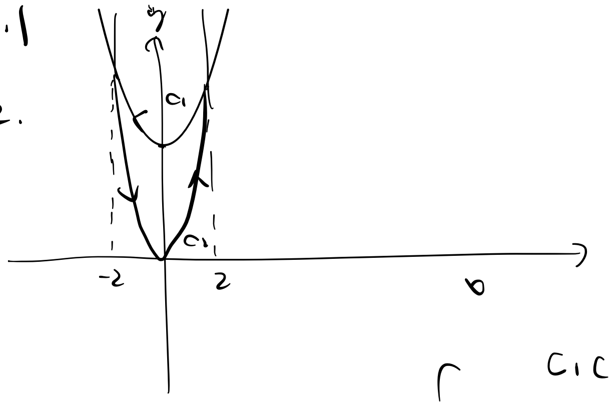
8.1

2.

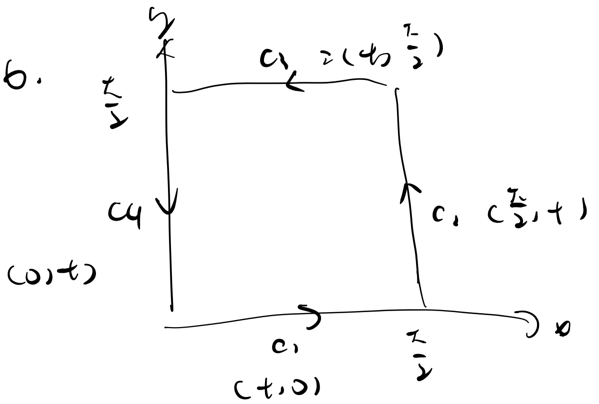


$x^2+4=2x^2$

$x=\pm 2$

$$C(t)=\begin{cases} C_1(t)=(4-t,\ (4-t)^2+4) & 2\le t\le 6 \\ C_2(t)=(t,\ 2t^2) & -2\le t\le 2 \end{cases}$$

6.



$t\in[0,\ 2\pi-2]$

$$\int_{C^+} P\,dx + Q\,dy =$$

$$\int_{C_1} P\,dx+Q\,dy + \int_{C_2} P\,dx+Q\,dy$$

$$+ \int_{C_3} P\,dx+Q\,dy + \int_{C_4} P\,dx+Q\,dy$$

$$= \int_0^{\frac{\pi}{2}} \sin t\,dt + \int_0^{\frac{\pi}{2}} \cos t\,dt - \underbrace{\int_0^{\frac{\pi}{2}} \sin t\,dt}_{\text{reverse orientation}} - \underbrace{\int_0^{\frac{\pi}{2}} \cos t\,dt}$$

$$=0$$

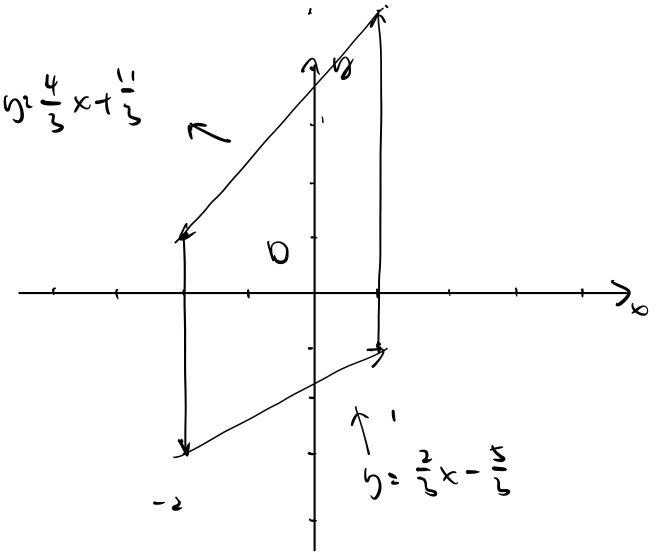
By Green's theorem.

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$$= \iint_D 0-0\,dx\,dy$$

Hence verified.

7. $y = \frac{4}{3}x + \frac{11}{3}$



$$\int_C 2xy\,dx + xy^2\,dy$$

$$= \int_{-2}^{1}\int_{\frac{2}{3}x-\frac{5}{3}}^{\frac{4}{3}x+\frac{11}{3}} \underbrace{\frac{\partial(xy^2)}{\partial x}}_{y^2} - \underbrace{\frac{\partial(2xy)}{\partial y}}_{-2x}\,dy\,dx$$

$$= \int_{-2}^{1} -2xy + \frac{1}{3}y^3 \Big|_{\frac{2}{3}x-\frac{5}{3}}^{\frac{4}{3}x+\frac{11}{3}} dx = \int_{-2}^{1} \frac{1456}{81} + \frac{140}{27}x + \frac{100}{27}x^2 + \frac{56}{81}x^3\,dx$$

$$= \frac{1456}{81}x + \frac{70}{27}x^2 + \frac{100}{81}x^3 + \frac{14}{81}x^4 \Big|_{-2}^{1}$$

$$= 61$$

9. $$\int_C y\,dx - x\,dy$$

$A(D)$

$$= \iint_D -1-1\,dx\,dy = -2\cdot(2\cdot 2) = -8$$

10 $$\iint_D dx\,dy = \iint_D \frac{\partial \overset{Q}{x}}{\partial x} - \frac{\partial \overset{P}{0}}{\partial y}\,dx\,dy$$

$$= \int_{C^+} \underbrace{P\,dx}_{0} + Q\,dy = \int_{C^+} x\,dy$$

$$= \int_0^{2\pi} \underbrace{a(\theta - \sin\theta)}_{x} \frac{d(a(1-\cos\theta))}{d\theta}\,d\theta$$

$$= \int_0^{2\pi} a(\theta - \sin\theta)\cdot a\sin\theta\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(-a^2 + a^2\cos 2\theta + 2a^2\theta\sin\theta\,d\theta$$

$$= -\frac{1}{2}a^2\theta - a^2\theta\cos\theta + a^2\sin\theta + \frac{1}{4}a^2\sin 2\theta \Big|_0^{2\pi}$$

$$= -3a^2\pi$$

Seems like at some step the orientation is flipped. But if we change the sign, we get $A(D) = 3a^2\pi$.

15. $$\int_C (2x^3 - y^3)\,dx + (x^3 + y^3)\,dy$$

$$\Downarrow \quad \begin{matrix} x = \cos t \\ y = \sin t \end{matrix}$$

$$\int_0^{2\pi} -(2\cos^3 t - \sin^3 t)\sin t + (\cos^3 t + \sin^3 t)\cos t \; dt$$

$$= \int_0^{2\pi} \cos^4 t - 2\cos^3 t\sin t + \cos t\sin^3 t + \sin^4 t \; dt$$

$$= \frac{3}{4}t + \frac{\cos t}{2} + \frac{1}{4}\sin 4t + \frac{1}{16}\sin 4t \Big|_0^{2\pi} = \frac{3}{2}\pi$$

By Green's Thm:

$$\int_C (2x^3 - y^3)\,dx + (x^3 + y^3)\,dy$$

$$= \iint_{x^2+y^2 \le 1} 3x^2 + 3y^2 \; dx\,dy$$

// same

$$= 3\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = 3\cdot\frac{\pi}{2}$$

20. Green's Thm requires both P and Q to be C^1

However, both P & Q in this question are not cont. and undefined at point $(0,0)$.

23 $$A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$$

$$\partial D: \begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \qquad \frac{dx}{dt} = -a\sin t \qquad \frac{dy}{dt} = a\cos t$$

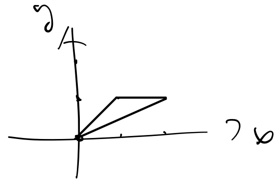
$$A = \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t \; dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2 t + \sin^2 t)\, dt = \frac{1}{2} ab\, 2\pi = ab\pi$$

8.2

8 First, we find a surface that includes the three points

$$\left.\begin{array}{ccc} Ax + By + & Cz + D = 0 & \\ 0 & 0 & 0 \\ 2 & 1 & 5 \\ 1 & 1 & 3 \end{array}\right\} \quad \begin{array}{l} A = 2 \\ B = 1 \\ C = -1 \\ D = 0 \end{array}$$



$y \in [0,1]$
$x = [y, 2y]$

$g(x,y) = z = 2x + y$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ xyz & xz & x \end{vmatrix} = (0,\ -1+xy,\ y - xz)$$

$$\int_C F\, ds = \iint_D \nabla \times F\, dS = \iint_D (0,\ -1+xy,\ y-xz)\cdot(-g_x, -g_y, 1)\, dx\, dy$$

$$= \iint (0, -1+xy, y-xz)\cdot(-2,-1,1)\, dx\, dy$$

$$= \int_0^1 \int_y^{2y} 1 - xy + y - x(2x+y)\, dx\, dy$$

$$= \int_0^1 x - \frac{2}{3}x^3 + xy - x^2 y \Big|_y^{2y} dy$$

$$= \int_0^1 -\frac{22}{3} y^3 + y(1+y)\, dy$$

$$= -\frac{13}{12}$$

10. $\iint_S \nabla \times F\, dS = \int_{\partial S} F\, ds =$

$c(t)$: $x = 4\cos t$ $\quad y = 4\sin t$ $\quad z = 0$ $\quad t \in [0, 2\pi]$

$$\int_0^{2\pi} ((4\cos t)^2 + 4\sin t - 4,\ 3\cdot 4\cos t \cdot 4\sin t,\ 0) \cdot (-4\sin t,\ 4\cos t,\ 0)\, dt$$

$$\int_0^{2\pi} -64\cos^2 t \sin t - 16\sin^2 t + 16\sin t + 48\cos^2 t \cdot 4\sin t \quad dt$$

$$= \int_0^{2\pi} 128\cos^2 t \sin t - 16\sin^2 t + 16\sin t\, dt$$

$$= -16\pi$$ < too many steps, skipped

11. ∂S : $c(t) = (\cos t,\ \sin t,\ 0)$ $\quad c'(t) = (-\sin t,\ \cos t,\ 0)$

$F(c(t)) = (\cos t,\ \sin t,\ 0)$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ (\ )_x & (\ )_y & (\ )_z \\ x & y & z \end{vmatrix} = (0, 0, 0)$$

$$\iint_S \nabla \times F\, dS = 0$$

$$\int_{\partial S} F\, ds = \int_0^{2\pi} (\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} dt = 0$$

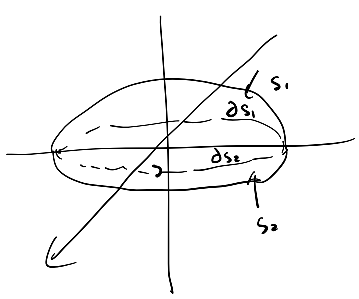
Hence verified.

13. $$\iint_S \nabla \times F\, dS = \int_{\partial S} F\, ds = \int_0^{2\pi} (\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0)\, dt$$

$= 0$ (Same as the previous questions)

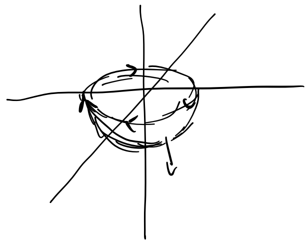
∂S: $c(t) = (\cos t,\ \sin t,\ 0)$ $\quad t \in [0, 2\pi]$

18.



Since ∂S_1 and ∂S_2 are of opposite orientations,

$$\iint_S \nabla \times F \cdot dS = \underbrace{\int_{\partial S_1} F\, ds + \int_{\partial S_2} F\, ds}_{\text{opposite sign}} = 0$$

19. 

$\partial S: \quad c(t) = (\cos t, \sin t, 0), \quad t \in [0, 2\pi)$

$c'(t) = (-\sin t, \cos t, 0)$

$$\iint_S (\nabla \times F)\cdot n\, dA = \iint_S \nabla \times F\, dS = \int_{\partial S} F\, ds$$

$$= \int_0^{2\pi} (\sin t, -\cos t, 0)\cdot(-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} -(\sin^2 t + \cos^2 t)\, dt = -2\pi.$$

31. a) $S: \quad \Phi(r, \theta) = (r\cos\theta, r\sin\theta, 0) \qquad r \in [0, 1], \ \theta \in [0, 2\pi]$

$T_r = (\cos\theta, \sin\theta, 0)$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$

$T_r \times T_\theta = (0, 0, r)$

$$\iint_S F\, dS = \int_0^1\int_0^{2\pi} \underbrace{(\dots, \dots, 0)}_{\text{doesn't matter}}\cdot(0, 0, r)\, d\theta\, dr = 0$$

b) $$\int_C F\, ds = \int_0^{2\pi} (\cos^2 t, 2\cos t\sin t + \cos t, 0)\cdot(-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} \cos^2 t(\sin t + 1)\, dt = \pi$$

c) $$\iint_S \nabla \times F\, dS = \iint_D (0, 0, 2y+1)\cdot(0, 0, r)\, dt\, dr$$

$$= \int_0^1\int_0^{2\pi} (2y+1)\, r\, dt\, dr$$

$$= \int_0^1\int_0^{2\pi} (2r\sin t + 1)\, r\, dt\, dr$$

$$= \pi$$

same

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & 2xy+x & z \end{vmatrix} = (0, 0, 2y+1)$$

33. S: $\Phi(r,\theta) = (r\cos\theta,\ r\sin\theta,\ \frac{1}{2}(r^2\cos^2\theta + r^2\sin\theta^2))$

$= (r\cos\theta,\ r\sin\theta,\ \frac{1}{2}r^2)$

$\frac{1}{2}r^2 \le 2 \Rightarrow r \in [0,2]$

$\theta \in [0, 2\pi)$

$T_r = (\cos\theta,\ \sin\theta,\ r)$ $\qquad T_\theta = (-r\sin\theta,\ r\cos\theta,\ 0)$

$T_r \times T_\theta = (-r^2\cos\theta,\ -r\sin^2\theta,\ r)$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 3y & -xz & -yz^2 \end{vmatrix} = (x - z^2, 0, -3 - z)$$

$= (r\cos\theta - \frac{1}{4}r^4,\ 0,\ -3 - \frac{r^2}{2})$

$$\int_0^2\int_0^{2\pi} (r\cos\theta - \frac{1}{4}r^4,\ 0,\ -3-\frac{r^2}{2}) \cdot (-r^2\cos\theta,\ -r\sin^2\theta,\ r)\, d\theta\, dr$$

$= \boxed{-20\pi}$ $\leftarrow$ skip steps - yay!

On the other hand $\quad \Phi(r,\theta)\big|_{r=2} = (2\cos\theta,\ 2\sin\theta,\ 2)$

$$\iint_S \nabla\times F\, dS = \int_{\partial S} F\, ds = \int_0^{2\pi} (6\sin\theta,\ -4\cos\theta,\ -8\sin\theta)\cdot(-2\sin\theta,\ 2\cos\theta,\ 0)\, dt$$

$\leftarrow$ skip steps

$= \boxed{-20\pi}$