x}(x,y)=e^x\cos y-x\cdot e^x-e^x=e^x(\cos y-x-1)$

$\frac{\partial f}{\partial y}(x,y)=(1+e^x)\cdot(-\sin y)$

$\nabla f(x,y)=(0,0)=\begin{cases} e^x(\cos y-x-1)=0\\ -\sin y(1+e^x)=0 \Rightarrow \sin y=0 \Rightarrow y=k\pi,\ k\in\mathbb{Z}\end{cases}$

* k par $\Rightarrow \cos(k\pi)=1 \Rightarrow e^x\cdot(1-x-1)=0 \Rightarrow x=0$
* k impar $\Rightarrow \cos(k\pi)=-1 \Rightarrow e^x(-2-x)=0 \Rightarrow x=-2$

Pct. critice sunt: $P_1(0, 2k\pi)$ și $P_2(-2, (2k+1)\pi)$, $k \in \mathbb{Z}$

II. $H(f)(x,y) = \begin{pmatrix} (\cos y - x - 1)\cdot e^x + (-1)\cdot e^x & -\sin y \cdot e^x \\ e^x \cdot (-\sin y) & (1+e^x)\cdot(-\cos y) \end{pmatrix}$

* P_1: $H(f)(0, 2k\pi) = \begin{pmatrix} (1-0-1)\cdot e^0 - e^0 & 0\cdot e^0 \\ e^0 \cdot 0 & (1+e^0)\cdot(-1) \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$

$\Delta_1 = -1 < 0$
$\Delta_2 = 2 > 0$ $\Rightarrow$ P_1 max. local

* P_2: $H(f)(-2, (2k+1)\pi) = \begin{pmatrix} (-1+2-1)\cdot e^{-2} - e^{-2} & 0 \\ 0 & (1+e^{-2})\cdot 1 \end{pmatrix}$

$= \begin{pmatrix} -e^{-2} & 0 \\ 0 & 1+e^{-2} \end{pmatrix}$

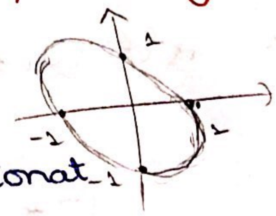
$\Delta_1 = -e^{-2} < 0$
$\Delta_2 = -e^{-2}(1+e^{-2}) < 0$ $\Rightarrow$ nu putem astfel decide

$d^2(f)(-2, (2k+1)\pi)(u_1, u_2) = -e^{-2}u_1^2 + (1+e^{-2})u_2^2$

$d^2(f)(\dots)(1,0) = -e^{-2} < 0$
$d^2(f)(\dots)(0,1) = 1+e^{-2} > 0$ $\Rightarrow$ $d^2 f$ indef. $\Rightarrow$ P_2 pct. șa

(1) Determinați pct. de extrem condiționat (specificând tipul acestora) și valorile extreme ale următoarelor funcții relativ la mulțimea S indicată (știind că e compactă)

a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x+y$, $S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + xy + y^2 = 1\}$



$$\left.\begin{array}{l} f \text{ cont.} \\ S \text{ compactă} \end{array}\right\} \xRightarrow{T.W} f \text{ are min. și max. condiționat relativ la } S.$$

Fie $F(x,y) = x^2 + xy + y^2 - 1$

$$S = \{(x,y) \in \mathbb{R}^2 \mid F(x,y) = 0\}$$

Det. multipl. lui Lagrange:

$$L(x,y,\lambda) = f(x,y) + \lambda \cdot F(x,y)$$
$$= x + y + \lambda(x^2 + xy + y^2 - 1)$$

$$\nabla L(x,y,\lambda) = (0,0,0) = \begin{cases} 1 + \lambda(2x + y) = 0 \\ 1 + \lambda(x + 2y) = 0 \\ x^2 + xy + y^2 - 1 = 0 \end{cases}$$

(1) și (2): $\begin{cases} 2x + y = -\frac{1}{\lambda} \\ x + 2y = -\frac{1}{\lambda} \;\; |\cdot 2 \end{cases}$ $\lambda \neq 0$ "$-$" $\Rightarrow -3y = \frac{1}{\lambda}$

$$\Rightarrow y = -\frac{1}{3\lambda}$$
$$x = -\frac{1}{\lambda} - 2y = \frac{-3}{3\lambda} + \frac{2}{3\lambda} = -\frac{1}{3\lambda}$$

(3) $\Rightarrow \frac{1}{9\lambda^2} + \frac{1}{9\lambda^2} + \frac{1}{9\lambda^2} = 1$

$$\frac{3}{9\lambda^2} = 1 \Rightarrow 9\lambda^2 = 3$$
$$\lambda^2 = \frac{1}{3} \Rightarrow \lambda = \pm\frac{\sqrt{3}}{3} \Rightarrow 3\lambda = \pm\sqrt{3}$$

$$\Rightarrow \begin{cases} x = -\frac{1}{\sqrt{3}}, \; y = -\frac{1}{\sqrt{3}} \\ x = \frac{1}{\sqrt{3}}, \; y = \frac{1}{\sqrt{3}} \end{cases}$$

Pct. critice sunt $P_1\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ şi $P_2\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

$$f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{2}{\sqrt{3}} = \min f|_S$$

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}} = \max f|_S.$$

b) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = xyz$, $S = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0, \; x^2+y^2+z^2=1\}$

$$\left.\begin{array}{l} S \text{ compactă} \\ f \text{ cont.} \end{array}\right\} \xRightarrow{T.W} f \text{ îşi atinge min. şi max. cond. pe } S.$$

Fie $F_1(x,y,z) = x+y+z$

$F_2(x,y,z) = x^2+y^2+z^2-1$

$\Rightarrow S = \{(x,y,z) \in \mathbb{R}^3 \mid F_1(x,y,z) = F_2(x,y,z) = 0\}$

$$L(x,y,z,\lambda_1,\lambda_2) = f(x,y) + \lambda_1 F_1(x,y) + \lambda_2 F_2(x,y)$$

$$= xyz + \lambda_1(x+y+z) + \lambda_2(x^2+y^2+z^2-1)$$

$$\nabla L(x,y,z,\lambda_1,\lambda_2) = 0 \Rightarrow \begin{cases} yz + \lambda_1 + 2\lambda_2 x = 0 \\ xz + \lambda_1 + 2\lambda_2 y = 0 \\ xy + \lambda_1 + 2\lambda_2 z = 0 \\ x+y+z = 0 \\ x^2+y^2+z^2-1 = 0 \end{cases}$$

$(1)-(2) \Rightarrow z(y-x) + 2\lambda_2(x-y)=0$

$(x-y)(2\lambda_2 - z)=0 \Rightarrow x=y$

sau $z=2\lambda_2$

$* \; x=y \overset{(4)}{\Rightarrow} z=-2x \overset{(5)}{\Rightarrow} 6x^2=1 \Rightarrow x^2=\frac{1}{6} \Rightarrow x=\pm\frac{\sqrt{6}}{6}$

$\Rightarrow y=\pm\frac{\sqrt{6}}{6}, \quad z=\mp\frac{\sqrt{6}}{3}$

$* \; z=2\lambda_2$

$(3): xy+\lambda_1+z^2=0$

$(2),(1): \begin{cases} yz+\lambda_1+2x=0 \\ xz+\lambda_1+2y=0 \end{cases} \Rightarrow xz+yz=-\lambda_1$

$(1)+(2)+(3): xy+xz+yz+3\lambda_1+2\underbrace{(x+y+z)}_{=0}=0$

$\Rightarrow -3\lambda_1 = xy+xz+yz$

$\underbrace{(x+y+z)^2}_{=0\,(\text{din}(4))} = \underbrace{x^2+y^2+z^2}_{1\,(\text{din}(5))}+2xy+2yz+2xz = 1-6\lambda_1$

$\Rightarrow 1-6\lambda_1=0 \Rightarrow \lambda_1=+\frac{1}{6}$

$\Rightarrow xz+yz=-\frac{1}{6} \Rightarrow z(x+y)=-\frac{1}{6} \Rightarrow x+y=-\frac{1}{6z}$

$\overset{(4)}{\Rightarrow} -\frac{1}{6z}+z=0 \Rightarrow z=+\frac{1}{6z} \Rightarrow +6z^2=1$

$\Rightarrow z=\pm\frac{1}{\sqrt{6}}=\pm\frac{\sqrt{6}}{6}$

$z=\frac{\sqrt{6}}{6} \overset{(4)}{\Rightarrow} x+y=-\frac{\sqrt{6}}{6} \Rightarrow x=-\frac{\sqrt{6}}{6}-y$

$\overset{(5)}{\Rightarrow} \left(-\frac{\sqrt{6}}{6}-y\right)^2+y^2+\frac{1}{6}-1=0 \Rightarrow$

$2y^2+y\frac{\sqrt{6}}{3}-\frac{2}{3}=0 \quad |\cdot 3 \Rightarrow 6y^2+\sqrt{6}y-2=0$

$\Delta = 6+48=54$

$$y_1 = \frac{-\sqrt{6}+3\sqrt{6}}{12} = \frac{2\sqrt{6}}{12} = \frac{\sqrt{6}}{6} \Rightarrow x = -\frac{\sqrt{6}}{3}$$

$$y_2 = \frac{-\sqrt{6}-3\sqrt{6}}{12} = \frac{-4\sqrt{6}}{12} = -\frac{\sqrt{6}}{3} \Rightarrow x = \frac{\sqrt{6}}{6}$$

$$z = -\frac{\sqrt{6}}{6} \overset{(4)}{\Rightarrow} x+y = \frac{\sqrt{6}}{6} \Rightarrow x = \frac{\sqrt{6}}{6} - y$$

$$\overset{(5)}{\Rightarrow} \left(\frac{\sqrt{6}}{6} - y\right)^2 + y^2 + \frac{1}{6} - 1 = 0$$

$$2y^2 - \frac{\sqrt{6}}{3}y - \frac{2}{3} = 0 \quad |\cdot 3$$

$$6y^2 - \sqrt{6}y - 2 = 0 \qquad \Delta = 6 + 48 = 54$$

$$y_1 = \frac{\sqrt{6}+3\sqrt{6}}{12} = \frac{4\sqrt{6}}{12} = \frac{\sqrt{6}}{3} \Rightarrow x_1 = -\frac{\sqrt{6}}{6}$$

$$y_2 = \frac{\sqrt{6}-3\sqrt{6}}{12} = \frac{-2\sqrt{6}}{12} = -\frac{\sqrt{6}}{6} \Rightarrow x_2 = \frac{\sqrt{6}}{3}$$

Pct. critice sunt: $P_1\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}, \lambda_1, \lambda_2\right)$

$P_2\left(-\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \lambda_1, \lambda_2\right)$

$P_3\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \lambda_1, \lambda_2\right)$

$P_4\left(\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, \lambda_1, \lambda_2\right)$

$P_5\left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}, \lambda_1, \lambda_2\right)$

$P_6\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}, \lambda_1, \lambda_2\right)$

Obs: Val. lui λ_1, λ_2 nu contează pt. rezolvarea exercițiului.

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$f\left(\frac{\sqrt6}{6}, \frac{\sqrt6}{6}, -\frac{\sqrt6}{3}\right) = \frac{\sqrt6}{6}\cdot\frac{\sqrt6}{6}\cdot\frac{-\sqrt6}{3} = -\frac{\sqrt6}{18} = \min f|_S$

$f\left(-\frac{\sqrt6}{6}, -\frac{\sqrt6}{6}, \frac{\sqrt6}{3}\right) = \frac{6}{36}\cdot\frac{\sqrt6}{3} = \frac{\sqrt6}{18} = \max f|_S$

$f\left(-\frac{\sqrt6}{3}, \frac{\sqrt6}{6}, \frac{\sqrt6}{6}\right) = -\frac{\sqrt6}{18} = \min f|_S$

$f\left(\frac{\sqrt6}{6}, -\frac{\sqrt6}{3}, \frac{\sqrt6}{6}\right) = -\frac{\sqrt6}{18} = \min f|_S$

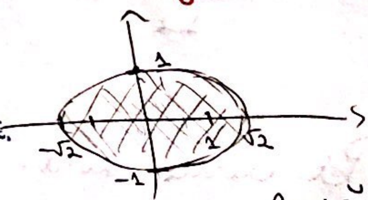
$f\left(-\frac{\sqrt6}{6}, \frac{\sqrt6}{3}, -\frac{\sqrt6}{6}\right) = \frac{\sqrt6}{18} = \max f|_S$

$f\left(\frac{\sqrt6}{3}, -\frac{\sqrt6}{6}, -\frac{\sqrt6}{6}\right) = \frac{\sqrt6}{18} = \max f|_S$

5. Det. valorile extreme ale fc. f relativ la mulțimea S indicată:

$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = xy$, $S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + 2y^2 \le 2\}$

$\left.\begin{array}{l} S \text{ compactă} \\ f \text{ cont.}\end{array}\right\} \xRightarrow{T.W.}$ f își atinge min. și max. relativ la S.



$x^2 + 2y^2 = 2 \to$ elipsă

$\text{int } S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + 2y^2 < 2\}$

$\text{fr } S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + 2y^2 = 2\}$

$S = \text{int } S \cup \text{fr } S.$

1) int S:

$\nabla f(x,y) = (0,0) \Rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \to P(0,0)$ pct. critic

$H(f)(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad H(f)(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \begin{array}{l}\Delta_1 = 0 \\ \Delta_2 = -1\end{array} \Rightarrow$ nu putem det. astfel natura

$d^2f(0,0)(u_1,u_2) = -2u_1u_2$ $\quad \begin{array}{l} d^2f(0,0)(1,1) = -2 < 0 \\ d^2f(0,0)(1,-1) = 2 > 0 \end{array} \Rightarrow d^2f$ indef. (12) $\Rightarrow P(0,0)$ pct. șa

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2) φ. S:

Fie $F(x,y) = x^2+2y^2-2$

și $S = \{ (x,y) \in \mathbb{R}^2 \mid F(x,y) = 0 \}$

$L(x,y,\lambda) = \varphi(x,y) + \lambda F(x,y) = xy + \lambda(x^2+2y^2-2)$

$$\left| \begin{array}{l} \frac{\partial L}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} y + 2x\lambda = 0 \Rightarrow y = -2x\lambda \\ x + 4y\lambda = 0 \Rightarrow x + 4\lambda(-2x\lambda) = 0 \\ x^2 + 2y^2 - 2 = 0 \end{array} \right.$$

$\nabla L(x,y,\lambda) = 0$

$x - 8x\lambda^2 = 0$

$x(1-8\lambda^2) = 0$

$\Rightarrow x = 0 \Rightarrow y = 0$

imposibil

$1-8\lambda^2 = 0 \Rightarrow 8\lambda^2 = 1 \Rightarrow \lambda^2 = \frac{1}{8} \Rightarrow$

$\lambda = \pm\frac{\sqrt{8}}{8}$

$\lambda = \frac{\sqrt{8}}{8} \Rightarrow y = -2x \cdot \frac{\sqrt{8}}{8} = -\frac{\sqrt{8}}{4} x = -\frac{\sqrt{2}}{2} x$

$x^2 + 2 \cdot \frac{2}{4} x^2 - 2 = 0$

$2x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow y = \mp \frac{\sqrt{2}}{2}$

$\lambda = -\frac{\sqrt{8}}{8} \Rightarrow y = \frac{\sqrt{2}}{2} x$

$x^2 + 2 \cdot \frac{2}{4} x^2 - 2 = 0 \Rightarrow x = \pm 1 \Rightarrow y = \pm \frac{\sqrt{2}}{2}$

Pct. critice: $P_1\left(1, \frac{\sqrt{2}}{2}, -\frac{\sqrt{8}}{8}\right)$, $P_2\left(-1, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{8}}{8}\right)$,

$P_3\left(1, -\frac{\sqrt{2}}{2}, \frac{\sqrt{8}}{8}\right)$, $P_4\left(-1, \frac{\sqrt{2}}{2}, \frac{\sqrt{8}}{8}\right)$

$\varphi\left(1, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} = \max \varphi|_S$ $\qquad$ $\varphi\left(1, -\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} = \min \varphi|_S$

$\varphi\left(-1, -\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} = \max \varphi|_S$ $\qquad$ $\varphi\left(-1, \frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} = \min \varphi|_S$

c) Determinați distanța minimă în plan de la punctul $a=(0,-1)$ la hiperbola de ecuație $xy=\sqrt{2}$, $x>0$, $y>0$.

Fie $A(0,-1)$ și $B(x,y)$ de pe hiperbolă, $x,y>0$ și $xy=\sqrt{2}$.

Avem de minimizat $AB=\sqrt{x^2+(y+1)^2}$, dar putem minimiza de fapt $AB^2=x^2+(y+1)^2$ (ca să nu lucrăm cu radicali).

Fie $f(x,y)=x^2+(y+1)^2$, $f:(0,\infty)^2\to\mathbb{R}$

Avem de minimizat f relativ la mulțimea

$S=\{(x,y)\in(0,\infty)^2 \mid xy=\sqrt{2}\}$

Avem o problemă de extrem condiționat.

Fie $F(x,y)=xy-\sqrt{2}$. $\Rightarrow S=\{(x,y)\in(0,\infty)^2 \mid F(x,y)=0\}$

$L(x,y,\lambda)=f(x,y)+\lambda F(x,y)=x^2+(y+1)^2+\lambda(xy-\sqrt{2})$

$$\begin{cases}\dfrac{\partial L}{\partial x}=0\\ \dfrac{\partial L}{\partial y}=0\\ \dfrac{\partial L}{\partial \lambda}=0\end{cases} \Rightarrow \begin{cases}2x+\lambda y=0\\ 2(y+1)+\lambda x=0\\ xy-\sqrt{2}=0\end{cases} \Rightarrow x=\frac{\sqrt{2}}{y},\ y\neq 0$$

$$\Rightarrow \begin{cases}\dfrac{2\sqrt{2}}{y}+\lambda y=0 \Rightarrow \lambda=-\dfrac{2\sqrt{2}}{y^2}\\ 2y+2+\dfrac{\lambda\sqrt{2}}{y}=0\end{cases} \Rightarrow 2y+2+\frac{\sqrt{2}}{y}\cdot\frac{-2\sqrt{2}}{y^2}=0 \quad \Big|\cdot\frac{y^3}{2} \Rightarrow$$

$y^4+y^3-2=0$

$y^4-y^3+2y^3-2=0$

$y^3(y-1)+2(y^3-1)=0$

$y^3(y-1)+2(y-1)(y^2+y+1)=0$

$(y-1)(\underbrace{y^3+2y^2+2y+2}_{\text{nu are sol}, >0})=0$

$\Rightarrow y=1 \Rightarrow x=\sqrt{2}$

$f(\sqrt{2},1)=2+4=6 \to$ minimul

$\Rightarrow d_{min}=\sqrt{6}$.