Dynamic Resource Allocation in Tullock Tug-of-war*

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Abstract

We examine a tug-of-war game between two players, employing the Tullock contest success function within the framework of dynamic resource allocation. Our aim is to establish a Markov perfect equilibrium of this game (if it exists) using pure strategies, and we show that there exist closed-form solutions for both players'strategies and winning probabilities. In each state, the optimal strategy for both players is to allocate the same proportion of their current endowment as their opponent, thereby maintaining a constant ratio of their endowments throughout the game. Furthermore, we discuss some extensions with a minimum score requirement and asymmetric score goal, respectively. In addition, We offer novel insights into the performance of the players in a dynamic contest under resource allocation framework, in contrast to that under cost structure framework.

Keywords: *Tug-of-war, Tullock contest, Blotto game, Resource allocation, Markov perfect equilibrium.*

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1 Introduction

Enduring competitions, prevalent across various domains such as economics, politics, sports, and the military, often bear a striking resemblance to the dynamics of tug-of-war games. This analogy is particularly evident in duopolistic competitions, exemplified by the rivalry between industry giants like Microsoft and Apple. The roles of leader and follower can interchange between periods, with the ultimate victor hinging upon the cumulative performance difference. In such situations, both players strive to allocate their resources effectively to attain advantage in the ongoing competition and become ultimate winner. A crucial question that arises in such scenarios is the optimal strategy for players throughout the enduring competition. Specifically, how should they allocate their finite resources over a possibly infinite number of competition periods?

In a typical tug-of-war contest, two players decide their effort levels in each round, incurring costs determined by their cost structure. The gainer of the score in each round is decided by a lottery contest success function, declaring one player as the ultimate winner once they accumulates scores exceeding their opponent's by a predetermined goal value. (Harris and Vickers, 1987). In this paper, we explore a novel version of the tug-of-war game, where players dynamically allocate resources across stages instead of exerting efforts subject to cost functions. Our setting exhibits generality in several aspects. Firstly, we permit the contest's winning criterion to follow a win-by-n rule for any arbitrary n, which implies that the target score difference is no longer restricted to two. Secondly, players can be asymmetric with respect to both resources and winning criteria. Thirdly, the contest success function is extended from lottery to Tullock. Fourthly, we also consider the requirement of a minimum number of scores before one can become the ultimate winner. In this paper, we derive a closed-form equilibrium solution and conduct a comprehensive comparative statics analysis, revealing that in cases of asymmetric resources, there is a virtual tie at the lowest point of the U-shaped competition intensity curve.

Previous studies on tug-of-war contests have predominantly focused on the analysis under effort-cost structure framework (Harris and Vickers, 1987; Karagozoglu et al., 2021), employing lottery contest success functions, or being considered as part of an all-pay auction (Konrad and Kovenock, 2005). Ewerhart and Teichgraber (2019) have studied a broad range of dynamic contests, including tug-of-war as a specific case, considering general contest success functions and effort cost structures. In contrast, our research explores the Tullock contest success function, which encompasses the lottery contest success function as a specific case.

Our research is also situated in the field of dynamic resource allocation contests. Klumpp et al. (2019) have analyzed a sequential Blotto game involving a two-player competition under a simple majority rule. Li and Zheng (2021a) and Anbarcı et al. (2020) both extended the framework of Klumpp et al. (2019) to encompass scenarios with more than two players and general objectives, but from different perspectives. Nevertheless, none of these prior studies have considered the "win-by-n" rule, a fundamental element of tug-of-war contests. Li and Zheng (2022) conducted research in a similar setting to ours, but they only considered the scenarios under the "win-by-2" rule, and their tug-of-war ended in finite rounds. In contrast, we examined tug-of-war contests that could potentially continue indefinitely, and we characterized the equilibrium for any n under the "win-by-n" rule.

Our research makes contributions to the tug-of-war contest literature in three main aspects. Firstly, we offer a clear definition of the tug-of-war contest under resource allocation framework and provide the closed-form solution of the pure strategy equilibrium (if it exists). Secondly, we improve the model's generality and applicability by considering a broad range of scenarios, including those with asymmetric players in terms of both resources and winning criteria, and that with a requirement on the minimum score of the ultimate winner. Lastly, we provide valuable insights, such as the existence of a "virtual tie" in the asymmetric scenario, the influence of the parameters on the competition intensity, and compare our findings under the resource allocation framework with those under the effort-cost structure framework.

The remainder of the paper is structured as follows. In Section 2, we introduce the details of the model. In Section 3, we derive our main results and discuss the properties about the solution. In Section 4, we make the discussion of the second order conditions, conduct comparative static analyses and demonstrate the expected periods before termination. In Section 5, we analyze two critical extensions, minimum score requirement and asymmetric score goal included. Finally, Section 6 concludes the paper.

2 Benchmark Model

We study a Tullock tug-of-war contest where two players strategically allocate their resources across multiple stages, considering their respective resource constraints. In each stage, both players participate in a competition with a Tullock contest success function, competing for a point. To explicate the model, we will introduce key notations and specify the details of the contest in the following paragraph.

There are two players, identified as A and B. With a slight abuse of notation, we will also use A and B to denote their initial total resources, while a and b represent players' available resources in a given stage. The whole contest consists of potentially infinitely many stages, terminating if and only if one player wins n more battles than the other, where $n \geq 1$ and $n \in \mathbb{Z}$. When $n = 1$, the contest is a trivial one-shot competition, and hence we focus on $n \geq 2$ in this paper. In each stage, the winner is determined by a Tullock contest success function, and these Tullock contest success functions employed in each stage share the same discriminatory power r, where $0 < r \leq 1$.

In summary, a Tullock tug-of-war contest can be characterized by a 4-tuple (n, r, A, B). In each stage, the following steps occur sequentially:

1. The current scores and resources of both sides are announced and become common knowledge for both players.

2. Both players determine the amount of resources allocated on competing for the current stage simultaneously.

3. The winner of this stage is determined by contest success function, and he or she gains one point.

Four more subtle aspects of settings need to be clarified carefully: (i) Stage states; (ii) Control variables; (iii) Contest success function; (iv)Objective function.

1. *Stage states*: *Each stage can be characterized by three variables*

$$s \equiv (k, a, b), s \in \mathbb{S} \equiv \mathbb{K} \times \mathbb{A} \times \mathbb{B}, \mathbb{K} = \mathbb{Z} \cap [-n, n], \mathbb{A} = [0, A], \mathbb{B} = [0, B] \tag{1}$$

where k is the score of player A minus that of player B, a and b are the current remaining resources for player A and player B respectively.

Note that the stage state has memoryless property, which means two stages are the same if they share the same (k, a, b), no matter what history information they have.

2. *Control variables*: *In each stage, player i decides how much to allocate on competing, denoted as x_i with the following expression*

$$x_i \equiv x_i(k, a, b), i \in \{A, B\}, k \in \mathbb{K}, a \in \mathbb{A}, b \in \mathbb{B} \tag{2}$$

3. *Contest success function*: *The winner of each stage is determined by the Tullock contest success function with discriminatory power r. Specifically, the winning probability of player i is denoted as p_i with the following expression*

$$p_i \equiv p_i\left(x_i\left(k,a,b\right),x_{-i}\left(k,a,b\right)\right) = \begin{cases} \dfrac{x_i^r\left(k,a,b\right)}{x_i^r\left(k,a,b\right)+x_{-i}^r\left(k,a,b\right)} & ,\left(x_i\left(k,a,b\right),x_{-i}\left(k,a,b\right)\right)\in\mathbb{A}\times\mathbb{B}\backslash\left\{(0,0)\right\} \\ \dfrac{1}{2} & ,\left(x_i\left(k,a,b\right),x_{-i}\left(k,a,b\right)\right)=(0,0) \end{cases},i\in\{A,B\} \quad (3)$$

When $r=1$, the contest success function becomes the lottery CSF which is commonly assumed in the literature. We relax its restriction by allowing $r\in(0,1]$.

4. *Objective function*: *For both players, the goal is to become the ultimate winner, thus they should maximize the expected probability to become the ultimate winner in each stage. Formally speaking, a player should maximize the state transition probability from (k,a,b) to $(n,\cdot,\cdot)$ for player A or $(-n,\cdot,\cdot)$ for player B, denoted as $\hat{P}_i\left(k,a,b\right)\equiv\max\limits_{x_i(k,a,b)}P_i\left(k,a,b\right)$. Thus, the following conditions hold.*

$$\begin{aligned}\hat{P}_A\left(k,a,b\right)=&\max_{x_A(k,a,b)}\frac{x_A^r\left(k,a,b\right)}{x_A^r\left(k,a,b\right)+x_B^r\left(k,a,b\right)}\hat{P}_A\left(k+1,a-x_A\left(k,a,b\right),b-x_B\left(k,a,b\right)\right)\\&+\frac{x_B^r\left(k,a,b\right)}{x_A^r\left(k,a,b\right)+x_B^r\left(k,a,b\right)}\hat{P}_A\left(k-1,a-x_A\left(k,a,b\right),b-x_B\left(k,a,b\right)\right)\end{aligned} \quad (4)$$

$$\begin{aligned}\hat{P}_B\left(k,a,b\right)=&\max_{x_B(k,a,b)}\frac{x_B^r\left(k,a,b\right)}{x_A^r\left(k,a,b\right)+x_B^r\left(k,a,b\right)}\hat{P}_B\left(k-1,a-x_A\left(k,a,b\right),b-x_B\left(k,a,b\right)\right)\\&+\frac{x_A^r\left(k,a,b\right)}{x_A^r\left(k,a,b\right)+x_B^r\left(k,a,b\right)}\hat{P}_B\left(k+1,a-x_A\left(k,a,b\right),b-x_B\left(k,a,b\right)\right)\end{aligned} \quad (5)$$

3 Main results

To start with, we consider a more general optimization problem to gain some intuitions.

Lemma 1. *For a Markov optimization problem $\hat{P}\left(k,a,b\right)=\max\limits_{x,y}\sum_{k'\in\mathbb{K}}f\left(x,y,k,k'\right)\hat{P}\left(k',a-x,b-y\right)$ on state space $\mathbb{K}\times\mathbb{A}\times\mathbb{B}$, where $\mathbb{A}\times\mathbb{B}\subset\mathbb{R}^2_{++}$ and $Card\left(\mathbb{K}\right)<\infty$, $\frac{x}{a}=\frac{y}{b}$ is a necessary condition for the first order conditions of the optimization problem if*

- *The Markov transition matrix $f(x, y, \cdot, \cdot)$ is differentiable and homogeneous of degree 0, i.e. $\forall (x, y) \in \mathbb{A} \times \mathbb{B}, k, k' \in \mathbb{K}, c > 0,\ f(x, y, k, k') = f(cx, cy, k, k')$ and f_x, f_y exists.*

- *The objective function $\hat{P}(\cdot, a, b)$ is also differentiable and homogeneous of degree 0, i.e. $\forall (a, b) \in \mathbb{A} \times \mathbb{B}, k, \in \mathbb{K}, c > 0,\ \hat{P}(k, a, b) = f(k, ca, cb)$ and $\hat{P}_a, \hat{P}_b$ exists.*

- $\forall (a, b) \in \mathbb{A} \times \mathbb{B}, k \in \mathbb{K},\ \sum_{k' \in \mathbb{K}} f'\left(k, k', 1, \frac{y}{x}\right) \hat{P}\left(k', 1, \frac{b-y}{a-x}\right) \neq 0.$

In other words, there does not exist inner solution that violates the requirement $\frac{x}{a} = \frac{y}{b}$.

In the benchmark model, the Markov transition matrix is formed by Tullock contest success functions in different states, rendering it differentiable and homogeneous of degree 0. Besides, the following assumption satisfies the second requirement in lemma 1.

Assumption 1. *The objective function is also differentiable and homogeneous of degree 0. Formally,*

$$\forall c > 0, \hat{P}_i(k, a, b) = \hat{P}_i(k, ca, cb), i = A, B$$

In fact, assumption 1 bears a resemblance to monetary neutrality assumption. The outcome will remain unchanged if both players convert their resources into another identical currency.

For the third requirement, $\sum_{k' \in \mathbb{K}} f'\left(k, k', 1, \frac{y}{x}\right) \hat{P}\left(k', 1, \frac{b-y}{a-x}\right) \neq 0$ can easily be verified by plugging in Tullock contest success function. Therefore, in our benchmark model, lemma 1 tells us $\frac{x_A}{a} = \frac{x_B}{b}$ is a necessary condition for the first order conditions of the optimization problem.

Given that the ratio of players' resources remains constant across all states, the following lemma is self-evident.

Lemma 2. *If $\frac{x_A}{a} = \frac{x_B}{b}$ is always satisfied, then the following relationship holds.*

$$\hat{P}_A(k, a, b) = \alpha \hat{P}_A(k+1, a, b) + (1-\alpha) \hat{P}_A(k-1, a, b), \forall k \in [-n+1, n-1] \quad (6)$$

where $\alpha = \frac{a^r}{a^r + b^r}$.

This lemma describes the transition relationship between adjacent states in the equilibrium state. For any given (a, b), the equations presented in lemma 2 constitute a set of linear equations, having an equal number of unknowns and equations.

Moreover, the parameter matrix of the set of linear equations is full-rank, so the set of linear equations has a unique solution.

Theorem 1. *If $\frac{x_A}{a} = \frac{x_B}{b}$ holds, then the equilibrium ultimate winning probabilities have the following expressions.*

$$\hat{P}_A\left(k,a,b\right) = \begin{cases} \dfrac{a^{(n-k)r}(b^{(n+k)r}-a^{(n+k)r})}{b^{2nr}-a^{2nr}}, & k\in[-n,n], a\neq b \\ \dfrac{n+k}{2n}, & k\in[-n,n], a=b \end{cases} \tag{7}$$

Since we pay no attention to the dynamic process before reaching the equilibrium state, the winning probability shown presented in theorem 1 demonstrates the equilibrium probability in any state (k,a,b). It is apparent that as the score difference increases, the probability of the player with the higher score becoming the ultimate winner also increases.

By the Envelop theorem, the optimization problems are equivalent to the following.

$$\begin{aligned}\hat{P}_A\left(k,a,b\right) =&\max_{x_A} \beta\left(\frac{\gamma^n}{\gamma^n+\left(1-\gamma\right)^n}+\frac{\left(1-\gamma\right)^n}{\gamma^n+\left(1-\gamma\right)^n}\left(\frac{\left(1-\gamma\right)^{k+1}-\gamma^{k+1}}{\left(1-\gamma\right)^n-\gamma^n}\gamma^{n-k-1}\right)\right)\\ &+\left(1-\beta\right)\left(\frac{\gamma^n}{\gamma^n+\left(1-\gamma\right)^n}+\frac{\left(1-\gamma\right)^n}{\gamma^n+\left(1-\gamma\right)^n}\left(\frac{\left(1-\gamma\right)^{k-1}-\gamma^{k-1}}{\left(1-\gamma\right)^n-\gamma^n}\gamma^{n-k+1}\right)\right)\\ =&\max_{x_A}\frac{\gamma^{n-k-1}}{(1-\gamma)^{2n}-\gamma^{2n}}\left(\gamma^2\left(1-\gamma\right)^{n+k-1}-\gamma^{n+k+1}+\beta\left(\left(1-\gamma\right)^{n+k+1}-\gamma^2\left(1-\gamma\right)^{n+k-1}\right)\right)\end{aligned} \tag{8}$$

$$\begin{aligned}\hat{P}_B\left(k,a,b\right) =&\max_{x_B} 1-\beta\left(\frac{\gamma^n}{\gamma^n+\left(1-\gamma\right)^n}+\frac{\left(1-\gamma\right)^n}{\gamma^n+\left(1-\gamma\right)^n}\left(\frac{\left(1-\gamma\right)^{k+1}-\gamma^{k+1}}{\left(1-\gamma\right)^n-\gamma^n}\gamma^{n-k-1}\right)\right)\\ &-\left(1-\beta\right)\left(\frac{\gamma^n}{\gamma^n+\left(1-\gamma\right)^n}+\frac{\left(1-\gamma\right)^n}{\gamma^n+\left(1-\gamma\right)^n}\left(\frac{\left(1-\gamma\right)^{k-1}-\gamma^{k-1}}{\left(1-\gamma\right)^n-\gamma^n}\gamma^{n-k+1}\right)\right)\\ =&1-\min_{x_B}\frac{\gamma^{n-k-1}}{\left(1-\gamma\right)^{2n}-\gamma^{2n}}\left(\gamma^2\left(1-\gamma\right)^{n+k-1}-\gamma^{n+k+1}+\beta\left(\left(1-\gamma\right)^{n+k+1}-\gamma^2\left(1-\gamma\right)^{n+k-1}\right)\right)\end{aligned} \tag{9}$$

where $\beta = \frac{x_A^r}{x_A^r+x_B^r}$, $\gamma = \frac{(a-x_A)^r}{(a-x_A)^r+(b-x_B)^r}$.

To solve the optimization problems, introducing the following auxiliary function offers convenience.

$$\begin{aligned}F\left(x_A,x_B;a,b\right) =& \left(n-k-1\right)\ln\gamma-\ln\left(\left(1-\gamma\right)^{2n}-\gamma^{2n}\right)\\ &+\ln\left(\left(\gamma^2\left(1-\gamma\right)^{n+k-1}-\gamma^{n+k+1}+\beta\left(\left(1-\gamma\right)^{n+k+1}-\gamma^2\left(1-\gamma\right)^{n+k-1}\right)\right)\right)\end{aligned}$$

The first order partial derivative of the auxiliary function $F(x_A, x_B; a, b)$ has the same zeros as the first order derivative of $\hat{P}_A(k, a, b)$ and $\hat{P}_B(k, a, b)$ respectively. Thus, we can study the following equation sets instead.

$$\frac{\partial F(x_A, x_B; a, b)}{\partial x_A} = 0 \tag{10}$$

$$\frac{\partial F(x_A, x_B; a, b)}{\partial x_B} = 0 \tag{11}$$

Theorem 2. *Under assumption 1, the following is an equilibrium strategy for both players.*

$$x_A(k, a, b) = \begin{cases} \dfrac{(b^r - a^r)(b^{2nr} - a^{2nr})}{(a^r + b^r)\left((n-k)b^{2nr} + (n+k)a^{2nr} - 2nb^{(n-k)r}a^{(n+k)r}\right)}a, & a \neq b \\ \dfrac{1}{n^2 - k^2}a, & a = b \end{cases} \tag{12}$$

$$x_B(k, a, b) = \begin{cases} \dfrac{(b^r - a^r)(b^{2nr} - a^{2nr})}{(a^r + b^r)\left((n-k)b^{2nr} + (n+k)a^{2nr} - 2nb^{(n-k)r}a^{(n+k)r}\right)}b, & a \neq b \\ \dfrac{1}{n^2 - k^2}b, & a = b \end{cases} \tag{13}$$

The outcome presented in theorem 2 depicts the optimal strategies for players in any state (k, a, b). It reveals a pattern where effort tends to be lower when the score difference is comparatively small. This pattern will be formally and comprehensively discussed in the subsequent section.

Consider the boundary solutions now, and we can find that there is only one possible boundary solution, i.e. when the player is going to win the game with one more point, he will choose to allocate all his resources in this state. Given that player spends all his resource, the other player will also spend almost all his resource, leaving few enough resources to win the rest of the game.

This will happen if and only if

$$\frac{a^r}{a^r + b^r} > \frac{a^r(b^{(2n-1)r} - a^{(2n-1)r})}{b^{2nr} - a^{2nr}} \tag{14}$$

i.e.

$$\left(\frac{a^r}{b^r}\right)\left(\frac{a^r}{b^r} + 1\right)\left(\left(\frac{a^r}{b^r}\right)^{2n} - 1\right)\left(\left(\frac{a^r}{b^r}\right)^{2n} - \left(\frac{a^r}{b^r}\right)^2\right) < 0 \tag{15}$$

which is impossible.

We have shown that (i) all possible inner solutions must always satisfy the requirement $\frac{x_A}{a} = \frac{x_B}{b}$, (ii) given the conditions that $\frac{x_A}{a} = \frac{x_B}{b}$, the solution in theorem 2 is the unique solution, (iii) boundary solutions are impossible. Therefore, the uniqueness of the solution is guaranteed, with the form in theorem 2

4 Discussion

In this section, we discuss four important aspects of the problem. Initially, we delve into the second-order conditions, a critical factor about the solution's existence. Subsequently, we conduct comparative statics analyses and study the competitive intensity of the contest in the second section. In the third part, our attention turns to the expected periods before termination, offering valuable insights into explicitly illustrating equilibrium traits. The final part juxtaposes the resource allocation framework with the classical cost structure framework, presenting noteworthy intuitions.

4.1 Second order conditions

To ensure that the solution presented in theorem 2 is exactly the maximum of the objective function, it is essential to examine the second order conditions. If the second order conditions are satisfied for specific parameter values, then the solution presented in theorem 2 is an equilibrium of the problem. However, if the second order conditions are violated for certain parameter values, then there are no pure strategy equilibria.

Nevertheless, the second order conditions do not always satisfy the requirements to be a maximum. The following is a counter-example.

Case 1. *Consider a case where $a = \frac{1}{2}b$, $n = 100$, $r = 1$, $k = 99$:*

If the solution presented in theorem 2 is the equilibrium, then the ratio of resources that player A and B allocates is as follows.

$$\frac{(2-1)\left(2^{200}-1^{200}\right)}{(1+2)\left(2^{200}+199\times 1^{200}-400\times 2\times 1^{199}\right)} \approx \frac{1}{3}$$

The ultimate winning probability for player A is

$$\frac{2^{199}-1^{199}}{2^{200}-1^{200}} \approx \frac{1}{2}$$

However, there is another choice for player A. If player A allocates all resources to this stage, then the probability to win the competition in the current stage, which *also means player A becomes the ultimate winner, is*

$$\frac{1}{1+\frac{2}{3}} = \frac{3}{5} > \frac{1}{2}$$

Therefore this is a profitable deviation and player A has incentive to deviate from the equilibrium strategy.

In order to show the feasible region of the parameter values for the solution presented in theorem 2, we provide the sufficient and necessary condition of the second order conditions here.[1]

$$M\left(\frac{b^r}{a^r}, r, n, k\right)\left(\frac{b^r}{a^r}\right)^{n+k} + N\left(\frac{b^r}{a^r}, r, n, k\right) > 0. \tag{16}$$

By the one-shot principal in Hendon E, Jacobsen H J, Sloth B. 1996, we provide the simulation result for one-shot deviation to explicitly show the feasible region of the parameter values.

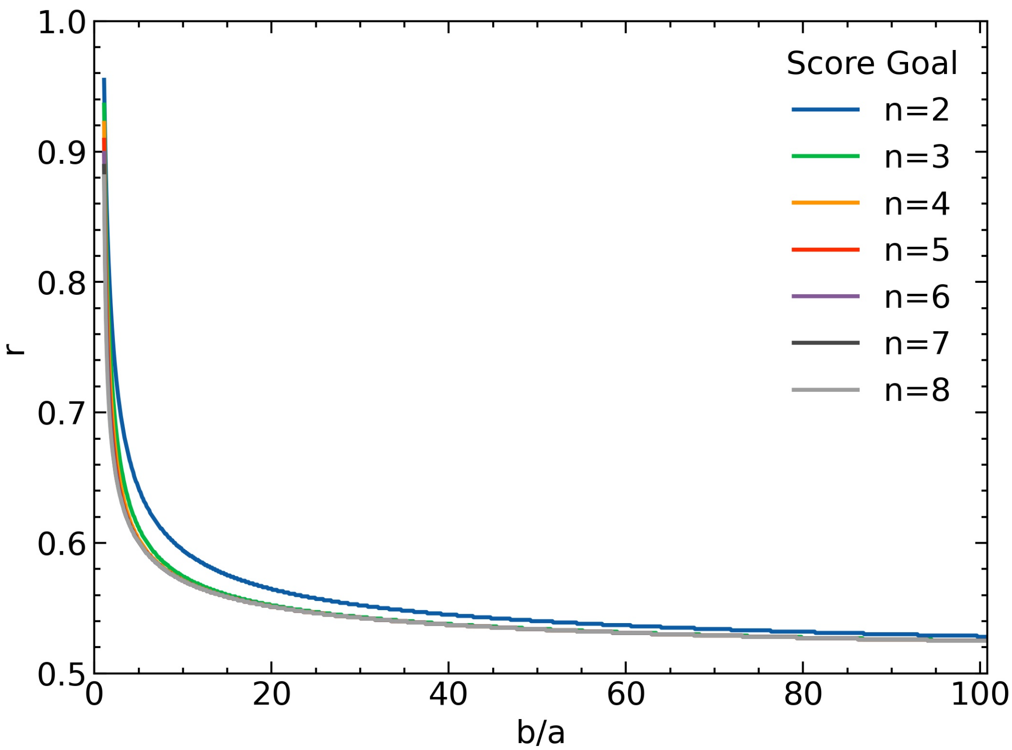

Figure 1: Simulation result for one-shot deviation.

Based on the simulation results, we conclude the following patterns.

[1]The specific forms of the functions are listed in Appendix.

Corollary 1. *1. Given n, a larger difference in resources necessitates a lower discriminatory power to ensure no profitable deviation.*

2. Given r, a higher required score difference necessitates a smaller difference in resources to ensure no profitable deviation.

3. Given b/a, a higher required score difference necessitates a lower discriminatory power to ensure no profitable deviation.

Furthermore, some conjectures can show that the result in theorem 2 is convincing and general.

1. When $b/a = 1$, $\forall r \in (0, 1]$, the equilibrium in theorem 2 always exists.

2. Even in extreme cases where $b/a \approx 100$ and $N = 9$, the range of r required to ensure no profitable deviation remains large, i.e., $r \leq \frac{1}{2}$.

4.2 Competition Intensity and Comparative Statics

Competition intensity is important for noncooperative competition, serving as a measurement of the players' motivation to become the winner in a stage. To be more specific, heightened competition intensity is indicative of a player's heightened motivation to become the winner. One method to measure competition intensity is the resource allocation ratio, denoted as $\frac{x_A}{a}$. This ratio is a useful indicator of the extent to which a player is willing to invest in their quest for scores.

The following empirical case illustrates performance patterns in a dynamic contest. Furthermore, we will conduct comparative statics analyses of the parameters k, r, $\frac{b}{a}$, and n to assess their respective impacts on the ultimate winning probability and competition intensity.

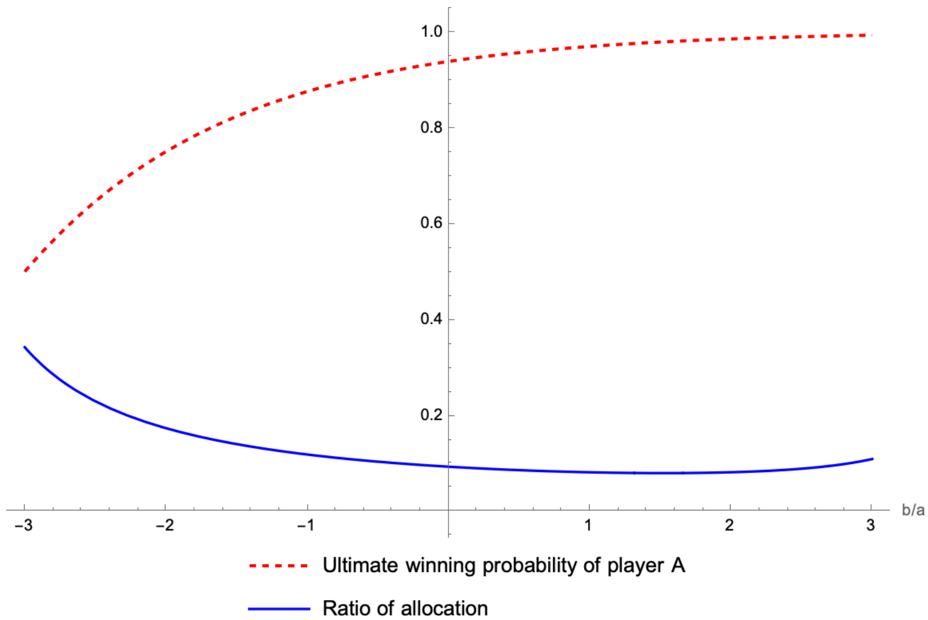

Figure 2: An empirical case under $n = 4$, $a/b = 2$, $r = 1$.

4.2.1 Comparative Statics about r

Note that in the equilibrium strategies and outcomes, parameter r only appears within expression $\frac{b^r}{a^r}$. Consequently, we can introduce the variable η as the modified resource ratio, defined as $\eta = \frac{b^r}{a^r}$. As the value of r increases, the modified resource ratio deviates from 1, signifying an amplification or mitigation in the impact of the resource difference. Furthermore, the influence of r on resource difference also provides a channel for social planners to adjust the resource difference between players.

Corollary 2. *For social planners, it is always available to select a feasible value of r^* to adjust the resource difference between players. Formally, considering the default discriminative parameter is r_0, if a social planner aims to have players with endowments a and b perform the same as those with a' and b', the choice of r^* should be as follows.*

$$r^* = \frac{\ln \frac{b}{a}}{\ln \frac{b'}{a'}} r_0 \tag{17}$$

Since the discriminative parameter can solely amplify or mitigate the impact of the resource difference, such method cannot reverse the resource advantage. Formally, the possible a^* and b^* must satisfy $(b - a)(b^* - a^*) > 0$ or $b - a = b^* - a^* = 0$.

4.2.2 Comparative Statics about k

k is the current score difference and is an important variable to measure the circumstance of the competition.

- **Ultimate winning probability**:

$$\hat{P}_A(k,a,b) = \begin{cases} \dfrac{\eta^{n+k}-1}{\eta^{2n}-1}, & \eta \neq 1 \\ \dfrac{n+k}{2n}, & \eta = 1 \end{cases} \tag{18}$$

It is evident that the ultimate winning probability strictly increases with an increase in the player's score. This outcome aligns with our intuition that a larger value of k implies a greater score superiority, which naturally translates into a higher ultimate winning probability.

- **Competition intensity**:

$$\frac{x_A(k,a,b)}{a} = \begin{cases} \dfrac{(\eta-1)(\eta^{2n}-1)}{(\eta+1)\left((n-k)\eta^{2n}+n+k-2n\eta^{n-k}\right)}, & \eta \neq 1 \\ \dfrac{1}{n^2-k^2}, & \eta = 1 \end{cases} \tag{19}$$

By calculating the partial derivative of $\frac{x_A(k,a,b)}{a}$, we can examine the impact of k on the competition intensity.

Corollary 3. *The competition intensity exhibits a pattern of initially decreasing, followed by an increase, as the scores increase. Notably, the competition intensity curve has a unique minimum point at $k(\eta) \in [-n,n]$.*

It is noteworthy that the value of k at the unique minimum point of competition intensity, named as "virtual tie" for short, may deviate from the actual tie, i.e. $k=0$.

Corollary 4. *The virtual tie, denoted as $k(\eta)$, will be situated on the negative side relative to the player with a greater endowment. Formally,*

1. *$\eta = 1$, $k(\eta) = 0$;*
2. *$\eta > 1$, $k(\eta) > 0$;*
3. *$\eta < 1$, $k(\eta) < 0$.*

The outcome in corollary 4 aligns with intuition, as when one player attains more resources, maintaining fairness, implied by the concept of the "virtual tie" , necessitates that the other player acquires additional scores as compensation.

Another reason for referring it as a "tie" is rooted in observations from symmetric resources case($a = b$), where players are impatient with large score difference and are most patient at the tie($k = 0$). The underlying reason for this pattern is that as the increase of the score difference, the whole contest is closer to its conclusion. Consequently, the future value of the resources shrinks, prompting both players to allocate more resources on current stage competition.

4.2.3 Comparative Statics about η

Modified resource ratio η measures the resource difference, taking the distortion of discriminative parameter into consideration. W.L.O.G., we assume $\eta \geq 1$ in section 4.2.3, in other words, player B has no fewer resources than player A. In the neighbourhood of $\eta = 1$, $\hat{P}_A$ and $\frac{x_A}{a}$ are both continuous, so we only need to focus on the case with $\eta > 1$.

- **Ultimate winning probability**:

$$\begin{aligned}\frac{\partial \hat{P}_B(k,\eta)}{\partial \eta} &= \frac{\eta^{n-1}\left(-2n\eta^n + (n+k)\eta^k + (n-k)\eta^{2n+k}\right)}{\left(\eta^{2n}-1\right)^2} \\ &= \frac{\eta^{n-1}\left(-2n\eta^n + 2n\eta^k\left(\frac{n+k}{2n}\eta^0 + \frac{n-k}{2n}\eta^{2n}\right)\right)}{\left(\eta^{2n}-1\right)^2} \\ &\geq \frac{\eta^{n-1}\left(-2n\eta^n + 2n\eta^k\eta^{\frac{n-k}{2n}\times 2n}\right)}{\left(\eta^{2n}-1\right)^2} \geq 0\end{aligned} \tag{20}$$

[2]

Consequently, a higher modified resource ratio will lead to a higher ultimate winning probability for the player with a resource advantage, as a higher modified resource ratio signifies a greater influence and power in the contest.

- **Competition intensity**:

$$\frac{\partial x_A(k,\eta)/a}{\partial \eta} = -\frac{(\eta-1)\eta^k\left(\left(\eta^{2n}-1\right)\left(\eta^k - 2\eta^n + \eta^{k+2n}\right) + 2n\eta^n\left(1+\eta^{2n}-2\eta^{n+k}\right)\ln(\eta)\right)}{(\eta+1)\left((n+k)\eta^k - 2n\eta^n + (n-k)\eta^{2n+k}\right)^2} < 0 \tag{21}$$

[2] Applying Jensen inequality to it

Therefore, a higher modified resource ratio leads to lower competition intensity. In other words, when both players have a balanced level of power, they will compete intensively because they are both confident about their chances of becoming the ultimate winner and, therefore, they invest more effort in the contest.

4.2.4 Comparative Statics about n

As the parameter n varies, the volume of the state set also varies. In order to make two cases with different goals where $n \neq n'$ comparable, it is essential to build a mapping from states in the case with n to states in the case with n'. Two possible mapping methods are available: one is to link states with the same score superiority, while the other is to link states with the same score needed to become the ultimate winner. W.L.O.G., we assume $\eta \geq 1$ in section 4.2.4, in other words, player B has no fewer resources than player A.

1. For $n < n'$, we map (k, a, b, n) to (k, a, b, n'). Note that two linked states share the same k, which means the score superiority of player A is the same. It is evident that when $\eta = 1$, the ultimate winning probability and competition intensity decrease with n. When $\eta > 1$, we discuss the problem in details in the following paragraph.

   - **Ultimate winning probability**:

$$\frac{\partial \hat{P}_A(n,k,a,b)}{\partial n} = \frac{\eta^n \ln \eta \left(-\eta^{2n+k} + 2\eta^n - \eta^k\right)}{\left(\eta^{2n} - 1\right)^2} \tag{22}$$

   (a) $k > 0$

$$\frac{\eta^n \ln \eta \left(-\eta^{2n+k} + 2\eta^n - \eta^k\right)}{\left(\eta^{2n} - 1\right)^2} < \frac{\eta^n \ln \eta \left(-2\eta^k \eta^n + 2\eta^n\right)}{\left(\eta^{2n} - 1\right)^2} < 0 \tag{23}$$

   (b) $k < 0$

$$\frac{\partial \left(-\eta^{2n+k} + 2\eta^n - \eta^k\right)}{\partial n} = 2\eta^n \left(1 - \eta^{n+k}\right) \ln \eta < 0 \tag{24}$$

$$\frac{\partial \hat{P}_A(n,k,a,b)}{\partial n}\Big|_{k=-n} = \frac{\eta^n \ln \eta \left(\eta^n - \eta^{-n}\right)}{\left(\eta^{2n} - 1\right)^2} > 0 \tag{25}$$

$$\frac{\partial \hat{P}_A(n,k,a,b)}{\partial n}\Big|_{k=0} = \frac{\eta^n \ln\eta\left(-\eta^{2n}+2\eta^n-1\right)}{\left(\eta^{2n}-1\right)^2} < \frac{\eta^n \ln\eta\left(-2\eta^n+2\eta^n\right)}{\left(\eta^{2n}-1\right)^2} < 0 \tag{26}$$

In all, $\exists \hat{k} \in [-n, 0)$, s.t. $\forall k < \hat{k}$, $\frac{\partial \hat{P}_A(n,k,a,b)}{\partial n} > 0$, $\forall k > \hat{k}$, $\frac{\partial \hat{P}_A(n,k,a,b)}{\partial n} < 0$. In other words, given the same score superiority, an increase in the required scores tends to reduce the probability of the player with the fewer resources becoming the ultimate winner, except in cases of extreme resource inferiority. The influence of an increase in the required scores can be separated into two aspects that contribute to the observed pattern: 1. Higher required scores works against the potential for a dark horse victory. 2. An increase in the required scores delays the forthcoming failure, offering an opportunity for a comeback. When k is low enough, i.e. the player with resource inferiority is close enough to the ultimate victory, the former factor becomes significant, while the latter factor dominates the contest otherwise.

- **Competition intensity**:

$$\frac{\partial \frac{x_A(n,k,a,b)}{a}}{\partial n} < 0 \tag{27}$$

When the required score increases while maintaining the same score superiority, the competition intensity decreases. For any given k, an increase in n suggests that the contest is further from its conclusion. Consequently, both players tend to reduce their competition intensity for future stages of the competition.

2. For $n < n'$, $k > 0$, we map (k, a, b, n) to $(n' - n + k, a, b, n')$ and for $k < 0$, we map (k, a, b, n) to $(-n' + n + k, a, b, n')$. Note that there is an implicit same variable between two states that $n - k = n' - (n' - n + k)$ for $k > 0$ and $-n - k = -n' - (-n' + n + k)$, which means the scores needed for player with scores superiority to become the ultimate winner is the same. It is evident that when $\eta = 1$, the ultimate winning probability of the player with scores superiority and competition intensity decrease with n. When $\eta > 1$, we discuss the problem in details in the following paragraph.

- **Ultimate winning probability**:

(a) $k > 0$

Rewrite the winning probability applying $m = n - k = n' - (n' - n + k)$ to it:

$$\hat{P}_A(n, m, a, b) = \frac{\eta^{2n-m} - 1}{\eta^{2n} - 1} \tag{28}$$

$$\frac{\partial \hat{P}_A(n, m, a, b)}{\partial n} = \frac{2\eta^{2n-m} \ln \eta \left(\eta^m - 1\right)}{\left(\eta^{2n} - 1\right)^2} > 0 \tag{29}$$

(b) $k < 0$

Rewrite the winning probability applying $m = -n - k = -n' - (-n' + n + k)$ to them:

$$\hat{P}_B(n, m, a, b) = \frac{\eta^{2n} - \eta^{-m}}{\eta^{2n} - 1} \tag{30}$$

$$\frac{\partial \hat{P}_B(n, m, a, b)}{\partial n} = \frac{2\eta^{2n-m} \left(1 - \eta^m\right) \ln \eta}{\left(\eta^{2n} - 1\right)^2} > 0 \tag{31}$$

In all, when required score increases while maintaining the same score needed to be the ultimate winner, the probability of the player with score superiority becoming the ultimate winner increases. In this part, the influence of delaying the forthcoming failure is controlled, and the sole effect of an increase in the required scores is reducing the chances of a dark horse victory, meaning that the player with score superiority is more likely to secure the ultimate victory.

- **Competition intensity**:

(a) $k > 0$

Rewrite the competition intensity applying $m = n - k = n' - (n' - n + k)$ to it:

$$\frac{x_A(n, m, a, b)}{a} = \frac{(\eta - 1)\left(\eta^{2n} - 1\right)}{(\eta + 1)\left(m\eta^{2n} + 2n - m - 2n\eta^m\right)} \tag{32}$$

$$\frac{\partial \frac{x_A(n, m, a, b)}{a}}{\partial n} = -\frac{2\left(\eta^m - 1\right)\left(\left(2n \ln \eta - 1\right)\eta^{2n} + 1\right)}{\left(m\eta^{2n} - 2n\eta^m + 2n - m\right)^2} \tag{33}$$

Let $g(n, \eta) = (2n \ln \eta - 1)\eta^{2n} + 1$, $g(n, 1) = 0$.

$$\frac{\partial g(n, \eta)}{\partial \eta} = 4n^2 \eta^{2n-1} \ln \eta > 0 \tag{34}$$

thus $g(n,\eta) > 0, \forall \eta > 1$, and then $\frac{\partial \frac{x_A(n,m,a,b)}{a}}{\partial n} < 0$.

(b) $k < 0$

Rewrite the competition intensity applying $m = -n - k = -n' - (-n' + n + k)$ to them:

$$\frac{x_A(n,m,a,b)}{a} = \frac{(\eta - 1)(\eta^{2n} - 1)}{(\eta + 1)((2n + m)\eta^{2n} - m - 2n\eta^{2n+m})} \tag{35}$$

$$\frac{\partial \frac{x_A(n,m,a,b)}{a}}{\partial n} = -\frac{2(\eta - 1)\eta^{2n}(1 - \eta^m)(\eta^{2n} - 1 - 2n\ln\eta)}{(\eta + 1)(m\eta^{2n} + 2n(1 - \eta^m)\eta^{2n} - m)^2} \tag{36}$$

Let $g(n,\eta) = \eta^{2n} - 1 - 2n\ln\eta,\ g(n,1) = 0$.

$$\frac{\partial g(n,\eta)}{\partial \eta} = 2n\left(\eta^{2n} - 1\right)\eta^{-1} > 0 \tag{37}$$

thus $g(n,\eta) > 0, \forall \eta > 1$, and then $\frac{\partial \frac{x_A(n,m,a,b)}{a}}{\partial n} < 0$.

When required score increases while maintaining the same score needed to be the ultimate winner, the competition intensity decreases because the contest is further from its conclusion.

4.3 Expected periods before termination

Another problem we may pay attention to is the expected periods before termination, which is another dimension to weigh the intensity of the game.

Denote the expected periods before termination at score state k by T_k, and the following relationships is natural.

$$T_{n-k} = 1 + P_{n-k}T_{n-k+1} + (1 - P_{n-k})T_{n-k-1} \tag{38}$$

$$T_n = T_{-n} = 0 \tag{39}$$

Thus the matrix form of the equation sets is as follows.

$$\begin{pmatrix} 1 & -P_{-n+1} & 0 & 0 & 0 & \cdots & 0 \\ -(1 - P_{-n+2}) & 1 & -P_{-n+2} & 0 & 0 & \cdots & 0 \\ 0 & -(1 - P_{-n+3}) & 1 & -P_{-n+3} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -(1 - P_{n-3}) & 1 & -P_{n-3} & 0 \\ 0 & \cdots & 0 & 0 & -(1 - P_{n-2}) & 1 & -P_{n-2} \\ 0 & \cdots & 0 & 0 & 0 & -(1 - P_{n-1}) & 1 \end{pmatrix} \times \begin{pmatrix} T_{-n+1} \\ T_{-n+2} \\ T_{-n+3} \\ \cdots \\ T_{n-3} \\ T_{n-2} \\ T_{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \cdots \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{40}$$

where $P_k = \frac{\eta^{n+k} - 1}{\eta^{2n} - 1}$.

The recursive relationships are about three adjacent states which is difficult to solve, thus we consider denoting $x_i = T_{i-1} - T_i$, $i \in [-n+1, n]$ to simplify the problem.

$$P_{n-k} x_{n-k+1} + (1 - P_{n-k}) x_{n-k} = 1 \tag{41}$$

$$\sum_{i=-n+1}^{n} x_i = 0 \tag{42}$$

$$\begin{pmatrix} 1 & \dots & 1 & 1 & 1 & 1 & 1 \\ 1-P_{-n+1} & P_{-n+1} & 0 & 0 & 0 & \dots & 0 \\ 0 & 1-P_{-n+2} & P_{-n+2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 1-P_{-n+3} & P_{-n+3} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & (1-P_{n-3}) & P_{n-3} & 0 & 0 \\ 0 & \dots & 0 & 0 & (1-P_{n-2}) & P_{n-2} & 0 \\ 0 & \dots & 0 & 0 & 0 & (1-P_{n-1}) & P_{n-1} \end{pmatrix} \times \begin{pmatrix} x_{-n+1} \\ x_{-n+2} \\ x_{-n+3} \\ \dots \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \dots \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \tag{43}$$

In order to eliminate the constant term on the right hand side, we denote $y_i = 1 - x_i$ to further simplify the problem.

$$P_{n-k} y_{n-k+1} + (1 - P_{n-k}) y_{n-k} = 0 \tag{44}$$

$$\sum_{i=-n+1}^{n} y_i = 2n - 1 \tag{45}$$

i.e.

$$-y_{-n+1}\left(1 + \frac{1-P_{-n+1}}{P_{-n+1}} + \frac{1-P_{-n+1}}{P_{-n+1}} \frac{1-P_{-n+2}}{P_{-n+2}} + \ldots\right) = 2n - 1 \tag{46}$$

$$-y_{-n+1}\left(1 + \frac{\eta^{2n} - \eta}{\eta - 1} + \frac{\eta^{2n} - \eta}{\eta - 1} \frac{\eta^{2n} - \eta^2}{\eta^2 - 1} + \ldots\right) = 2n - 1 \tag{47}$$

Notice that $T_0 = \sum_{i=-n+1}^{0} x_i = n - \sum_{i=-n+1}^{0} y_i$.

$$T_0 = n + (2n-1) \frac{\sum_{i=0}^{n-1} \prod_{j=1}^{i} \frac{\eta^{2n} - \eta^j}{\eta^j - 1}}{\sum_{i=0}^{2n-1} \prod_{j=1}^{i} \frac{\eta^{2n} - \eta^j}{\eta^j - 1}} \tag{48}$$

After that, we analyze the characteristics of T_0.

Theorem 3.

- $\lim_{\eta \to \infty} T_0 = \lim_{\eta \to 0} T_0 = n$, $T_0|_{\eta=1} = 2n - \frac{1}{2}$.
- *T_0 increases with the increase of η when $\eta < 1$, decreases when $\eta > 1$.*
- *T_0 increases with the increase of n.*

Theorem 3 demonstrates the characteristics of expected periods before termination: it increases with the increase of the goals n and decreases with the increase of the moderated resource ratio η. Besides, it has a maximum with $2n - \frac{1}{2}$ when $\eta = 1$ and converge to n when η approaches to infinity. Therefore, if we want to increase the intensity of the contest, i.e. increase the expected periods before termination, we should set a high goal and narrow the resource gap between players, which is consistent with the conclusions in the analysis of $\frac{x_A}{a}$.

4.4 Resource Allocation versus Cost Structure

As mentioned previously, the Tug-of-war contest has been well studied via a cost structure framework in the existing literature, making it natural to compare our results under the resource allocation framework with those under the cost structure framework. In Konrad&Kovenock(2005), they demonstrate that the unique subgame perfect Markov equilibrium for an all-pay auction in each period is to have a strictly positive effort distribution on j_0 and $j_0 - 1$, where j_0 is the cutoff to change the advantaged player, whose discounted prize value is higher, while pay no effort otherwise.

The difference between the results under the cost structure framework and the resource allocation framework in Tug-of-War can be attributed to the choice of tie-breaking rule. In Konrad&Kovenock(2005), the rule was to let the advantaged player win in case of a tie. Consequently, once a player becomes the advantaged player, maintaining their effort no less than their opponent is a straightforward and effective strategy to secure victory. This results in a positive feedback system where the competition is focused on the advantaged player position, and the other player is likely to give up paying effort once they fall behind. Even the slightest deviation from the "tipping state" prompts both players to immediately give up paying effort to change the game, sitting by and watching the game comes to the end by inertia.

Another important work on tug-of-war contest is Karagozoglu&Saglam&Turan(2021) which provides a closed-form solution for a lottery auction in each period under quadratic cost structure. The following is an example where the ultimate winner receives 1 unit, and 0 otherwise in a win-by-3 tug-of-war contest. To facilitate comparison, we present the results under the resource allocation framework in symmetric case($a = b$). It is important to note that there is no comparability between two rows because, in the first row, the numbers represent the exact effort both players will exert, whereas in the second row, the numbers represent the ratio of their present resources that both players will allocate. Paying attention to the variation tendency within each rows may provide more meaningful insights.

Under resource allocation framework, the game is in a different shape where players allocate more of their resources when the advantage/disadvantage is greater. This aligns with intuition, as a larger advantage/disadvantage signifies that the contest

Framework \ Score	-2	-1	0	1	2
Effort Cost	0.311	0.3681	0.335	0.234	0.125
Resource Allocation	0.200	0.125	0.111	0.125	0.200

Table 1: Effort/Resource allocation in a win-by-3 tug-of-war contest.

is closer to its conclusion, and the future value of the remaining resources shrinks. This reflects the real-world scenario where comebacks are not uncommon, and players never give up pursuing the victory.

The difference between the two frameworks, effort cost and resource allocation, arises from their different attitudes towards effort/resources. Under the effort cost framework, expending more effort leads to negative payoffs, and as the contest approaches its conclusion, players are more likely to cut their losses and stop expending effort. In contrast, under the resource allocation framework, any unspent resources become worthless, once the ultimate winner is determined. Consequently, players tend to allocate their remaining resources in a manner that enhances their prospects of winning as the contest draws to a close.

5 Extensions

In this section, our benchmark model is extended to analyze several scenarios that appear in the real life frequently.

5.1 Minimum score requirement

A natural extension involves the introduction of a minimum score requirement, inspired by games like badminton, where a player cannot win until their total score surpasses a predetermined threshold. In the following paragraphs, we will discuss the details of this extension and explore its implications.

Now stage states must be characterized by four variables

$$s \equiv (k, a, b, K) \tag{49}$$

where K is the higher score between player A and B. $K \in [0, \bar{K}], \bar{K} > n$, where $\bar{K}$ is the predetermined threshold, above which a player can become the ultimate winner.

By lemma 1, $\frac{x_A}{a} = \frac{x_B}{b}$ is still satisfied under assumption 1, because we can extend the state transition function to one with more dimensions.

Take the probability for player A for display, and denote $P_{x,y} = \hat{P}_A\left(x, a, b, y\right)$ for short. Then the following matrix as an extension of benchmark model.

$$\mathbb{L} \equiv \left(l_{i,j}\right)|_{\bar{K}+1,2\bar{K}+1} = \begin{pmatrix} P_{-\bar{K},\bar{K}} & \cdots & P_{\bar{K},\bar{K}} \\ \vdots & \ddots & \vdots \\ P_{-\bar{K},0} & \cdots & P_{\bar{K},0} \end{pmatrix} \tag{50}$$

We start by examining a special part of this problem that $s = \left(k, a, b, \bar{K}\right)$, where $-n < k < n$. It is important to note that, starting from the state $\left(k, a, b, \bar{K}\right)$ on, we no longer need to worry about the minimum score requirement because, once satisfied, it will continue to be satisfied. Consequently, the probabilities should remain the same as that without minimum score requirement, i.e. $l_{i,j} = \frac{\eta^{n+j-\bar{K}+1}-1}{\eta^{2n}-1}, i = 1, j \in [\bar{K}-n, n+\bar{K}-2]$.

Another special part of this problem is $s = (k, a, b, K)$, where either $k > K > 0$ or $k < K < 0$, which means the score difference is strictly higher than the higher score between player A and B. However, this scenario can never occur, because if one player has k more scores than the other, then they must have at least k scores, which in turn means the higher scores between the two players also exceeds k. To make the matrix tidy, we assign a placeholder value of -1 to $l_{i,j}$ where $i > j$ and $i > 2\bar{K} + 2 - j$, $\forall j \in [1, 2\bar{K} + 1]$.

After that, we consider the part $\hat{P}_A\left(k, a, b, \bar{K}\right)$, where $-\bar{K} + 1 \leq k \leq -n$ and $\bar{K} - 1 \geq k \geq n$. In this part, the ultimate winner is already determined since the minimum score requirement is satisfied. Formally, $l_{i,j} = 0$, for $i = 1, j \in [1, \bar{K}-n-1]$, and $l_{i,j} = 1$, for $i = 1, j \in [n + \bar{K} - 1, 2\bar{K} + 1]$.

Then, the rest elements can be written out recursively, i.e. $l_{i,j} = \frac{1}{1+\eta} l_{i,j+1} + \frac{\eta}{1+\eta} l_{i-1,j-1}$, for $2 \leq i \leq j$, $2 \leq j \leq \bar{K} + 1$, and $l_{i,j} = \frac{1}{1+\eta} l_{i-1,j+1} + \frac{\eta}{1+\eta} l_{i,j-1}$, for $2 \leq i \leq -j + 2\bar{K} + 2$, $\bar{K} + 1 \leq j \leq 2\bar{K}$.

In summary, the expression of the probability matrix as described in equation 50 is as follows.

$$
l_{i,j}=\begin{cases}
\dfrac{\eta^{n+j-\bar{K}+1}-1}{\eta^{2n}-1}, & i=1, j\in[\bar{K}-n,n+\bar{K}-2]\\
-1, & i>j \text{ and } i>2\bar{K}+2-j, j\in[1,2\bar{K}+1]\\
0, & i=1, j\in[1,\bar{K}-n-1]\\
1, & i=1, j\in[n+\bar{K}-1,2\bar{K}+1]\\
\dfrac{1}{1+\eta}l_{i,j+1}+\dfrac{\eta}{1+\eta}l_{i-1,j-1}, & 2\le i\le j, 2\le j\le\bar{K}\\
\dfrac{1}{1+\eta}l_{i-1,j+1}+\dfrac{\eta}{1+\eta}l_{i,j-1}, & 2\le i\le -j+2\bar{K}+2, \bar{K}+2\le j\le 2\bar{K}\\
\dfrac{1}{1+\eta}l_{i-1,j+1}+\dfrac{\eta}{1+\eta}l_{i-1,j-1}, & 2\le i, j=\bar{K}+1
\end{cases}
\tag{51}
$$

Here is a numerical example with $n=2,\bar{K}=4$. The winning probability matrix of player A is as follows.

$$
\mathbb{L}=\Big(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6,\phi_7,\phi_8,\phi_9\Big) \tag{52}
$$

where

$$
(\phi_1,\phi_2,\phi_3)=\begin{pmatrix}
0, & 0,0\\
-1, & \dfrac{\eta^2-1}{(\eta^4-1)(1+\eta)^3},\dfrac{\eta^2-1}{(\eta^4-1)(1+\eta)^2}\\
-1, & -1,\dfrac{\eta^2+4\eta+1}{(\eta+1)^4(\eta^2+1)},\\
-1, & -1,-1\\
-1, & -1,-1
\end{pmatrix} \tag{53}
$$

$$
(\phi_4,\phi_5,\phi_6)=\begin{pmatrix}
\dfrac{\eta-1}{\eta^4-1}, & \dfrac{\eta^2-1}{\eta^4-1},\dfrac{\eta^3-1}{\eta^4-1}\\
\dfrac{\eta^2-1}{(\eta^4-1)(1+\eta)}, & \dfrac{\eta^2-1}{\eta^4-1},\dfrac{\eta^2+\eta+1}{\eta^3+\eta^2+\eta+1}\\
\dfrac{\eta^2+3\eta+1}{(\eta+1)^3(\eta^2+1)}, & \dfrac{\eta^2-1}{\eta^4-1},\dfrac{3\eta^3+4\eta^2+3\eta+1}{(\eta+1)^3(\eta^2+1)}\\
\dfrac{5\eta^3+11\eta^2+5\eta+1}{(\eta+1)^5(\eta^2+1)}, & \dfrac{4\eta^2+3\eta+1}{(\eta+1)^3(\eta^2+1)},\dfrac{(2\eta+1)(5\eta^3+5\eta^2+3\eta+1)}{(\eta+1)^5(\eta^2+1)}\\
-1, & \dfrac{(3\eta+1)(5\eta^2+2\eta+1)}{(\eta+1)^5(\eta^2+1)},-1
\end{pmatrix} \tag{54}
$$

$$
(\phi_7,\phi_8,\phi_9) = \begin{pmatrix}
1, & 1{,}1 \\
\dfrac{2\eta^3+2\eta^2+2\eta+1}{(\eta+1)^2(\eta^2+1)}, & \dfrac{3\eta^4+4\eta^3+4\eta^2+3\eta+1}{(\eta+1)^3(\eta^2+1)}, -1 \\
\dfrac{6\eta^4+8\eta^3+7\eta^2+4\eta+1}{(\eta+1)^4(\eta^2+1)}, & -1,-1 \\
-1, & -1,-1 \\
-1, & -1,-1
\end{pmatrix} \tag{55}
$$

The strategy matrix of $\frac{x_A}{a}$ is as follows.

$$
\mathbb{S} = \Big(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8,\psi_9\Big) \tag{56}
$$

where

$$
(\psi_1,\psi_2,\psi_3) = \begin{pmatrix}
0, & 0{,}0 \\
-1, & \dfrac{\eta^2+1}{5\eta^2+2\eta+3}, \dfrac{\eta^2+1}{4\eta^2+2\eta+2} \\
-1, & -1, \dfrac{\eta^3+3\eta^2+\eta+3}{4\eta^3+20\eta^2+12\eta+12} \\
-1, & -1,-1 \\
-1, & -1,-1
\end{pmatrix} \tag{57}
$$

$$
(\psi_4,\psi_5,\psi_6) = \begin{pmatrix}
\dfrac{\eta^2+1}{3\eta^2+2\eta+1}, & \dfrac{\eta^2+1}{2(\eta+1)^2}, \dfrac{\eta^2+1}{\eta^2+2\eta+3} \\
\dfrac{\eta^2+1}{3\eta^2+2\eta+1}, & \dfrac{\eta^2+1}{2(\eta+1)^2}, \dfrac{\eta^2+1}{\eta^2+2\eta+3} \\
\dfrac{\eta^3+2\eta^2+\eta+2}{3\eta^3+12\eta^2+9\eta+6}, & \dfrac{\eta^2+1}{2(\eta+1)^2}, \dfrac{2\eta^3+\eta^2+2\eta+1}{6\eta^3+9\eta^2+12\eta+3} \\
\dfrac{2\eta^3+3\eta^2+2\eta+3}{10\eta^3+25\eta^2+20\eta+15}, & \dfrac{\eta^2+1}{4\eta^2+4\eta+4}, \dfrac{3\eta^3+2\eta^2+3\eta+2}{15\eta^3+20\eta^2+25\eta+10} \\
-1, & \dfrac{\eta^2+1}{6\eta^2+4\eta+6}, -1
\end{pmatrix} \tag{58}
$$

$$(\psi_7,\psi_8,\psi_9)=\begin{pmatrix}0, & 0{,}0\\ \dfrac{\eta^2+1}{2\eta^2+2\eta+4}, & \dfrac{\eta^2+1}{3\eta^2+2\eta+5},-1\\ \dfrac{3\eta^3+\eta^2+3\eta+1}{12\eta^3+12\eta^2+20\eta+4}, & -1,-1\\ -1, & -1,-1\\ -1, & -1,-1\end{pmatrix} \tag{59}$$

The special case with $\eta=1$ is as follows.

$$\mathbb{S}=\begin{pmatrix}0, & 0{,}0, & \frac{1}{3},\frac{1}{4}, & \frac{1}{3},0, & 0{,}0\\ -1, & \frac{1}{5},\frac{1}{4}, & \frac{1}{3},\frac{1}{4}, & \frac{1}{3},\frac{1}{4}, & \frac{1}{5},-1\\ -1, & -1,\frac{1}{6}, & \frac{1}{5},\frac{1}{4}, & \frac{1}{5},\frac{1}{6}, & -1,-1\\ -1, & -1,-1, & \frac{1}{7},\frac{1}{6}, & \frac{1}{7},-1, & -1,-1\\ -1, & -1,-1, & -1,\frac{1}{8}, & -1,-1, & -1,-1\end{pmatrix} \tag{60}$$

The following is the winning probability heat graph of a numeric example with $n=2,\bar{K}=10,r=1,a=1,b=2$.

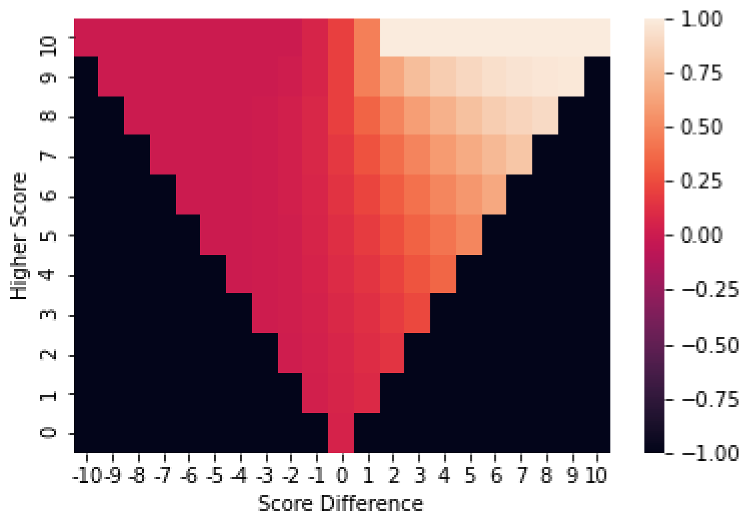

Figure 3: Winning probability for player A with minimum score requirement.

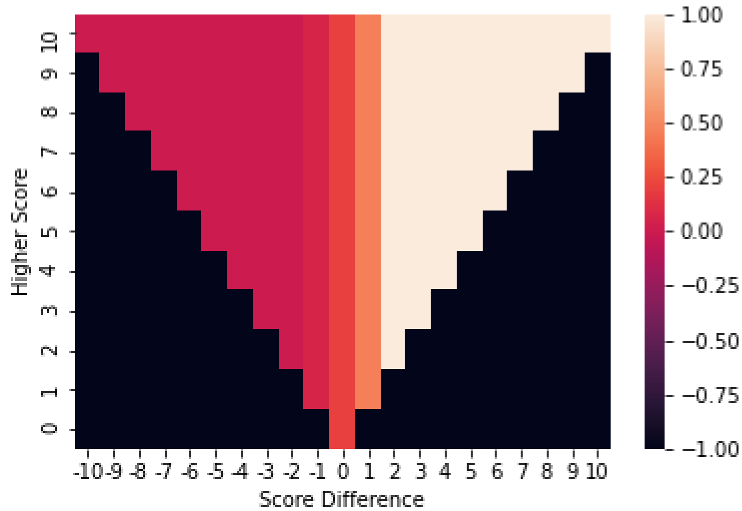

Figure 4: Winning probability for player A without minimum score requirement.

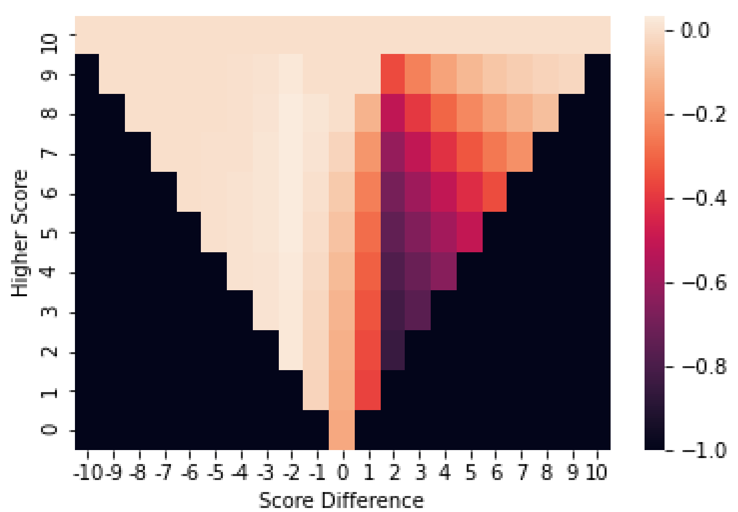

Figure 5: Difference of winning probability for player A between case with minimum score requirement and case without.

The following is the winning probability heat graph of a numeric example with $n = 2, \bar{K} = 4, r = 1, a = 1, b = 2$.

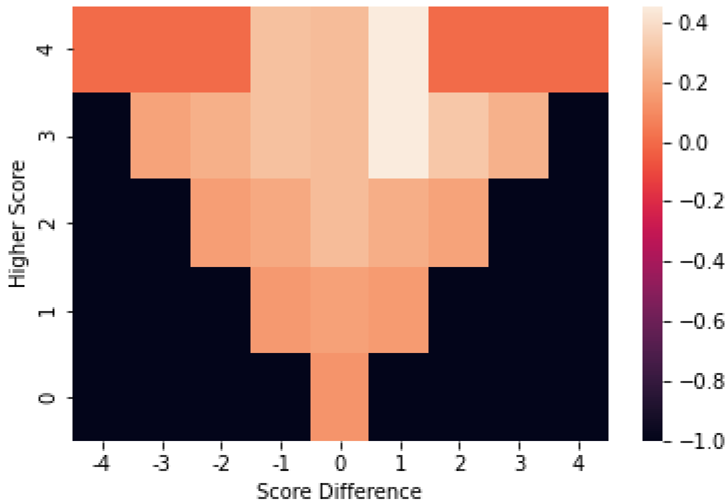

Figure 6: Strategy with minimum score requirement.

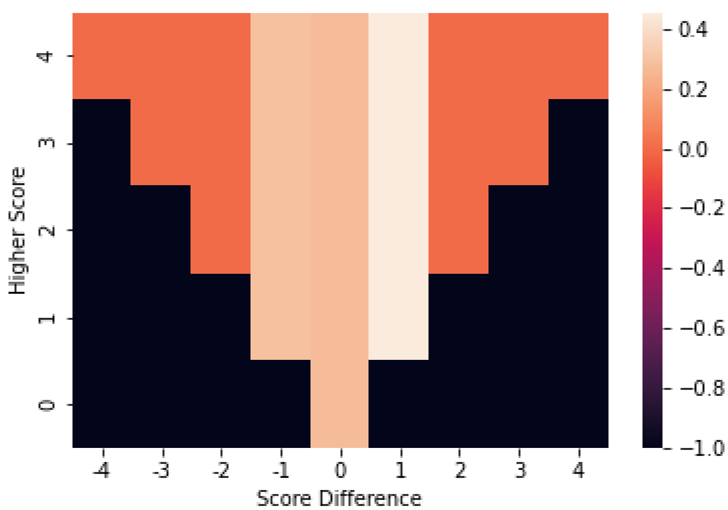

Figure 7: Strategy without minimum score requirement.

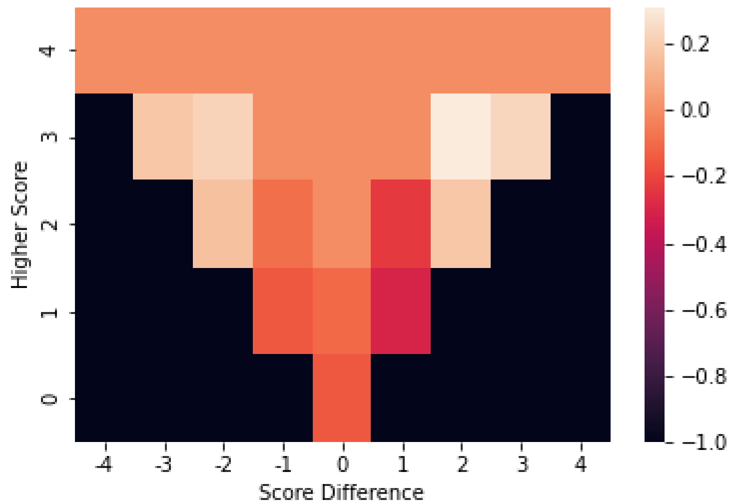

Figure 8: Difference of strategy between case with minimum score requirement and case without.

From the perspective of strategy, the change from introducing minimum score requirement seems chaotic, because the relationships among states change. In short, $2:1$ and $1:0$ in the case with minimum score requirement are both corresponded to $1:0$ in the case without, however, these two states are quite different, because the successors of the former is $3:1$ and $2:2$ while those of the latter is $2:0$ and $1:1$.

From the perspective of winning probabilities, significant patterns is clear. It can be seen that for the player with fewer resource, the imposition of a minimum score requirement diminishes their chances of winning, particularly when they has an advantage in scores. This results in a reduced probability of black horse victory in the competition, making it more probable that the player with a higher endowment will become the ultimate winner. Therefore, the utilization of a minimum score requirement can serve as a useful tool in screening for high-quality participants.

5.2 Asymmetric score goal

Another common scenario arises in contests where the players have different goal. For instance, in a local war, the attacker aims to take control of the entire region, while the defender's goal is to safeguard certain crucial sites and wait for reinforcements. Formally, the scores required to be the ultimate winner for A, B are n_A, n_B respectively, i.e. $\hat{\hat{P}}_A(n_A, \cdot, \cdot) = 1, \hat{\hat{P}}_A(-n_B, \cdot, \cdot) = 0$.

To address the problem, we need to partition the problem into two cases based on whether the sum of n_A and n_B is odd or even.

1. $n_A + n_B$ **is even**:

   The state set is $\tilde{\mathbb{S}} = \tilde{\mathbb{K}} \times \mathbb{R}_+^2$, where $\tilde{\mathbb{K}} = \mathbb{Z} \cap [-n_B, n_A]$. We construct another state set $\mathbb{S} \equiv \mathbb{K} \times \mathbb{R}_+^2$, $\mathbb{K} = \mathbb{Z} \cap [-n, n]$, where $n = \frac{n_A+n_B}{2}$, and then we can construct a bijection from $\tilde{\mathbb{S}}$ to $\mathbb{S}$,

$$\begin{aligned} f : \tilde{\mathbb{S}} &\to \mathbb{S} \\ k &\mapsto f(k), \end{aligned}$$

   where $f(k) = n + k - n_A$. The state $(k, a, b) \in \tilde{\mathbb{S}}$ in the asymmetric score goal cases and $(f(k), a, b) \in \mathbb{S}$ in the symmetric score goal cases share the same state transition relationships in the bijection f sense. Consequently, the optimization problems remain the same in the bijection f sense, and therefore the final solutions also remain unchanged, i.e. $\{\hat{\tilde{P}}_i(k, a, b)\} = \{\hat{P}_i(n + k - n_A, a, b)\}$.

2. $n_A + n_B$ **is odd**:

   The approach employed in the previous section is no longer applicable because, even though we can construct a similar state set $\mathbb{S}$ and a similar bijection f, the state transition relationships are different between $\tilde{\mathbb{S}}$ to $\mathbb{S}$.

   However, if we focus on the essence of the state set, we can adopt an alternative approach. In fact, the parameter k represents the difference in scores, which means the actual score values are irrelevant. Therefore, we can "shift " k leftward or rightward to construct another k' that still represents the score difference. Denote $\mathbb{S} = \mathbb{K}' \times \mathbb{R}_+^2, \mathbb{K}' = \mathbb{Z} \cap [0, 2n-1]$, where $n = \frac{n_A+n_B+1}{2}$. Here $k' \in \mathbb{K}'$ has a new interpretation, signifying the number of scores that player B needs to become the ultimate winner. The following theorem demonstrates the equilibrium in asymmetric score goal case.

   Theorem 4. *In asymmetric score goal case, the ultimate winning probabilities are as follows.*

$$\hat{P}'_A(k, a, b) = \begin{cases} \dfrac{a^{(n_A+n_B-k')r}(b^{k'r} - a^{k'r})}{b^{(n_A+n_B)r} - a^{(n_A+n_B)r}}, & k' \in [0, n_A + n_B], a \neq b \\ \dfrac{k'}{n_A + n_B}, & k' \in [0, n_A + n_B], a = b \end{cases} \tag{61}$$

In asymmetric score goal case, the equilibrium strategies are as follows.

$$x'_A\left(k',a,b\right)=\begin{cases}\dfrac{\left(b^r-a^r\right)\left(b^{(n_A+n_B)r}-a^{(n_A+n_B)r}\right)}{\left(a^r+b^r\right)\left(\left(n_A+n_B-k'\right)b^{(n_A+n_B)r}+k'a^{(n_A+n_B)r}-\left(n_A+n_B\right)b^{(n_A+n_B-k')r}a^{k'r}\right)}a, & a\neq b\\ \dfrac{1}{k'\left(n_A+n_B-k'\right)}a, & a=b\end{cases} \tag{62}$$

If we compare the results above, we can find that the form of the equilibrium is exactly the same, no matter n_A+n_B is even or not and no matter $n_A=n_B$ or not.

Theorem 5. *In both symmetric and asymmetric score goal cases, the ultimate winning probabilities are as follows.*

$$\hat{P}'_A(k,a,b)=\begin{cases}\dfrac{a^{(n_A+n_B-k')r}(b^{k'r}-a^{k'r})}{b^{(n_A+n_B)r}-a^{(n_A+n_B)r}}, & k'\in[0,n_A+n_B],a\neq b\\ \dfrac{k'}{n_A+n_B}a, & k'\in[0,n_A+n_B],a=b\end{cases} \tag{63}$$

In both symmetric and asymmetric score goal cases, the equilibrium strategies are as follows.

$$x'_A\left(k',a,b\right)=\begin{cases}\dfrac{\left(b^r-a^r\right)\left(b^{(n_A+n_B)r}-a^{(n_A+n_B)r}\right)}{\left(a^r+b^r\right)\left(\left(n_A+n_B-k'\right)b^{(n_A+n_B)r}+k'a^{(n_A+n_B)r}-\left(n_A+n_B\right)b^{(n_A+n_B-k')r}a^{k'r}\right)}a, & a\neq b\\ \dfrac{1}{k'\left(n_A+n_B-k'\right)}a, & a=b\end{cases} \tag{64}$$

In summary, even if we permit asymmetric score goals for players to become ultimate winner, we can still derive solutions with same form, shifting the score state either leftward or rightward.

6 Conclusion

In this paper, we study a tug-of-war contest where two players dynamically allocate resources across stages, following the Tullock form contest success function with discriminative power $r\in(0,1]$. Our research demonstrates that, under certain conditions, there exists a Markov perfect equilibrium for any tug-of-war resource allocation contest, wherein the ratio of allocation in remaining resources is the same for both players in any given states. Our closed-form equilibrium characterization allows us to conduct comparative statics analysis, and our findings reveal the following: (1) A player's winning probability increases with their scores. Furthermore, the player with resource superiority has a higher winning probability with larger modified resource difference, while the player with score superiority has a higher winning probability with a higher score goal. (2) The competition intensity decreases with both the modified resource difference and the score goal. Additionally, as the scores increase, the competition intensity exhibits a pattern of initially decreasing, followed by an increase.

Our research makes several progresses based on present research of the game. We construct a comprehensive framework that provides generality in five aspects: (1) We allow for two players with asymmetric resource, while also including the symmetric setup as a special case. (2) We provide a closed-form solution for a Tullock contest success function with $r \in (0, 1]$, which encompasses the lottery contest success function as a special case. (3) Although many real-world tug-of-way contests implement a "win-by-2 " rule, we fully characterize the equilibrium for any n. (4) We discuss the influence of a minimum score requirement on the equilibrium of the game. (5) We allow for two players to have different score goals, while including the symmetric score target setup as a special case.

To give impetus to this line of literature, potential directions for future research include discussing the structure of all the solutions, mixed strategy equilibria included, studying situation with more than two players. These open questions are remained to be solved in future studies.

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A Proof of Lemma 1

Proof. For any state $k \in \mathbb{K}$,

$$\begin{aligned} \hat{P}(k,a,b) =& \max_{x,y} \sum_{k' \in \mathbb{K}} f(k,k',x,y)\, \hat{P}(k',a-x,b-y) \\ =& \max_{x,y} \sum_{k' \in \mathbb{K}} f\left(k,k',1,\frac{y}{x}\right) \hat{P}\left(k',1,\frac{b-y}{a-x}\right) \end{aligned} \tag{65}$$

The partial derivatives of $\hat{P}(k,a,b)$ are as follows.

$$\begin{aligned} \frac{\partial \sum_{k' \in \mathbb{K}} f\left(k,k',1,\frac{y}{x}\right) \hat{P}\left(k',1,\frac{b-y}{a-x}\right)}{\partial x} =& \sum_{k' \in \mathbb{K}} \left[f'\left(k,k',1,\frac{y}{x}\right) \hat{P}\left(k',1,\frac{b-y}{a-x}\right) \left(-\frac{y}{x^2}\right) \right. \\ & \left. + f\left(k,k',1,\frac{y}{x}\right) \hat{P}'\left(k',1,\frac{b-y}{a-x}\right) \frac{b-y}{(a-x)^2} \right] = 0 \end{aligned} \tag{66}$$

$$\begin{aligned} \frac{\partial \sum_{k' \in \mathbb{K}} f\left(k,k',1,\frac{y}{x}\right) \hat{P}\left(k',1,\frac{b-y}{a-x}\right)}{\partial y} =& \sum_{k' \in \mathbb{K}} \left[f'\left(k,k',1,\frac{y}{x}\right) \hat{P}\left(k',1,\frac{b-y}{a-x}\right) \frac{1}{x} \right. \\ & \left. + f\left(k,k',1,\frac{y}{x}\right) \hat{P}'\left(k',1,\frac{b-y}{a-x}\right) \left(-\frac{1}{a-x}\right) \right] = 0 \end{aligned} \tag{67}$$

Since $\sum_{k' \in \mathbb{K}} f'\left(k,k',1,\frac{y}{x}\right) \hat{P}\left(k',1,\frac{b-y}{a-x}\right) \neq 0$, and it is evident that $\frac{y}{x^2}, \frac{b-y}{(a-x)^2}, \frac{1}{x}, \frac{1}{a-x} \neq 0$, the following equation is satisfied.

$$\frac{y/x^2}{(b-y)/(a-x)^2} = \frac{1/x}{1/(a-x)} \tag{68}$$

Therefore, we prove that $\frac{x}{a} = \frac{y}{b}$ is a necessary condition of the first order conditions of the optimization problem.

$\square$

B Proof of Theorem 1

In order to facilitate the calculation of ultimate winning probabilities, we partition the equation set into three parts based on $k > 0, k = 0, k < 0$. Beginning with the $k > 0$ part, we can integrate these functions into matrix form:

$$\begin{pmatrix} 1 & -\alpha & 0 & 0 & 0 & ... & 0 \\ -(1-\alpha) & 1 & -\alpha & 0 & 0 & ... & 0 \\ 0 & -(1-\alpha) & 1 & -\alpha & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & 0 & -(1-\alpha) & 1 & -\alpha \\ 0 & ... & 0 & 0 & 0 & -(1-\alpha) & 1 \\ & & & & & & \end{pmatrix} \times \begin{pmatrix} \hat{P}_A(1,a,b) \\ \hat{P}_A(2,a,b) \\ \hat{P}_A(3,a,b) \\ ... \\ \hat{P}_A(n-2,a,b) \\ \hat{P}_A(n-1,a,b) \end{pmatrix} = \begin{pmatrix} (1-\alpha)\hat{P}_A(0,a,b) \\ 0 \\ 0 \\ ... \\ 0 \\ \alpha\hat{P}_A(n,a,b) \end{pmatrix} \tag{69}$$

Solve its inverse matrix and thus solve the probability in all states, noting that $\hat{P}_A(n,a,b) = 1$.

$$\hat{P}_A(k,a,b) = \hat{P}_A(0,a,b) + \frac{(1-\alpha)^k - \alpha^k}{(1-\alpha)^n - \alpha^n}\alpha^{n-k}\left(1 - \hat{P}_A(0,a,b)\right), k > 0 \tag{70}$$

Symmetrically, we can solve the equation set in $k < 0$ part.

$$1 - \hat{P}_A(k,a,b) = 1 - \hat{P}_A(0,a,b) + \frac{(1-\alpha)^{-k} - \alpha^{-k}}{(1-\alpha)^n - \alpha^n}(1-\alpha)^{n+k}\hat{P}_A(0,a,b), k < 0 \tag{71}$$

Finally, the equation set in $k = 0$ part is straightforward, i.e. $\hat{P}_A(0,a,b) = \alpha\hat{P}_A(1,a,b) + (1-\alpha)\hat{P}_A(-1,a,b)$. Combining three parts above together, we can solve all the ultimate winning probabilities.

$$\hat{P}_A(k,a,b)=\frac{a^{(n-k)r}\left(b^{(n+k)r}-a^{(n+k)r}\right)}{b^{2nr}-a^{2nr}},k\in[-n,n],a\neq b \tag{72}$$

Note that denominator and numerator are both zero when $a=b$, which means the expression is not well-defined. To be rigorous, define $\hat{P}_A(k,a,a)=\lim_{b\to a}\hat{P}_A(k,a,b)=\frac{n+k}{2n}$ by L'Hospital's rule.

C Detailed form of equation 16

The equation 16 is the second order condition of winning probability $\hat{P}_A(k,a,b)$.

$$\begin{aligned}
\frac{\partial^2\hat{P}_A(k,a,b)}{\partial x_A^2}=&\frac{r}{T^2\left(x^{2n}-1\right)^2(x+1)^2}\times\Big[-4nr(x+1)\Big[Tx^{n+k+1}-Tx^{n+k}-(n-k-1)\,x^{n+k+1}\\
&+2n(x+1)-(n-k+1)\,x^{n+k}\Big]+2n(x+1)^2\left(2nr-1+\frac{4nr}{x^{2n}-1}\right)\left(x^{n+k}-1\right)\\
&+\left(x^{2n}-1\right)\Big[2(n-k+1)\,rTx^{n+k}-2\left(T^2-(n-k-1)\,T\right)rx^{n+k+1}\\
&+(2r-(r-1)(x+1))\,T^2x^{n+k}-2n(2nr-1)(x+1)^2\\
&+(n-k+1)((n-k+1)\,r-1)(x+1)\,x^{n+k}+(x+1)\Big((r-1)\,T^2x^{n+k+1}\\
&-2(n-k-1)\,rTx^{n+k+1}+(n-k-1)((n-k-1)\,r-1)\,x^{n+k+1}\Big)\Big]\Big]\\
=&\frac{r}{\left(x^{2n}-1\right)^4(x+1)^2(x-1)^2}\left(Mx^{n+k}+N\right)
\end{aligned} \tag{73}$$

where $x=\left(\frac{b}{a}\right)^r$, $T=\frac{(x-1)\left(x^{2n}-1\right)}{(x+1)\left((n-k)x^{2n}-2nx^{n-k}+n+k\right)}$, and the expressions of M and N are as follows.

$$\begin{aligned}
M=&(x+1)^3\left((n-k)\,x^{2n}-2nx^{n-k}+n+k\right)^2\Big[4nr((n-k-1)\,x+n-k+1)\\
&+2n(x+1)\left(2nr-1+\frac{4nr}{x^{2n}-1}\right)+\left(x^{2n}-1\right)\Big((n-k+1)((n-k+1)\,r-1)\\
&+(n-k-1)((n-k-1)\,r-1)\,x\Big)\Big]+2r(x+1)(x-1)\left(x^{2n}-1\right)\left((n-k)\,x^{2n}-2nx^{n-k}+n+k\right)\\
&[2n(x+1)(1-x)+\left(x^{2n}-1\right)((n-k+1)-(n-k-1)\,x)\Big]+(x-1)^3\left(x^{2n}-1\right)^3\\
&((r-1)(x+1)-2r)
\end{aligned} \tag{74}$$

$$N = -\frac{2n\left(x+1\right)^4}{x^{2n}-1}\left(\left(n-k\right)x^{2n}-2nx^{n-k}+n+k\right)^2 x^{2n}\left(4nr+\left(2nr-1\right)\left(x^{2n}-1\right)\right) \tag{75}$$

D Proof of corollary 3 and 4

Proof. When $\eta = 1$, $\frac{x_A(k,a,b)}{a} = \frac{1}{n^2-k^2}$, which satisfies the result in corollary 3 and 4.

When $\eta \neq 1$, W.L.O.G. consider the case that $\eta > 1$.

To study the sign of the partial derivative of $\frac{x_A}{a}$ over k, we an auxiliary function $G\left(\eta,k\right) = -ln[\left(n-k\right)\eta^{2n} + \left(n+k\right) - 2n\eta^{(n-k)}]$, which contains all terms related to k in the natural logarithm of $\frac{x_A}{a}$, thus partial derivative of $G\left(\eta,k\right)$ over k has the same sign as that of $\frac{x_A}{a}$ over k.

$$\begin{aligned}\frac{\partial G\left(\eta,k\right)}{\partial k} =& \frac{1}{\left(n-k\right)\eta^{2n}+\left(n+k\right)-2n\eta^{n-k}} \\ &\times[\eta^{2n}-1-2n\eta^{n-k}\ln\eta],\end{aligned} \tag{76}$$

In order to examine the sign of $\frac{\partial G(\eta,k)}{\partial k}$, we introduce auxiliary functions $H_1\left(\eta,k\right) = \left(n-k\right)\eta^{2n} + \left(n+k\right) - 2n\eta^{n-k}$ and $H_2\left(\eta,k\right) = \eta^{2n} - 1 - 2n\eta^{n-k}\ln\eta$.

1. H_1:

   $H_1\left(\eta,n\right) = 0,\ H_1\left(\eta,-n\right) = 0.$

   $$\begin{aligned}\frac{\partial^2 H_1\left(\eta,k\right)}{\partial k^2} =& \frac{\partial - \eta^{2n} + 1 + 2n\eta^{n-k}\ln\eta}{\partial k} \\ =& -2n\eta^{n-k}\left(ln\eta\right)^2 < 0\end{aligned} \tag{77}$$

   Therefore, $H_1\left(\eta,k\right)$ must be in inverse U-shape form under any given η. Consequently, $\forall k \in [-n,n],\ H_1\left(\eta,k\right) \geq 0$.

2. H_2:

   $$\frac{\partial H_2\left(\eta,k\right)}{\partial k} = 2n\eta^{n-k}\left(\ln\eta\right)^2 > 0 \tag{78}$$

   Therefore, H_2 increases strictly in k, and the sign of H_2 is determined by $H_2\left(\eta,-n\right)$ and $H_2\left(\eta,n\right)$.

$$H_2(\eta, -n) = \eta^{2n} - 1 - 2n\eta^{2n} \ln \eta \tag{79}$$

$$H_2(\eta, n) = \eta^{2n} - 1 - 2n \ln \eta \tag{80}$$

Then we explore the properties of $H_2(\eta, -n)$ and $H_2(\eta, n)$.

(a) $H_2(\eta, -n)$:

$$\begin{aligned} \frac{\partial H_2(\eta, -n)}{\partial \eta} =& 2n\eta^{2n-1} - 2n\eta^{2n}\left(\frac{1}{\eta} + \frac{2n}{\eta}\ln \eta\right) \\ =& -4n^2\eta^{2n-1}\ln \eta < 0 \end{aligned} \tag{81}$$

$$\lim_{\eta \to 1} H_2(\eta, -n) = 1 - 1 - 0 = 0 \tag{82}$$

$$H_2(e^n, -n) = (e^n)^{2n} - 1 - 2n^2 (e^n)^{2n} < 0 \tag{83}$$

Therefore, $H_2(\eta, -n)$ decreases in η, and $H_2(\eta, -n) < 0$.

(b) $H_2(\eta, n)$:

$$\begin{aligned} \frac{\partial H_2(\eta, n)}{\partial \eta} =& 2n\eta^{2n-1} - \frac{2n}{\eta} \\ =& \frac{2n}{\eta}\left(\eta^{2n} - 1\right) > 0 \end{aligned} \tag{84}$$

$$\lim_{\eta \to 1} H_2(\eta, n) = 1 - 1 - 0 = 0 \tag{85}$$

$$H_2(e^n, n) = (e^n)^{2n} - 1 - 2n^2 \geq \left(e^2\right)^4 - 1 - 8 > 0 \tag{86}$$

Therefore, $H_2(\eta, n)$ increases in η, and $H_2(\eta, n) > 0$.

In summary, for any given η, $H_2(\eta, k)$ increases in k and $\exists k(\eta) \in (-n, n)$, s.t. $\forall k \in (-n, k(\eta)), H_2(\eta, k) < 0$, $\forall k \in (k(\eta), n), H_2(\eta, k) > 0$, and $H_2(k(\eta), \eta) = 0$.

To conclude, $\forall \eta > 1$, $\exists k(\eta)$, s.t. $\forall k \in (-n, k(\eta))$, $G(\eta, k)$ decreases with k, $\forall k \in (k(\eta), n)$, $G(\eta, k)$ increases with k. Ultimately, the competition intensity also performs like U-shape function.

For corollary 4, the only thing is to verify $H_2(\eta, 0) = \eta^{2n} - 1 - 2n\eta^n \ln \eta$

$$\begin{aligned} \frac{\partial H_2(\eta, 0)}{\partial \eta} =& 2n\eta^{2n-1} - 2n^2\eta^{n-1}\ln \eta - 2n\eta^{n-1} \\ =& 2n\eta^{n-1}\left(\eta^n - n\ln \eta - 1\right) \end{aligned} \tag{87}$$

$$\frac{\partial\left(\eta^{n}-n\ln\eta-1\right)}{\partial\eta}=n\eta^{n-1}-n\frac{1}{\eta}>0 \tag{88}$$

Considering that $\lim\limits_{\eta\to 1}\eta^{n}-n\ln\eta-1=1-0-1=0$ and $\eta^{n}-n\ln\eta-1$ increases in η, we can deduce $\eta^{n}-n\ln\eta-1>0$ and $\frac{\partial H_2(\eta,0)}{\partial\eta}>0$. Considering that $\lim\limits_{\eta\to 1}H_2\left(\eta,0\right)=1-1-0=0$, $H_2\left(e^{n},0\right)=\left(e^{n}\right)^{2n}-1-2n^{2}\left(e^{n}\right)^{2}\geq\left(e^{2}\right)^{4}-1-8\left(e^{2}\right)^{2}>0$ and $H_2\left(\eta,0\right)$ increases in η, we can deduce $H_2\left(\eta,0\right)>0$, which means $k\left(\eta\right)<0$.

□

E Calculation of equation 21

To verify the sign of $\frac{\partial x_A(k,\eta)}{\partial\eta}$, we introduce an auxiliary function $d\left(\eta,k\right)$ with the following expression, the sign of which is opposite to $\frac{\partial x_A(k,\eta)}{\eta}$.

$$d\left(\eta,k\right)\equiv\left(\eta^{2n}-1\right)\left(\eta^{k}-2\eta^{n}+\eta^{k+2n}\right)+2n\eta^{n}\left(1+\eta^{2n}-2\eta^{n+k}\right)\ln\eta \tag{89}$$

- When $\eta>2$, $\eta^{2n+k}-2\eta^{n}=\eta^{n}\left(\eta^{n+k}-2\right)>0$, $\eta^{2n}-2\eta^{n+k}=\eta^{n+k}\left(\eta^{n-k}-2\right)>0$, thus $d\left(\eta,k\right)>0$.
- When $1<\eta<2$, further analysis is needed.
  - When $k\geq 0$, $\eta^{k}-1>0$,

$$\begin{aligned} d\left(\eta,k\right)>&\left(\eta^{2n}-1\right)\left(2\eta^{n+k}-2\eta^{n}\right)+2n\eta^{n}\left(2\eta^{n}-2\eta^{n+k}\right)\ln\eta\\ =&2\eta^{n}\left(\eta^{k}-1\right)\left(\eta^{2n}-1+2n\eta^{n}\ln\eta\right)\end{aligned} \tag{90}$$

    [3] Therefore $d\left(\eta,k\right)>0$.
  - When $k<0$,

$$\begin{aligned}\frac{\partial d\left(\eta,k\right)}{\partial k}=&-4n\eta^{2n+k}\left(\ln\eta\right)^{2}+\left(\eta^{2n}-1\right)\left(\eta^{2n+k}+\eta^{k}\right)\ln\eta\\ =&\eta^{k}\ln\eta\left(-4n\eta^{2n}\ln\eta+\eta^{4n}-1\right)\\ =&\eta^{k}\ln\eta\left(\left(\eta^{2n}\right)^{2}-1-2\eta^{2n}\ln\left(\eta^{2n}\right)\right)\end{aligned} \tag{91}$$

    To discuss the sign of $\frac{\partial d(\eta,k)}{\partial k}$, we introduce an auxiliary function $g\left(x\right)=x^{2}-1-2x\ln x, x>1$. Considering that $g''\left(x\right)=2-\frac{2}{x}>0$ and $g'\left(1\right)=0$, we can deduce $g\left(x\right)>g\left(1\right)=0$, i.e.$\left(\eta^{2n}\right)^{2}-1-2\eta^{2n}\ln\left(\eta^{2n}\right)>0$. Therefore,

[3] Applying am-gm inequality to it

$d(\eta, k)$ increases in k.

$$
\begin{aligned}
d(\eta, k) > & d(\eta, k)\big|_{k=-n} \\
& = \left(\eta^{2n} - 1\right)\left(\eta^{-n} - 2\eta^{n} + \eta^{n}\right) + 2n\eta^{n}\left(1 + \eta^{2n} - 2\right)\ln\eta \\
& = \left(\eta^{2n} - 1\right)\left((2n\ln\eta - 1)\,\eta^{n} + \eta^{-n}\right)
\end{aligned} \tag{92}
$$

Considering that $\frac{\partial(2n\ln\eta-1)\eta^{n}+\eta^{-n}}{\partial\eta} = 2n^{2}\eta^{n-1}\ln\eta + n\eta^{-n-1}\left(\eta^{2}n - 1\right) > 0$, we can deduce $(2n\ln\eta - 1)\,\eta^{n} + \eta^{-n} > (2n\ln\eta - 1)\,\eta^{n} + \eta^{-n}|_{\eta=1} = 0$, i.e. $d(\eta, k) > 0$.

In all, $\forall\eta > 1, -n < k < n$, $d(\eta, k) > 0$, in other words, $\frac{\partial x_{A}(k,\eta)}{\partial\eta} < 0$.

F Calculation of equation 27

We list the specific form of equation 27 first.

$$
\begin{aligned}
\frac{\partial\frac{x_{A}(n,k,a,b)}{a}}{\partial n} &= \frac{\partial\frac{\left(\eta^{2n}-1\right)}{\left((n-k)\eta^{2n}+n+k-2n\eta^{n-k}\right)}}{\partial n} \\
&= -\frac{\eta^{4n} + 2(-1 + n\ln\eta)\,\eta^{3n-k} - 4n\eta^{2n}\ln\eta + 2(1 + n\ln\eta)\,\eta^{n-k} - 1}{\left((n-k)\,\eta^{2n} + n + k - 2n\eta^{n-k}\right)^{2}}
\end{aligned} \tag{93}
$$

Focusing on the sign of the partial derivative, we introduce an auxiliary function $f(n, k, \eta)$, which has the same sign as $\frac{\partial\frac{x_{A}(n,k,a,b)}{a}}{\partial n}$.

$$
f(n, k, \eta) \equiv \eta^{4n} + 2(-1 + n\ln\eta)\,\eta^{3n-k} - 4n\eta^{2n}\ln\eta + 2(1 + n\ln\eta)\,\eta^{n-k} - 1 \tag{94}
$$

$f(n, k, \eta)$ has the following properties.

$$
f'(n, k, 1) = 0 \tag{95}
$$

$$
f''(n, k, 1) = 0 \tag{96}
$$

$$
\begin{aligned}
\frac{\partial^3 f(n,k,\eta)}{\partial \eta^3} =& \eta^{n-k-3}\Big[-2k(k+1)(k+2)+4(2+3k(k+2))n-18(k+1)n^2+8n^3 \\
&+8n\left(8n^2-6n+1\right)\eta^{3n+k}+2(n-k)(n-k-1)(n-k-2)n\ln\eta \\
&-8n\eta^{n+k}(1+6n(n-1)+2n(n-1)(2n-1)\ln\eta) \\
&+2\eta^{2n}\Big(k(k+1)(k+2)-2(2+3k(k+2))n+9(k+1)n^2 \\
&+n(3n-k-2)(3n-k-1)(3n-k)\ln\eta\Big)\Big] \\
\equiv& \eta^{n-k-3}(M_1+M_2+M_3)
\end{aligned}
\tag{97}
$$

where $M_1 = -2k(k+1)(k+2)+4(2+3k(k+2))n-18(k+1)n^2+8n^3$, $M_2 = 2(n-k)(n-k-1)(n-k-2)n\ln\eta - 8n\eta^{n+k}(2n(n-1)(2n-1)\ln\eta)$ $+2n\eta^{2n}(3n-k-2)(3n-k-1)(3n-k)\ln\eta$ and $M_3 = 8n(8n^2-6n+1)\eta^{3n+k} - 8n\eta^{n+k}(1+6n(n-1))+2\eta^{2n}(k(k+1)(k+2)-2(2+3k(k+2))n+9(k+1)n^2)$.

Then we will prove that $M_1+M_2+M_3>0$, which is a sufficient condition of $f(n,k,\eta)>0$.

1. $k\in[-1,n-1]$:

$$
\begin{aligned}
M_1 =& -2k(k+1)(k+2)+4(2+3k(k+2))n-18(k+1)n^2+8n^3 \\
=& 2(n-k-1)^2(4n-k-1)-2(2n-k-1) \\
\geq& -4n
\end{aligned}
\tag{98}
$$

$$
\begin{aligned}
M_2 =& 2(n-k)(n-k-1)(n-k-2)n\ln\eta - 8n\eta^{n+k}(2n(n-1)(2n-1)\ln\eta) \\
&+2n\eta^{2n}(3n-k-2)(3n-k-1)(3n-k)\ln\eta \\
\geq& -16n^2(n-1)(2n-1)\eta^{n+k}\ln\eta + 2n\eta^{2n}(3n-k-2)(3n-k-1)(3n-k)\ln\eta \\
>& 2n\eta^{2n}\ln\eta[-8n(n-1)(2n-1)+(3n-k-2)(3n-k-1)(3n-k)] \\
\geq& 2n\eta^{2n}\ln\eta[-8n(n-1)(2n-1)+(2n-1)2n(2n+1)] \\
=& 4n^2(2n-1)(-2n+5)\eta^{2n}\ln\eta \\
>& -16n^4\eta^{2n}\ln\eta
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
M_3 =&8n\left(8n^2-6n+1\right)\eta^{3n+k}-8n\eta^{n+k}\left(1+6n\left(n-1\right)\right)\\
&+2\eta^{2n}\left(k\left(k+1\right)\left(k+2\right)-2\left(2+3k\left(k+2\right)\right)n+9\left(k+1\right)n^2\right)\\
=&8n\left(8n^2-6n+1\right)\eta^{3n+k}-8n\eta^{n+k}\left(6n^2-6n+1\right)+[2\left(k+1\right)\left(\left(3n-k-1\right)^2-1\right)+4n]\eta^{2n}\\
>&8n\left(8n^2-6n+1\right)\eta^{3n-1}-8n\eta^{n+2n-1}\left(6n^2-6n+1\right)+4n\eta^{2n}\\
=&16n^3\eta^{3n-1}+4n\eta^{2n}\\
\geq&16n^3\eta^{3n-1}+4n
\end{aligned}
\tag{100}
$$

$$
M_1+M_2+M_3>16n^3\eta^{2n}\ln\eta\left(\frac{\eta^{n-1}}{\ln\eta}-n\right) \tag{101}
$$

To verify the sign of $M_1+M_2+M_3$, we introduce an auxiliary function $m\left(x\right)=\frac{x^{n-1}}{\ln x}, x>1$.

$$
m'\left(x\right)=\frac{\left(\left(n-1\right)\ln x-1\right)x^{n-2}}{\ln^2 x} \tag{102}
$$

i.e. $m'\left(x\right)<0$ if and only if $x\in\left(1,e^{\frac{1}{n-1}}\right)$, thus the minimum point of $m\left(x\right)$ is $m\left(e^{frac1n-1}\right)$.

$$
\min_{x\in(1,\infty)}m\left(x\right)=m\left(e^{\frac{1}{n-1}}\right)=e\left(n-1\right)>n \tag{103}
$$

Therefore, $M_1+M_2+M_3>0$.

2. $k\in[-n+1,-2]$:

$$
\begin{aligned}
M_1 =&2\left(n-k-1\right)^2\left(4n-k-1\right)-2\left(2n-k-1\right)\\
\geq&2\left(n-k-1\right)^2\left(4n-k-1\right)-6n
\end{aligned}
\tag{104}
$$

$$
\begin{aligned}
M_2 =&2\left(n-k\right)\left(n-k-1\right)\left(n-k-2\right)n\ln\eta-8n\eta^{n+k}\left(2n\left(n-1\right)\left(2n-1\right)\ln\eta\right)\\
&+2n\eta^{2n}\left(3n-k-2\right)\left(3n-k-1\right)\left(3n-k\right)\ln\eta\\
\geq&-16n^2\left(n-1\right)\left(2n-1\right)\eta^{n+k}\ln\eta+2n\eta^{2n}\left(3n-k-2\right)\left(3n-k-1\right)\left(3n-k\right)\ln\eta\\
>&-32n^4\eta^{n+k}\ln\eta+2n\eta^{2n}\left(3n\right)\left(3n+1\right)\left(3n+2\right)\ln\eta\\
>&0
\end{aligned}
\tag{105}
$$

$$
\begin{aligned}
M_3 =&8n\left(8n^2-6n+1\right)\eta^{3n+k}-8n\eta^{n+k}\left(6n^2-6n+1\right)+[2\left(k+1\right)\left(\left(3n-k-1\right)^2-1\right)+4n]\eta^{2n}\\
>&8n\left(8n^2-6n+1\right)\eta^{3n-n+1}-8n\eta^{n-2}\left(6n^2-6n+1\right)-2\left(k+1\right)\eta^{2n}\\
&+2\left(k+1\right)\left(3n-k-1\right)^2\eta^{2n}+4n\eta^{2n}\\
>&8n\left(8n^2-6n+1\right)\eta^{2n}-8n\eta^{2n}\left(6n^2-6n+1\right)-2\left(-2+1\right)\eta^{2n}\\
&+2\left(k+1\right)\left(3n-k-1\right)^2\eta^{2n}+4n\eta^{2n}\\
>&16n^3\eta^{2n}+2\left(k+1\right)\left(3n-k-1\right)^2\eta^{2n}+4n\eta^{2n}
\end{aligned}
\tag{106}
$$

To facilitate the discussion, we partition the problem into two parts.

(a) $k\in[-\frac{n}{2}-1,-2]$:

$$
M_1\geq-6n \tag{107}
$$

$$
\begin{aligned}
M_3>&\left(16n^3-n\left(\frac{7}{2}n\right)^2+4n\right)\eta^{2n}\\
=&\left(\frac{15}{4}n^3+4n\right)\eta^{2n}\\
\geq&19n
\end{aligned}
\tag{108}
$$

Therefore, $M_1+M_2+M_3>0$.

(b) $k\in\left[-n+1,-\frac{n}{2}-1\right)$.

$$
\begin{aligned}
M_1\geq&2\left(n-k-1\right)^2\left(4n-k-1\right)-6n\\
\geq&2\times\left(\frac{3}{2}n\right)^2\frac{9}{2}n-6n\\
=&\frac{81}{4}n^3-6n\\
>&16n^3
\end{aligned}
\tag{109}
$$

$$
\begin{aligned}
M_3 =&8n\left(8n^2-6n+1\right)\eta^{3n+k}-8n\eta^{n+k}\left(6n^2-6n+1\right)\\
&+[2\left(k+1\right)\left(\left(3n-k-1\right)^2-1\right)+4n]\eta^{2n}\\
>&16n^3\eta^{3n+k}+8n\eta^{n+k}\left(\eta^{2n}-1\right)\left(6n^2-6n+1\right)+2\left(k+1\right)\left(3n-k-1\right)^2\eta^{2n}+4n\eta^{2n}\\
>&16n^3\eta^{3n+k}+8n\left(\eta^{2n}-1\right)\left(3n^2+1\right)+2\left(-n\right)\left(4n\right)^2\eta^{2n}\\
>&16n^3\eta^{2n}+8n\left(\eta^{2n}-1\right)\left(3n^2+1\right)-32n^3\eta^{2n}\\
>&-16n^3\eta^{2n}+16n^3\left(\eta^{2n}-1\right)\\
=&-16n^3
\end{aligned}
\tag{110}
$$

Therefore $M_1+M_2+M_3>0$.

In summary, for any k, $\frac{\partial^3 f(n,k,\eta)}{\partial\eta^3}>0$. Thus $\frac{\partial^2 f(n,k,\eta)}{\partial\eta^2}$ increases in η, i.e. $\frac{\partial^2 f(n,k,\eta)}{\partial\eta^2}>\frac{\partial^2 f(n,k,\eta)}{\partial\eta^2}\big|_{\eta=1}=0$. Then $\frac{\partial f(n,k,\eta)}{\partial\eta}$ increases in η, i.e. $\frac{\partial f(n,k,\eta)}{\partial\eta}>\frac{\partial f(n,k,\eta)}{\partial\eta}\big|_{\eta=1}=0$. Ultimately, $f\left(n,k,\eta\right)$ increases in η, i.e. $f\left(n,k,\eta\right)>\ f\left(n,k,1\right)=0$, i.e. $\frac{\partial x_A(n,k,a,b)}{\partial n}<0$.

G Proof of theorem 3

We start from the first part of the theorem. By the expression of T_0, we can find that when η converges to infinity, $T_0=n+(2n-1)\frac{O(\eta^{\frac{3}{2}n(n-1)})}{O(\eta^{n(2n-1)})}$. Therefore, by L'Hôpital's rule, $\lim\limits_{\eta\to\infty}T_0=n$ is obvious to prove. When η coverges to zero, $T_0=n+(2n-1)\frac{O((\frac{1}{\eta}^{\frac{3}{2}n(n-1)})}{O(\frac{1}{\eta}^{n(2n-1)})}$. Therefore, by L'Hôpital's rule, $\lim\limits_{\eta\to 0}T_0=n$ is obvious to prove.

On the other hand, we can calculate the special case $T_0|_{\eta=1}$.

$$
\begin{aligned}
T_0|_{\eta=1}=&n+(2n-1)\frac{\sum_{i=0}^{n-1}\Pi_{j=1}^{i}\frac{2n-j}{j}}{\sum_{i=0}^{2n-1}\Pi_{j=1}^{i}\frac{2n-j}{j}}\\
=&n+(2n-1)\frac{\sum_{i=0}^{n-1}\frac{(2n-1)!}{(2n-1-i)!i!}}{\sum_{i=0}^{2n-1}\frac{(2n-1)!}{(2n-1-i)!i!}}\\
=&n+(2n-1)\frac{\sum_{i=0}^{n-1}C_{2n-1}^{i}}{\sum_{i=0}^{2n-1}C_{2n-1}^{i}}\\
=&n+(2n-1)\frac{2^{2n-2}}{2^{2n-1}}=2n-\frac{1}{2}
\end{aligned}
\tag{111}
$$

Then we continue to prove the second part of the theorem. W.L.O.G., we consider the situation $\eta > 1$, because of the duality between η and $\frac{1}{\eta}$. The problem can be solved by calculating the sign of the derivative, but it is too complex and here we give another way to prove it.

We denote $t_i(\eta) = \Pi_{j=1}^{i} \frac{\eta^{2n}-\eta^j}{\eta^j-1}$, then $T_0(\eta) = n + (2n-1)\frac{\sum_{i=0}^{n-1} t_i(\eta)}{\sum_{i=0}^{2n-1} t_i(\eta)}$.

$\forall \eta > 1, i \geq 1, n \geq 2$, $\frac{\partial \frac{(\eta)^{2n}-(\eta)^i}{(\eta)^i-1}}{\partial \eta} = \frac{i\eta^i(\eta^{2n}-1)+2n\eta^{2n}(\eta^i-1)}{\eta(\eta^i-1)^2} > 0$, thus $\forall \eta' > \eta > 1$, $\frac{(\eta')^{2n}-(\eta')^i}{(\eta')^i-1} > \frac{(\eta)^{2n}-(\eta)^i}{(\eta)^i-1} > 1$.

After that, $\frac{t_i(\eta')-t_i(\eta)}{t_i(\eta)} = \frac{\frac{(\eta')^{2n}-(\eta')^i}{(\eta')^i-1} t_{i-1}(\eta') - \frac{(\eta)^{2n}-(\eta)^i}{(\eta)^i-1} t_{i-1}(\eta)}{\frac{(\eta)^{2n}-(\eta)^i}{(\eta)^i-1} t_{i-1}(\eta)} > \frac{t_{i-1}(\eta')-t_{i-1}(\eta)}{t_{i-1}(\eta)}$.

Therefore, $\forall i' > i, \eta' > \eta > 1$, $\frac{t_{i'}(\eta')-t_{i'}(\eta)}{t_{i'}(\eta)} > \frac{t_i(\eta')-t_i(\eta)}{t_i(\eta)}$. Furthermore, $\frac{t_{i'}(\eta')-t_{i'}(\eta)}{t_{i'}(\eta)} > \frac{t_{i'}(\eta')-t_{i'}(\eta)+t_i(\eta')-t_i(\eta)}{t_{i'}(\eta)+t_i(\eta)} > \frac{t_i(\eta')-t_i(\eta)}{t_i(\eta)}$. Naturally, $\frac{t_l(\eta')-t_l(\eta)}{t_l(\eta)} > \frac{\sum_{i=m}^{l} t_i(\eta')-\sum_{i=0}^{n-1} t_i(\eta)}{\sum_{i=0}^{n-1} t_i(\eta)} > \frac{t_m(\eta')-t_m(\eta)}{t_m(\eta)}$.

W.T.S. $\forall \eta' > \eta > 1$, $\frac{\sum_{i=0}^{n-1} t_i(\eta')}{\sum_{i=0}^{2n-1} t_i(\eta')} < \frac{\sum_{i=0}^{n-1} t_i(\eta)}{\sum_{i=0}^{2n-1} t_i(\eta)}$, i.e. $\frac{\sum_{i=0}^{2n-1} t_i(\eta')-\sum_{i=0}^{2n-1} t_i(\eta)}{\sum_{i=0}^{2n-1} t_i(\eta)} > \frac{\sum_{i=0}^{n-1} t_i(\eta')-\sum_{i=0}^{n-1} t_i(\eta)}{\sum_{i=0}^{n-1} t_i(\eta)}$, which can be proved by the statement above.

Lastly, we prove the third part of the theorem. From the analysis above, we have proved that $\frac{\sum_{i=0}^{n-1} \Pi_{j=1}^{i} \frac{\eta^{2n}-\eta^j}{\eta^j-1}}{\sum_{i=0}^{2n-1} \Pi_{j=1}^{i} \frac{\eta^{2n}-\eta^j}{\eta^j-1}} > \frac{\sum_{i=0}^{n-2} \Pi_{j=1}^{i} \frac{\eta^{2n}-\eta^j}{\eta^j-1}}{\sum_{i=0}^{2n-3} \Pi_{j=1}^{i} \frac{\eta^{2n}-\eta^j}{\eta^j-1}}$. Thus $T_0(n) - T_0(n-1) = 1 + (2n-1)\frac{\sum_{i=0}^{n-1} \Pi_{j=1}^{i} \frac{\eta^{2n}-\eta^j}{\eta^j-1}}{\sum_{i=0}^{2n-1} \Pi_{j=1}^{i} \frac{\eta^{2n}-\eta^j}{\eta^j-1}} - (2n-3)\frac{\sum_{i=0}^{n-2} \Pi_{j=1}^{i} \frac{\eta^{2n}-\eta^j}{\eta^j-1}}{\sum_{i=0}^{2n-3} \Pi_{j=1}^{i} \frac{\eta^{2n}-\eta^j}{\eta^j-1}} > 1$. Ultimately, T_0 increases with the increase of n.