1. (a) $Ax + b$

(b) $\frac{\partial f(x)}{\partial x_1} = \frac{d\, g(h(x))}{d\, h(x)} \cdot \frac{\partial h(x)}{\partial x_1}$

Thus $\nabla f(x) = g'(h(x)) \cdot \nabla h(x)$

(c) $\nabla f(x) = Ax + b$

$\nabla^2 f(x) = A$

(d) $\nabla f(x) = g'(a^T x) \cdot a$

suppose that $a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ then $\nabla f(x) = \begin{pmatrix} a_1 g'(a^T x) \\ \vdots \\ a_n g'(a^T x) \end{pmatrix}$

that is to say $\nabla f(x) = \begin{pmatrix} a_1 g'(a_1 x_1 + \cdots + a_n x_n) \\ \vdots \\ a_n g'(a_1 x_1 + \cdots + a_n x_n) \end{pmatrix} = g'(\sum_{i=1}^{n} a_i x_i) \cdot a = g'(a^T x) \cdot a$

thus $\nabla^2 f(x) = \nabla^2 f(x) = (g_{ij}(x))_{n \times n}$. $g_{ij}(x) = a_i a_j g''(a^T x)$

2. (a) $\forall x \in \mathbb{R}^n$, $x^T A x = x^T z z^T x = (x^T z)(x^T z)^T$

$= (x^T z)^2 \geq 0$

Thus A is positive semidefinite

(b) if $x \in \mathbb{R}^n$

the null space of A : $Ax = 0$, which means $z z^T x = 0$

that is to say $z^T x = 0$

the dimension of this null-space equals $n-1$

(c) $x^T B A B^T x = (B^T x)^T A (B^T x)$

since A is positive semidefinite, $(B^T x)^T A (B^T x) \geq 0$

thus BAB^T is PSD

3.

(a) $A = T \Lambda T^{-1} \Rightarrow AT = T\Lambda$

that is to say $A(t^{(1)}, t^{(2)}, \cdots, t^{(n)}) = (\lambda_1 t^{(1)}, \lambda_2 t^{(2)}, \cdots, \lambda_n t^{(n)})$

which means $A t^{(i)} = \lambda_i t^{(i)}$

(b) Claim: If $T \in \mathbb{R}^{n \times n}$ and T is invertible

then the eigenvalues of $T^{-1} A T$ equal those of A

the proof of claim: $|\lambda I - T^{-1} A T| = |T^{-1}| \cdot |\lambda I - A| \cdot |T| = |\lambda I - A|$. done

since $u \cdot u^T = I$, $u^T = u^{-1}$

that is to say $A = u \Lambda u^{-1} \Leftrightarrow Au = u\Lambda$

Apparently, by quoting the conclusion of (a)
(b) is done

(c) If A is PSD $\Rightarrow$ $A = U \Lambda U^T$. $\forall x \in \mathbb{R}^n$
there is $x^T U \Lambda U^T x \geq 0$ $\Rightarrow$ $(U^T x)^T \Lambda (U^T x) \geq 0$
since U is orthogonal, the rank of U is n
Due to the arbitrarity of x, $U^T x$ can attain arbitrary vector
That is to say, $\forall y \in \mathbb{R}^n$. $y^T \Lambda y \geq 0$
Suppose that Λ is $\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$
$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, $y^T \Lambda y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \geq 0$
let $y_1 = 1$, $\{y_i\}_{i=2}^n = 0$, it is easy to see $\lambda_1 \geq 0$
then let $y_1 = 0$, $y_2 = 1$ $y_3 = 0$, $\cdots$, $y_n = 0$, then you can obtain $\lambda_2 \geq 0$
$\vdots$
Thus $\lambda_i(A) \geq 0$, for each i