Implementation of Wilcox kOmega 1988 and 2006 in OpenFOAM

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1 The Wilcox kOmega turbulence model (1988)

The governing equations match with NASA Turbulence Modeling Resource (TMR):

$$\frac{\partial(\rho k)}{\partial t}+\frac{\partial(\rho u_j k)}{\partial x_j}=\mathscr{P}-\beta^*\rho\omega k+\frac{\partial}{\partial x_j}\left[\left(\mu+\sigma_k\frac{\rho k}{\omega}\right)\frac{\partial k}{\partial x_j}\right] \tag{1}$$

$$\frac{\partial(\rho\omega)}{\partial t}+\frac{\partial(\rho u_j\omega)}{\partial x_j}=\frac{\gamma\omega}{k}\mathscr{P}-\beta\rho\omega^2+\frac{\partial}{\partial x_j}\left[\left(\mu+\sigma_\omega\frac{\rho k}{\omega}\right)\frac{\partial\omega}{\partial x_j}\right] \tag{2}$$

In these equations, the production term is

$$\mathscr{P}=\tau_{ij}\frac{\partial u_i}{\partial x_j} \tag{3}$$

The Boussinesq assumption for linear eddy viscosity models (for the compressible flow) is

$$\tau_{ij}=\mu_t\left(2S_{ij}-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)-\frac{2}{3}\rho k\delta_{ij} \tag{4}$$

Notice in the Boussinesq assumption for linear eddy viscosity models for the incompressible flow is the following, in which the divergence term $-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}$ is zero.

$$\tau_{ij}=\mu_t 2S_{ij}-\frac{2}{3}\rho k\delta_{ij} \tag{5}$$

In this document, we work with the compressible form to be consistent with the OpenFOAM implementation.

The deviatoric/strain-rate stress tensor is

$$S_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{6}$$

The eddy viscosity is calculated as

$$\mu_t=\frac{\rho k}{\omega} \tag{7}$$

1.1 Reformulation

Substitute the Boussinesq assumption of Eq. 4 and deviatoric stress tensor of Eq. 6 the into the production term, $\mathscr{P}$.

$$\mathscr{P} = \tau_{ij}\frac{\partial u_i}{\partial x_j} = \mu_t\left(\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\frac{\partial u_i}{\partial x_j} - \frac{2}{3}\rho k\delta_{ij}\frac{\partial u_i}{\partial x_j} \tag{8}$$

Now remove all the Kronecker delta

$$\frac{\partial u_k}{\partial x_k}\delta_{ij}\frac{\partial u_i}{\partial x_j} = \left(\frac{\partial u_i}{\partial x_j}\right)^2$$

Arrive at

$$\mathscr{P} = \tau_{ij}\frac{\partial u_i}{\partial x_j} = \mu_t\left(\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\frac{\partial u_i}{\partial x_j} - \frac{2}{3}\left[\frac{\partial u_i}{\partial x_j}\right]^2\right) - \frac{2}{3}\rho k\frac{\partial u_i}{\partial x_j} \tag{9}$$

1.2 Native implementation in OpenFOAM 5.0

Move the diffusion term from the RHS to the LHS in Eq. 2

$$\frac{\partial(\rho\omega)}{\partial t} + \frac{\partial(\rho u_j\omega)}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_\omega\frac{\rho k}{\omega}\right)\frac{\partial\omega}{\partial x_j}\right] = \frac{\gamma\omega}{k}\mathscr{P} - \beta\rho\omega^2 \tag{10}$$

The LHS of Eq. 10 is implemented in OpenFOAM as

```
fvm::ddt(alpha, rho, omega_)
+ fvm::div(alphaRhoPhi, omega_)
- fvm::laplacian(alpha*rho*DomegaEff(), omega_)
```

Substitute the above production term of Eq. 9 into RHS of the transport equation of Omega in Eq. 9

$$RHS = \frac{\gamma\omega}{k}\mu_t\left(\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\frac{\partial u_i}{\partial x_j} - \frac{2}{3}\left[\frac{\partial u_i}{\partial x_j}\right]^2\right) - \frac{\gamma\omega}{k}\frac{2}{3}\rho k\frac{\partial u_i}{\partial x_j} - \beta\rho\omega^2 \tag{11}$$

Separate the RHS equation further into RHS1-3

$$RHS1 = \frac{\gamma\omega}{k}\mu_t\left(\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\frac{\partial u_i}{\partial x_j} - \frac{2}{3}\left[\frac{\partial u_i}{\partial x_j}\right]^2\right) = \frac{\gamma G\omega\rho}{k} \tag{12}$$

where

$$G = \frac{\mu_t}{\rho}\left(\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\frac{\partial u_i}{\partial x_j} - \frac{2}{3}\left[\frac{\partial u_i}{\partial x_j}\right]^2\right) \tag{13}$$

TheRHS1 is calculated explicitly

```
gamma_*alpha*rho*G*omega_/k_
```

With the G is defined as

```
nut*(tgradU() && dev(twoSymm(tgradU())))
```

The notation and deviation can be found in the appendix.

$$RHS2 = -\frac{\gamma\omega}{k}\frac{2}{3}\rho k\frac{\partial u_i}{\partial x_j} = -\left[\frac{2}{3}\gamma\rho\frac{\partial u_i}{\partial x_j}\right]\omega \tag{14}$$

The last term in the RHS2 is calculated explicitly or implicitly depends on the the sign of the $\left[\frac{2}{3}\gamma\rho\frac{\partial u_i}{\partial x_j}\right]$ through the fvm::SuSp()

```
- fvm::SuSp(((2.0/3.0)*gamma_)*alpha*rho*divU, omega_)
```

$$RHS3 = -\beta\rho\omega^2 = -\left[\beta\rho\omega\right]\omega \tag{15}$$

The last term in the RHS3 is calculated implicitly through fvm::Sp() to enhance the diagonal dominance as it is (negative) sink term.

```
- fvm::Sp(beta_*alpha*rho*omega_, omega_)
```

2 The Wilcox kOmega turbulence model (2006)

$$\frac{\partial(\rho k)}{\partial t} + \frac{\partial(\rho u_j k)}{\partial x_j} = \mathscr{P} - \beta^*\rho\omega k + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_k\frac{\rho k}{\omega}\right)\frac{\partial k}{\partial x_j}\right] \tag{16}$$

$$\frac{\partial(\rho\omega)}{\partial t} + \frac{\partial(\rho u_j\omega)}{\partial x_j} = \frac{\gamma\omega}{k}\mathscr{P} - \beta\rho\omega^2 + \frac{\partial}{\partial x_j}\left[\left(\mu + \sigma_\omega\frac{\rho k}{\omega}\right)\frac{\partial\omega}{\partial x_j}\right] + \frac{\rho\sigma_d}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial\omega}{\partial x_j} \tag{17}$$

The model has extra cross diffusion term in the transport of the Omega equation. Meanwhile, a limiter is applied to the omega.

$$\mu_t = \frac{\rho k}{\hat{\omega}} \tag{18}$$

$$\hat{\omega} = \max\left[\omega, C_{lim}\sqrt{\frac{2\bar{S}_{ij}\bar{S}_{ij}}{\beta^*}}\right] \tag{19}$$

where,

$$\bar{S}_{ij} = S_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{20}$$

2.1 Implementation in OpenFOAM 5.0

The implemntation of kOmega 2006 is patterned after the native kOmega 1988 solver. Here only the major modifications are illustrated.

Stress limiter of Eq. 19 is implemented as

```
this->nut_ = k_/ max(omega_, Clim_*sqrt(2.0/0.09*magSqr(symm(fvc::grad(this->U_)))));
```

The corss diffusion term,$\frac{\rho\sigma_d}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}$ is impletmented as

```
- fvm::SuSp(alpha*rho*(- scalar(1))*CDkOmega()/omega_(),omega_)
```

With the CDkOmega defined a, which is similar to the implementation of the cross diffusion of Menter kOmega SST 2003 model in OpenFOAM 5.0.

```
(1*sigmaD_)*(fvc::grad(k_) & fvc::grad(omega_))/omega_
```

3 Appendix

3.1 Expression of the G in the production term

The velocity gradient tensor, tgradU() can be decomposed into symmetric (S_{ij},strain-rate tensor, sometime people also use D as symbol) and anti-symmetric parts (Ω_{ij}, rotation-rate tensor, sometime people also use W as symbol.)

$$\nabla u = \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right] = S_{ij} + \Omega_{ij} = \mathbf{S} + \Omega$$

The following derivation uses mixed notation.

$$(\mathrm{tgradU()}\,\&\&\,\mathrm{dev(twoSymm(tgradU()))} = (\mathbf{S}+\Omega) : \left[2\mathbf{S} - \frac{2}{3}(tr\mathbf{S})\mathbf{I}\right] = 2\mathbf{S}:\mathbf{S} - \frac{2}{3}(tr\mathbf{S})^2$$

$$2\mathbf{S}:\mathbf{S} = 2S_{ij}S_{ij} = \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\frac{\partial u_i}{\partial x_j}$$

$$-\frac{2}{3}(tr\mathbf{S})^2 = -\frac{2}{3}\left[\frac{\partial u_i}{\partial x_i}\right]^2$$

$$(\mathrm{tgradU()}\,\&\&\,\mathrm{dev(twoSymm(tgradU()))} = \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\frac{\partial u_i}{\partial x_j} - \frac{2}{3}\left[\frac{\partial u_i}{\partial x_i}\right]^2$$