Special unitary particle pusher for extreme fields

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Introduction

In this notebook we reproduce some results from the paper.

Prove equation 4->3

In[2032]:=
```
(* prove equation 4 implies equation 3 *)
Clear[ζ, u0, u1, u2, u3]
ζ = {{u0 + u3, u1 - I u2}, {u1 + I u2, u0 - u3}}
{0.5 Tr[PauliMatrix[1].ζ],
  0.5 Tr[PauliMatrix[2].ζ], 0.5 Tr[PauliMatrix[3].ζ]} // Simplify
```

Out[2033]=
```
{{u0 + u3, u1 - ⅈ u2}, {u1 + ⅈ u2, u0 - u3}}
```

Out[2034]=
```
{1. u1, 1. u2, 1. u3}
```

Prove equation 6

eq5: $\zeta(s+\Delta s)=\Lambda(\Delta s)\,\zeta(s)\,\Lambda^{\dagger}(\Delta s)$

define generator λ through: $\Lambda(\Delta s) = 1+\lambda\,\Delta s$

Replace in eq5: $\zeta(s+\Delta s)=(1+\lambda\,\Delta s)\,\zeta(s)\,\left(1+\lambda^{\dagger}\,\Delta s\right) = \lambda\,\zeta(s)\,\Delta s + \zeta(s)\,\lambda^{\dagger}\,\Delta s + \lambda\,\zeta(s)\,\lambda^{\dagger}\,\Delta s^{2}$

$d\zeta/ds = (\zeta(s+\Delta s)-\zeta(s))/\Delta s = \lambda\,\zeta(s) + \zeta(s)\,\lambda^{\dagger} + O(\Delta s) = \lambda\,\zeta + \zeta\,\lambda^{\dagger}$

Prove equation 9

In[3181]:=

```
Clear[E, Ex, Ey, Ez, B, Bx, By, Bz, Ω, e, m, c, σ, Ωnrm, Λ9, Λ, Δs]
E = {Ex, Ey, Ez};
B = {Bx, By, Bz};
σ = {PauliMatrix[1], PauliMatrix[2], PauliMatrix[3]};
(* eq8 *)
Ω = e/(2 m c) (E + I B);
(* eq7 *)
λ = σ⟦1⟧ × Ω⟦1⟧ + σ⟦2⟧ × Ω⟦2⟧ + σ⟦3⟧ × Ω⟦3⟧;
(* eq10 *)
Ωnrm = Refine[Sqrt[Ω.Ω] // Simplify, {m > 0, c > 0, e > 0}];
(*Ωnrm=Refine[Norm[Ω]//Simplify,{m>0,c>0,e>0}];*)
(* eq11 *)
ω = Ω / Ωnrm;

(* matrix exponential form of Λ*)
Λ = MatrixExp[λ Δs];
(* eq9 form of Λ *)
Λ9 = IdentityMatrix[2] Cosh[Ωnrm Δs] + (λ / Ωnrm) Sinh[Ωnrm Δs] // Simplify;

(* choose random values for fields and physical parameters *)
{Ex, Ey, Ez} = RandomReal[{-1, +1}, 3];
{Bx, By, Bz} = RandomReal[{-1, +1}, 3];
c = RandomReal[];
m = RandomReal[];
e = RandomReal[];
Δs = RandomReal[{0, 0.2}];
Λ9 - Λ // Simplify // N // Chop
```

Out[3197]= {{0, 0}, {0, 0}}

In[3198]:=

```
(* Ψ.Ψ=0 for a PW *)
Clear[E, Ex, Ey, Ez, B, Bx, By, Bz, Ω, e, m, c, σ, Ωnrm, Λ9, Λ, E0, Δs]
E = {Ex, Ey, Ez};
B = {Bx, By, Bz};
{Ex, Ey, Ez} = {0, E0, 0};
{Bx, By, Bz} = {0, 0, E0};
σ = {PauliMatrix[1], PauliMatrix[2], PauliMatrix[3]};
Ω = e/(2 m c) (E + I B);
Ψ = Ω Δs;
Ψ.Ψ
```

Out[3206]= 0

In[3346]:=
```
(* Ψ.Ψ = (σ.Ψ)^2 only for E^2-B^2=0 and E.B=0*)
Clear[E, Ex, Ey, Ez, B, Bx, By, Bz, Ω, e, m, c, σ, Ωnrm, Λ9, Λ, E0, ΨΨ, Ψσ2, Δs]
E = {Ex, Ey, Ez};
B = {Bx, By, Bz};
σ = {PauliMatrix[1], PauliMatrix[2], PauliMatrix[3]};
Ω = e/(2 m c) (E + I B);
Ψ = Ω Δs;
ΨΨ = (Ψ[[1]] × Ψ[[1]] + Ψ[[2]] × Ψ[[2]] + Ψ[[3]] × Ψ[[3]]) // Simplify;
Ψσ2 = (Ψ[[1]] × σ[[1]] + Ψ[[2]] × σ[[2]] + Ψ[[3]] × σ[[3]]).(Ψ[[1]] × σ[[1]] + Ψ[[2]] × σ[[2]] + Ψ[[3]] × σ[[3]]);
ΨΨ - Ψσ2 // Expand // Simplify
```

Out[3354]=

$$\left\{\left\{0, \frac{e^2 \Delta s^2 \left(-Bx^2 - By^2 - Bz^2 + 2 i\, Bx\, Ex + Ex^2 + 2 i\, By\, Ey + Ey^2 + 2 i\, Bz\, Ez + Ez^2\right)}{4 c^2 m^2}\right\}, \left\{\frac{e^2 \Delta s^2 \left(-Bx^2 - By^2 - Bz^2 + 2 i\, Bx\, Ex + Ex^2 + 2 i\, By\, Ey + Ey^2 + 2 i\, Bz\, Ez + Ez^2\right)}{4 c^2 m^2}, 0\right\}\right\}$$

```
(* Sqrt[Ψ.Ψ] is real for pure electric field → Lorentz boost *)
Clear[E, Ex, Ey, Ez, B, Bx, By, Bz, Ω, e, m, c, σ, Ωnrm, Λ9, Λ, E0, sqΨΨ, Ψσ2, Δs]
E = {Ex, Ey, Ez};
B = {0, 0, 0};
σ = {PauliMatrix[1], PauliMatrix[2], PauliMatrix[3]};
Ω = e/(2 m c) (E + I B);
Ψ = Ω Δs;
sqΨΨ = Refine[Sqrt[(Ψ[[1]] × Ψ[[1]] + Ψ[[2]] × Ψ[[2]] + Ψ[[3]] × Ψ[[3]]) // Simplify],
  {c > 0, m > 0, e > 0, Ex ∈ Reals, Ey ∈ Reals, Ez ∈ Reals, Δs > 0}]
```

Out[4146]=

$$\frac{e\, \Delta s \sqrt{Ex^2 + Ey^2 + Ez^2}}{2 c m}$$

```
(* Sqrt[Ψ.Ψ] is imaginary for pure magnetic field → rotation *)
Clear[E, Ex, Ey, Ez, B, Bx, By, Bz, Ω, e, m, c, σ, Ωnrm, Λ9, Λ, E0, sqΨΨ, Ψσ2, Δs]
E = {0, 0, 0};
B = {Bx, By, Bz};
σ = {PauliMatrix[1], PauliMatrix[2], PauliMatrix[3]};
Ω = e/(2 m c) (E + I B);
Ψ = Ω Δs;
sqΨΨ = Refine[Sqrt[(Ψ[[1]] × Ψ[[1]] + Ψ[[2]] × Ψ[[2]] + Ψ[[3]] × Ψ[[3]]) // Simplify],
  {c > 0, m > 0, e > 0, Bx ∈ Reals, By ∈ Reals, Bz ∈ Reals, Δs > 0}]
```

Out[4160]=

$$\frac{e \sqrt{-Bx^2 - By^2 - Bz^2}\, \Delta s}{2 c m}$$

Prove equation 19 ...

In[4076]:=
```
Clear[E, Ex, Ey, Ez, B, Bx, By, Bz, Ω, e, m, c, σ, Ωnrm, Λ9, Λ, Δs, Ψσ, ΨΨ]
E = {Ex, Ey, Ez};
B = {Bx, By, Bz};
σ = {PauliMatrix[1], PauliMatrix[2], PauliMatrix[3]};
(* eq8 *)
Ω = e/(2 m c) (E + I B);
(* eq7 *)
λ = σ[[1]] × Ω[[1]] + σ[[2]] × Ω[[2]] + σ[[3]] × Ω[[3]];
(* eq10 *)
Ωnrm = Refine[Sqrt[Ω.Ω] // Simplify, {m > 0, c > 0, e > 0}];
(*Ωnrm=Refine[Norm[Ω]//Simplify,{m>0,c>0,e>0}];*)
(* eq11 *)
ω = Ω / Ωnrm;
Ψ = Ω Δs;
Ψσ = (Ψ[[1]] × σ[[1]] + Ψ[[2]] × σ[[2]] + Ψ[[3]] × σ[[3]]);
ΨΨ = (Ψ[[1]] × Ψ[[1]] + Ψ[[2]] × Ψ[[2]] + Ψ[[3]] × Ψ[[3]]) // Simplify;

(* Λ(n) equation 17 *)
Λn = (1 + Ψσ / 2).Inverse[1 - Ψσ / 2] // Simplify;
(* eq19 form of Λ *)
Λ19 = (1 + Ψσ + ΨΨ / 4)/(1 - ΨΨ / 4) // Simplify;

(* choose random values for fields and physical parameters *)
{Ex, Ey, Ez} = RandomReal[{-1, +1}, 3];
{Bx, By, Bz} = RandomReal[{-1, +1}, 3];
c = RandomReal[];
m = RandomReal[];
e = RandomReal[];
Δs = RandomReal[{0, 0.2}];
Det[Λn]
Det[(1 + Ψσ / 2)] - Det[(1 - Ψσ / 2)]
Λn - Λ19 // Simplify // N // Chop
```

Out[4095]= $-1.01609 + 0.0110517\, i$

Out[4096]= $0.0098092 - 0.0295714\, i$

Out[4097]= {{0.955963 + 0.11057 i, -1.95288 - 0.116914 i},
{1.97862 + 0.0757111 i, -2.94004 - 0.122202 i}}

Prove equation 24

In[2195]:=
```
(* prove equation 24: trucate equation 23 at order s^2,
solve for s as function of Δt, Taylor expand *)
Clear[Δs, t, Δt, eEu, m, c, u0]
Refine[Series[Solve[Δt == u0 s + eEu/(2 m c) s^2, s][[2, 1, 2]], {Δt, 0, 2}] // Simplify,
 {m > 0, c > 0, u0 > 0}]
```

Out[2196]=
$$\frac{\Delta t}{u0} - \frac{eEu\, \Delta t^2}{2\left(c\, m\, u0^3\right)} + O[\Delta t]^3$$

Figure 1...

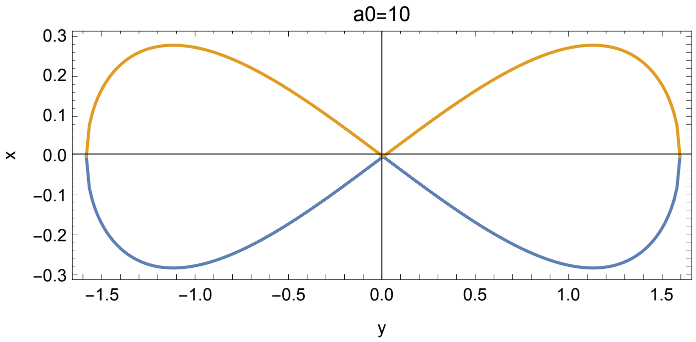
In[4414]:=

```
Clear[ω0, λ, c, a0, ωp, ρpll, ρprp, tab010, tab100]
ωp = ω0 / Sqrt[1 + 0.5 a0^2];
ρpll = c / (8 ωp) a0^2 / (1 + 0.5 a0^2);
ρprp = c / ωp a0 / Sqrt[1 + 0.5 a0^2];
λ = 1;
c = 1;
ω0 = 2 π c / λ;

tab010 = {ParallelTable[
     {y, -2 ρpll Sqrt[(y / ρprp)^2 - (y / ρprp)^4]}, {y, -ρprp, +ρprp, ρprp / 100}],
    ParallelTable[{y, +2 ρpll Sqrt[(y / ρprp)^2 - (y / ρprp)^4]},
     {y, -ρprp, +ρprp, ρprp / 100}]} /. {a0 → 10};
ListPlot[tab010, Joined → True, Frame → True, FrameLabel → {"y", "x"},
 PlotLabel → "a0=10", AspectRatio → 0.4]

tab100 = {ParallelTable[
     {y, -2 ρpll Sqrt[(y / ρprp)^2 - (y / ρprp)^4]}, {y, -ρprp, +ρprp, ρprp / 100}],
    ParallelTable[{y, +2 ρpll Sqrt[(y / ρprp)^2 - (y / ρprp)^4]},
     {y, -ρprp, +ρprp, ρprp / 100}]} /. {a0 → 1000};
ListPlot[tab100, Joined → True, Frame → True, FrameLabel → {"y", "x"},
 PlotLabel → "a0=1000", AspectRatio → 0.4]
```

Out[4422]=



Out[4424]=