CSE 521 HW1

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1. Let $G = (V, E)$ be an undirected graph with $n = |V|$ vertices, let $k = \min_{S \subset V} |E(S, V \setminus S)|$ be the size of the min cut, and let α be fixed and sufficiently small (i.e., $k\alpha \leqslant |E|$, note that this forces $\alpha \leqslant n/2$ by the hand-shake lemma). Let $(S, V \setminus S)$ be an arbitrary α-approximate min cut. We show that

$$\mathbb{P}[\text{Karger's algorithm finds } (S, V \setminus S)] \geqslant \frac{1}{n^{2\alpha - 1}}$$

First, notice that

$$\mathbb{P}[e \in E(S, V \setminus S)] = \frac{|E(S, V \setminus S)|}{|E|} \leqslant \frac{\alpha k}{|E|} \leqslant \frac{\alpha k}{nk/2} = \frac{2\alpha}{n}$$

Letting A_i denote the event that the uniformly random edge chosen by Krager's algorithm is not in $E(S, V \setminus S)$, we see that

$$\begin{aligned}
\mathbb{P}[\text{alg finds } (S, V \setminus S)] &= \mathbb{P}[A_1 \mid A_2] \cdot \mathbb{P}[A_2 \mid A_1] \cdots \mathbb{P}[A_{n-3} \mid A_1, A_2, \ldots, A_{n-3}] \\
&= \left(1 - \frac{2\alpha}{n}\right) \cdot \left(1 - \frac{2\alpha - 1}{n - 1}\right) \cdots \left(1 - \frac{2\alpha}{4}\right)\left(1 - \frac{2\alpha}{3}\right) \\
&= \frac{n - 2\alpha}{n} \cdot \frac{n - 2\alpha - 1}{n - 1} \cdots \frac{4 - 2\alpha}{4} \cdot \frac{3 - 2\alpha}{3}
\end{aligned}$$

Extracting the first $2\alpha - 1$ factors from the bottom, we can cancel $n - 2 - (2\alpha - 1)$ factors that become $\geqslant 1$, and we have the remaining $2\alpha - 1$ factors on top that equate to something $O(\alpha!)$. We get that this probability is $\geqslant O(\alpha!)/n^{2\alpha - 1}$. We see there must be at most $\leqslant n^{2\alpha - 1}/O(\alpha!) \leqslant n^{2\alpha}$ α-approximate min cuts.

More detail:

$$
\begin{aligned}
&= \prod_{i=0}^{n-3} \frac{n-2\alpha-i}{n-i} \\
&= \prod_{0\leqslant i<2\alpha} \frac{1}{n-i} \cdot \prod_{0\leqslant i\leqslant n-3-2\alpha} \frac{n-2\alpha-i}{n-\lfloor 2\alpha\rfloor-i} \prod_{0\leqslant i<2\alpha} (i+3-2\alpha) \\
&\geqslant \frac{O(\alpha!)}{n^{2\alpha-1}}
\end{aligned}
$$

Basically, we have around $2\alpha-1$ factors on the bottom at the start that don't cancel, we have $n-2-2\alpha$ factors in the top/bottom that cancel to something $\geqslant 1$, and we have the remaining $2\alpha-1$ factors on top that turn into $O(\alpha!)$. Since probabilities are $\leqslant 1$ we must have $\leqslant \frac{n^{2\alpha}}{o(\alpha!)} \leqslant n^{2\alpha}$ α-approximate min cuts for large enough α. For the reader/grader, does the first level of detail work? This was sort of a pain to work through the algebra.

2. We shall construct a graph where, if the algorithm cuts one particular edge, the algorithm fails to find the min s-t cut. The first type of graph that came to my mind is the following: