CSE 521 HW2

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1. For primes $p \neq 2$ and $t \geqslant p/4$,

$$\mathbb{P}[a = 0, b < p/2] \approx \frac{1}{p} \cdot \frac{p/2}{p} = \frac{1}{2p} \geqslant \frac{2}{t}$$

When $a = 0$ and $b < p/2$, $r_i : \mathbb{N} \to \mathbb{N}$ is a constant $< p/2$, and hence mod p it is in $\{0, \ldots, p/2 - 1\}$. When $t < p/4$, we see

$$\mathbb{P}[a \leqslant p/4t, b < p/4] \approx \frac{p/4t}{p} \cdot \frac{1}{4} = \frac{1}{16t}$$

Note also in this case, $r_i < p/4t \cdot t + p/4 = p/2$ reduces to something below $p/2$ mod p, meaning it is contained in $\{0, \ldots, p/2 - 1\}$. In any case the probability is $\Omega\left(\frac{1}{t}\right)$, completing the proof.

2. (a) We make the first observation that a graph is connected iff every cut of the graph contains an edge. Fix a cut $C = (S, V \setminus S)$ with $|S| = k$ and $|V \setminus S| = n - k$. We wish to find $\mathbb{P}[E(S, V \setminus S) \geqslant 1]$. We can instead find $\mathbb{P}[E(S, V \setminus S) = 0]$. The only way this is possible is if each perfect matching contains 0 edges connecting a vertex of S to a vertex of $V \setminus S$. First, we make the observation that if $E(S, V \setminus S) = 0$, then the perfect matching restricted to S will also be a perfect matching. Since perfect matchings are only possible on graphs with an even number of vertices we conclude that k must be even. The number of perfect matchings on $2l$ vertices is $(2l - 1)!!$, which can be found by picking a vertex v_1, notice it has $2l - 1$ choices for its endpoint v_2 (cannot be itself), then find another vertex v_3 and see it has $2l - 3$ vertexes for its endpoint

(cannot be any of the ones previously), and so on. Thus, to find the number of perfect matchings on n vertices which don't have an edge connecting an element of S to $V \setminus S$, we simply have to find a perfect matching on S and a perfect matching on $V \setminus S$. By the formula above this can be done in $(k-1)!! \cdot (n-k-1)!!$ ways. The probability all 3 perfect matchings have no edges crossing this cut is therefore

$$\mathbb{P}[E(S, V \setminus S) = 0] = \left(\frac{(k-1)!!(n-k-1)!!}{(n-1)!!} \right)^3$$

since the perfect matchings are drawn u.a.r. Letting $k = 2l$ and remembering that $(2l-1)!! = (2l)!/(2l)!! = (2l)!/2^l l!$, we see that

$$\begin{aligned} \frac{(k-1)!!(n-k-1)!!}{(n-1)!!} &= \frac{(2l)!(n-2l)!n!!}{(2l)!!(n-2l)!!n!} = \frac{1}{\binom{n}{2l}} \cdot \frac{n!!}{(2l)!!(n-2l)!!} = \frac{1}{\binom{n}{2l}} \cdot \frac{2^n (n/2)!}{2^l l! 2^{n-l}(n/2-l)!} \\ &= \frac{\binom{n/2}{l}}{\binom{n}{2l}} \end{aligned}$$

The union bound tells us that $\mathbb{P}[\bigcap_i A_i^c] = 1 - \mathbb{P}[\bigcup_i A_i] \geqslant 1 - \sum_i \mathbb{P}[A_i]$, and hence

$$\begin{aligned} \mathbb{P}[G \text{ connected}] &= \mathbb{P}\left[\bigcap_{l=1}^{n/4} \text{all cuts of size } 2l \text{ have an edge} \right] \\ &\geqslant 1 - \sum_{l=1}^{n/4} \binom{n}{2l} \left(\frac{\binom{n/2}{l}}{\binom{n}{2l}} \right)^3 = 1 - \sum_{l=1}^{n/4} \frac{\binom{n/2}{l}^3}{\binom{n}{2l}^2} \end{aligned}$$

Using that $\binom{n}{k} \approx \frac{n^n}{k^k (n-k)^{n-k}}$ (stirling...) we see that

$$\begin{aligned} \frac{\binom{n/2}{l}^3}{\binom{n}{2l}^2} &\approx \left(\frac{(n/2)^{n/2}}{l^l (n/2-l)^{n/2-l}} \right)^3 \cdot \left(\frac{(2l)^{2l}(n-2l)^{n-2l}}{n^n} \right)^2 \\ &= \frac{(n/2)^{3n/2}}{l^{3l}(n/2-l)^{3(n/2-l)}} \cdot \frac{2^{4l} l^{4l} 2^{2(n-2l)}(n/2-l)^{4(n/2-l)}}{n^{2n}} \\ &= 2^{n/2} \frac{l^l (n/2-l)^{n/2-l}}{n^{n/2}} \end{aligned}$$

Now for some math. The function $f : [2, a/2] \to \mathbb{R}$ defined by $f(x) = x^x (a-x)^{a-x}$ is bounded (being continuous), and, taking logs on both sides and differentiating, yields $f'/f = \log x + 1 - \log(a-x) - 1$, thus $\log x = \log(a-x)$ or $x = a/2$. Plugging this in gives $f(a/2) = (a/2)^{a/2}$, while plugging in at $x = 2$ gives $4(a-2)^{a-2}$. So

$f(x) \leqslant 4(a-2)^{a-2}$ (this one is clearly bigger for large values of a). We conclude

$$2^{n/2}\frac{l^l(n/2-l)^{n/2-l}}{n^{n/2}} \leqslant 2^{n/2}\frac{(n/2-1)^{n/2-2}}{n^{n/2}} = \frac{2^{n/2}}{2^{n/2-2}} \cdot \frac{(n/2-1)^{n-2}}{n^{n/2}} \leqslant \frac{4}{n^2}$$

Thus,

$$\begin{aligned} \sum_{l=1}^{n/4} \frac{\binom{n/2}{l}^3}{\binom{n}{2l}^2} &\approx \frac{n^3}{2n^2(n-1)^2} + \sum_{l=2}^{n/4} 2^{n/2}\frac{l^l(n/2-l)^{n/2-l}}{n^{n/2}} \\ &\leqslant \frac{n^3}{2n^2(n-1)^2} + \left(\frac{n}{4}-1\right)\frac{4}{n^2} \to 0 \end{aligned}$$

This completes the proof. $\square$

(b) In the case where $k=2$, we can see that the multigraph formed by the union of two perfect matchings will be so that each vertex has degree 2. In this case, it is clear that the resulting multigraph is the union of disjoint cycles. Thus, G is connected iff if it has a Hamiltonian cycle. So now fix the first perfect matching. We need to find the probability the second perfect matching does not close a shorter cycle. Start with the first vertex v_1 and let (v_1, u_1) be the first edge. At this point, to not close this cycle we just have to have the edge coming out of u_1 not go to v_1, and since we have n vertices, this leaves $n-2$ choices. Thus there is a $(n-2)/(n-1)$ chance of that happening (you can't have self loops). Calling this new vertex v_2, and its associated edge (v_2, u_2), we need the second edge of u_2 to not go to any of v_2, v_1, u_1, or itself. This gives $(n-4)/(n-3)$ choices (denominator since it can't go to itself or any edge with 2 vertices, which right now is just v_1, v_2). Continuing this process yields

$$\mathbb{P}[\text{G connected }] = \frac{(n-2)!!}{(n-1)!!} = \frac{1}{n-1}\cdot\frac{n-2}{n-3}\cdots\frac{4}{3}\frac{2}{1}$$

For $n = 2l$, this equals

$$\begin{aligned} \frac{(2l-2)!!}{(2l-1)!!} &= \frac{2^{l-1}(l-1)!}{(2l)!/(2l)!!} = \frac{(2l)!!2^{l-1}(l-1)!}{(2l)!} \\ &\approx 2^{2l-1}\frac{l!(l-1)!}{(2l)!} = 2^{2l-1}\frac{(l/e)^l((l-1)/e)^{l-1}}{(2l/e)^{2l}} = \frac{e}{2}\frac{l^l(l-1)^{l-1}}{l^{2l}} = \frac{e}{2}\frac{(l-1)^{l-1}}{l^l} \\ &\leqslant \frac{e}{2}\frac{1}{l} = \frac{e}{n} \to 0 \end{aligned}$$