Visualizing the kinematics of relativistic wave packets

Bernd Thaller, arXiv:quant-ph/0409079v1 (2004)
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Example 1

In[1993]:=
```
Clear[h0, m, c, p, λ, k, ψhat0, ϕposk, ϕnegk,
 int, t, ρ, kdim, kmax, dk, ψx1, ψx2, ψk1, ψk2, ψ, ψ1, ψ2]
ψ[t, x] = {ψ1[t, x], ψ2[t, x]}
D[ψ[t, x], t]
(-I PauliMatrix[1].D[ψ[t, x], x] + PauliMatrix[3].ψ[t, x]) / I // Expand
```

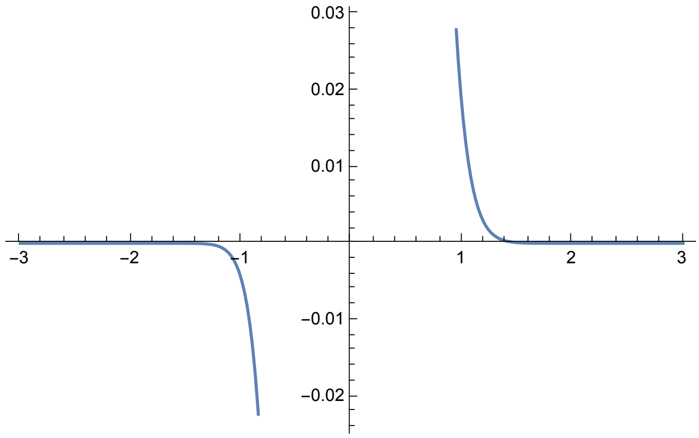
Out[1994]= $\{\psi 1[t, x], \psi 2[t, x]\}$

Out[1995]= $\{\psi 1^{(1,0)}[t, x], \psi 2^{(1,0)}[t, x]\}$

Out[1996]= $\{-i\, \psi 1[t, x] - \psi 2^{(0,1)}[t, x], i\, \psi 2[t, x] - \psi 1^{(0,1)}[t, x]\}$

```
Clear[h0, m, c, p, λ, k, ψhat0, ϕposk, ϕnegk, int, t, ρ, kdim,
  kmax, dk, ψx1, ψx2, ψk1, ψk2, uposрxt, unegpxt, unegp, uposp, ψ0,
  ψ0hat, ψposhat, ψneghat, integrand, ψ13, ρ, xmax, xdim, pmax, tab0]
xmax = 20;
xdim = 21;
pmax = 4;

(* natural units *)
m = c = ℏ = 1;

(* equation7: free Dirac hamiltonian *)
h0 = {{m c^2, c p}, {c p, -m c^2}};
(* eigenvalue *)
λ = Eigenvalues[h0][[2]];
(* eigenvectos *)
{unegp, uposp} = Eigenvectors[h0] // Simplify;

(* check equation 9 *)
h0.uposp - λ uposp // Simplify;
h0.unegp + λ unegp // Simplify;

(* equation 8: build plane wave solutions *)
uposрxt = 1/Sqrt[2 π] uposp Exp[I p x - I λ t];
unegpxt = 1/Sqrt[2 π] unegp Exp[I p x + I λ t];

(*** example 1 ***)
(* equation 15: initial wave packet *)
ψ0 = (1/(32 π))^(1/4) Exp[-x^2/16] {1, 1};
(* equation 16: Fourier transform *)
ψ0hat = FourierTransform[ψ0, x, p];

(* equation 14: Fourier coefficient functions *)
ψposhat = (1/2 (IdentityMatrix[2] + h0/λ).ψ0hat // Simplify)[[1]];
ψneghat = (1/2 (IdentityMatrix[2] - h0/λ).ψ0hat // Simplify)[[2]];

integrand = ψposhat uposрxt + ψneghat unegpxt;

(* singularity at p=0 *)
Plot[Re[integrand[[1]]] /. {t → 0, x → 0}, {p, -3, 3},
  PlotRange → Automatic, PlotPoints → 5, Exclusions → {0}]
```

Out[1939]=



Example 2

```
Clear[h0, m, c, p, λ, k, ψhat0, ϕposk, ϕnegk,
 int, t, ρ, kdim, kmax, dk, ψx1, ψx2, ψk1, ψk2, ψ, ψ1, ψ2]

(* natural units *)
m = c = ℏ = 1;

(* equation7: free Dirac hamiltonian *)
h0 = {{m c^2, c p}, {c p, -m c^2}};
(* eigenvalue *)
λ = Eigenvalues[h0][[2]];
(* eigenvectos *)
{unegp, uposp} = Eigenvectors[h0] // Simplify;

(*** example 2 ***)
(* equation 17: initial wave packet *)
ψ0 = (1/(32 π))^(1/4) Exp[-x^2/16 - I 3 x/4] {1, 1};
(* Fourier transform *)
ψ0hat = FourierTransform[ψ0, x, p]

(* equation 14: Fourier coefficient functions *)
ψposhat = (1/2 (IdentityMatrix[2] + h0/λ).ψ0hat // Simplify)[[1]]
ψneghat = (1/2 (IdentityMatrix[2] - h0/λ).ψ0hat // Simplify)[[2]]
```

Out[871]= $\left\{e^{-\frac{1}{4}(3-4p)^2}\left(\frac{2}{\pi}\right)^{1/4},\ e^{-\frac{1}{4}(3-4p)^2}\left(\frac{2}{\pi}\right)^{1/4}\right\}$

Out[872]= $\dfrac{e^{-\frac{1}{4}(3-4p)^2}\left(1+p+\sqrt{1+p^2}\right)}{2^{3/4}\sqrt{1+p^2}\,\pi^{1/4}}$

Out[873]= $\dfrac{e^{-\frac{1}{4}(3-4p)^2}\left(1-p+\sqrt{1+p^2}\right)}{2^{3/4}\sqrt{1+p^2}\,\pi^{1/4}}$

Example 3

In[1997]:=
```
Clear[h0, m, c, p, λ, k, ψhat0, ϕposk, ϕnegk,
  int, t, ρ, kdim, kmax, dk, ψx1, ψx2, ψk1, ψk2, ψ, ψ1, ψ2]

(*** example 3 ***)
(* equation 23: initial wave packet *)
```
$$\psi 0 = \left(\frac{1}{4\,\pi}\right)^{1/4} \mathrm{Exp}\left[-x^2/8\right] \{1, 0\};$$
```
(* Fourier transform *)
ψ0hat = FourierTransform[ψ0, x, p]

(* equation 14: Fourier coefficient functions *)
```
$$\psi\mathrm{poshat} = \left(\frac{1}{2}\left(\mathrm{IdentityMatrix}[2] + \frac{h0}{\lambda}\right).\psi 0\mathrm{hat} \mathbin{//} \mathrm{Simplify}\right)[\![1]\!]$$
$$\psi\mathrm{neghat} = \left(\frac{1}{2}\left(\mathrm{IdentityMatrix}[2] - \frac{h0}{\lambda}\right).\psi 0\mathrm{hat} \mathbin{//} \mathrm{Simplify}\right)[\![2]\!]$$

Out[1999]= $\left\{\frac{\sqrt{2}\, e^{-2 p^2}}{\pi^{1/4}}, 0\right\}$

Out[2000]= $\frac{e^{-2 p^2}\,(h0 + \lambda)}{\sqrt{2}\,\pi^{1/4}\,\lambda}$

Out[2001]= $-\frac{e^{-2 p^2}\, h0}{\sqrt{2}\,\pi^{1/4}\,\lambda}$