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$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \underbrace{\ln\left(x+\sqrt{1+x^2}\right)}$$

I. 1) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\ln\left(x+\sqrt{1+x^2}\right)$

a) $f$ strict crescătoare

$f$ derivabilă pe $\mathbb{R}$ (compunere + operații cu fct. derivabile)

$$f'(x) = (\ln u)' = \frac{u'}{u} = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}$$

$$= \frac{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{\cancel{\left(\sqrt{1+x^2}+x\right)}}{\sqrt{1+x^2}\cdot\cancel{\left(x+\sqrt{1+x^2}\right)}} =$$

$$\left(\sqrt{1+x^2}\right)' = \left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$$

$$= \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$$

$$f'(x) = \frac{1}{\sqrt{1+x^2}} > 0, \ \forall x\in\mathbb{R}$$

$\Rightarrow f$ strict crescătoare pe $\mathbb{R}$

3p

5

b) $(a_n)$; $a_1 = 1$; $a_{n+1} = f(a_n)$, $\forall n\in\mathbb{N}^*$

$\left.\begin{array}{l} a_{n+1} = f(a_n) \\ f \text{ strict crescătoare} \end{array}\right\} \Rightarrow (a_n)$ este → strict crescător, dacă $a_1 < a_2$ ; → strict descrescător, dacă $a_1 > a_2$

$\underline{a_1 < a_2} \xRightarrow{f\ \text{cresc}} f(a_1) < f(a_2) \Leftrightarrow \underline{a_2 < a_3} \Rightarrow \ldots a_3 < a_4 \ldots$

$a_1 = 1$, $a_2 = f(a_1) = f(1) = \ln(1+\sqrt{2})$

$e \simeq 2{,}71$

$\sqrt{2} \simeq 1{,}41$

$1+\sqrt{2} \simeq 2{,}41 < e$

$1+\sqrt{2} < e \Rightarrow \ln(1+\sqrt{2}) < \ln e = 1$

$a_2 < a_1$

$\Rightarrow (a_n)$ strict descrescător (1)

3p

$(a_n)$ mărginit inferior

$\underline{a_n > 0, \ \forall n}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \ln\left(x+\sqrt{1+x^2}\right)^{x-\sqrt{1+x^2}}$$

$$= \lim_{x\to-\infty} \ln\left(\frac{\cancel{x^2}-(1+\cancel{x^2})}{x-\sqrt{1+x^2}}\right) = \ln 0 = -\infty.$$

$\sqrt{1+x^2}+x > 0$ pt $x>0$

$P(n)$: $a_n > 0$.

$P(1)$: $a_1 = 1 > 0$ $\quad P(1)$ adev.

Pp. $P(k)$ adev. și dem $P(k+1)$ adev.

$\left.\begin{array}{l} \text{Pp. } a_k > 0 \text{ atunci } f(a_k) = a_{k+1} > 0 \\ \sqrt{1+x^2}+x > 0 \text{ pt } x>0 \end{array}\right\} \Rightarrow P(k+1)$ adev.

$\underline{a_k > 0} \Rightarrow \sqrt{1+a_k^2}+a_k > 0$

Cf inducției matem, $a_n > 0$, $\forall n \geq 1$. (2)

(1)+(2) ⇒ $(a_n)$ convergent

c) $f(x+1) - f(x) \leq 1$, $\forall x\in\mathbb{R}$.

$\forall x\in\mathbb{R}$

c) $f(x+1) - f(x) \leq 1, \ \forall x \in \mathbb{R}$

$\ln\left(x+1+\sqrt{1+(x+1)^2}\right) - \ln\left(x+\sqrt{1+x^2}\right) \leq 1, \ \forall x \in \mathbb{R}$

3p

$g: \mathbb{R} \to \mathbb{R}; \quad g(x) = 1 + \ln\left(x+\sqrt{1+x^2}\right) - \ln\left(x+1+\sqrt{1+(x+1)^2}\right)$

g derivabilă, $g'(x) = 0 + \dfrac{1}{\sqrt{1+x^2}} - \dfrac{1}{\sqrt{1+(x+1)^2}} > 0$

$\Leftrightarrow \dfrac{1}{\sqrt{1+x^2}} > \dfrac{1}{\sqrt{1+(x+1)^2}}$

$\Leftrightarrow \sqrt{1+(x+1)^2} > \sqrt{1+x^2} \Leftrightarrow \cancel{1}+(x+1)^2 > \cancel{1}+x^2$

$\Leftrightarrow (x+1)^2 > x^2$

$\Leftrightarrow (x+1)^2 - x^2 > 0$

$\Leftrightarrow (\cancel{x}+1-\cancel{x})(x+1+x) > 0$

$\Leftrightarrow 2x+1 > 0$

$\Leftrightarrow \boxed{x > -\frac{1}{2}}$

| $x$ | $-\infty$ | | $-\frac{1}{2}$ | | $+\infty$ |
|---|---|---|---|---|---|
| $g'(x)$ | $-$ | $- - - - -$ | $0$ | $+ + + +$ | |
| $g(x)$ | | $\searrow$ | $g\left(-\frac{1}{2}\right)$ $>0$ | $\nearrow$ | |

$\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$

$g\left(-\frac{1}{2}\right) = 1 + \ln\left(-\frac{1}{2}+\sqrt{1+\frac{1}{4}}\right) - \ln\left(\frac{1}{2}+\sqrt{1+\frac{1}{4}}\right)$

$= 1 + \ln\left(\dfrac{-1+\sqrt{5}}{2}\right) - \ln\left(\dfrac{1+\sqrt{5}}{2}\right)$

$= 1 + \ln\left(\dfrac{-1+\sqrt{5}}{\cancel{2}} \cdot \dfrac{\cancel{2}}{1+\sqrt{5}}\right) = 1 + \ln\left(\dfrac{\sqrt{5}-1}{\sqrt{5}+1}\right) > 0$

$\ln\left(\dfrac{\sqrt{5}-1}{\sqrt{5}+1}\right) > -1 \quad / e^{(\ )}$

$\sqrt{5}-1$

$\dfrac{\sqrt{5}-1}{\sqrt{5}+1} > \dfrac{1}{e} \Leftrightarrow \dfrac{(\sqrt{5}-1)^2}{5-1} > \dfrac{1}{e} \Leftrightarrow \dfrac{5+1-2\sqrt{5}}{4} > \dfrac{1}{e}$

$3+\sqrt{5}$

$\Leftrightarrow \dfrac{3-\sqrt{5}}{2} > \dfrac{1}{e} \Leftrightarrow e > \dfrac{2}{3-\sqrt{5}}$

$\Leftrightarrow e > \dfrac{6+2\sqrt{5}}{4}$

$\Leftrightarrow \quad e > \frac{3+\sqrt{5}}{2}$ (A)

$\sqrt{5} \simeq 2,23$

$3+\sqrt{5} \simeq 5,23$

$e \simeq 2,7\ldots$ $\quad 2,6\ldots$

$\frac{3+\sqrt{5}}{2} \simeq 2,6\ldots$

Deci, $\quad g(x) > 0, \quad \forall\, x \in \mathbb{R}$

2. a) $\underbrace{\sum_{n=1}^{\infty} \sin\left(\frac{n^{2019}}{n^{2021}+2020}\right)}_{a_n \downarrow 0}$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\sin(\ ) \sim (\ )$

$b_n = \frac{n^{2019}}{n^{2021}+2020} \downarrow 0$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin b_n}{b_n} = 1 \in (0, +\infty) \xrightarrow[\text{comparație cu limită}]{\text{criteriu de}} \underset{(C)}{\sum a_n} \sim \underset{(C)}{\sum b_n}$

$\sum_{n=1}^{\infty} \frac{n^{2019}}{n^{2021}+2020} \sim \sum_{n=1}^{\infty} \frac{1}{n^2}$ (C) ( serie armonică generaliz. cu $\alpha = 2 > 1$)

$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \begin{cases} (C), \text{ pt } \alpha > 1 \\ (D), \text{ pt } \alpha \le 1. \end{cases}$

$\lim_{n\to\infty} \frac{b_n}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^{2019}}{n^{2021}+2020} \cdot \frac{n^2}{1} = 1 \in (0, +\infty) \xRightarrow{CCL} \underset{(C)}{\sum b_n} \sim \underset{(C)}{\sum \frac{1}{n^2}}$

$\Rightarrow \sum a_n$ (C)

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$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{n\cdot(\ln n)(\ln\ln n)^{2020}}}_{a_n}$$

criteriul condensării

$$\sum a_n,\quad a_n \downarrow 0 \Rightarrow \sum a_n \sim \sum 2^n \cdot a_{2^n}$$

$$2^n\cdot a_{2^n} = \cancel{2^n}\cdot \frac{1}{\cancel{2^n}\cdot \ln 2^n \cdot \left(\ln \boxed{\ln 2^n}\right)^{2020}} =$$

$$= \frac{1}{n\cdot \ln 2\,\left(\ln(n\cdot\ln 2)\right)^{2020}}$$

$\ln 2^n = n\cdot\ln 2$

$$\sum b_n \sim \sum(\lambda\cdot b_n)$$

pt $\lambda\neq 0$

$$= \frac{1}{\ln 2}\cdot\frac{1}{n\cdot(\ln n+\ln\ln 2)^{2020}}$$

$\lambda=$

Deci, $\sum a_n \sim \sum \underbrace{\frac{1}{n\cdot(\ln n+\ln\ln 2)^{2020}}}_{b_n}$

$$\sum b_n \sim \sum 2^n\cdot b_{2^n}$$

$$2^n\cdot b_{2^n} = \cancel{2^n}\cdot\frac{1}{\cancel{2^n}\cdot(\ln 2^n + \underbrace{\ln\ln 2}_{k})^{2020}}$$

$$= \frac{1}{(n\ln 2+k)^{2020}}$$

$$\sum a_n \sim \sum b_n \sim \sum \underbrace{\frac{1}{(n\ln 2+k)^{2020}}}\ \underset{\#}{\sim}\ \sum\frac{1}{n^{2020}} \quad (C)$$

(C)

(s. ar m. gen $\alpha = 2020 > 1$)

$$\lim_{n\to\infty}\frac{\frac{1}{(n\ln 2+k)^{2020}}}{\frac{1}{n^{2020}}} = \lim_{n\to\infty}\left(\frac{1\cdot n^1}{n^1\cdot\ln 2+k}\right)^{2020}$$

$$=\left(\frac{1}{\ln 2}\right)^{2020}\in(0,+\infty)$$

b) Mulțimea de convergență și suma seriei de puteri

$$\sum_{n=1}^{\infty} \frac{3^n}{n} \cdot x^{2n} = \frac{3}{1} \cdot x^2 + \frac{3^2}{2} \cdot x^4 + \frac{3^3}{3} \cdot x^6 + \dots$$

$\sum_{n=1} a_n x^n$;  $\lambda = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$  I.

sau

$\lambda = \lim_{n\to\infty} \sqrt[n]{|a_n|}$  II.

$R = \frac{1}{\lambda}$ raza de convergență.

sau

$\lambda = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$  III.

$$|a_n| = a_n = \begin{cases} \frac{3^k}{k}, & n = 2k \\ 0, & n = 2k+1 \end{cases}$$

$\sum a_n x^n = a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$, $a_1 = 0$, $a_3 = 0$

$\sqrt[2k]{\dots} = (\dots)^{\frac{1}{2k}} = \left(\frac{3^k}{k}\right)^{\frac{1}{2k}}$   $(k^{\frac{1}{k}})^{\frac{1}{2}}$

I. nu se poate aplica

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[2k]{\frac{3^k}{k}}, & n = 2k \\ 0, & n = 2k+1 \end{cases} = \begin{cases} \frac{\sqrt{3}}{(\sqrt[k]{k})^{\frac{1}{2}}}, & n = 2k \\ 0, & n = 2k+1 \end{cases} \longrightarrow \begin{cases} \to \sqrt{3}, & n = 2k \\ \to 0, & n = 2k+1 \end{cases}$$

$\sqrt[k]{k} \to 1$

Știm că $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{k\to\infty} \sqrt[k]{k} = 1$

Dacă $\exists \lim \frac{x_{n+1}}{x_n} = l \Rightarrow \exists \lim \sqrt[n]{x_n} = l$

$\sqrt[n]{n}$   $\frac{n+1}{n} \to 1 \Rightarrow \sqrt[n]{n} \to 1$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \sqrt{3} = \lambda \Rightarrow R = \frac{1}{\sqrt{3}}$$

$$\boxed{I\ C = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

MC : putem adăuga capetele ?

$x = -\frac{1}{\sqrt{3}}$;   $x = \frac{1}{\sqrt{3}}$

$$\sum_{n=1}^{\infty} \frac{3^n}{n} \cdot \left(\pm\frac{1}{\sqrt{3}}\right)^{2n} = \sum_{n=1}^{\infty} \frac{\cancel{3^n}}{n} \cdot \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{n} \quad (D)$$

Deci, $MC = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

---

Metodă alternativă:

$$\sum_{n=1}^{\infty} \frac{3^n \cdot x^{2n}}{n} = \sum_{n=1}^{\infty} \frac{(3x^2)^n}{n} \qquad y = 3x^2$$

$$\underbrace{\sum_{n=1}^{\infty} \frac{y^n}{n}}_{R=1}; \quad a_n = \frac{1}{n}; \quad \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1 = \lambda = R$$

$-1 < y < 1$

$\sum_{n=1} \frac{\phantom{x}}{n}$

$R=1$

$-1 < y < 1$

$$\sum_{n=1}^{\infty} \frac{(3x^2)^n}{n}$$

$-1 < 3x^2 < 1$

$x^2 < \frac{1}{3}$

$$\boxed{x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

Suma seriei de puteri:

$$\sum_{n=1}^{\infty} \frac{3^n}{n} \cdot x^{2n} = \sum_{n=1}^{\infty} \frac{(3x^2)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{y^n}{n}$$

Seria geometrică:

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}, \quad y \in (-1,1) \quad \Big/ \int$$

$$\sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1} = -\ln(1-y) + C \qquad (\ln u)' = \frac{u'}{u}$$

$$\underbrace{\sum_{n=1}^{\infty} \frac{y^n}{n}}_{} = -\ln(1-y) + C$$

$$y = 0 \Rightarrow 0 = \underbrace{-\ln 1}_{0} + C \Rightarrow C = 0$$

$$\sum_{n=1}^{\infty} \frac{y^n}{n} = -\ln(1-y), \qquad y \in (-1,1)$$

$$y \mapsto 3x^2$$

$$\sum_{n=1}^{\infty} \frac{3^n x^{2n}}{n} = \underline{-\ln(1-3x^2)}, \qquad 3x^2 \in (-1,1) \Leftrightarrow x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

suma seriei de puteri

II. 1) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} (x^2+y^2) \cdot \sin \dfrac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

a) Cont. fct. f.

f cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$ fiind obținută prin operații și compuneri cu fct. elementare, continue.

Studiem cont. în $(0,0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \underbrace{(x^2+y^2)}_{\to 0} \cdot \underbrace{\sin \frac{1}{\sqrt{x^2+y^2}}}_{\text{mărginită}} = 0 = f(0,0)$$

$\Rightarrow$ f cont în $(0,0)$, deci pe $\mathbb{R}^2$.

b) $\exists$ deriv. parțiale și diferenț. lui $f$ în $(0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{x^2 \cdot \sin\frac{1}{\sqrt{x^2}} - 0}{\cancel{x}} =$$

$$= \lim_{x\to 0} \underbrace{x}_{\to 0} \cdot \underbrace{\sin\frac{1}{|x|}}_{\text{mărg}} = 0 \in \mathbb{R} \Rightarrow f \text{ deriv. parțial în raport cu } x$$

în $(0,0)$, $\frac{\partial f}{\partial x}(0,0) = 0$.

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{y^2 \sin\frac{1}{\sqrt{y^2}} - 0}{y-0} = 0 \Rightarrow$$

cu

$$\frac{\partial f}{\partial y}(0,0) = 0.$$

Studiul diferențiabilității:

Candidat la diferențială: dacă există,

$$T = df(0,0) = \underbrace{\frac{\partial f}{\partial x}(0,0)}_{0} \cdot dx + \underbrace{\frac{\partial f}{\partial y}(0,0)}_{0} \cdot dy = 0$$

$$f \text{ diferențiabilă în } (0,0) \Leftrightarrow \lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-T(x,y)}{\sqrt{x^2+y^2}} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \overbrace{f(0,0)}^{0} - \overbrace{T(x,y)}^{0}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\overset{\sqrt{x^2+y^2}}{\cancel{(x^2+y^2)}} \cdot \sin\frac{1}{\sqrt{x^2+y^2}} - 0 - 0}{\cancel{\sqrt{x^2+y^2}}}$$

$$= \lim_{(x,y)\to(0,0)} \underbrace{\sqrt{x^2+y^2}}_{\to 0} \cdot \underbrace{\sin\frac{1}{\sqrt{x^2+y^2}}}_{\text{mărg}} = 0 \Rightarrow f \text{ diferenț. în } (0,0).$$

2) $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=(x-2y)\cdot e^{-x^2-4y^2}$

a) Deriv. part. de ord. 1 și 2 și diferenț. de ord 1 și 2.

$$\frac{\partial f}{\partial x}=1\cdot e^{-x^2-4y^2}+(x-2y)\cdot e^{-x^2-4y^2}\cdot(-2x)=e^{-x^2-4y^2}\cdot(1-2x^2+4xy)$$

$$\frac{\partial f}{\partial y}=-2\cdot e^{-x^2-4y^2}+(x-2y)\cdot e^{-x^2-4y^2}\cdot(-8y)=e^{-x^2-4y^2}\cdot(-2-8xy+16y^2)$$

$$\frac{\partial^2 f}{\partial x^2}=e^{-x^2-4y^2}\cdot(-2x)\cdot(1-2x^2+4xy)+e^{-x^2-4y^2}\cdot(-4x+4y)$$

$$\frac{\partial^2 f}{\partial y^2}=e^{-x^2-4y^2}\cdot(-8y)\cdot(-2-8xy+16y^2)+e^{-x^2-4y^2}\cdot(-8x+32y)$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=e^{-x^2-4y^2}\cdot(-8y)\cdot(1-2x^2+4xy)+e^{-x^2-4y^2}\cdot 4x$$

$$\frac{\partial^2 f}{\partial x^2}\left(-\tfrac12,\tfrac14\right)=e^{-\frac14-\frac14}\cdot(2+1)=\frac{1}{\sqrt e}\cdot 3$$

$$\frac{\partial^2 f}{\partial y^2}\left(-\tfrac12,\tfrac14\right)=\frac{1}{\sqrt e}(4+8)=\frac{1}{\sqrt e}\cdot 12$$

$$\frac{\partial^2 f}{\partial x\partial y}\left(-\tfrac12,\tfrac14\right)=\frac{1}{\sqrt e}\cdot(-2)$$

$$df(x,y)=\frac{\partial f}{\partial x}\cdot dx+\frac{\partial f}{\partial y}\cdot dy \qquad (\qquad)^{(2)\ \text{formal}}$$

$$d^2f(x,y)=\frac{\partial^2 f}{\partial x^2}\cdot(dx)^2+\frac{\partial^2 f}{\partial y^2}\cdot(dy)^2+2\frac{\partial^2 f}{\partial x\partial y}\cdot dx\cdot dy.$$

b) Pct. critice

$$\begin{cases}\frac{\partial f}{\partial x}=0\\ \frac{\partial f}{\partial y}=0\end{cases}\iff\begin{cases}\underbrace{e^{-x^2-4y^2}}_{\neq 0}\cdot(1-2x^2+4xy)=0\\ \underbrace{e^{-x^2-4y^2}}_{\neq 0}(-2-8xy+16y^2)=0\end{cases}$$

$$\begin{cases}1-2x^2+4xy=0\\ -2-8xy+16y^2=0\ \ |:2\end{cases}\Rightarrow\begin{cases}-2x^2+4xy=-1\Rightarrow 4xy=-1+2x^2\\ -4xy+8y^2=1\Rightarrow 4xy=-1+8y^2\end{cases}$$

$-1-4xy+8y^2=0$

$\Downarrow$

$2x^2=8y^2$

$x^2=4y^2 \qquad x=\pm 2y$

1) $x=2y$; $-\cancel{8y^2}+\cancel{8y^2}=-1$

$0=-1$ Fals

2) $x=-2y$; $-8y^2-8y^2=-1\Rightarrow 16y^2=1\Rightarrow y^2=\frac{1}{16}\Rightarrow y=\pm\frac14$

$x=\mp\frac12$

Pct. critice: $\left(-\frac12,\frac14\right)$; $\left(\frac12,-\frac14\right)$

$$H_f\left(-\tfrac12,\tfrac14\right)=\begin{pmatrix}\frac{\partial^2 f}{\partial x^2}\left(-\frac12,\frac14\right) & \frac{\partial^2 f}{\partial x\partial y}\left(-\frac12,\frac14\right)\\ \frac{\partial^2 f}{\partial y\partial x}\left(-\frac12,\frac14\right) & \frac{\partial^2 f}{\partial y^2}\left(-\frac12,\frac14\right)\end{pmatrix}=\frac{1}{\sqrt e}\begin{pmatrix}3 & -2\\ -2 & 12\end{pmatrix}$$

$\Delta_0=1$ (din oficiu)

$\Delta_1=3$

$\Delta_2=\begin{vmatrix}3 & -2\\ -2 & 12\end{vmatrix}=32$

$$d^2f\left(-\tfrac12,\tfrac14\right)=\underbrace{\frac{1}{3}}_{>0}h_1^2+\underbrace{\frac{3}{32}}_{>0}h_2^2$$

pozitiv def $\Rightarrow\left(-\frac12,\frac14\right)$ minim local pt $f$.

$$H_f\left(\frac{1}{2}, -\frac{1}{4}\right) = \frac{1}{\sqrt{3}}\begin{pmatrix} -3 & 2 \\ 2 & -12 \end{pmatrix}$$

$d^2 f\left(\frac{1}{2}, -\frac{1}{4}\right)$ negativ def

$\Rightarrow \left(\frac{1}{2}, -\frac{1}{4}\right)$ maxim local pt $f$.

$$\left(\frac{1}{x^2+2}\right)' = \left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$$

$$= \underbrace{-\frac{2x}{(x^2+2)^2}}$$

$$\underline{\int \frac{1}{(x^2+a)^2}\,dx}$$

III. 1) a)

$$\int \frac{dx}{(x^2+2)^2} = \frac{1}{2}\cdot\int \frac{2+x^2-x^2}{(x^2+2)^2}\,dx =$$

$$= \frac{1}{2}\cdot\int \frac{\cancel{(2+x^2)}}{(x^2+2)^{\cancel{2}}}\,dx - \frac{1}{2}\int \frac{x^2}{(x^2+2)^2}\,dx$$

$$= \frac{1}{2}\cdot\underbrace{\int \frac{1}{x^2+\sqrt{2}^2}\,dx}_{\downarrow} + \frac{1}{\cancel{2}4}\int x\cdot\underbrace{\frac{-2x}{(x^2+2)^2}}\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\,\mathrm{arctg}\,\frac{x}{\sqrt{2}} + \frac{1}{4}\int x\cdot\left(\frac{1}{x^2+2}\right)'\,dx$$

$$= \frac{1}{2\sqrt{2}}\,\mathrm{arctg}\,\frac{x}{\sqrt{2}} + \frac{1}{4}\cdot\left(x\cdot\frac{1}{x^2+2} - \int\frac{1}{x^2+\sqrt{2}^2}\,dx\right)$$

$$= \frac{1}{\underbrace{2\sqrt{2}}}\,\mathrm{arctg}\,\frac{x}{\sqrt{2}} + \frac{x}{4(x^2+2)} - \underbrace{\frac{1}{4}\cdot\frac{1}{\sqrt{2}}}\,\mathrm{arctg}\,\frac{x}{\sqrt{2}} + \mathcal{C}$$

$$= \frac{1}{4\sqrt{2}}\,\mathrm{arctg}\,\frac{x}{\sqrt{2}} + \frac{x}{4(x^2+2)} + \mathcal{C}.$$

$$(x^2+2)^2 = x^4+4x^2+4$$

b) $f:(-3,+\infty)\to\mathbb{R}$;

$$f(x)=\frac{2x^4+6x^2-5x+11}{x^5+3x^4+4x^3+12x^2+4x+12}$$

$\pm1 \quad \pm2 \quad \pm3 \quad \pm4 \quad \pm6 \quad \pm12$

$\cancel{-1} \quad \cancel{-2} \quad -3 \quad -4 \quad -6 \quad -12$

$$\begin{array}{l|l} x^5+3x^4+4x^3+12x^2+4x+12 & x^4+4x^2+4 \\ -x^5 \quad -4x^3 \quad -4x & \overline{x+3} \\ \hline / \quad 3x^4 \quad / \quad +12x^2 \quad / \quad +12 & \\ -3x^4 \quad -12x^2 \quad -12 & \\ \hline / \quad / \quad / & \end{array}$$

$3x^4+12x^2+12$

$=3(x^4+4x^2+4)$

$=3(x^2+2)^2$

$$f(x)=\frac{2x^4+6x^2-5x+11}{(x+3)(x^2+2)^2}=\frac{\overset{(x^2+2)^2}{A}}{x+3}+\frac{\overset{(x^2+2)(x+3)}{Bx+C}}{x^2+2}+\frac{\overset{x+3}{Dx+E}}{(x^2+2)^2}$$

$$2x^4+6x^2-5x+11=A(x^2+2)^2+(Bx+C)(x^2+2)(x+3)+(Dx+E)(x+3)$$

$\boxed{x=-3}$

$2\cdot81+6\cdot9+15+11 = A\cdot121$

$242=A\cdot121 \Rightarrow \boxed{A=2}$

$162+$
$54$
$26$
$242$

$$2\cancel{x^4}+6x^2-5x+11-2(\cancel{x^4}+4x^2+4)=(Bx+C)(x^2+2)(x+3)+(Dx+E)(x+3)$$

$$-2x^2-5x+3=(Bx+C)(x^2+2)(x+3)+(Dx+E)(x+3)$$

$-2x^2-6x+x+3$

$$(-2x+1)\cancel{(x+3)}=(Bx+C)(x^2+2)\cancel{(x+3)}+(Dx+E)\cancel{(x+3)}$$

$$-2x+1=Bx^3+Cx^2+2Bx+2C+Dx+E$$

$$\boxed{\begin{array}{ll} B=0 & ;\ A=2 \\ C=0 & \\ D=-2 & \\ E=1 & \end{array}}$$

$$f(x)=\frac{2}{x+3}+\frac{-2x+1}{(x^2+2)^2}.$$

c) Aria $=\displaystyle\int_0^1 f(x)\,dx=\int_0^1 \frac{2}{x+3}+\frac{-2x+1}{(x^2+2)^2}\,dx$

$$=2\ln|x+3|\,\Big|_0^1+\underbrace{\int_0^1\frac{-2x}{(x^2+2)^2}\,dx}_{f=x^2+2}+\int_0^1\frac{1}{(x^2+2)^2}\,dx$$

calculată la pct a)

$\left(\frac{1}{f}\right)'=-\frac{f'}{\ldots} \quad 2x$

$f = x^2+2$ calculată la pct a)

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2} = -\frac{2x}{(x^2+2)^2}$$

$$= 2\ln 4 - 2\ln 3 + \frac{1}{x^2+2}\Big|_0^1 + \frac{1}{4\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}\Big|_0^1 + \frac{x}{4(x^2+2)}\Big|_0^1$$

$$= 2\ln\frac{4}{3} + \frac{1}{3} - \frac{1}{2} + \frac{1}{4\sqrt{2}}\left(\operatorname{arctg}\frac{1}{\sqrt{2}} - 0\right) + \frac{1}{4\cdot 3}$$

Speța I interval nemărginit

$\left( \int_a^{\infty}, \int_{-\infty}^{b}, \int_{-\infty}^{+\infty} \right)$

2. a) $I = \int_{2019}^{2020} \sqrt{\frac{2020-x}{x-2019}}\, dx$

Integrală improprie de speța a II-a

Speța II (fct. nemărg.)

$\int_{a^+}^{b} f(x)\, dx$

Criteriul în $\lambda$

$\lim_{x \searrow a} (x-a)^{\lambda} \cdot f(x) = \ell$

$\lambda < 1, \ \ell \in [0, +\infty)$ (C)

$\lambda > 1, \ \ell \in (0, +\infty]$ (D).

$\lim_{x \searrow 2019} (x-2019)^{\frac{1}{2}} \cdot f(x) = \lim_{x \searrow 2019} \cancel{\sqrt{x-2019}} \cdot \frac{\sqrt{2020-x}}{\cancel{\sqrt{x-2019}}} = 1$

$\left. \begin{array}{l} \lambda = \frac{1}{2} < 1 \\ \ell = 1 \in [0, +\infty) \end{array} \right\} \xrightarrow{\text{Crit în } \lambda}$ integrala este (C).

b) Determinați valoarea lui I.

$$I=\int_a^b \sqrt{\frac{b-x}{x-a}}\,dx$$

$x=b \Rightarrow y=0$

$x \searrow a \Rightarrow y=\sqrt{\frac{b-a}{0_+}}=+\infty$

$$\underbrace{\sqrt{\frac{b-x}{x-a}}=y,\ y>0}\ \Rightarrow \frac{b-x}{x-a}=y^2$$

$$\Rightarrow b-x=x\cdot y^2-a\cdot y^2 \Rightarrow b+ay^2=xy^2+x$$

$$\Rightarrow x(y^2+1)=b+ay^2 \Rightarrow x=\frac{ay^2+b}{y^2+1}$$

$$dx=\left(\frac{ay^2+b}{y^2+1}\right)'dy=\frac{2ay(\cancel{y^2}+1)-(a\cancel{y^2}+b)\cdot 2y}{(y^2+1)^2}\,dy$$

$$=\frac{2y\cdot(a-b)}{(y^2+1)^2}\,dy$$

$$I=\int_\infty^0 y\cdot\frac{2y\,(a-b)}{(y^2+1)^2}\,dy=+(a-b)\cdot\int_0^\infty \frac{-2y}{(y^2+1)^2}\,dy$$

$$=(a-b)\cdot\left.\frac{1}{y^2+1}\right|_0^\infty=(a-b)(0-1)=b-a$$

$$\left(\frac{1}{f}\right)'=-\frac{f'}{f^2}=-\frac{2y}{(y^2+1)^2} \qquad \int\left(\frac{1}{f}\right)'dy=\frac{1}{f}$$

$f=y^2+1$ ; $f'=2y$

$b=2020$, $a=2019$

$I=1$