CSE 521 HW1

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1. Let $G = (V, E)$ be an undirected graph with $n = |V|$ vertices, let $k = \min_{S \subset V} |E(S, V \setminus S)|$ be the size of the min cut, and let α be fixed and sufficiently small (i.e., $k\alpha \leqslant |E|$, note that this forces $\alpha \leqslant n/2$ by the hand-shake lemma). Let $(S, V \setminus S)$ be an arbitrary α-approximate min cut. We show that

$$\mathbb{P}[\text{Karger's algorithm finds } (S, V \setminus S)] \geqslant \frac{\Theta(\alpha!)}{n^{2\alpha-1}}$$

First, notice that

$$\mathbb{P}[e \in E(S, V \setminus S)] = \frac{|E(S, V \setminus S)|}{|E|} \leqslant \frac{\alpha k}{|E|} \leqslant \frac{\alpha k}{nk/2} = \frac{2\alpha}{n}$$

Letting A_i denote the event that the uniformly random edge chosen by Krager's algorithm is not in $E(S, V \setminus S)$, we see that

$$\begin{aligned}
\mathbb{P}[\text{alg finds } (S, V \setminus S)] &= \mathbb{P}[A_1 \mid A_2] \cdot \mathbb{P}[A_2 \mid A_1] \cdots \mathbb{P}[A_{n-3} \mid A_1, A_2, \dots, A_{n-3}] \\
&= \left(1 - \frac{2\alpha}{n}\right) \cdot \left(1 - \frac{2\alpha - 1}{n-1}\right) \cdots \left(1 - \frac{2\alpha}{4}\right)\left(1 - \frac{2\alpha}{3}\right) \\
&= \frac{n - 2\alpha}{n} \cdot \frac{n - 2\alpha - 1}{n-1} \cdots \frac{4 - 2\alpha}{4} \cdot \frac{3 - 2\alpha}{3}
\end{aligned}$$

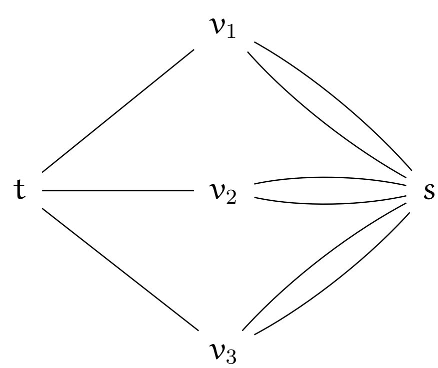
Extracting the first $2\alpha - 1$ factors from the bottom, we can cancel $n - 2 - (2\alpha - 1)$ factors that become $\geqslant 1$, and we have the remaining $2\alpha - 1$ factors on top that equate to something $\Theta(\alpha!)$. We get that this probability is $\geqslant \Theta(\alpha!)/n^{2\alpha-1}$. We see there must be at most $\leqslant n^{2\alpha-1}/\Theta(\alpha!) \leqslant n^{2\alpha}$ α-approximate min cuts.

More detail:

$$
\begin{aligned}
&= \prod_{i=0}^{n-3} \frac{n-2\alpha-i}{n-i} \\
&= \prod_{0 \leqslant i < 2\alpha} \frac{1}{n-i} \cdot \prod_{0 \leqslant i \leqslant n-3-2\alpha} \frac{n-2\alpha-i}{n-\lfloor 2\alpha \rfloor - i} \prod_{0 \leqslant i < 2\alpha} (i+3-2\alpha) \\
&\geqslant \frac{\Theta(\alpha!)}{n^{2\alpha-1}}
\end{aligned}
$$

Basically, we have around $2\alpha - 1$ factors on the bottom at the start that don't cancel, we have $n-2-2\alpha$ factors in the top/bottom that cancel to something $\geqslant 1$, and we have the remaining $2\alpha - 1$ factors on top that turn into $\Theta(\alpha!)$. Since probabilities are $\leqslant 1$ we must have $\leqslant \frac{n^{2\alpha}}{\Theta(\alpha!)} \leqslant n^{2\alpha}$ α-approximate min cuts for large enough α. For the reader/grader, does the first level of detail work? This was sort of a pain to work through the algebra.

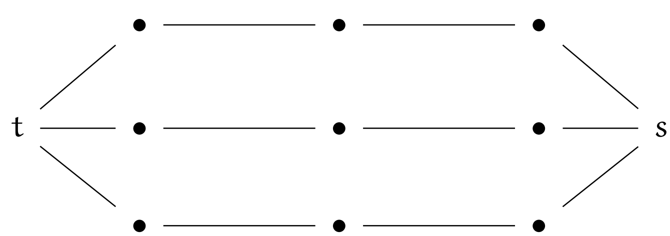
2. We shall construct a graph where, if the algorithm cuts one particular edge, the algorithm fails to find the min s-t cut. The first type of graph that came to my mind is the following:



We could have any number of the v_n's. The crucial part to the proof is that if we contract an edge of the form tv_i, 2 new edges go from s to the new supernode tv_i. This is a contradiction however as you could've broken the path by simply cutting the tv_i edge, not contracting it (which would result in 1 less edge). We see that the algorithm fails in this case iff it contracts an edge of the form tv_i, of which there are $n/3$ of those (if the number of edges in the graph is n). So, defining A_i to be the event that the algorithm doesn't contract any of the tv_α edges in step i, for $i = 1$ to $n/3$, we would get that $\mathbb{P}[A_1] = 1 - \frac{n/3}{n}$, $\mathbb{P}[A_2 \mid A_1] = 1 - \frac{n/3}{n-1}$, and in general $\mathbb{P}[A_i \mid A_1, \ldots, A_{i-1}] = 1 - \frac{n/3}{n-i+1} = \frac{2n/3-i+1}{n-i+1}$. Putting this together gives us

$$
\begin{aligned}
\mathbb{P}[\text{alg doesn't fail}] &= \mathbb{P}[A_1] \cdot \mathbb{P}[A_2 \mid A_1] \cdots \mathbb{P}\big[A_{n/3} \mid A_1, \ldots, A_{n/3-1}\big] \\
&= \frac{2n/3}{n} \cdot \frac{2n/3-1}{n-1} \cdots \frac{n/3+1}{2n/3+1} \\
&= \frac{(2n/3)!}{n!} \cdot \frac{(2n/3)!}{(n/3)!} \approx \frac{(2n/3e)^{4n/3}}{(n/e)^n (n/3e)^{n/3}} \\
&= \frac{(2n)^{4n/3} \cdot (1/3e)^{4n/3}}{n^{4n/3} \cdot e^n \cdot (1/3e)^{n/3}} \\
&= \frac{2^{4n/3}}{3^n} \approx 0.839947665^n
\end{aligned}
$$

Indeed, a quite small exponential decay. Also, by utilizing the following graph:



We can show that the probability this algorithm finds a specific $s-t$ cut is very small. For this graph, an $s-t$ cut can be found by cutting 1 edge on the 1st path, 1 on the second, and one on the third. If we let n be the number of edges, we can do this in $(n/3)^3$ ways. In general, for any n and any k, we could get at least $(n/k)^k = \Theta(n^k)$ (the only important variable being n) $s-t$ cuts for the generalized version of this graph. Thus the probability our algorithm finds one specific $s-t$ cut is no more than $1/\Theta(n^k)$ for every k. Although I am not sure, I believe this tells us that the probability it finds a specific $s-t$ cut is necessarily exponential since it is below all polynomials.