

1 Contest

2 Mathematics

3 Data structures

4 Numerical

5 Number theory

6 Combinatorial

7 Graph

8 Geometry

9 Strings

10 Various

## Contest (1)

template.cpp15 lines

```
#include <bits/stdc++.h>
using namespace std;

#define rep(i, a, b) for(int i = a; i < (b); ++i)
#define trav(a, x) for(auto& a : x)
#define all(x) begin(x), end(x)
#define sz(x) (int)(x).size()
typedef long long ll;
typedef pair<int, int> pii;
typedef vector<int> vi;

int main() {
    cin.sync_with_stdio(0); cin.tie(0);
    cin.exceptions(cin.failbit);
}
```

.bashrc3 lines

```
alias c='g++ -Wall -Wconversion -Wfatal-errors -g -std=c++14 \
-fsanitize=undefined,address'
xmodmap -e 'clear lock' -e 'keycode 66=less greater' #caps =<
```

hash.sh3 lines

```
# Hashes a file, ignoring all whitespace and comments. Use for
# verifying that code was correctly typed.
cpp -dD -P -fpreprocessed | tr -d '[:space:]' | md5sum |cut -c-6
```

## Mathematics (2)

### 2.1 Equations

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1 The extremum is given by  $x = -b/2a$ .

1

3
$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \Rightarrow \begin{aligned} x &= \frac{ed - bf}{ad - bc} \\ y &= \frac{af - ec}{ad - bc} \end{aligned}$$

5

9In general, given an equation  $Ax = b$ , the solution to a variable  $x_i$  is given by

11

12
$$x_i = \frac{\det A'_i}{\det A}$$

18where  $A'_i$  is  $A$  with the  $i$ 'th column replaced by  $b$ .

### 2.2 Recurrences

23 If  $a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}$ , and  $r_1, \dots, r_k$  are distinct roots of  $x^k + c_1 x^{k-1} + \dots + c_k$ , there are  $d_1, \dots, d_k$  s.t.

$$a_n = d_1 r_1^n + \dots + d_k r_k^n.$$

Non-distinct roots  $r$  become polynomial factors, e.g.  
 $a_n = (d_1 n + d_2) r^n$ .

### 2.3 Trigonometry

$$\begin{aligned} \sin(v + w) &= \sin v \cos w + \cos v \sin w \\ \cos(v + w) &= \cos v \cos w - \sin v \sin w \end{aligned}$$

$$\begin{aligned} \tan(v + w) &= \frac{\tan v + \tan w}{1 - \tan v \tan w} \\ \sin v + \sin w &= 2 \sin \frac{v + w}{2} \cos \frac{v - w}{2} \\ \cos v + \cos w &= 2 \cos \frac{v + w}{2} \cos \frac{v - w}{2} \end{aligned}$$

$$(V + W) \tan(v - w)/2 = (V - W) \tan(v + w)/2$$

where  $V, W$  are lengths of sides opposite angles  $v, w$ .

$$\begin{aligned} a \cos x + b \sin x &= r \cos(x - \phi) \\ a \sin x + b \cos x &= r \sin(x + \phi) \end{aligned}$$

where  $r = \sqrt{a^2 + b^2}, \phi = \text{atan2}(b, a)$ .

### 2.4 Geometry

#### 2.4.1 Triangles

Side lengths:  $a, b, c$   
Semiperimeter:  $p = \frac{a + b + c}{2}$   
Area:  $A = \sqrt{p(p - a)(p - b)(p - c)}$   
Circumradius:  $R = \frac{abc}{4A}$   
Inradius:  $r = \frac{A}{p}$   
Length of median (divides triangle into two equal-area triangles):  $m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$   
Length of bisector (divides angles in two):

$$\begin{aligned} s_a &= \sqrt{bc \left[ 1 - \left( \frac{a}{b + c} \right)^2 \right]} \\ \text{Law of sines: } \frac{\sin \alpha}{a} &= \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R} \\ \text{Law of cosines: } a^2 &= b^2 + c^2 - 2bc \cos \alpha \\ \text{Law of tangents: } \frac{a + b}{a - b} &= \frac{\tan \frac{\alpha + \beta}{2}}{\tan \frac{\alpha - \beta}{2}} \end{aligned}$$

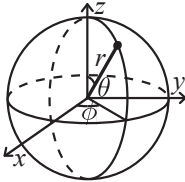
#### 2.4.2 Quadrilaterals

With side lengths  $a, b, c, d$ , diagonals  $e, f$ , diagonals angle  $\theta$ , area  $A$  and magic flux  $F = b^2 + d^2 - a^2 - c^2$ :

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is  $180^\circ$ ,  $ef = ac + bd$ , and  $A = \sqrt{(p - a)(p - b)(p - c)(p - d)}$ .

#### 2.4.3 Spherical coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z / \sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \text{atan2}(y, x) \end{aligned}$$

## 2.5 Derivatives/Integrals

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx} \tan x = 1 + \tan^2 x \quad \frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$
$$\int \tan ax = -\frac{\ln|\cos ax|}{a} \quad \int x \sin ax = \frac{\sin ax - ax \cos ax}{a^2}$$
$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) \quad \int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1)$$

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

## 2.6 Sums

$$c^a + c^{a+1} + \dots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$$
$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$
$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

## 2.7 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

## 2.8 Probability theory

Let  $X$  be a discrete random variable with probability  $p_X(x)$  of assuming the value  $x$ . It will then have an expected value (mean)  $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$  and variance  $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$  where  $\sigma$  is the standard deviation. If  $X$  is instead continuous it will have a probability density function  $f_X(x)$  and the sums above will instead be integrals with  $p_X(x)$  replaced by  $f_X(x)$ .

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent  $X$  and  $Y$ ,

$$V(aX + bY) = a^2V(X) + b^2V(Y).$$

### 2.8.1 Discrete distributions

#### Binomial distribution

The number of successes in  $n$  independent yes/no experiments, each which yields success with probability  $p$  is  $\text{Bin}(n, p)$ ,  $n = 1, 2, \dots$ ,  $0 \leq p \leq 1$ .

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$
$$\mu = np, \sigma^2 = np(1-p)$$

$\text{Bin}(n, p)$  is approximately  $\text{Po}(np)$  for small  $p$ .

#### First success distribution

The number of trials needed to get the first success in independent yes/no experiments, each wich yields success with probability  $p$  is  $\text{Fs}(p)$ ,  $0 \leq p \leq 1$ .

$$p(k) = p(1-p)^{k-1}, k = 1, 2, \dots$$
$$\mu = \frac{1}{p}, \sigma^2 = \frac{1-p}{p^2}$$

#### Poisson distribution

The number of events occurring in a fixed period of time  $t$  if these events occur with a known average rate  $\kappa$  and independently of the time since the last event is  $\text{Po}(\lambda)$ ,  $\lambda = t\kappa$ .

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, 2, \dots$$

$$\mu = \lambda, \sigma^2 = \lambda$$

### 2.8.2 Continuous distributions

#### Uniform distribution

If the probability density function is constant between  $a$  and  $b$  and 0 elsewhere it is  $\text{U}(a, b)$ ,  $a < b$ .

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{a+b}{2}, \sigma^2 = \frac{(b-a)^2}{12}$$

#### Exponential distribution

The time between events in a Poisson process is  $\text{Exp}(\lambda)$ ,  $\lambda > 0$ .

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mu = \frac{1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}$$

#### Normal distribution

Most real random values with mean  $\mu$  and variance  $\sigma^2$  are well described by  $\mathcal{N}(\mu, \sigma^2)$ ,  $\sigma > 0$ .

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If  $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$  and  $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$  then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$

## 2.9 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let  $X_1, X_2, \dots$  be a sequence of random variables generated by the Markov process. Then there is a transition matrix  $\mathbf{P} = (p_{ij})$ , with  $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$ , and  $\mathbf{p}^{(n)} = \mathbf{P}^n \mathbf{p}^{(0)}$  is the probability distribution for  $X_n$  (i.e.,  $p_i^{(n)} = \Pr(X_n = i)$ ), where  $\mathbf{p}^{(0)}$  is the initial distribution.

$\pi$  is a stationary distribution if  $\pi = \pi \mathbf{P}$ . If the Markov chain is *irreducible* (it is possible to get to any state from any state), then  $\pi_i = \frac{1}{\mathbb{E}(T_i)}$  where  $\mathbb{E}(T_i)$  is the expected time between two visits in state  $i$ .  $\pi_j/\pi_i$  is the expected number of visits in state  $j$  between two visits in state  $i$ .

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors,  $\pi_i$  is proportional to node  $i$ ’s degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1).  $\lim_{k \rightarrow \infty} \mathbf{P}^k = \mathbf{1}\pi$ .

A Markov chain is an A-chain if the states can be partitioned into two sets **A** and **G**, such that all states in **A** are absorbing ( $p_{ii} = 1$ ), and all states in **G** leads to an absorbing state in **A**. The probability for absorption in state  $i \in \mathbf{A}$ , when the initial state is  $j$ , is  $a_{ij} = p_{ij} + \sum_{k \in \mathbf{G}} a_{ik} p_{kj}$ . The expected time until absorption, when the initial state is  $i$ , is  $t_i = 1 + \sum_{k \in \mathbf{G}} p_{ki} t_k$ .

## Data structures (3)

OrderStatisticTree.h

**Description:** A set (not multiset!) with support for finding the n'th element, and finding the index of an element. To get a map, change null\_type.

**Time:**  $\mathcal{O}(\log N)$

782797, 16 lines

```
#include <bits/extc++.h>
using namespace __gnu_pbds;

template<class T>
using Tree = tree<T, null_type, less<T>, rb_tree_tag,
    tree_order_statistics_node_update>;

void example() {
    Tree<int> t, t2; t.insert(8);
    auto it = t.insert(10).first;
    assert(it == t.lower_bound(9));
    assert(t.order_of_key(10) == 1);
    assert(t.order_of_key(11) == 2);
    assert(*t.find_by_order(0) == 8);
```

t.join(t2); // assuming T < T2 or T > T2, merge t2 into t

}

HashMap.h

**Description:** Hash map with the same API as unordered\_map, but ~3x faster. Initial capacity must be a power of 2 (if provided).

1443bc, 7 lines

```
#include <bits/extc++.h>
// To use most bits rather than just the lowest ones:
struct chash {
    const uint64_t C = 1l(2e18 * M_PI) + 7l; // large odd number
    ll operator()(ll x) const { return __builtin_bswap64(x*C); }
};
__gnu_pbds::gp_hash_table<ll,int,chash> h({}, {}, {}, {}, {}, {1<<16});
```

UnionFind.h

**Description:** Disjoint-set data structure.

**Time:**  $\mathcal{O}(\alpha(N))$

b5bfc3, 14 lines

```
struct UF {
    vi e;
    UF(int n) : e(n, -1) {}
    bool same_set(int a, int b) { return find(a) == find(b); }
    int size(int x) { return -e[find(x)]; }
    int find(int x) { return e[x] < 0 ? x : e[x] = find(e[x]); }
    bool join(int a, int b) {
        a = find(a), b = find(b);
        if (a == b) return false;
        if (e[a] > e[b]) swap(a, b);
        e[a] += e[b]; e[b] = a;
        return true;
    }
};
```

Splay.h

f385e5, 101 lines

```
struct Node {
    Node *l, *r, *p;
    int v, size;
    Node(): l(NULL), r(NULL), p(NULL), v(0), size(0) {};
    Node(int v): l(NULL), r(NULL), p(NULL), v(v), size(1) {};
};
Node *root;
void update(Node *P) {
    P->size = (P->l ? P->l->size : 0) + (P->r ? P->r->size : 0)
        + 1;
}
void rightRotate(Node *P) {
    Node *T = P->l, *B = T->r, *D = P->p;
    if (D)
        if (D->r == P) D->r = T;
        else D->l = T;
    if (B) B->p = P;
    T->p = D, T->r = P;
    P->p = T, P->l = B;
    update(P); update(T);
}
void leftRotate(Node *P) {
    Node *T = P->r, *B = T->l, *D = P->p;
    if (D)
        if (D->r == P) D->r = T;
        else D->l = T;
    if (B) B->p = P;
    T->p = D, T->l = P;
    P->p = T, P->r = B;
    update(P); update(T);
}
// Make T root and Balance tree
void Splay(Node *T) {
```

```
while (true) {
    Node *p = T->p; if (!p) break;
    Node *pp = p->p;
    if (!pp) { //Zig
        if (p->l == T) rightRotate(p); else leftRotate(p);
        break;
    }
    if (pp->l == p) {
        if (p->l == T) { rightRotate(pp); rightRotate(p);
            } //ZigZig
        else { leftRotate(p);rightRotate(pp); } //ZigZag
    } else {
        if (p->l == T) { rightRotate(p);leftRotate(pp); }
            //ZigZag
        else { leftRotate(pp);leftRotate(p); } //ZigZig
    }
}
root = T;
```

// Insert to the right of the root, and return reference to the new created node

Node \*Insert(int v) {

Node \*newNode = new Node(v);

if (!root) return root = newNode;

Node \*R = root->r;

root->r = newNode, newNode->p = root;

if (R) newNode->r = R, R->p = newNode;

update(newNode);

update(root);

return newNode;

}

// Delete v

bool Erase(Node \*node) {

Splay(node);

Node \*L = node->l, \*R = node->r;

if (L) L->p = NULL; if (R) R->p = NULL;

free(node);

if (!L) return root = R, true;

if (!R) return root = L, true;

while (L->r) L = L->r;

Splay(L);

root->r = R;

update(root);

return true;

}

// Find the node at position p and make it root (position start from 1)

void Find(int p) {

Node \*node = root;

while (true) {

if (node->l) {

if (node->l->size >= p) {

node = node->l;

continue;

}

else

p -= node->l->size;

}

}

if (p == 1) break; p -= 1;

if (node->r) node = node->r;

else break;

}

Splay(node);

}

void Inorder(Node \*R) {

if (!R) return;

```
Inorder(R->l);
printf("%d ", R->v);
Inorder(R->r);
}
```

SubMatrix.h  
**Description:** Calculate submatrix sums quickly, given upper-left and lower-right corners (half-open).  
**Usage:** SubMatrix<int> m(matrix);  
m.sum(0, 0, 2, 2); // top left 4 elements  
**Time:**  $\mathcal{O}(N^2 + Q)$

c59ada, 13 lines

```
template<class T>
struct SubMatrix {
    vector<vector<T>> p;
    SubMatrix(vector<vector<T>>& v) {
        int R = sz(v), C = sz(v[0]);
        p.assign(R+1, vector<T>(C+1));
        rep(r,0,R) rep(c,0,C)
            p[r+1][c+1] = v[r][c] + p[r][c+1] + p[r+1][c] - p[r][c];
    }
    T sum(int u, int l, int d, int r) {
        return p[d][r] - p[d][l] - p[u][r] + p[u][l];
    }
};
```

Matrix.h  
**Description:** Basic operations on square matrices.  
**Usage:** Matrix<int, 3> A;  
A.d = {{{{1,2,3}}, {{4,5,6}}, {{7,8,9}}}}};  
vector<int> vec = {1,2,3};  
vec = (A^N) \* vec;

c43c7d, 26 lines

```
template<class T, int N> struct Matrix {
    typedef Matrix M;
    array<array<T, N>, N> d{};
    M operator*(const M& m) const {
        M a;
        rep(i,0,N) rep(j,0,N)
            rep(k,0,N) a.d[i][j] += d[i][k]*m.d[k][j];
        return a;
    }
    vector<T> operator*(const vector<T>& vec) const {
        vector<T> ret(N);
        rep(i,0,N) rep(j,0,N) ret[i] += d[i][j] * vec[j];
        return ret;
    }
    M operator^(ll p) const {
        assert(p >= 0);
        M a, b(*this);
        rep(i,0,N) a.d[i][i] = 1;
        while (p) {
            if (p&1) a = a*b;
            b = b*b;
            p >>= 1;
        }
        return a;
    }
};
```

LineContainer.h  
**Description:** Container where you can add lines of the form  $kx+m$ , and query maximum values at points  $x$ . Useful for dynamic programming (“convex hull trick”).  
**Time:**  $\mathcal{O}(\log N)$

95e223, 30 lines

```
struct Line {
    mutable ll k, m, p;
    bool operator<(const Line& o) const { return k < o.k; }
```

```
    bool operator<(ll x) const { return p < x; }
};

struct LineContainer : multiset<Line, less<>> {
    // (for doubles, use inf = 1/.0, div(a,b) = a/b)
    const ll inf = LLONG_MAX;
    ll div(ll a, ll b) { // floored division
        return a / b - ((a ^ b) < 0 && a % b); }
    bool isect(iterator x, iterator y) {
        if (y == end()) { x->p = inf; return false; }
        if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
        else x->p = div(y->m - x->m, x->k - y->k);
        return x->p >= y->p;
    }
    void add(ll k, ll m) {
        auto z = insert({k, m, 0}), y = z++, x = y;
        while (isect(y, z)) z = erase(z);
        if (x != begin() && isect(--x, y)) isect(x, y = erase(y));
        while ((y = x) != begin() && (--x)->p >= y->p)
            isect(x, erase(y));
    }
    ll query(ll x) {
        assert(!empty());
        auto l = *lower_bound(x);
        return l.k * x + l.m;
    }
};
```

Treap.h  
**Description:** A short self-balancing tree. It acts as a sequential container with log-time splits/joins, and is easy to augment with additional data.  
**Time:**  $\mathcal{O}(\log N)$

9556fc, 55 lines

```
struct Node {
    Node *l = 0, *r = 0;
    int val, y, c = 1;
    Node(int val) : val(val), y(rand()) {}
    void recalc();
};

int cnt(Node* n) { return n ? n->c : 0; }
void Node::recalc() { c = cnt(l) + cnt(r) + 1; }

template<class F> void each(Node* n, F f) {
    if (n) { each(n->l, f); f(n->val); each(n->r, f); }
}

pair<Node*, Node*> split(Node* n, int k) {
    if (!n) return {};
    if (cnt(n->l) >= k) { // "n->val >= k" for lower_bound(k)
        auto pa = split(n->l, k);
        n->l = pa.second;
        n->recalc();
        return {pa.first, n};
    } else {
        auto pa = split(n->r, k - cnt(n->l) - 1); // and just "k"
        n->r = pa.first;
        n->recalc();
        return {n, pa.second};
    }
}
```

```
Node* merge(Node* l, Node* r) {
    if (!l) return r;
    if (!r) return l;
    if (l->y > r->y) {
        l->r = merge(l->r, r);
        l->recalc();
        return l;
    }
```

```
    } else {
        r->l = merge(l, r->l);
        r->recalc();
        return r;
    }
}
```

```
Node* ins(Node* t, Node* n, int pos) {
    auto pa = split(t, pos);
    return merge(merge(pa.first, n), pa.second);
}
```

```
// Example application: move the range [l, r) to index k
void move(Node*& t, int l, int r, int k) {
    Node *a, *b, *c;
    tie(a,b) = split(t, l); tie(b,c) = split(b, r - l);
    if (k <= l) t = merge(ins(a, b, k), c);
    else t = merge(a, ins(c, b, k - r));
}
```

FenwickTree.h  
**Description:** Computes partial sums  $a[0] + a[1] + \dots + a[\text{pos} - 1]$ , and updates single elements  $a[i]$ , taking the difference between the old and new value.  
**Time:** Both operations are  $\mathcal{O}(\log N)$ .

e62fac, 22 lines

```
struct FT {
    vector<ll> s;
    FT(int n) : s(n) {}
    void update(int pos, ll dif) { // a[pos] += dif
        for (; pos < sz(s); pos |= pos + 1) s[pos] += dif;
    }
    ll query(int pos) { // sum of values in [0, pos)
        ll res = 0;
        for (; pos > 0; pos &= pos - 1) res += s[pos-1];
        return res;
    }
    int lower_bound(ll sum) { // min pos st sum of [0, pos] >= sum
        // Returns n if no sum is >= sum, or -1 if empty sum is.
        if (sum <= 0) return -1;
        int pos = 0;
        for (int pw = 1 << 25; pw; pw >>= 1) {
            if (pos + pw <= sz(s) && s[pos + pw-1] < sum)
                pos += pw, sum -= s[pos-1];
        }
        return pos;
    }
};
```

FenwickTree2d.h  
**Description:** Computes sums  $a[i,j]$  for all  $i < I, j < J$ , and increases single elements  $a[i,j]$ . Requires that the elements to be updated are known in advance (call `fakeUpdate()` before `init()`).  
**Time:**  $\mathcal{O}(\log^2 N)$ . (Use persistent segment trees for  $\mathcal{O}(\log N)$ .)

b28c27, 22 lines

```
"FenwickTree.h"
struct FT2 {
    vector<vi> ys; vector<FT> ft;
    FT2(int limx) : ys(limx) {}
    void fakeUpdate(int x, int y) {
        for (; x < sz(ys); x |= x + 1) ys[x].push_back(y);
    }
    void init() {
        trav(v, ys) sort(all(v)), ft.emplace_back(sz(v));
    }
    int ind(int x, int y) {
        return (int)(lower_bound(all(ys[x]), y) - ys[x].begin()); }
    void update(int x, int y, ll dif) {
        for (; x < sz(ys); x |= x + 1)
```

```
ft[x].update(ind(x, y), dif);
}
ll query(int x, int y) {
    ll sum = 0;
    for (; x; x &= x - 1)
        sum += ft[x-1].query(ind(x-1, y));
    return sum;
}
};
```

RMQ.h  
**Description:** Range Minimum Queries on an array. Returns min(V[a], V[a + 1], ... V[b - 1]) in constant time.  
**Usage:** RMQ rmq(values);  
rmq.query(inclusive, exclusive);  
**Time:**  $\mathcal{O}(|V| \log |V| + Q)$

1f8996, 17 lines

```
template<class T>
struct RMQ {
    vector<vector<T>>> jmp;
    RMQ(const vector<T>& V) {
        int N = sz(V), on = 1, depth = 1;
        while (on < N) on *= 2, depth++;
        jmp.assign(depth, V);
        rep(i, 0, depth-1) rep(j, 0, N)
            jmp[i+1][j] = min(jmp[i][j],
                               jmp[i][min(N - 1, j + (1 << i))]);
    }
    T query(int a, int b) {
        assert(a < b); // or return inf if a == b
        int dep = 31 - __builtin_clz(b - a);
        return min(jmp[dep][a], jmp[dep][b - (1 << dep)]);
    }
};
```

PersistentSegTree.h  
<iostream>, <vector>

44b362, 77 lines

```
using namespace std;

/* persistent segment tree w/ sum query */
struct per_seg {
    vector<int> lc, rc, lx, rx, val, roots = {0};

    //CHANGE ME
    inline static int combine(int a, int b) {
        return a+b;
    }

    //build PST on indices i ... j of array a
    int build(int i, int j, int* a) {
        int v = lc.size();
        lc.push_back(-1); rc.push_back(-1);
        lx.push_back(i); rx.push_back(j);
        val.push_back(a[i]);
        if(i != j) {
            int l = build(i, (i+j)/2, a), r = build((i+j)/2+1, j, a);
            lc[v] = l; rc[v] = r;
            val[v] = combine(val[l], val[r]);
        }
        return v;
    }

    int q(int v, int i, int j) {
        if(j < lx[v] || rx[v] < i)
            return 0;
        if(i <= lx[v] && rx[v] <= j)
            return val[v];
    }
};
```

```
return combine(q(lc[v], i, j), q(rc[v], i, j));
}

int u(int v, int i, int a) {
    if(i < lx[v] || rx[v] < i)
        return v;
    int w = lc.size();
    lc.push_back(lc[v]); rc.push_back(rc[v]);
    lx.push_back(lx[v]); rx.push_back(rx[v]);
    //CHANGE ME
    val.push_back(val[v]+a);
    if(lx[v] != rx[v]) {
        int l = u(lc[v], i, a), r = u(rc[v], i, a);
        lc[w] = l;
        rc[w] = r;
        val[w] = combine(val[l], val[r]);
    }
    return w;
}

//sum from i to j after t updates
int query(int i, int j, int t = -1) {
    if(t == -1) t = roots.size()-1;
    return q(roots[t], i, j);
}

//add a to position i at time t
void update(int i, int a, int t) {
    roots.push_back(u(roots[t], i, a));
}

};

/* USAGE */

int main() {
    int a[5] = {1,1,1,1,1};
    per_seg p;
    p.build(0, 4, a);
    p.update(1, 1);
    p.update(3, 2);
    cout << p.query(2, 2, 5) << "\n";
    cout << p.query(0, 2, 5) << "\n";
    cout << p.query(1, 0, 3) << "\n"
    return 0;
}
```

WaveletTree.h

1cd45c, 55 lines

```
template <class T>
struct wavelet{
    struct node {
        vector<int> b;
        T lo, hi, md;
    };
    vector<node> t;

    void build(const vector<T> &c, T *A, T *B, int v, int i,
               int j) {
        t[v].b.resize(B-A+1);
        t[v].lo = c[i], t[v].hi = c[j], t[v].md = c[(i+j)/2];
        for(int i = 0; A+i != B; ++i)
            t[v].b[i+1] = t[v].b[i] + (A[i]<=t[v].md);
        if(i == j) return;
        T *p = stable_partition(A, B, [=](int x){return x <= t[v].md;});
        build(c, A, p, v<<1, i, (i+j)/2);
        build(c, p, B, v<<1|1, (i+j)/2+1, j);
    }
};
```

```
void init(T *A, int n) {
    vector<T> c(A, A+n);
    sort(c.begin(), c.end());
    c.erase(unique(c.begin(), c.end()), c.end());
    int N = c.size();
    t.resize(N<<2);
    build(c, A, A+n, 1, 0, N-1);
}

wavelet(){}
wavelet(T *A, int n){init(A, n);}

//kth smallest element in [l, r]
T kth(int l, int r, int k, int v = 1) {
    if(t[v].lo == t[v].hi) return t[v].lo;
    int lb = t[v].b[l-1], rb = t[v].b[r], il = rb-lb;
    return (k < il) ? kth(lb+1, rb, k, v<<1) : kth(l-lb, r-rb, k-il, v<<1|1);
}

//number of elements in [l, r] <= to a
int leq(int l, int r, T a, int v = 1) {
    if(a < t[v].lo) return 0;
    if(t[v].hi <= a) return r-l+1;
    int lb = t[v].b[l-1], rb = t[v].b[r];
    return leq(lb+1, rb, a, v<<1) + leq(l-lb, r-rb, a, v<<1|1);
}

//number of elements in [l, r] equal to a
int count(int l, int r, T a, int v = 1) {
    if(a < t[v].lo || a > t[v].hi) return 0;
    if(t[v].lo == t[v].hi) return r-l+1;
    int lb = t[v].b[l-1], rb = t[v].b[r];
    if(a <= t[v].md) return count(lb+1, rb, a, v<<1);
    return count(l-lb, r-rb, a, v<<1|1);
}
};
```

## Numerical (4)

### 4.1 Polynomials and recurrences

Polynomial.h

c9b7b0, 17 lines

```
struct Poly {
    vector<double> a;
    double operator()(double x) const {
        double val = 0;
        for(int i = sz(a); i--;) (val *= x) += a[i];
        return val;
    }
    void diff() {
        rep(i, 1, sz(a)) a[i-1] = i*a[i];
        a.pop_back();
    }
    void divroot(double x0) {
        double b = a.back(), c; a.back() = 0;
        for(int i=sz(a)-1; i--;) c = a[i], a[i] = a[i+1]*x0+b, b=c;
        a.pop_back();
    }
};
```

PolyRoots.h  
**Description:** Finds the real roots to a polynomial.  
**Usage:** poly.roots({{2,-3,1}}, -1e9, 1e9) // solve x<sup>2</sup>-3x+2 = 0

<b>Time:</b> $\mathcal{O}(n^2 \log(1/\epsilon))$	
"Polynomial.h"	2cf190, 23 lines
<pre>vector&lt;double&gt; poly_roots(Poly p, double xmin, double xmax) {     if (sz(p.a) == 2) { return {-p.a[0]/p.a[1]}; }     vector&lt;double&gt; ret;     Poly der = p;     der.diff();     auto dr = poly_roots(der, xmin, xmax);     dr.push_back(xmin-1);     dr.push_back(xmax+1);     sort(all(dr));     rep(i,0,sz(dr)-1) {         double l = dr[i], h = dr[i+1];         bool sign = p(l) &gt; 0;         if (sign ^ (p(h) &gt; 0)) {             rep(it,0,60) { // while (h - l &gt; 1e-8)                 double m = (l + h) / 2, f = p(m);                 if ((f &lt;= 0) ^ sign) l = m;                 else h = m;             }             ret.push_back((l + h) / 2);         }     }     return ret; }</pre>	

### PolyInterpolate.h

**Description:** Given  $n$  points  $(x[i], y[i])$ , computes an  $n$ -1-degree polynomial  $p$  that passes through them:  $p(x) = a[0] * x^0 + \dots + a[n-1] * x^{n-1}$ . For numerical precision, pick  $x[k] = c * \cos(k/(n-1) * \pi), k = 0 \dots n-1$ .  
**Time:**  $\mathcal{O}(n^2)$

	08bf48, 13 lines
<pre>typedef vector&lt;double&gt; vd; vd interpolate(vd x, vd y, int n) {     vd res(n), temp(n);     rep(k,0,n-1) rep(i,k+1,n)         y[i] = (y[i] - y[k]) / (x[i] - x[k]);     double last = 0; temp[0] = 1;     rep(k,0,n) rep(i,0,n) {         res[i] += y[k] * temp[i];         swap(last, temp[i]);         temp[i] -= last * x[k];     }     return res; }</pre>	

### BerlekampMassey.h

**Description:** Recovers any  $n$ -order linear recurrence relation from the first  $2n$  terms of the recurrence. Useful for guessing linear recurrences after brute-forcing the first terms. Should work on any field, but numerical stability for floats is not guaranteed. Output will have size  $\leq n$ .  
**Usage:** BerlekampMassey({0, 1, 1, 3, 5, 11}) // {1, 2}  
**Time:**  $\mathcal{O}(N^2)$

"../number-theory/ModPow.h"	40387d, 20 lines
<pre>vector&lt;ll&gt; BerlekampMassey(vector&lt;ll&gt; s) {     int n = sz(s), L = 0, m = 0;     vector&lt;ll&gt; C(n), B(n), T;     C[0] = B[0] = 1;      ll b = 1;     rep(i,0,n) { ++m;         ll d = s[i] % mod;         rep(j,1,L+1) d = (d + C[j] * s[i - j]) % mod;         if (!d) continue;         T = C; ll coef = d * modpow(b, mod-2) % mod;         rep(j,m,n) C[j] = (C[j] - coef * B[j - m]) % mod;         if (2 * L &gt; i) continue;         L = i + 1 - L; B = T; b = d; m = 0;     }</pre>	

<pre>}  C.resize(L + 1); C.erase(C.begin()); trav(x, C) x = (mod - x) % mod; return C; }</pre>	

### LinearRecurrence.h

**Description:** Generates the  $k$ 'th term of an  $n$ -order linear recurrence  $S[i] = \sum_j S[i-j-1]tr[j]$ , given  $S[0 \dots \geq n-1]$  and  $tr[0 \dots n-1]$ . Faster than matrix multiplication. Useful together with Berlekamp-Massey.  
**Usage:** linearRec({0, 1}, {1, 1}, k) //  $k$ 'th Fibonacci number  
**Time:**  $\mathcal{O}(n^2 \log k)$

	f4e444, 26 lines
<pre>typedef vector&lt;ll&gt; Poly; ll linearRec(Poly S, Poly tr, ll k) {     int n = sz(tr);      auto combine = [&amp;](Poly a, Poly b) {         Poly res(n * 2 + 1);         rep(i,0,n+1) rep(j,0,n+1)             res[i + j] = (res[i + j] + a[i] * b[j]) % mod;         for (int i = 2 * n; i &gt; n; --i) rep(j,0,n)             res[i - 1 - j] = (res[i - 1 - j] + res[i] * tr[j]) % mod;         res.resize(n + 1);         return res;     };      Poly pol(n + 1), e(pol);     pol[0] = e[1] = 1;      for (++k; k; k /= 2) {         if (k % 2) pol = combine(pol, e);         e = combine(e, e);     }      ll res = 0;     rep(i,0,n) res = (res + pol[i + 1] * S[i]) % mod;     return res; }</pre>	

## 4.2 Optimization

### GoldenSectionSearch.h

**Description:** Finds the argument minimizing the function  $f$  in the interval  $[a,b]$  assuming  $f$  is unimodal on the interval, i.e. has only one local minimum. The maximum error in the result is  $\epsilon$ ps. Works equally well for maximization with a small change in the code. See TernarySearch.h in the Various chapter for a discrete version.  
**Usage:** double func(double x) { return 4+x+.3\*x\*x; }  
double xmin = gss(-1000,1000,func);  
**Time:**  $\mathcal{O}(\log((b-a)/\epsilon))$

	31d45b, 14 lines
<pre>double gss(double a, double b, double (*f)(double)) {     double r = (sqrt(5)-1)/2, eps = 1e-7;     double x1 = b - r*(b-a), x2 = a + r*(b-a);     double f1 = f(x1), f2 = f(x2);     while (b-a &gt; eps)         if (f1 &lt; f2) { //change to &gt; to find maximum             b = x2; x2 = x1; f2 = f1;             x1 = b - r*(b-a); f1 = f(x1);         } else {             a = x1; x1 = x2; f1 = f2;             x2 = a + r*(b-a); f2 = f(x2);         }     return a; }</pre>	

### HillClimbing.h

<b>Description:</b> Poor man's optimization for unimodal functions	f40e55, 16 lines
<pre>typedef array&lt;double, 2&gt; P;  double func(P p);  pair&lt;double, P&gt; hillClimb(P start) {     pair&lt;double, P&gt; cur(func(start), start);     for (double jmp = 1e9; jmp &gt; 1e-20; jmp /= 2) {         rep(j,0,100) rep(dx,-1,2) rep(dy,-1,2) {             P p = cur.second;             p[0] += dx*jmp;             p[1] += dy*jmp;             cur = min(cur, make_pair(func(p), p));         }     }     return cur; }</pre>	

### Integrate.h

**Description:** Simple integration of a function over an interval using Simpson's rule. The error should be proportional to  $h^4$ , although in practice you will want to verify that the result is stable to desired precision when epsilon changes.

	4756fc, 7 lines
<pre>template&lt;class F&gt; double quad(double a, double b, F f, const int n = 1000) {     double h = (b - a) / 2 / n, v = f(a) + f(b);     rep(i,1,n*2)         v += f(a + i*h) * (i&amp;1 ? 4 : 2);     return v * h / 3; }</pre>	
<b>IntegrateAdaptive.h</b>	
<p><b>Description:</b> Fast integration using an adaptive Simpson's rule. <b>Usage:</b> double sphereVolume = quad(-1, 1, [](double x) { return quad(-1, 1, [&amp;](double y) { return quad(-1, 1, [&amp;](double z) { return x*x + y*y + z*z &lt; 1; }}});});</p>	
	92dd79, 15 lines

<pre>typedef double d; #define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (b-a) / 6  template &lt;class F&gt; d rec(F&amp; f, d a, d b, d eps, d S) {     d c = (a + b) / 2;     d S1 = S(a, c), S2 = S(c, b), T = S1 + S2;     if (abs(T - S) &lt;= 15 * eps    b - a &lt; 1e-10)         return T + (T - S) / 15;     return rec(f, a, c, eps / 2, S1) + rec(f, c, b, eps / 2, S2); }  template&lt;class F&gt; d quad(d a, d b, F f, d eps = 1e-8) {     return rec(f, a, b, eps, S(a, b)); }</pre>	
--	--

### Simplex.h

**Description:** Solves a general linear maximization problem: maximize  $c^T x$  subject to  $Ax \leq b, x \geq 0$ . Returns -inf if there is no solution, inf if there are arbitrarily good solutions, or the maximum value of  $c^T x$  otherwise. The input vector is set to an optimal  $x$  (or in the unbounded case, an arbitrary solution fulfilling the constraints). Numerical stability is not guaranteed. For better performance, define variables such that  $x = 0$  is viable.  
**Usage:** vvd A = {{1,-1}, {-1,1}, {-1,-2}};  
vd b = {1,1,-4}, c = {-1,-1}, x;  
T val = LPSolver(A, b, c).solve(x);

**Time:**  $\mathcal{O}(NM \cdot \#\text{pivots})$ , where a pivot may be e.g. an edge relaxation.  $\mathcal{O}(2^n)$  in the general case.

aa8530, 68 lines

```
typedef double T; // long double, Rational, double + mod<P>...
typedef vector<T> vd;
typedef vector<vd> vvd;
```

```
const T eps = 1e-8, inf = 1/.0;
#define MP make_pair
#define ltj(X) if(s == -1 || MP(X[j],N[j]) < MP(X[s],N[s])) s=j
```

```
struct LPSolver {
    int m, n;
    vi N, B;
    vvd D;
```

```
    LPSolver(const vvd& A, const vd& b, const vd& c) :
        m(sz(b)), n(sz(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
        rep(i,0,m) rep(j,0,n) D[i][j] = A[i][j];
        rep(i,0,m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[i]; }
        rep(j,0,n) { N[j] = j; D[m][j] = -c[j]; }
        N[n] = -1; D[m+1][n] = 1;
    }
```

```
    void pivot(int r, int s) {
        T *a = D[r].data(), inv = 1 / a[s];
        rep(i,0,m+2) if (i != r && abs(D[i][s]) > eps) {
            T *b = D[i].data(), inv2 = b[s] * inv;
            rep(j,0,n+2) b[j] -= a[j] * inv2;
            b[s] = a[s] * inv2;
        }
        rep(j,0,n+2) if (j != s) D[r][j] *= inv;
        rep(i,0,m+2) if (i != r) D[i][s] *= -inv;
        D[r][s] = inv;
        swap(B[r], N[s]);
    }
}
```

```
bool simplex(int phase) {
    int x = m + phase - 1;
    for (;;) {
        int s = -1;
        rep(j,0,n+1) if (N[j] != -phase) ltj(D[x]);
        if (D[x][s] >= -eps) return true;
        int r = -1;
        rep(i,0,m) {
            if (D[i][s] <= eps) continue;
            if (r == -1 || MP(D[i][n+1] / D[i][s], B[i])
                < MP(D[r][n+1] / D[r][s], B[r])) r = i;
        }
        if (r == -1) return false;
        pivot(r, s);
    }
}
```

```
T solve(vd &x) {
    int r = 0;
    rep(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
    if (D[r][n+1] < -eps) {
        pivot(r, n);
        if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
        rep(i,0,m) if (B[i] == -1) {
            int s = 0;
            rep(j,1,n+1) ltj(D[i]);
            pivot(i, s);
        }
    }
    bool ok = simplex(1); x = vd(n);
    rep(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
    return ok ? D[m][n+1] : inf;
}
```

```
    }
};
```

### 4.3 Matrices

#### Determinant.h

**Description:** Calculates determinant of a matrix. Destroys the matrix.  
**Time:**  $\mathcal{O}(N^3)$

bd5cec, 15 lines

```
double det(vector<vector<double>>& a) {
    int n = sz(a); double res = 1;
    rep(i,0,n) {
        int b = i;
        rep(j,i+1,n) if (fabs(a[j][i]) > fabs(a[b][i])) b = j;
        if (i != b) swap(a[i], a[b]), res *= -1;
        res *= a[i][i];
        if (res == 0) return 0;
        rep(j,i+1,n) {
            double v = a[j][i] / a[i][i];
            if (v != 0) rep(k,i+1,n) a[j][k] -= v * a[i][k];
        }
    }
    return res;
}
```

#### IntDeterminant.h

**Description:** Calculates determinant using modular arithmetics. Modulos can also be removed to get a pure-integer version.  
**Time:**  $\mathcal{O}(N^3)$

3313dc, 18 lines

```
const ll mod = 12345;
ll det(vector<vector<ll>>& a) {
    int n = sz(a); ll ans = 1;
    rep(i,0,n) {
        rep(j,i+1,n) {
            while (a[j][i] != 0) { // gcd step
                ll t = a[i][i] / a[j][i];
                if (t) rep(k,i,n)
                    a[i][k] = (a[i][k] - a[j][k] * t) % mod;
                swap(a[i], a[j]);
                ans *= -1;
            }
        }
        ans = ans * a[i][i] % mod;
        if (!ans) return 0;
    }
    return (ans + mod) % mod;
}
```

#### SolveLinear.h

**Description:** Solves  $A * x = b$ . If there are multiple solutions, an arbitrary one is returned. Returns rank, or -1 if no solutions. Data in  $A$  and  $b$  is lost.  
**Time:**  $\mathcal{O}(n^2m)$

44c9ab, 38 lines

```
typedef vector<double> vd;
const double eps = 1e-12;

int solveLinear(vector<vd>& A, vd& b, vd& x) {
    int n = sz(A), m = sz(x), rank = 0, br, bc;
    if (n) assert(sz(A[0]) == m);
    vi col(m); iota(all(col), 0);

    rep(i,0,n) {
        double v, bv = 0;
        rep(r,i,n) rep(c,i,m)
            if ((v = fabs(A[r][c])) > bv)
                br = r, bc = c, bv = v;
        if (bv <= eps) {
```

```
            rep(j,i,n) if (fabs(b[j]) > eps) return -1;
            break;
        }
        swap(A[i], A[br]);
        swap(b[i], b[br]);
        swap(col[i], col[bc]);
        rep(j,0,n) swap(A[j][i], A[j][bc]);
        bv = 1/A[i][i];
        rep(j,i+1,n) {
            double fac = A[j][i] * bv;
            b[j] -= fac * b[i];
            rep(k,i+1,m) A[j][k] -= fac*A[i][k];
        }
        rank++;
    }

    x.assign(m, 0);
    for (int i = rank; i--;) {
        b[i] /= A[i][i];
        x[col[i]] = b[i];
        rep(j,0,i) b[j] -= A[j][i] * b[i];
    }
    return rank; // (multiple solutions if rank < m)
}
```

#### SolveLinear2.h

**Description:** To get all uniquely determined values of  $x$  back from SolveLinear, make the following changes:

08e495, 7 lines

```
"SolveLinear.h"
rep(j,0,n) if (j != i) // instead of rep(j,i+1,n)
// ... then at the end:
x.assign(m, undefined);
rep(i,0,rank) {
    rep(j,rank,m) if (fabs(A[i][j]) > eps) goto fail;
    x[col[i]] = b[i] / A[i][i];
fail:; }
```

#### SolveLinearBinary.h

**Description:** Solves  $Ax = b$  over  $\mathbb{F}_2$ . If there are multiple solutions, one is returned arbitrarily. Returns rank, or -1 if no solutions. Destroys  $A$  and  $b$ .  
**Time:**  $\mathcal{O}(n^2m)$

fa2d7a, 34 lines

```
typedef bitset<1000> bs;

int solveLinear(vector<bs>& A, vi& b, bs& x, int m) {
    int n = sz(A), rank = 0, br;
    assert(m <= sz(x));
    vi col(m); iota(all(col), 0);
    rep(i,0,n) {
        for (br=i; br<n; ++br) if (A[br].any()) break;
        if (br == n) {
            rep(j,i,n) if(b[j]) return -1;
            break;
        }
        int bc = (int)A[br]._Find_next(i-1);
        swap(A[i], A[br]);
        swap(b[i], b[br]);
        swap(col[i], col[bc]);
        rep(j,0,n) if (A[j][i] != A[j][bc]) {
            A[j].flip(i); A[j].flip(bc);
        }
        rep(j,i+1,n) if (A[j][i]) {
            b[j] ^= b[i];
            A[j] ^= A[i];
        }
        rank++;
    }
}
```

```
x = bs();
for (int i = rank; i--;) {
    if (!b[i]) continue;
    x[col[i]] = 1;
    rep(j,0,i) b[j][i] ^= A[j][i];
}
return rank; // (multiple solutions if rank < m)
}
```

MatrixInverse.h

**Description:** Invert matrix  $A$ . Returns rank; result is stored in  $A$  unless singular ( $\text{rank} < n$ ). Can easily be extended to prime moduli; for prime powers, repeatedly set  $A^{-1} = A^{-1}(2I - AA^{-1}) \pmod{p^k}$  where  $A^{-1}$  starts as the inverse of  $A \bmod p$ , and  $k$  is doubled in each step.

**Time:**  $\mathcal{O}(n^3)$

ebfff6, 35 lines

```
int matInv(vector<vector<double>>& A) {
    int n = sz(A); vi col(n);
    vector<vector<double>> tmp(n, vector<double>(n));
    rep(i,0,n) tmp[i][i] = 1, col[i] = i;

    rep(i,0,n) {
        int r = i, c = i;
        rep(j,i,n) rep(k,i,n)
            if (fabs(A[j][k]) > fabs(A[r][c]))
                r = j, c = k;
        if (fabs(A[r][c]) < 1e-12) return i;
        A[i].swap(A[r]); tmp[i].swap(tmp[r]);
        rep(j,0,n) swap(A[j][i], A[j][c]), swap(tmp[j][i], tmp[j][c]);
        swap(col[i], col[c]);
        double v = A[i][i];
        rep(j,i+1,n) {
            double f = A[j][i] / v;
            A[j][i] = 0;
            rep(k,i+1,n) A[j][k] -= f*A[i][k];
            rep(k,0,n) tmp[j][k] -= f*tmp[i][k];
        }
        rep(j,i+1,n) A[i][j] /= v;
        rep(j,0,n) tmp[i][j] /= v;
        A[i][i] = 1;
    }

    for (int i = n-1; i > 0; --i) rep(j,0,i) {
        double v = A[j][i];
        rep(k,0,n) tmp[j][k] -= v*tmp[i][k];
    }

    rep(i,0,n) rep(j,0,n) A[col[i]][col[j]] = tmp[i][j];
    return n;
}
```

Tridiagonal.h

**Description:**  $x = \text{tridiagonal}(d, p, q, b)$  solves the equation system

$b_0$

$b_1$

$b_2$

$b_3$

$\vdots$

$b_{n-1}$

$\left( \begin{array}{c} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{array} \right) = \left( \begin{array}{cccccc} d_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_0 & d_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_1 & d_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n-3} & d_{n-2} & p_{n-2} \\ 0 & 0 & \cdots & 0 & q_{n-2} & d_{n-1} \end{array} \right) \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{array} \right).$

This is useful for solving problems on the type

$$a_i = b_i a_{i-1} + c_i a_{i+1} + d_i, 1 \leq i \leq n,$$

where  $a_0, a_{n+1}, b_i, c_i$  and  $d_i$  are known.  $a$  can then be obtained from

$$\{a_i\} = \text{tridiagonal}(\{1, -1, -1, \dots, -1, 1\}, \{0, c_1, c_2, \dots, c_n\}, \{b_1, b_2, \dots, b_n, 0\}, \{a_0, d_1, d_2, \dots, d_n, a_{n+1}\}).$$

Fails if the solution is not unique.

If  $|d_i| > |p_i| + |q_{i-1}|$  for all  $i$ , or  $|d_i| > |p_{i-1}| + |q_i|$ , or the matrix is positive definite, the algorithm is numerically stable and neither `tr` nor the check for `diag[i] == 0` is needed.

**Time:**  $\mathcal{O}(N)$

8f9fa8, 26 lines

```
typedef double T;
vector<T> tridiagonal(vector<T> diag, const vector<T>& super,
    const vector<T>& sub, vector<T> b) {
    int n = sz(b); vi tr(n);
    rep(i,0,n-1) {
        if (abs(diag[i]) < 1e-9 * abs(super[i])) { // diag[i] == 0
            b[i+1] -= b[i] * diag[i+1] / super[i];
            if (i+2 < n) b[i+2] -= b[i] * sub[i+1] / super[i];
            diag[i+1] = sub[i]; tr[++i] = 1;
        } else {
            diag[i+1] -= super[i]*sub[i]/diag[i];
            b[i+1] -= b[i]*sub[i]/diag[i];
        }
    }
    for (int i = n; i--;) {
        if (tr[i]) {
            swap(b[i], b[i-1]);
            diag[i-1] = diag[i];
            b[i] /= super[i-1];
        } else {
            b[i] /= diag[i];
            if (i) b[i-1] -= b[i]*super[i-1];
        }
    }
    return b;
}
```

#### 4.4 Fourier transforms

FastFourierTransform.h

**Description:** `fft(a)` computes  $\hat{f}(k) = \sum_x a[x] \exp(2\pi i \cdot kx/N)$  for all  $k$ . Useful for convolution: `conv(a, b) = c`, where  $c[x] = \sum a[i]b[x-i]$ . For convolution of complex numbers or more than two vectors: FFT, multiply pointwise, divide by  $n$ , reverse(`start+1`, `end`), FFT back. Rounding is safe if  $(\sum a_i^2 + \sum b_i^2) \log_2 N < 9 \cdot 10^{14}$  (in practice  $10^{16}$ ; higher for random inputs). Otherwise, use long doubles/NTT/FFTMod.

**Time:**  $\mathcal{O}(N \log N)$  with  $N = |A| + |B|$  ( $\sim 1s$  for  $N = 2^{22}$ )

aca9fa, 35 lines

```
typedef complex<double> C;
typedef vector<double> vd;
void fft(vector<C>& a) {
    int n = sz(a), L = 31 - __builtin_clz(n);
    static vector<complex<long double>> R(2, 1);
    static vector<C> rt(2, 1); // (^ 10% faster if double)
    for (static int k = 2; k < n; k *= 2) {
        R.resize(n); rt.resize(n);
        auto x = polar(1.0L, M_PI1 / k); // M_PI, lower-case L
        rep(i,k,2*k) rt[i] = R[i] = i&1 ? R[i/2] * x : R[i/2];
    }
    vi rev(n);
    rep(i,0,n) rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
    rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
    for (int k = 1; k < n; k *= 2)
        for (int i = 0; i < n; i += 2 * k) rep(j,0,k) {
            C z = rt[j+k] * a[i+j+k]; // (25% faster if hand-rolled)
            a[i + j + k] = a[i + j] - z;
            a[i + j] += z;
        }
}
```

```
    }
}
vd conv(const vd& a, const vd& b) {
    if (a.empty() || b.empty()) return {};
    vd res(sz(a) + sz(b) - 1);
    int L = 32 - __builtin_clz(sz(res)), n = 1 << L;
    vector<C> in(n), out(n);
    copy(all(a), begin(in));
    rep(i,0,sz(b)) in[i].imag(b[i]);
    fft(in);
    trav(x, in) x *= x;
    rep(i,0,n) out[i] = in[-i & (n - 1)] - conj(in[i]);
    fft(out);
    rep(i,0,sz(res)) res[i] = imag(out[i]) / (4 * n);
    return res;
}
```

FastFourierTransformMod.h

**Description:** Higher precision FFT, can be used for convolutions modulo arbitrary integers as long as  $N \log_2 N \cdot \text{mod} < 8.6 \cdot 10^{14}$  (in practice  $10^{16}$  or higher). Inputs must be in  $[0, \text{mod})$ .

**Time:**  $\mathcal{O}(N \log N)$ , where  $N = |A| + |B|$  (twice as slow as NTT or FFT)

b82773, 22 lines

```
typedef vector<ll> vl;
template<int M> vl convMod(const vl &a, const vl &b) {
    if (a.empty() || b.empty()) return {};
    vl res(sz(a) + sz(b) - 1);
    int B=32-__builtin_clz(sz(res)), n=1<<B, cut=int(sqrt(M));
    vector<C> L(n), R(n), outs(n), outl(n);
    rep(i,0,sz(a)) L[i] = C((int)a[i] / cut, (int)a[i] % cut);
    rep(i,0,sz(b)) R[i] = C((int)b[i] / cut, (int)b[i] % cut);
    fft(L), fft(R);
    rep(i,0,n) {
        int j = -i & (n - 1);
        outl[j] = (L[i] + conj(L[j])) * R[i] / (2.0 * n);
        outs[j] = (L[i] - conj(L[j])) * R[i] / (2.0 * n) / 1i;
    }
    fft(outl), fft(outs);
    rep(i,0,sz(res)) {
        ll av = ll(real(outl[i])+.5), cv = ll(imag(outs[i])+.5);
        ll bv = ll(imag(outl[i])+.5) + ll(real(outs[i])+.5);
        res[i] = ((av % M * cut + bv) % M * cut + cv) % M;
    }
    return res;
}
```

NumberTheoreticTransform.h

**Description:** Can be used for convolutions modulo specific nice primes of the form  $2^a b + 1$ , where the convolution result has size at most  $2^a$ . Inputs must be in  $[0, \text{mod})$ .

**Time:**  $\mathcal{O}(N \log N)$

d75aad, 32 lines

```
const ll mod = (119 << 23) + 1, root = 62; // = 998244353
// For p < 2^30 there is also e.g. 5 << 25, 7 << 26, 479 << 21
// and 483 << 21 (same root). The last two are > 10^9.

typedef vector<ll> vl;
void ntt(vl& a, vl& rt, vl& rev, int n) {
    rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
    for (int k = 1; k < n; k *= 2)
        for (int i = 0; i < n; i += 2 * k) rep(j,0,k) {
            ll z = rt[j+k] * a[i+j+k] % mod, &ai = a[i+j];
            ai + j + k] = (z > ai ? ai - z + mod : ai - z);
            ai += (ai + z >= mod ? z - mod : z);
        }
}
```



```

vl conv(const vl& a, const vl& b) {
    if (a.empty() || b.empty())
        return {};
    int s = sz(a)+sz(b)-1, B = 32 - __builtin_clz(s), n = 1 << B;
    vl L(a), R(b), out(n), rt(n, 1), rev(n);
    L.resize(n), R.resize(n);
    rep(i,0,n) rev[i] = (rev[i / 2] | (i & 1) << B) / 2;
    ll curL = mod / 2, inv = modpow(n, mod - 2);
    for (int k = 2; k < n; k *= 2) {
        ll z[] = {1, modpow(root, curL /= 2)};
        rep(i,k,2*k) rt[i] = rt[i / 2] * z[i & 1] % mod;
    }
    ntt(L, rt, rev, n); ntt(R, rt, rev, n);
    rep(i,0,n) out[-i & (n-1)] = L[i] * R[i] % mod * inv % mod;
    ntt(out, rt, rev, n);
    return {out.begin(), out.begin() + s};
}
```

### FastSubsetTransform.h

**Description:** Transform to a basis with fast convolutions of the form  $c[z] = \sum_{z=x\oplus y} a[x] \cdot b[y]$ , where  $\oplus$  is one of AND, OR, XOR. The size of  $a$  must be a power of two.  
**Time:**  $\mathcal{O}(N \log N)$

```

void FST(vi& a, bool inv) {
    for (int n = sz(a), step = 1; step < n; step *= 2) {
        for (int i = 0; i < n; i += 2 * step) rep(j,i,i+step) {
            int &u = a[j], &v = a[j + step]; tie(u, v) =
                inv ? pii(v - u, u) : pii(v, u + v); // AND
                inv ? pii(v, u - v) : pii(u + v, u); // OR
                pii(u + v, u - v); // XOR
        }
    }
    if (inv) trav(x, a) x /= sz(a); // XOR only
}
vi conv(vi a, vi b) {
    FST(a, 0); FST(b, 0);
    rep(i,0,sz(a)) a[i] *= b[i];
    FST(a, 1); return a;
}
```

## Number theory (5)

### 5.1 Modular arithmetic

#### ModularArithmetic.h

**Description:** Operators for modular arithmetic. You need to set mod to some number first and then you can use the structure.

```

"euclid.h"
35bfea, 18 lines

const ll mod = 17; // change to something else
struct Mod {
    ll x;
    Mod(ll xx) : x(xx) {}
    Mod operator+(Mod b) { return Mod((x + b.x) % mod); }
    Mod operator-(Mod b) { return Mod((x - b.x + mod) % mod); }
    Mod operator*(Mod b) { return Mod((x * b.x) % mod); }
    Mod operator/(Mod b) { return *this * invert(b); }
    Mod invert(Mod a) {
        ll x, y, g = euclid(a.x, mod, x, y);
        assert(g == 1); return Mod((x + mod) % mod);
    }
    Mod operator^(ll e) {
        if (!e) return Mod(1);
        Mod r = *this ^ (e / 2); r = r * r;
        return e&1 ? *this * r : r;
    }
}
```

```
};
```

#### ModInverse.h

**Description:** Pre-computation of modular inverses. Assumes  $\text{LIM} \leq \text{mod}$  and that mod is a prime.

```

6f684f, 3 lines

const ll mod = 1000000007, LIM = 200000;
ll* inv = new ll[LIM] - 1; inv[1] = 1;
rep(i,2,LIM) inv[i] = mod - (mod / i) * inv[mod % i] % mod;
```

#### ModPow.h

```

b83e45, 8 lines

const ll mod = 1000000007; // faster if const

ll modpow(ll b, ll e) {
    ll ans = 1;
    for (; e; b = b * b % mod, e /= 2)
        if (e & 1) ans = ans * b % mod;
    return ans;
}
```

#### ModLog.h

**Description:** Returns the smallest  $x \geq 0$  s.t.  $a^x = b \pmod m$ . a and m must be coprime.  
**Time:**  $\mathcal{O}(\sqrt{m})$

```

49d606, 10 lines

ll modLog(ll a, ll b, ll m) {
    assert(__gcd(a, m) == 1);
    ll n = (ll) sqrt(m) + 1, e = 1, x = 1, res = LLONG_MAX;
    unordered_map<ll, ll> f;
    rep(i,0,n) e = e * a % m;
    rep(i,0,n) x = x * e % m, f.emplace(x, i + 1);
    rep(i,0,n) if (f.count(b = b * a % m))
        res = min(res, f[b] * n - i - 1);
    return res;
}
```

#### ModSum.h

**Description:** Sums of mod'ed arithmetic progressions.  
 $\text{modsum}(\text{to}, c, k, m) = \sum_{i=0}^{\text{to}-1} (ki + c) \% m$ . divsum is similar but for floored division.  
**Time:**  $\log(m)$ , with a large constant.

```

5c5bc5, 16 lines

typedef unsigned long long ull;
ull sumsq(ull to) { return to / 2 * ((to-1) | 1); }

ull divsum(ull to, ull c, ull k, ull m) {
    ull res = k / m * sumsq(to) + c / m * to;
    k %= m; c %= m;
    if (!k) return res;
    ull to2 = (to * k + c) / m;
    return res + (to - 1) * to2 - divsum(to2, m-1 - c, m, k);
}

ll modsum(ull to, ll c, ll k, ll m) {
    c = ((c % m) + m) % m;
    k = ((k % m) + m) % m;
    return to * c + k * sumsq(to) - m * divsum(to, c, k, m);
}
```

#### ModMulLL.h

**Description:** Calculate  $a \cdot b \bmod c$  (or  $a^b \bmod c$ ) for  $0 \leq a, b < c < 2^{63}$ .  
**Time:**  $\mathcal{O}(1)$  for mod\_mul,  $\mathcal{O}(\log b)$  for mod\_pow

```

88c37a, 12 lines

typedef unsigned long long ull;
typedef long double ld;
ull mod_mul(ull a, ull b, ull M) {
    ll ret = a * b - M * ull(ld(a) * ld(b) / ld(M));
}
```

```

return ret + M * (ret < 0) - M * (ret >= (ll)M);
}
ull mod_pow(ull b, ull e, ull mod) {
    ull ans = 1;
    for (; e; b = mod_mul(b, b, mod), e /= 2)
        if (e & 1) ans = mod_mul(ans, b, mod);
    return ans;
}
```

#### ModSqrt.h

**Description:** Tonelli-Shanks algorithm for modular square roots. Finds  $x$  s.t.  $x^2 = a \pmod p$  ( $-x$  gives the other solution).  
**Time:**  $\mathcal{O}(\log^2 p)$  worst case,  $\mathcal{O}(\log p)$  for most  $p$

```

"ModPow.h"
19a793, 24 lines

ll sqrt(ll a, ll p) {
    a %= p; if (a < 0) a += p;
    if (a == 0) return 0;
    assert(modpow(a, (p-1)/2, p) == 1); // else no solution
    if (p % 4 == 3) return modpow(a, (p+1)/4, p);
    // a^(n+3)/8 or 2^(n+3)/8 * 2^(n-1)/4 works if p % 8 == 5
    ll s = p - 1, n = 2;
    int r = 0, m;
    while (s % 2 == 0)
        ++r, s /= 2;
    while (modpow(n, (p - 1) / 2, p) != p - 1) ++n;
    ll x = modpow(a, (s + 1) / 2, p);
    ll b = modpow(a, s, p), g = modpow(n, s, p);
    for (; r = m) {
        ll t = b;
        for (m = 0; m < r && t != 1; ++m)
            t = t * t % p;
        if (m == 0) return x;
        ll gs = modpow(g, 1LL << (r - m - 1), p);
        g = gs * gs % p;
        x = x * gs % p;
        b = b * g % p;
    }
}
```

## 5.2 Primality

#### eratosthenes.h

**Description:** Prime sieve for generating all primes up to a certain limit.  $\text{isprime}[i]$  is true iff  $i$  is a prime.  
**Time:**  $\text{lim}=100'000'000 \approx 0.8$  s. Runs 30% faster if only odd indices are stored.

```

29cd0a, 11 lines

const int MAX_PR = 5'000'000;
bitset<MAX_PR> isprime;
vi eratosthenes_sieve(int lim) {
    isprime.set(); isprime[0] = isprime[1] = 0;
    for (int i = 4; i < lim; i += 2) isprime[i] = 0;
    for (int i = 3; i*i < lim; i += 2) if (isprime[i])
        for (int j = i*i; j < lim; j += i*2) isprime[j] = 0;
    vi pr;
    rep(i,2,lim) if (isprime[i]) pr.push_back(i);
    return pr;
}
```

#### MillerRabin.h

**Description:** Deterministic Miller-Rabin primality test. Guaranteed to work for numbers up to  $2^{64}$ ; for larger numbers, extend A randomly.  
**Time:** 7 times the complexity of  $a^b \bmod c$ .

```

"ModMulLL.h"
6ab8e1, 12 lines

bool isPrime(ull n) {
    if (n < 2 || n % 6 % 4 != 1) return n - 2 < 2;
```

```
ull A[] = {2, 325, 9375, 28178, 450775, 9780504, 1795265022},
s = __builtin_ctzll(n-1), d = n >> s;
trav(a, A) { // ^ count trailing zeroes
    ull p = mod_pow(a, d, n), i = s;
    while (p != 1 && p != n - 1 && a % n && i--)
        p = mod_mul(p, p, n);
    if (p != n-1 && i != s) return 0;
}
return 1;
}
```

Factor.h

**Description:** Pollard-rho randomized factorization algorithm. Returns prime factors of a number, in arbitrary order (e.g. 2299 -> {11, 19, 11}).

**Time:**  $\mathcal{O}\left(n^{1/4}\right)$  gcd calls, less for numbers with small factors.

```
"ModMullL.h", "MillerRabin.h" f5adaa, 18 lines

ull pollard(ull n) {
    auto f = [n](ull x) { return (mod_mul(x, x, n) + 1) % n; };
    if (!(n & 1)) return 2;
    for (ull i = 2;; i++) {
        ull x = i, y = f(x), p;
        while ((p = __gcd(n + y - x, n)) == 1)
            x = f(x), y = f(f(y));
        if (p != n) return p;
    }
}

vector<ull> factor(ull n) {
    if (n == 1) return {};
    if (isPrime(n)) return {n};
    ull x = pollard(n);
    auto l = factor(x), r = factor(n / x);
    l.insert(l.end(), all(r));
    return l;
}
```

5.3 Divisibility

euclid.h

**Description:** Finds two integers  $x$  and  $y$ , such that  $ax + by = \gcd(a, b)$ . If you just need gcd, use the built in `__gcd` instead. If  $a$  and  $b$  are coprime, then  $x$  is the inverse of  $a$  (mod  $b$ ).

```
ll euclid(ll a, ll b, ll &x, ll &y) {
    if (b) { ll d = euclid(b, a % b, y, x);
        return y -= a/b * x, d; }
    return x = 1, y = 0, a;
}
```

Euclid.java

**Description:** Finds {x, y, d} s.t.  $ax + by = d = \gcd(a, b)$ .

```
static BigInteger[] euclid(BigInteger a, BigInteger b) {
    BigInteger x = BigInteger.ONE, yy = x;
    BigInteger y = BigInteger.ZERO, xx = y;
    while (b.signum() != 0) {
        BigInteger q = a.divide(b), t = b;
        b = a.mod(b); a = t;
        t = xx; xx = x.subtract(q.multiply(xx)); x = t;
        t = yy; yy = y.subtract(q.multiply(yy)); y = t;
    }
    return new BigInteger[]{x, y, a};
}
```

CRT.h

**Description:** Chinese Remainder Theorem.

`crt(a, m, b, n)` computes  $x$  such that  $x \equiv a \pmod{m}$ ,  $x \equiv b \pmod{n}$ . If  $|a| < m$  and  $|b| < n$ ,  $x$  will obey  $0 \leq x < \text{lcm}(m, n)$ . Assumes  $mn < 2^{62}$ .

```
"euclid.h" 04d93a, 7 lines

ll crt(ll a, ll m, ll b, ll n) {
    if (n > m) swap(a, b), swap(m, n);
    ll x, y, g = euclid(m, n, x, y);
    assert((a - b) % g == 0); // else no solution
    x = (b - a) % n * x % n / g * m + a;
    return x < 0 ? x + m*n/g : x;
}
```

5.3.1 Bézout’s identity

For  $a \neq, b \neq 0$ , then  $d = \gcd(a, b)$  is the smallest positive integer for which there are integer solutions to

$$ax + by = d$$

If  $(x, y)$  is one solution, then all solutions are given by

$$\left(x + \frac{kb}{\gcd(a,b)}, y - \frac{ka}{\gcd(a,b)}\right), \quad k \in \mathbb{Z}$$

phiFunction.h

**Description:** *Euler’s*  $\phi$  function is defined as  $\phi(n) := \#$  of positive integers  $\leq n$  that are coprime with  $n$ .  $\phi(1) = 1$ ,  $p$  prime  $\Rightarrow \phi(p^k) = (p - 1)p^{k-1}$ ,  $m, n$  coprime  $\Rightarrow \phi(mn) = \phi(m)\phi(n)$ . If  $n = p_1^{k_1}p_2^{k_2}...p_r^{k_r}$  then  $\phi(n) = (p_1 - 1)p_1^{k_1-1}...(p_r - 1)p_r^{k_r-1}$ .  $\phi(n) = n \cdot \prod_{p|n}(1 - 1/p)$ .  $\sum_{d|n} \phi(d) = n$ ,  $\sum_{1 \leq k \leq n, \gcd(k, n)=1} k = n\phi(n)/2, n > 1$

**Euler’s thm:**  $a, n$  coprime  $\Rightarrow a^{\phi(n)} \equiv 1 \pmod{n}$ .

**Fermat’s little thm:**  $p$  prime  $\Rightarrow a^{p-1} \equiv 1 \pmod{p} \ \forall a$ .

```
const int LIM = 5000000;
int phi[LIM];

void calculatePhi() {
    rep(i, 0, LIM) phi[i] = i&1 ? i : i/2;
    for(int i = 3; i < LIM; i += 2) if(phi[i] == i)
        for(int j = i; j < LIM; j += i) phi[j] -= phi[j] / i;
}
```

5.4 Fractions

ContinuedFractions.h

**Description:** Given  $N$  and a real number  $x \geq 0$ , finds the closest rational approximation  $p/q$  with  $p, q \leq N$ . It will obey  $|p/q - x| \leq 1/qN$ . For consecutive convergents,  $p_{k+1}q_k - q_{k+1}p_k = (-1)^k$ . ( $p_k/q_k$  alternates between  $> x$  and  $< x$ .) If  $x$  is rational,  $y$  eventually becomes  $\infty$ ; if  $x$  is the root of a degree 2 polynomial the  $a$ ’s eventually become cyclic.

```
"Time: O(log N) dd6c5e, 21 lines

typedef double d; // for N ~ 1e7; long double for N ~ 1e9
pair<ll, ll> approximate(d x, ll N) {
    ll LP = 0, LQ = 1, P = 1, Q = 0, inf = LLONG_MAX; d y = x;
    for (;;) {
        ll lim = min(P ? (N-LP) / P : inf, Q ? (N-LQ) / Q : inf),
            a = (ll)floor(y), b = min(a, lim),
            NP = b*P + LP, NQ = b*Q + LQ;
        if (a > b) {
            // If b > a/2, we have a semi-convergent that gives us a
```

```
// better approximation; if b = a/2, we *may* have one.
// Return {P, Q} here for a more canonical approximation.
return (abs(x - (d)NP / (d)NQ) < abs(x - (d)P / (d)Q)) ?
    make_pair(NP, NQ) : make_pair(P, Q);

}
if (abs(y = 1/(y - (d)a)) > 3*N) {
    return {NP, NQ};
}
LP = P; P = NP;
LQ = Q; Q = NQ;
}
}
```

FracBinarySearch.h

**Description:** Given  $f$  and  $N$ , finds the smallest fraction  $p/q \in [0, 1]$  such that  $f(p/q)$  is true, and  $p, q \leq N$ . You may want to throw an exception from  $f$  if it finds an exact solution, in which case  $N$  can be removed.

**Usage:** `fracBS([f](Frac f) { return f.p>=3*f.q; }, 10);` // {1,3}

```
"Time: O(log(N)) 27ab3e, 25 lines

struct Frac { ll p, q; };

template<class F>
Frac fracBS(F f, ll N) {
    bool dir = 1, A = 1, B = 1;
    Frac lo{0, 1}, hi{1, 1}; // Set hi to 1/0 to search (0, N]
    if (f(lo)) return lo;
    assert(f(hi));
    while (A || B) {
        ll adv = 0, step = 1; // move hi if dir, else lo
        for (int si = 0; step; (step *= 2) >= si) {
            adv += step;
            Frac mid{lo.p * adv + hi.p, lo.q * adv + hi.q};
            if (abs(mid.p) > N || mid.q > N || dir == !f(mid)) {
                adv -= step; si = 2;
            }
        }
        hi.p += lo.p * adv;
        hi.q += lo.q * adv;
        dir = !dir;
        swap(lo, hi);
        A = B; B = !adv;
    }
    return dir ? hi : lo;
}
```

5.5 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2),$$

with  $m > n > 0, k > 0, m \perp n$ , and either  $m$  or  $n$  even.

5.6 Primes

$p = 962592769$  is such that  $2^{21} \mid p - 1$ , which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power  $p^a$ , except for  $p = 2, a > 2$ , and there are  $\phi(\phi(p^a))$  many. For  $p = 2, a > 2$ , the group  $\mathbb{Z}_{2^a}^\times$  is instead isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$ .

### 5.7 Estimates

$$\sum_{d|n} d = O(n \log \log n).$$

The number of divisors of  $n$  is at most around 100 for  $n < 5e4$ , 500 for  $n < 1e7$ , 2000 for  $n < 1e10$ , 200 000 for  $n < 1e19$ .

## Combinatorial (6)

### 6.1 Permutations

#### 6.1.1 Factorial

$n$	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
$n$	11	12	13	14	15	16	17			
$n!$	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14			
$n$	20	25	30	40	50	100	150	171		
$n!$	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBLMAX		

IntPerm.h

**Description:** Permutation -> integer conversion. (Not order preserving.) Integer -> permutation can use a lookup table.

**Time:**  $\mathcal{O}(n)$

```
int permToInt(vi& v) {
    int use = 0, i = 0, r = 0;
    trav(x, v) r = r * ++i + __builtin_popcount(use & -(1 << x)),
        use |= 1 << x;
    return r;
}
```

e1b8ea, 6 lines

#### 6.1.2 Cycles

Let  $g_S(n)$  be the number of  $n$ -permutations whose cycle lengths all belong to the set  $S$ . Then

$$\sum_{n=0}^\infty g_S(n) \frac{x^n}{n!} = \exp\left(\sum_{n \in S} \frac{x^n}{n}\right)$$

#### 6.1.3 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1)+D(n-2)) = nD(n-1)+(-1)^n = \left\lfloor \frac{n!}{e} \right\rfloor$$

IntPerm binomialModPrime multinomial

**6.1.4 Burnside’s lemma**

Given a group  $G$  of symmetries and a set  $X$ , the number of elements of  $X$  *up to symmetry* equals

$$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where  $X^g$  are the elements fixed by  $g$  ( $g.x = x$ ).

If  $f(n)$  counts “configurations” (of some sort) of length  $n$ , we can ignore rotational symmetry using  $G = \mathbb{Z}_n$  to get

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k|n} f(k) \phi(n/k).$$

**6.2 Partitions and subsets**

**6.2.1 Partition function**

Number of ways of writing  $n$  as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \quad p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k - 1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

$n$	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	$\sim 2e5$	$\sim 2e8$

**6.2.2 Binomials**

binomialModPrime.h

**Description:** Lucas’ thm: Let  $n, m$  be non-negative integers and  $p$  a prime. Write  $n = n_k p^k + \dots + n_1 p + n_0$  and  $m = m_k p^k + \dots + m_1 p + m_0$ . Then  $\binom{n}{m} \equiv \prod_{i=0}^k \binom{n_i}{m_i} \pmod{p}$ . fact and invfact must hold pre-computed factorials / inverse factorials, e.g. from ModInverse.h.

**Time:**  $\mathcal{O}(\log_p n)$

```
ll chooseModP(ll n, ll m, int p, vi& fact, vi& invfact) {
    ll c = 1;
    while (n || m) {
        ll a = n % p, b = m % p;
        if (a < b) return 0;
        c = c * fact[a] % p * invfact[b] % p * invfact[a - b] % p;
        n /= p; m /= p;
    }
    return c;
}
```

81845f, 10 lines

multinomial.h

**Description:** Computes  $\binom{k_1 + \dots + k_n}{k_1, k_2, \dots, k_n} = \frac{(\sum k_i)!}{k_1! k_2! \dots k_n!}$ .

a0a312, 6 lines

```
ll multinomial(vi& v) {
```

```
ll c = 1, m = v.empty() ? 1 : v[0];
rep(i, 1, sz(v)) rep(j, 0, v[i])
    c = c * ++m / (j+1);
return c;
}
```

### 6.3 General purpose numbers

#### 6.3.1 Bernoulli numbers

EGF of Bernoulli numbers is  $B(t) = \frac{t}{e^t - 1}$  (FFT-able).  
 $B[0, \dots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots]$

Sums of powers:

$$\sum_{i=1}^n i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^\infty f(i) = \int_m^\infty f(x) dx - \sum_{k=1}^\infty \frac{B_k}{k!} f^{(k-1)}(m)$$

$$\approx \int_m^\infty f(x) dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

#### 6.3.2 Stirling numbers of the first kind

Number of permutations on  $n$  items with  $k$  cycles.

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k), \quad c(0, 0) = 1$$
$$\sum_{k=0}^n c(n, k) x^k = x(x+1) \dots (x+n-1)$$

$$c(8, k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$$
$$c(n, 2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \dots$$

#### 6.3.3 Eulerian numbers

Number of permutations  $\pi \in S_n$  in which exactly  $k$  elements are greater than the previous element.  $k$   $j$ :s s.t.  $\pi(j) > \pi(j+1)$ ,  $k+1$   $j$ :s s.t.  $\pi(j) \geq j$ ,  $k$   $j$ :s s.t.  $\pi(j) > j$ .

$$E(n, k) = (n-k)E(n-1, k-1) + (k+1)E(n-1, k)$$

$$E(n, 0) = E(n, n-1) = 1$$

$$E(n, k) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$$

### 6.3.4 Stirling numbers of the second kind

Partitions of  $n$  distinct elements into exactly  $k$  groups.

$$S(n,k)=S(n-1,k-1)+kS(n-1,k)$$

$$S(n,1)=S(n,n)=1$$

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^k(-1)^{k-j}\binom{k}{j}j^n$$

### 6.3.5 Bell numbers

Total number of partitions of  $n$  distinct elements.  $B(n)=1,1,2,5,15,52,203,877,4140,21147,\dots$  For  $p$  prime,

$$B(p^m+n)\equiv mB(n)+B(n+1)\pmod{p}$$

### 6.3.6 Labeled unrooted trees

- # on  $n$  vertices:  $n^{n-2}$
- # on  $k$  existing trees of size  $n_i$ :  $n_1n_2\cdots n_kn^{k-2}$
- # with degrees  $d_i$ :  $(n-2)!/((d_1-1)!\cdots(d_n-1)!)$

### 6.3.7 Catalan numbers

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n+1}=\frac{(2n)!}{(n+1)!n!}$$

$$C_0=1,\;C_{n+1}=\frac{2(2n+1)}{n+2}C_n,\;C_{n+1}=\sum C_iC_{n-i}$$

$$C_n=1,1,2,5,14,42,132,429,1430,4862,16796,58786,\dots$$

- sub-diagonal monotone paths in an  $n\times n$  grid.
- strings with  $n$  pairs of parenthesis, correctly nested.
- binary trees with with  $n+1$  leaves (0 or 2 children).
- ordered trees with  $n+1$  vertices.
- ways a convex polygon with  $n+2$  sides can be cut into triangles by connecting vertices with straight lines.
- permutations of  $[n]$  with no 3-term increasing subseq.

## Graph (7)

### 7.1 Fundamentals

BellmanFord.h

**Description:** Calculates shortest paths from  $s$  in a graph that might have negative edge weights. Unreachable nodes get `dist = inf`; nodes reachable through negative-weight cycles get `dist = -inf`. Assumes  $V^2\max|w_i|<\sim 2^{63}$ .  
**Time:**  $\mathcal{O}(VE)$

```
const ll inf = LLONG_MAX;
struct Ed { int a, b, w, s() { return a < b ? a : -a; } };
struct Node { ll dist = inf; int prev = -1; };

void bellmanFord(vector<Node>& nodes, vector<Ed>& eds, int s) {
    nodes[s].dist = 0;
    sort(all(eds), [](Ed a, Ed b) { return a.s() < b.s(); });

    int lim = sz(nodes) / 2 + 2; // /3+100 with shuffled vertices
    rep(i,0,lim) trav(ed, eds) {
        Node cur = nodes[ed.a], &dest = nodes[ed.b];
        if (abs(cur.dist) == inf) continue;
        ll d = cur.dist + ed.w;
        if (d < dest.dist) {
            dest.prev = ed.a;
            dest.dist = (i < lim-1 ? d : -inf);
        }
    }
    rep(i,0,lim) trav(e, eds) {
        if (nodes[e.a].dist == -inf)
            nodes[e.b].dist = -inf;
    }
}
```

#### FloydWarshall.h

**Description:** Calculates all-pairs shortest path in a directed graph that might have negative edge weights. Input is an distance matrix  $m$ , where  $m[i][j] = \text{inf}$  if  $i$  and  $j$  are not adjacent. As output,  $m[i][j]$  is set to the shortest distance between  $i$  and  $j$ , `inf` if no path, or `-inf` if the path goes through a negative-weight cycle.  
**Time:**  $\mathcal{O}(N^3)$

```
const ll inf = 1LL << 62;
void floydWarshall(vector<vector<ll>>& m) {
    int n = sz(m);
    rep(i,0,n) m[i][i] = min(m[i][i], 0LL);
    rep(k,0,n) rep(i,0,n) rep(j,0,n)
        if (m[i][k] != inf && m[k][j] != inf) {
            auto newDist = max(m[i][k] + m[k][j], -inf);
            m[i][j] = min(m[i][j], newDist);
        }
    rep(k,0,n) if (m[k][k] < 0) rep(i,0,n) rep(j,0,n)
        if (m[i][k] != inf && m[k][j] != inf) m[i][j] = -inf;
}
```

#### TopoSort.h

**Description:** Topological sorting. Given is an oriented graph. Output is an ordering of vertices, such that there are edges only from left to right. If there are cycles, the returned list will have size smaller than  $n$  – nodes reachable from cycles will not be returned.  
**Time:**  $\mathcal{O}(|V|+|E|)$

```
vi topo_sort(const vector<vi>& gr) {
    vi indeg(sz(gr)), ret;
    trav(li, gr) trav(x, li) indeg[x]++;
    queue<int> q; // use priority queue for lexic. smallest ans.
    rep(i,0,sz(gr)) if (indeg[i] == 0) q.push(-i);
    while (!q.empty()) {
        int i = -q.front(); // top() for priority queue
        ret.push_back(i);
        q.pop();
        trav(x, gr[i])
            if (--indeg[x] == 0) q.push(-x);
    }
}
```

```
return ret;
}
```

### 7.2 Euler walk

#### EulerWalk.h

**Description:** Eulerian undirected/directed path/cycle algorithm. Input should be a vector of (dest, global edge index), where for undirected graphs, forward/backward edges have the same index. Returns a list of nodes in the Eulerian path/cycle with `src` at both start and end, or empty list if no cycle/path exists. To get edge indices back, add `.second` to `s` and `ret`.  
**Time:**  $\mathcal{O}(V+E)$

```
vi eulerWalk(vector<vector<pii>>& gr, int nedges, int src=0) {
    int n = sz(gr);
    vi D(n), its(n), eu(nedges), ret, s = {src};
    D[src]++; // to allow Euler paths, not just cycles
    while (!s.empty()) {
        int x = s.back(), y, e, &it = its[x], end = sz(gr[x]);
        if (it == end){ ret.push_back(x); s.pop_back(); continue; }
        tie(y, e) = gr[x][it++];
        if (!eu[e]) {
            D[x]--, D[y]++;
            eu[e] = 1; s.push_back(y);
        }
    }
    trav(x, D) if (x < 0 || sz(ret) != nedges+1) return {};
    return {ret.rbegin(), ret.rend()};
}
```

### 7.3 Network flow

#### PushRelabel.h

**Description:** Push-relabel using the highest label selection rule and the gap heuristic. Quite fast in practice. To obtain the actual flow, look at positive values only.  
**Time:**  $\mathcal{O}(V^2\sqrt{E})$

```
typedef ll Flow;
struct Edge {
    int dest, back;
    Flow f, c;
};

struct PushRelabel {
    vector<vector<Edge>>> g;
    vector<Flow> ec;
    vector<Edge*> cur;
    vector<vi> hs; vi H;
    PushRelabel(int n) : g(n), ec(n), cur(n), hs(2*n), H(n) {}

    void add_edge(int s, int t, Flow cap, Flow rcap=0) {
        if (s == t) return;
        g[s].push_back({t, sz(g[t]), 0, cap});
        g[t].push_back({s, sz(g[s])-1, 0, rcap});
    }

    void add_flow(Edge& e, Flow f) {
        Edge &back = g[e.dest][e.back];
        if (!ec[e.dest] && f) hs[H[e.dest]].push_back(e.dest);
        e.f += f; e.c -= f; ec[e.dest] += f;
        back.f -= f; back.c += f; ec[back.dest] -= f;
    }
    Flow maxflow(int s, int t) {
        int v = sz(g); H[s] = v; ec[t] = 1;
        vi co(2*v); co[0] = v-1;
        rep(i,0,v) cur[i] = g[i].data();
    }
}
```

```
trav(e, g[s]) add_flow(e, e.c);

for (int hi = 0;;) {
    while (hs[hi].empty()) if (!hi--) return -ec[s];
    int u = hs[hi].back(); hs[hi].pop_back();
    while (ec[u] > 0) // discharge u
        if (cur[u] == g[u].data() + sz(g[u])) {
            H[u] = le9;
            trav(e, g[u]) if (e.c && H[u] > H[e.dest]+1)
                H[u] = H[e.dest]+1, cur[u] = &e;
            if (++co[H[u]],!--co[hi] && hi < v)
                rep(i,0,v) if (hi < H[i] && H[i] < v)
                    --co[H[i]], H[i] = v + 1;
            hi = H[u];
        } else if (cur[u]->c && H[u] == H[cur[u]->dest]+1)
            add_flow(*cur[u], min(ec[u], cur[u]->c));
        else ++cur[u];
    }
}

bool leftOfMinCut(int a) { return H[a] >= sz(g); }
};
```

**MinCostMaxFlow.h**  
**Description:** Min-cost max-flow.  $\text{cap}[i][j] \neq \text{cap}[j][i]$  is allowed; double edges are not. If costs can be negative, call `setpi` before `maxflow`, but note that negative cost cycles are not supported. To obtain the actual flow, look at positive values only.  
**Time:** Approximately  $\mathcal{O}(E^2)$

6915ce, 81 lines

```
#include <bits/extc++.h>

const ll INF = numeric_limits<ll>::max() / 4;
typedef vector<ll> VL;

struct MCMF {
    int N;
    vector<vi> ed, red;
    vector<VL> cap, flow, cost;
    vi seen;
    VL dist, pi;
    vector<pii> par;

    MCMF(int N) :
        N(N), ed(N), red(N), cap(N, VL(N)), flow(cap), cost(cap),
        seen(N), dist(N), pi(N), par(N) {}

    void addEdge(int from, int to, ll cap, ll cost) {
        this->cap[from][to] = cap;
        this->cost[from][to] = cost;
        ed[from].push_back(to);
        red[to].push_back(from);
    }

    void path(int s) {
        fill(all(seen), 0);
        fill(all(dist), INF);
        dist[s] = 0; ll di;

        __gnu_pbds::priority_queue<pair<ll, int>> q;
        vector<decltype(q)::point_iterator> its(N);
        q.push({0, s});

        auto relax = [&](int i, ll cap, ll cost, int dir) {
            ll val = di - pi[i] + cost;
            if (cap && val < dist[i]) {
                dist[i] = val;
                par[i] = {s, dir};
                if (its[i] == q.end()) its[i] = q.push({-dist[i], i});
            } else q.modify(its[i], {-dist[i], i});
        };
    };
};
```

```
    }
};

while (!q.empty()) {
    s = q.top().second; q.pop();
    seen[s] = 1; di = dist[s] + pi[s];
    trav(i, ed[s]) if (!seen[i])
        relax(i, cap[s][i] - flow[s][i], cost[s][i], 1);
    trav(i, red[s]) if (!seen[i])
        relax(i, flow[i][s], -cost[i][s], 0);
}
rep(i,0,N) pi[i] = min(pi[i] + dist[i], INF);
}

pair<ll, ll> maxflow(int s, int t) {
    ll totflow = 0, totcost = 0;
    while (path(s), seen[t]) {
        ll fl = INF;
        for (int p,r,x = t; tie(p,r) = par[x], x != s; x = p)
            fl = min(fl, r ? cap[p][x] - flow[p][x] : flow[x][p]);
        totflow += fl;
        for (int p,r,x = t; tie(p,r) = par[x], x != s; x = p)
            if (r) flow[p][x] += fl;
            else flow[x][p] -= fl;
    }
    rep(i,0,N) rep(j,0,N) totcost += cost[i][j] * flow[i][j];
    return {totflow, totcost};
}

// If some costs can be negative, call this before maxflow:
void setpi(int s) { // (otherwise, leave this out)
    fill(all(pi), INF); pi[s] = 0;
    int it = N, ch = 1; ll v;
    while (ch-- && it--)
        rep(i,0,N) if (pi[i] != INF)
            trav(to, ed[i]) if (cap[i][to])
                if ((v = pi[i] + cost[i][to]) < pi[to])
                    pi[to] = v, ch = 1;
    assert(it >= 0); // negative cost cycle
}
};
```

**EdmondsKarp.h**  
**Description:** Flow algorithm with guaranteed complexity  $\mathcal{O}(VE^2)$ . To get edge flow values, compare capacities before and after, and take the positive values only.

979bb9, 35 lines

```
template<class T> T edmondsKarp(vector<unordered_map<int, T>>&
    graph, int source, int sink) {
    assert(source != sink);
    T flow = 0;
    vi par(sz(graph)), q = par;

    for (;;) {
        fill(all(par), -1);
        par[source] = 0;
        int ptr = 1;
        q[0] = source;

        rep(i,0,ptr) {
            int x = q[i];
            trav(e, graph[x]) {
                if (par[e.first] == -1 && e.second > 0) {
                    par[e.first] = x;
                    q[ptr++] = e.first;
                    if (e.first == sink) goto out;
                }
            }
        }
    }
}
```

```
return flow;

out:
    T inc = numeric_limits<T>::max();
    for (int y = sink; y != source; y = par[y])
        inc = min(inc, graph[par[y]][y]);

    flow += inc;
    for (int y = sink; y != source; y = par[y]) {
        int p = par[y];
        if ((graph[p][y] -= inc) <= 0) graph[p].erase(y);
        graph[y][p] += inc;
    }
}
}
```

**MinCut.h**  
**Description:** After running max-flow, the left side of a min-cut from  $s$  to  $t$  is given by all vertices reachable from  $s$ , only traversing edges with positive residual capacity.

**GlobalMinCut.h**  
**Description:** Find a global minimum cut in an undirected graph, as represented by an adjacency matrix.  
**Time:**  $\mathcal{O}(V^3)$

03261f, 31 lines

```
pair<int, vi> GetMinCut(vector<vi>& weights) {
    int N = sz(weights);
    vi used(N), cut, best_cut;
    int best_weight = -1;

    for (int phase = N-1; phase >= 0; phase--) {
        vi w = weights[0], added = used;
        int prev, k = 0;
        rep(i,0,phase){
            prev = k;
            k = -1;
            rep(j,1,N)
                if (!added[j] && (k == -1 || w[j] > w[k])) k = j;
            if (i == phase-1) {
                rep(j,0,N) weights[prev][j] += weights[k][j];
                rep(j,0,N) weights[j][prev] = weights[prev][j];
                used[k] = true;
                cut.push_back(k);
                if (best_weight == -1 || w[k] < best_weight) {
                    best_cut = cut;
                    best_weight = w[k];
                }
            } else {
                rep(j,0,N)
                    w[j] += weights[k][j];
                added[k] = true;
            }
        }
    }
    return {best_weight, best_cut};
}
```

## 7.4 Matching

**hopcroftKarp.h**  
**Description:** Fast bipartite matching algorithm. Graph  $g$  should be a list of neighbors of the left partition, and  $btoa$  should be a vector full of  $-1$ 's of the same size as the right partition. Returns the size of the matching.  $btoa[i]$  will be the match for vertex  $i$  on the right side, or  $-1$  if it's not matched.  
**Usage:** `vi btoa(m, -1); hopcroftKarp(g, btoa);`

Time:  $\mathcal{O}(\sqrt{VE})$ 536939, 42 lines

```
bool dfs(int a, int L, vector<vi>& g, vi& btoa, vi& A, vi& B) {
    if (A[a] != L) return 0;
    A[a] = -1;
    trav(b, g[a]) if (B[b] == L + 1) {
        B[b] = 0;
        if (btoa[b] == -1 || dfs(btoa[b], L + 1, g, btoa, A, B))
            return btoa[b] = a, 1;
    }
    return 0;
}

int hopcroftKarp(vector<vi>& g, vi& btoa) {
    int res = 0;
    vi A(g.size()), B(btoa.size()), cur, next;
    for (;;) {
        fill(all(A), 0);
        fill(all(B), 0);
        cur.clear();
        trav(a, btoa) if(a != -1) A[a] = -1;
        rep(a,0,sz(g)) if(A[a] == 0) cur.push_back(a);
        for (int lay = 1;; lay++) {
            bool islast = 0;
            next.clear();
            trav(a, cur) trav(b, g[a]) {
                if (btoa[b] == -1) {
                    B[b] = lay;
                    islast = 1;
                }
                else if (btoa[b] != a && !B[b]) {
                    B[b] = lay;
                    next.push_back(btoa[b]);
                }
            }
            if (islast) break;
            if (next.empty()) return res;
            trav(a, next) A[a] = lay;
            cur.swap(next);
        }
        rep(a,0,sz(g))
            res += dfs(a, 0, g, btoa, A, B);
    }
}
```

DFSMatching.hDescription: Simple bipartite matching algorithm. Graph  $g$  should be a list of neighbors of the left partition, and  $btoa$  should be a vector full of -1's of the same size as the right partition. Returns the size of the matching.  $btoa[i]$  will be the match for vertex  $i$  on the right side, or  $-1$  if it's not matched. Usage: vi btoa(m, -1); dfsMatching(g, btoa); Time:  $\mathcal{O}(VE)$ 6a3472, 22 lines

```
bool find(int j, vector<vi>& g, vi& btoa, vi& vis) {
    if (btoa[j] == -1) return 1;
    vis[j] = 1; int di = btoa[j];
    trav(e, g[di])
        if (!vis[e] && find(e, g, btoa, vis)) {
            btoa[e] = di;
            return 1;
        }
    return 0;
}

int dfsMatching(vector<vi>& g, vi& btoa) {
    vi vis;
    rep(i,0,sz(g)) {
        vis.assign(sz(btoa), 0);
        trav(j,g[i])
            if (find(j, g, btoa, vis)) {
```

```
        btoa[j] = i;
        break;
    }
}

return sz(btoa) - (int)count(all(btoa), -1);
}

MinimumVertexCover.hDescription: Finds a minimum vertex cover in a bipartite graph. The size is the same as the size of a maximum matching, and the complement is a maximum independent set. "DFSMatching.h"d0b3f2, 20 lines

vi cover(vector<vi>& g, int n, int m) {
    vi match(m, -1);
    int res = dfsMatching(g, match);
    vector<bool> lfound(n, true), seen(m);
    trav(it, match) if (it != -1) lfound[it] = false;
    vi q, cover;
    rep(i,0,n) if (lfound[i]) q.push_back(i);
    while (!q.empty()) {
        int i = q.back(); q.pop_back();
        lfound[i] = 1;
        trav(e, g[i]) if (!seen[e] && match[e] != -1) {
            seen[e] = true;
            q.push_back(match[e]);
        }
    }
    rep(i,0,n) if (!lfound[i]) cover.push_back(i);
    rep(i,0,m) if (seen[i]) cover.push_back(n+i);
    assert(sz(cover) == res);
    return cover;
}
```

WeightedMatching.hDescription: Min cost bipartite matching. Negate costs for max cost. Time:  $\mathcal{O}(N^3)$ 055ca9, 75 lines

```
typedef vector<double> vd;
bool zero(double x) { return fabs(x) < 1e-10; }
double MinCostMatching(const vector<vd>& cost, vi& L, vi& R) {
    int n = sz(cost), mated = 0;
    vd dist(n), u(n), v(n);
    vi dad(n), seen(n);

    rep(i,0,n) {
        u[i] = cost[i][0];
        rep(j,1,n) u[i] = min(u[i], cost[i][j]);
    }
    rep(j,0,n) {
        v[j] = cost[0][j] - u[0];
        rep(i,1,n) v[j] = min(v[j], cost[i][j] - u[i]);
    }

    L = R = vi(n, -1);
    rep(i,0,n) rep(j,0,n) {
        if (R[j] != -1) continue;
        if (zero(cost[i][j] - u[i] - v[j])) {
            L[i] = j;
            R[j] = i;
            mated++;
            break;
        }
    }

    for (; mated < n; mated++) { // until solution is feasible
        int s = 0;
        while (L[s] != -1) s++;
        fill(all(dad), -1);
```

```
        fill(all(seen), 0);
        rep(k,0,n)
            dist[k] = cost[s][k] - u[s] - v[k];

        int j = 0;
        for (;;) {
            j = -1;
            rep(k,0,n) {
                if (seen[k]) continue;
                if (j == -1 || dist[k] < dist[j]) j = k;
            }
            seen[j] = 1;
            int i = R[j];
            if (i == -1) break;
            rep(k,0,n) {
                if (seen[k]) continue;
                auto new_dist = dist[j] + cost[i][k] - u[i] - v[k];
                if (dist[k] > new_dist) {
                    dist[k] = new_dist;
                    dad[k] = j;
                }
            }
        }

        rep(k,0,n) {
            if (k == j || !seen[k]) continue;
            auto w = dist[k] - dist[j];
            v[k] += w, u[R[k]] -= w;
        }
        u[s] += dist[j];

        while (dad[j] >= 0) {
            int d = dad[j];
            R[j] = R[d];
            L[R[j]] = j;
            j = d;
        }
        R[j] = s;
        L[s] = j;
    }

    auto value = vd(1)[0];
    rep(i,0,n) value += cost[i][L[i]];
    return value;
}
```

GeneralMatching.hDescription: Matching for general graphs. Fails with probability  $N/mod$ . Time:  $\mathcal{O}(N^3)$ bb8be4, 40 lines

```
vector<pii> generalMatching(int N, vector<pii>& ed) {
    vector<vector<ll>> mat(N, vector<ll>(N)), A;
    trav(pa, ed) {
        int a = pa.first, b = pa.second, r = rand() % mod;
        mat[a][b] = r, mat[b][a] = (mod - r) % mod;
    }

    int r = matInv(A = mat), M = 2*N - r, fi, fj;
    assert(r % 2 == 0);

    if (M != N) do {
        mat.resize(M, vector<ll>(M));
        rep(i,0,M) {
            mat[i].resize(M);
            rep(j,N,M) {
                int r = rand() % mod;
                mat[i][j] = r, mat[j][i] = (mod - r) % mod;
            }
        }
    }
```

```
    } while (matInv(A = mat) != M);

vi has(M, 1); vector<pii> ret;
rep(it,0,M/2) {
    rep(i,0,M) if (has[i])
        rep(j,i+1,M) if (A[i][j] && mat[i][j]) {
            fi = i; fj = j; goto done;
        }
    assert(0); done:
    if (fj < N) ret.emplace_back(fi, fj);
    has[fi] = has[fj] = 0;
    rep(sw,0,2) {
        ll a = modpow(A[fi][fj], mod-2);
        rep(i,0,M) if (has[i] && A[i][fj]) {
            ll b = A[i][fj] * a % mod;
            rep(j,0,M) A[i][j] = (A[i][j] - A[fi][j] * b) % mod;
        }
        swap(fi,fj);
    }
}
return ret;
}
```

7.5 DFS algorithms

SCC.h

**Description:** Finds strongly connected components in a directed graph. If vertices  $u, v$  belong to the same component, we can reach  $u$  from  $v$  and vice versa.

**Usage:** `scc(graph, [&](vi& v) { ... })` visits all components in reverse topological order. `comp[i]` holds the component index of a node (a component only has edges to components with lower index). `ncomps` will contain the number of components.

**Time:**  $\mathcal{O}(E + V)$

bb2963, 24 lines

```
vi val, comp, z, cont;
int Time, ncomps;
template<class G, class F> int dfs(int j, G& g, F& f) {
    int low = val[j] = ++Time, x; z.push_back(j);
    trav(e,g[j]) if (comp[e] < 0)
        low = min(low, val[e] ?: dfs(e,g,f));

    if (low == val[j]) {
        do {
            x = z.back(); z.pop_back();
            comp[x] = ncomps;
            cont.push_back(x);
        } while (x != j);
        f(cont); cont.clear();
        ncomps++;
    }
    return val[j] = low;
}
template<class G, class F> void scc(G& g, F f) {
    int n = sz(g);
    val.assign(n, 0); comp.assign(n, -1);
    Time = ncomps = 0;
    rep(i,0,n) if (comp[i] < 0) dfs(i, g, f);
}
```

BiconnectedComponents.h

**Description:** Finds all biconnected components in an undirected graph, and runs a callback for the edges in each. In a biconnected component there are at least two distinct paths between any two nodes. Note that a node can be in several components. An edge which is not in a component is a bridge, i.e., not part of any cycle.

Usage:

`int eid = 0; ed.resize(N);`  
for each edge (a,b) {  
 `ed[a].emplace_back(b, eid);`  
 `ed[b].emplace_back(a, eid++);` }  
`bicomps([&](const vi& edgelist) {...});`  
**Time:**  $\mathcal{O}(E + V)$

cca7e6, 33 lines

```
vi num, st;
vector<vector<pii>> ed;
int Time;
template<class F>
int dfs(int at, int par, F& f) {
    int me = num[at] = ++Time, e, y, top = me;
    trav(pa, ed[at]) if (pa.second != par) {
        tie(y, e) = pa;
        if (num[y]) {
            top = min(top, num[y]);
            if (num[y] < me)
                st.push_back(e);
        } else {
            int si = sz(st);
            int up = dfs(y, e, f);
            top = min(top, up);
            if (up == me) {
                st.push_back(e);
                f(vi(st.begin() + si, st.end()));
                st.resize(si);
            }
            else if (up < me) st.push_back(e);
            else /* e is a bridge */
        }
    }
    return top;
}

template<class F>
void bicomps(F f) {
    num.assign(sz(ed), 0);
    rep(i,0,sz(ed)) if (!num[i]) dfs(i, -1, f);
}
```

2sat.h

**Description:** Calculates a valid assignment to boolean variables  $a, b, c, \dots$  to a 2-SAT problem, so that an expression of the type  $(a \vee b) \wedge \neg (a \vee c) \wedge (d \vee \neg b) \wedge \dots$  becomes true, or reports that it is unsatisfiable. Negated variables are represented by bit-inversions ( $\sim x$ ).

**Usage:** `TwoSat ts(number of boolean variables);`  
`ts.either(0, ~3);` // Var 0 is true or var 3 is false  
`ts.set_value(2);` // Var 2 is true  
`ts.at_most_one({0, ~1, 2});` //  $\leq 1$  of vars 0, ~1 and 2 are true  
`ts.solve();` // Returns true iff it is solvable  
`ts.values[0..N-1]` holds the assigned values to the vars

**Time:**  $\mathcal{O}(N + E)$ , where  $N$  is the number of boolean variables, and  $E$  is the number of clauses.

0911c1, 56 lines

```
struct TwoSat {
    int N;
    vector<vi> gr;
    vi values; // 0 = false, 1 = true

    TwoSat(int n = 0) : N(n), gr(2*n) {}

    int add_var() { // (optional)
        gr.emplace_back();
        gr.emplace_back();
        return N++;
    }

    void either(int f, int j) {
```

```
        f = max(2*f, -1-2*f);
        j = max(2*j, -1-2*j);
        gr[f].push_back(j^1);
        gr[j].push_back(f^1);
    }
    void set_value(int x) { either(x, x); }

    void at_most_one(const vi& li) { // (optional)
        if (sz(li) <= 1) return;
        int cur = ~li[0];
        rep(i,2,sz(li)) {
            int next = add_var();
            either(cur, ~li[i]);
            either(cur, next);
            either(~li[i], next);
            cur = ~next;
        }
        either(cur, ~li[1]);
    }

    vi val, comp, z; int time = 0;
    int dfs(int i) {
        int low = val[i] = ++time, x; z.push_back(i);
        trav(e, gr[i]) if (!comp[e])
            low = min(low, val[e] ?: dfs(e));
        if (low == val[i]) do {
            x = z.back(); z.pop_back();
            comp[x] = low;
            if (values[x>>1] == -1)
                values[x>>1] = x&1;
        } while (x != i);
        return val[i] = low;
    }

    bool solve() {
        values.assign(N, -1);
        val.assign(2*N, 0); comp = val;
        rep(i,0,2*N) if (!comp[i]) dfs(i);
        rep(i,0,N) if (comp[2*i] == comp[2*i+1]) return 0;
        return 1;
    }
};

7.6 Heuristics

MaximalCliques.h
Description: Runs a callback for all maximal cliques in a graph (given as a symmetric bitset matrix; self-edges not allowed). Callback is given a bitset representing the maximal clique.
Time:  $\mathcal{O}\left(3^{n/3}\right)$ , much faster for sparse graphs
b0d5b1, 12 lines

typedef bitset<128> B;
template<class F>
void cliques(vector<B>& eds, F f, B P = ~B(), B X={}, B R={}) {
    if (!P.any()) { if (!X.any()) f(R); return; }
    auto q = (P | X)._Find_first();
    auto cands = P & ~eds[q];
    rep(i,0,sz(eds)) if (cands[i]) {
        R[i] = 1;
        cliques(eds, f, P & eds[i], X & eds[i], R);
        R[i] = P[i] = 0; X[i] = 1;
    }
}
```

### MaximumClique.h

**Description:** Quickly finds a maximum clique of a graph (given as symmetric bitset matrix; self-edges not allowed). Can be used to find a maximum independent set by finding a clique of the complement graph.

**Time:** Runs in about 1s for n=155 and worst case random graphs (p=.90).  
Runs faster for sparse graphs.

```
typedef vector<bitset<200>> vb;
struct Maxclique {
    double limit=0.025, pk=0;
    struct Vertex { int i, d=0; };
    typedef vector<Vertex> vv;
    vb e;
    vv V;
    vector<vi> C;
    vi qmax, q, S, old;
    void init(vv& r) {
        trav(V,r) v.d = 0;
        trav(V, r) trav(j, r) v.d += e[v.i][j.i];
        sort(all(r), [](auto a, auto b) { return a.d > b.d; });
        int mxD = r[0].d;
        rep(i,0,sz(r)) r[i].d = min(i, mxD) + 1;
    }
    void expand(vv& R, int lev = 1) {
        S[lev] += S[lev - 1] - old[lev];
        old[lev] = S[lev - 1];
        while (sz(R)) {
            if (sz(q) + R.back().d <= sz(qmax)) return;
            q.push_back(R.back().i);
            vv T;
            trav(V,R) if (e[R.back().i][v.i]) T.push_back({v.i});
            if (sz(T)) {
                if (S[lev]++ / ++pk < limit) init(T);
                int j = 0, mxk = 1, mnk = max(sz(qmax) - sz(q) + 1, 1);
                C[1].clear(), C[2].clear();
                trav(v, T) {
                    int k = 1;
                    auto f = [&](int i) { return e[v.i][i]; };
                    while (any_of(all(C[k]), f)) k++;
                    if (k > mxk) mxk = k, C[mxk + 1].clear();
                    if (k < mnk) T[j++].i = v.i;
                    C[k].push_back(v.i);
                }
                if (j > 0) T[j - 1].d = 0;
                rep(k,mnk,mxk + 1) trav(i, C[k])
                    T[j].i = i, T[j++].d = k;
                expand(T, lev + 1);
            } else if (sz(q) > sz(qmax)) qmax = q;
            q.pop_back(), R.pop_back();
        }
    }
    vi maxClique() { init(V), expand(V); return qmax; }
    Maxclique(vb conn) : e(conn), C(sz(e)+1), S(sz(C)), old(S) {
        rep(i,0,sz(e)) V.push_back({i});
    }
};
```

### MaximumIndependentSet.h

**Description:** To obtain a maximum independent set of a graph, find a max clique of the complement. If the graph is bipartite, see MinimumVertexCover.

## 7.7 Trees

### TreePower.h

**Description:** Calculate power of two jumps in a tree, to support fast upward jumps and LCAs. Assumes the root node points to itself.

```
Time: construction  $\mathcal{O}(N \log N)$ , queries  $\mathcal{O}(\log N)$ 
bfce85, 25 lines

vector<vi> treeJump(vi& P) {
    int on = 1, d = 1;
    while(on < sz(P)) on *= 2, d++;
    vector<vi> jmp(d, P);
    rep(i,1,d) rep(j,0,sz(P))
        jmp[i][j] = jmp[i-1][jmp[i-1][j]];
    return jmp;
}

int jmp(vector<vi>& tbl, int nod, int steps) {
    rep(i,0,sz(tbl))
        if(steps&(1<<i)) nod = tbl[i][nod];
    return nod;
}

int lca(vector<vi>& tbl, vi& depth, int a, int b) {
    if (depth[a] < depth[b]) swap(a, b);
    a = jmp(tbl, a, depth[a] - depth[b]);
    if (a == b) return a;
    for (int i = sz(tbl); i--;) {
        int c = tbl[i][a], d = tbl[i][b];
        if (c != d) a = c, b = d;
    }
    return tbl[0][a];
}

LCA.h
Description: Data structure for computing lowest common ancestors in a tree (with 0 as root). C should be an adjacency list of the tree, either directed or undirected. Can also find the distance between two nodes.
Usage: LCA lca(undirGraph);
lca.query(firstNode, secondNode);
lca.distance(firstNode, secondNode);
Time:  $\mathcal{O}(N \log N + Q)$ 
../data-structures/RMQ.h
aa0d4d, 37 lines

typedef vector<pii> vpi;
typedef vector<vpi> graph;

struct LCA {
    vi time;
    vector<ll> dist;
    RMQ<pii> rmq;

    LCA(graph& C) : time(sz(C), -99), dist(sz(C)), rmq(dfs(C)) {}

    vpi dfs(graph& C) {
        vector<tuple<int, int, int, ll>> q(1);
        vpi ret;
        int T = 0, v, p, d; ll di;
        while (!q.empty()) {
            tie(v, p, d, di) = q.back();
            q.pop_back();
            if (d) ret.emplace_back(d, p);
            time[v] = T++;
            dist[v] = di;
            trav(e, C[v]) if (e.first != p)
                q.emplace_back(e.first, v, d+1, di + e.second);
        }
        return ret;
    }

    int query(int a, int b) {
        if (a == b) return a;
        a = time[a], b = time[b];
        return rmq.query(min(a, b), max(a, b)).second;
    }
    ll distance(int a, int b) {
```

```
int lca = query(a, b);
return dist[a] + dist[b] - 2 * dist[lca];
}
};

CompressTree.h
Description: Given a rooted tree and a subset S of nodes, compute the minimal subtree that contains all the nodes by adding all (at most |S| - 1) pairwise LCA's and compressing edges. Returns a list of (par, orig_index) representing a tree rooted at 0. The root points to itself.
Time:  $\mathcal{O}(|S| \log |S|)$ 
LCA.h
dabd75, 20 lines

vpi compressTree(LCA& lca, const vi& subset) {
    static vi rev; rev.resize(sz(lca.dist));
    vi li = subset, &T = lca.time;
    auto cmp = [&](int a, int b) { return T[a] < T[b]; };
    sort(all(li), cmp);
    int m = sz(li)-1;
    rep(i,0,m) {
        int a = li[i], b = li[i+1];
        li.push_back(lca.query(a, b));
    }
    sort(all(li), cmp);
    li.erase(unique(all(li)), li.end());
    rep(i,0,sz(li)) rev[li[i]] = i;
    vpi ret = {pii(0, li[0])};
    rep(i,0,sz(li)-1) {
        int a = li[i], b = li[i+1];
        ret.emplace_back(rev[lca.query(a, b)], b);
    }
    return ret;
}

HLD.h
Description: Decomposes a tree into vertex disjoint heavy paths and light edges such that the path from any leaf to the root contains at most log(n) light edges. The function of the HLD can be changed by modifying T, LOW and f. f is assumed to be associative and commutative.
Usage: HLD hld(G);
hld.update(index, value);
tie(value, lca) = hld.query(n1, n2);
../data-structures/SegmentTree.h
d952a9, 93 lines

typedef vector<pii> vpi;

struct Node {
    int d, par, val, chain = -1, pos = -1;
};

struct Chain {
    int par, val;
    vector<int> nodes;
    Tree tree;
};

struct HLD {
    typedef int T;
    const T LOW = -(1<<29);
    void f(T& a, T b) { a = max(a, b); }

    vector<Node> V;
    vector<Chain> C;

    HLD(vector<vpi>& g) : V(sz(g)) {
        dfs(0, -1, g, 0);
        trav(c, C) {
            c.tree = {sz(c.nodes), 0};
            for (int ni : c.nodes)
```



```

        c.tree.update(V[ni].pos, V[ni].val);
    }
}

void update(int node, T val) {
    Node& n = V[node]; n.val = val;
    if (n.chain != -1) C[n.chain].tree.update(n.pos, val);
}

int pard(Node& nod) {
    if (nod.par == -1) return -1;
    return V[nod.chain == -1 ? nod.par : C[nod.chain].par].d;
}

// query all *edges* between n1, n2
pair<T, int> query(int i1, int i2) {
    T ans = LOW;
    while (i1 != i2) {
        Node n1 = V[i1], n2 = V[i2];
        if (n1.chain != -1 && n1.chain == n2.chain) {
            int lo = n1.pos, hi = n2.pos;
            if (lo > hi) swap(lo, hi);
            f(ans, C[n1.chain].tree.query(lo, hi));
            i1 = i2 = C[n1.chain].nodes[hi];
        } else {
            if (pard(n1) < pard(n2))
                n1 = n2, swap(i1, i2);
            if (n1.chain == -1)
                f(ans, n1.val), i1 = n1.par;
            else {
                Chain& c = C[n1.chain];
                f(ans, n1.pos ? c.tree.query(n1.pos, sz(c.nodes))
                    : c.tree.s[1]);
                i1 = c.par;
            }
        }
    }
    return make_pair(ans, i1);
}

// query all *nodes* between n1, n2
pair<T, int> query2(int i1, int i2) {
    pair<T, int> ans = query(i1, i2);
    f(ans.first, V[ans.second].val);
    return ans;
}

pii dfs(int at, int par, vector<vpi>& g, int d) {
    V[at].d = d; V[at].par = par;
    int sum = 1, ch, nod, sz;
    tuple<int, int, int> mx(-1, -1, -1);
    trav(e, g[at]) {
        if (e.first == par) continue;
        tie(sz, ch) = dfs(e.first, at, g, d+1);
        V[e.first].val = e.second;
        sum += sz;
        mx = max(mx, make_tuple(sz, e.first, ch));
    }
    tie(sz, nod, ch) = mx;
    if (2*sz < sum) return pii(sum, -1);
    if (ch == -1) { ch = sz(C); C.emplace_back(); }
    V[nod].pos = sz(C[ch].nodes);
    V[nod].chain = ch;
    C[ch].par = at;
    C[ch].nodes.push_back(nod);
    return pii(sum, ch);
}
};

```

## LinkCutTree.h

**Description:** Represents a forest of unrooted trees. You can add and remove edges (as long as the result is still a forest), and check whether two nodes are in the same tree.

**Time:** All operations take amortized  $\mathcal{O}(\log N)$ .

693483, 90 lines

```

struct Node { // Splay tree. Root's pp contains tree's parent.
    Node *p = 0, *pp = 0, *c[2];
    bool flip = 0;
    Node() { c[0] = c[1] = 0; fix(); }
    void fix() {
        if (c[0]) c[0]->p = this;
        if (c[1]) c[1]->p = this;
        // (+ update sum of subtree elements etc. if wanted)
    }
    void push_flip() {
        if (!flip) return;
        flip = 0; swap(c[0], c[1]);
        if (c[0]) c[0]->flip ^= 1;
        if (c[1]) c[1]->flip ^= 1;
    }
    int up() { return p ? p->c[1] == this : -1; }
    void rot(int i, int b) {
        int h = i ^ b;
        Node *x = c[i], *y = b == 2 ? x : x->c[h], *z = b ? y : x;
        if ((y->p = p) p->c[up()] = y;
        c[i] = z->c[i ^ 1];
        if (b < 2) {
            x->c[h] = y->c[h ^ 1];
            z->c[h ^ 1] = b ? x : this;
        }
        y->c[i ^ 1] = b ? this : x;
        fix(); x->fix(); y->fix();
        if (p) p->fix();
        swap(pp, y->pp);
    }
    void splay() {
        for (push_flip(); p; ) {
            if (p->p) p->p->push_flip();
            p->push_flip(); push_flip();
            int c1 = up(), c2 = p->up();
            if (c2 == -1) p->rot(c1, 2);
            else p->p->rot(c2, c1 != c2);
        }
    }
    Node* first() {
        push_flip();
        return c[0] ? c[0]->first() : (splay(), this);
    }
};

struct LinkCut {
    vector<Node> node;
    LinkCut(int N) : node(N) {}

    void link(int u, int v) { // add an edge (u, v)
        assert(!connected(u, v));
        make_root(&node[u]);
        node[u].pp = &node[v];
    }
    void cut(int u, int v) { // remove an edge (u, v)
        Node *x = &node[u], *top = &node[v];
        make_root(top); x->splay();
        assert(top == (x->pp ? x->c[0]));
        if (x->pp) x->pp = 0;
        else {
            x->c[0] = top->p = 0;
            x->fix();
        }
    }
};

```

```

}
bool connected(int u, int v) { // are u, v in the same tree?
    Node* nu = access(&node[u])->first();
    return nu == access(&node[v])->first();
}
void make_root(Node* u) {
    access(u);
    u->splay();
    if (u->c[0]) {
        u->c[0]->p = 0;
        u->c[0]->flip ^= 1;
        u->c[0]->pp = u;
        u->c[0] = 0;
        u->fix();
    }
}
Node* access(Node* u) {
    u->splay();
    while (Node* pp = u->pp) {
        pp->splay(); u->pp = 0;
        if (pp->c[1]) {
            pp->c[1]->p = 0; pp->c[1]->pp = pp; }
        pp->c[1] = u; pp->fix(); u = pp;
    }
    return u;
}
};

```

## DirectedMST.h

**Description:** Edmonds' algorithm for finding the weight of the minimum spanning tree/arborescence of a directed graph, given a root node. If no MST exists, returns -1.

**Time:**  $\mathcal{O}(E \log V)$

"/data-structures/UnionFind.h"

a69883, 48 lines

```

struct Edge { int a, b; ll w; };
struct Node {
    Edge key;
    Node *l, *r;
    ll delta;
    void prop() {
        key.w += delta;
        if (l) l->delta += delta;
        if (r) r->delta += delta;
        delta = 0;
    }
    Edge top() { prop(); return key; }
};
Node *merge(Node *a, Node *b) {
    if (!a || !b) return a ? b;
    a->prop(), b->prop();
    if (a->key.w > b->key.w) swap(a, b);
    swap(a->l, (a->r = merge(b, a->r)));
    return a;
}
void pop(Node& a) { a->prop(); a = merge(a->l, a->r); }

ll dmst(int n, int r, vector<Edge>& g) {
    UF uf(n);
    vector<Node*> heap(n);
    trav(e, g) heap[e.b] = merge(heap[e.b], new Node{e});
    ll res = 0;
    vi seen(n, -1), path(n);
    seen[r] = r;
    rep(s, 0, n) {
        int u = s, qi = 0, w;
        while (seen[u] < 0) {
            path[qi++] = u, seen[u] = s;
            if (!heap[u]) return -1;
        }
    }
};

```

```
Edge e = heap[u]->top();
heap[u]->delta -= e.w, pop(heap[u]);
res += e.w, u = uf.find(e.a);
if (seen[u] == s) {
    Node* cyc = 0;
    do cyc = merge(cyc, heap[w = path[--qi]]);
    while (uf.join(u, w));
    u = uf.find(u);
    heap[u] = cyc, seen[u] = -1;
}
}
}
return res;
}
```

7.8 Math

7.8.1 Number of Spanning Trees

Create an  $N \times N$  matrix  $\text{mat}$ , and for each edge  $a \rightarrow b \in G$ , do  $\text{mat}[a][b]--$ ,  $\text{mat}[b][b]++$  (and  $\text{mat}[b][a]--$ ,  $\text{mat}[a][a]++$  if  $G$  is undirected). Remove the  $i$ th row and column and take the determinant; this yields the number of directed spanning trees rooted at  $i$  (if  $G$  is undirected, remove any row/column).

7.8.2 Erdős–Gallai theorem

A simple graph with node degrees  $d_1 \geq \dots \geq d_n$  exists iff  $d_1 + \dots + d_n$  is even and for every  $k = 1 \dots n$ ,

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(d_i, k).$$

Geometry (8)

8.1 Geometric primitives

Point.h

Description: Class to handle points in the plane. T can be e.g. double or long long. (Avoid int.)

```
47ec0a, 28 lines
template <class T> int sgn(T x) { return (x > 0) - (x < 0); }
template<class T>
struct Point {
    typedef Point P;
    T x, y;
    explicit Point(T x=0, T y=0) : x(x), y(y) {}
    bool operator<(P p) const { return tie(x,y) < tie(p.x,p.y); }
    bool operator==(P p) const { return tie(x,y)==tie(p.x,p.y); }
    P operator+(P p) const { return P(x+p.x, y+p.y); }
    P operator-(P p) const { return P(x-p.x, y-p.y); }
    P operator*(T d) const { return P(x*d, y*d); }
    P operator/(T d) const { return P(x/d, y/d); }
    T dot(P p) const { return x*p.x + y*p.y; }
    T cross(P p) const { return x*p.y - y*p.x; }
    T cross(P a, P b) const { return (a-*this).cross(b-*this); }
    T dist2() const { return x*x + y*y; }
```

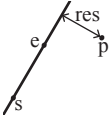
```
double dist() const { return sqrt((double)dist2()); }
// angle to x-axis in interval [-pi, pi]
double angle() const { return atan2(y, x); }
P unit() const { return *this/dist(); } // makes dist()==1
P perp() const { return P(-y, x); } // rotates +90 degrees
P normal() const { return perp().unit(); }
// returns point rotated 'a' radians ccw around the origin
P rotate(double a) const {
    return P(x*cos(a)-y*sin(a),x*sin(a)+y*cos(a)); }
friend ostream& operator<<(ostream& os, P p) {
    return os << "(" << p.x << "," << p.y << ")"; }
};
```

lineDistance.h

Description:

Returns the signed distance between point p and the line containing points a and b. Positive value on left side and negative on right as seen from a towards b. a==b gives nan. P is supposed to be Point<T> or Point3D<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long. Using Point3D will always give a non-negative distance. For Point3D, call .dist on the result of the cross product.

```
"Point.h"
6bf6b, 4 lines
template<class P>
double lineDist(const P& a, const P& b, const P& p) {
    return (double) (b-a).cross(p-a) / (b-a).dist();
}
```



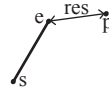
SegmentDistance.h

Description:

Returns the shortest distance between point p and the line segment from point s to e.

Usage: Point<double> a, b(2,2), p(1,1); bool onSegment = segDist(a,b,p) < 1e-10;

```
"Point.h"
5c88f4, 6 lines
typedef Point<double> P;
double segDist(P& s, P& e, P& p) {
    if (s==e) return (p-s).dist();
    auto d = (e-s).dist2(), t = min(d,max(.0, (p-s).dot(e-s)));
    return ((p-s)*d-(e-s)*t).dist()/d;
}
```



SegmentIntersection.h

Description:

If a unique intersection point between the line segments going from s1 to e1 and from s2 to e2 exists then it is returned. If no intersection point exists an empty vector is returned. If infinitely many exist a vector with 2 elements is returned, containing the endpoints of the common line segment. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or long long.

Usage: vector<P> inter = segInter(s1,e1,s2,e2); if (sz(inter)==1) cout << "segments intersect at " << inter[0] << endl;

```
"Point.h", "OnSegment.h"
9d57f2, 13 lines
template<class P> vector<P> segInter(P a, P b, P c, P d) {
    auto oa = c.cross(d, a), ob = c.cross(d, b),
        oc = a.cross(b, c), od = a.cross(b, d);
    // Checks if intersection is single non-endpoint point.
    if (sgn(oa) * sgn(ob) < 0 && sgn(oc) * sgn(od) < 0)
        return {(a * ob - b * oa) / (ob - oa)};
    set<P> s;
    if (onSegment(c, d, a)) s.insert(a);
    if (onSegment(c, d, b)) s.insert(b);
```

```
if (onSegment(a, b, c)) s.insert(c);
if (onSegment(a, b, d)) s.insert(d);
return {all(s)};
}
```

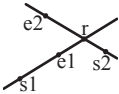
lineIntersection.h

Description:

If a unique intersection point of the lines going through s1,e1 and s2,e2 exists {1, point} is returned. If no intersection point exists {0, (0,0)} is returned and if infinitely many exists {-1, (0,0)} is returned. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or ll.

Usage: auto res = lineInter(s1,e1,s2,e2); if (res.first == 1) cout << "intersection point at " << res.second << endl;

```
"Point.h"
a01f81, 8 lines
template<class P>
pair<int, P> lineInter(P s1, P e1, P s2, P e2) {
    auto d = (e1 - s1).cross(e2 - s2);
    if (d == 0) // if parallel
        return {-(s1.cross(e1, s2) == 0), P(0, 0)};
    auto p = s2.cross(e1, e2), q = s2.cross(e2, s1);
    return {1, (s1 * p + e1 * q) / d};
}
```



sideOf.h

Description:

Returns where p is as seen from s towards e. 1/0/-1 ⇔ left/on line/right. If the optional argument eps is given 0 is returned if p is within distance eps from the line. P is supposed to be Point<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long.

Usage: bool left = sideOf(p1,p2,q)==1;

```
"Point.h"
3af81c, 9 lines
template<class P>
int sideOf(P s, P e, P p) { return sgn(s.cross(e, p)); }
```

```
template<class P>
int sideOf(const P& s, const P& e, const P& p, double eps) {
    auto a = (e-s).cross(p-s);
    double l = (e-s).dist()*eps;
    return (a > l) - (a < -l);
}
```

OnSegment.h

Description:

Returns true iff p lies on the line segment from s to e. Use (segDist(s,e,p)<=epsilon) instead when using Point<double>.

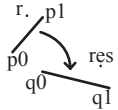
```
"Point.h"
c597e8, 3 lines
template<class P> bool onSegment(P s, P e, P p) {
    return p.cross(s, e) == 0 && (s - p).dot(e - p) <= 0;
}
```

linearTransformation.h

Description:

Apply the linear transformation (translation, rotation and scaling) which takes line p0-p1 to line q0-q1 to point r.

```
"Point.h"
03a306, 6 lines
typedef Point<double> P;
P linearTransformation(const P& p0, const P& p1,
    const P& q0, const P& q1, const P& r) {
    P dp = p1-p0, dq = q1-q0, num(dp.cross(dq), dp.dot(dq));
    return q0 + P((r-p0).cross(num), (r-p0).dot(num))/dp.dist2();
}
```





```
// search closest side first, other side if needed
auto best = search(f, p);
if (bsec < best.first)
    best = min(best, search(s, p));
return best;
}
```

```
// find nearest point to a point, and its squared distance
// (requires an arbitrary operator< for Point)
pair<T, P> nearest(const P& p) {
    return search(root, p);
}
};
```

FastDelaunay.h

**Description:** Fast Delaunay triangulation. Each circumcircle contains none of the input points. There must be no duplicate points. If all points are on a line, no triangles will be returned. Should work for doubles as well, though there may be precision issues in 'circ'. Returns triangles in order {t[0][0], t[0][1], t[0][2], t[1][0], ... }, all counter-clockwise.

**Time:**  $\mathcal{O}(n \log n)$

"Point.h"	bf87ec, 88 lines
-----------	------------------

```
typedef Point<ll> P;
typedef struct Quad* Q;
typedef __int128_t lll; // (can be ll if coords are < 2e4)
P arb(LLONG_MAX,LLONG_MAX); // not equal to any other point
```

```
struct Quad {
    bool mark; Q o, rot; P p;
    P F() { return r()->p; }
    Q r() { return rot->rot; }
    Q prev() { return rot->o->rot; }
    Q next() { return r()->prev(); }
};
```

```
bool circ(P p, P a, P b, P c) { // is p in the circumcircle?
    lll p2 = p.dist2(), A = a.dist2()-p2,
        B = b.dist2()-p2, C = c.dist2()-p2;
    return p.cross(a,b)*C + p.cross(b,c)*A + p.cross(c,a)*B > 0;
}
```

```
Q makeEdge(P orig, P dest) {
    Q q[] = {new Quad{0,0,0,orig}, new Quad{0,0,0,arb},
            new Quad{0,0,0,dest}, new Quad{0,0,0,arb}};
    rep(i,0,4)
        q[i]->o = q[-i & 3], q[i]->rot = q[(i+1) & 3];
    return *q;
}
```

```
void splice(Q a, Q b) {
    swap(a->o->rot->o, b->o->rot->o); swap(a->o, b->o);
}
```

```
Q connect(Q a, Q b) {
    Q q = makeEdge(a->F(), b->p);
    splice(q, a->next());
    splice(q->r(), b);
    return q;
}
```

```
pair<Q,Q> rec(const vector<P>& s) {
    if (sz(s) <= 3) {
        Q a = makeEdge(s[0], s[1]), b = makeEdge(s[1], s.back());
        if (sz(s) == 2) return { a, a->r() };
        splice(a->r(), b);
        auto side = s[0].cross(s[1], s[2]);
        Q c = side ? connect(b, a) : 0;
        return {side < 0 ? c->r() : a, side < 0 ? c : b->r() };
    }
}
```

```
#define H(e) e->F(), e->p
#define valid(e) (e->F().cross(H(base)) > 0)
Q A, B, ra, rb;
int half = sz(s) / 2;
tie(ra, A) = rec({all(s) - half});
tie(B, rb) = rec({sz(s) - half + all(s)});
while ((B->p.cross(H(A)) < 0 && (A = A->next())) ||
        (A->p.cross(H(B)) > 0 && (B = B->r()->o)));
```

FastDelaunay PolyhedronVolume Point3D 3dHull

```
Q base = connect(B->r(), A);
if (A->p == ra->p) ra = base->r();
if (B->p == rb->p) rb = base;

#define DEL(e, init, dir) Q e = init->dir; if (valid(e)) \
    while (circ(e->dir->F(), H(base), e->F())) { \
        Q t = e->dir; \
        splice(e, e->prev()); \
        splice(e->r(), e->r()->prev()); \
        e = t; \
    }
for (;;) {
    DEL(LC, base->r(), o); DEL(RC, base, prev());
    if (!valid(LC) && !valid(RC)) break;
    if (!valid(LC) || (valid(RC) && circ(H(RC), H(LC))))
        base = connect(RC, base->r());
    else
        base = connect(base->r(), LC->r());
}
return { ra, rb };
}
```

```
vector<P> triangulate(vector<P> pts) {
    sort(all(pts)); assert(unique(all(pts)) == pts.end());
    if (sz(pts) < 2) return {};
    Q e = rec(pts).first;
    vector<Q> q = {e};
    int qi = 0;
    while (e->o->F().cross(e->F(), e->p) < 0) e = e->o;
    #define ADD { Q c = e; do { c->mark = 1; pts.push_back(c->p); \
        q.push_back(c->r()); c = c->next(); } while (c != e); }
    ADD; pts.clear();
    while (qi < sz(q)) if (!(e = q[qi++])->mark) ADD;
    return pts;
}
```

8.5 3D

PolyhedronVolume.h

**Description:** Magic formula for the volume of a polyhedron. Faces should point outwards.

template<class V, class L> double signed_poly_volume(const V& p, const L& trilst) {     double v = 0;     trav(i, trilst) v += p[i.a].cross(p[i.b]).dot(p[i.c]);     return v / 6; }	1ec4d3, 6 lines
--	-----------------

Point3D.h

**Description:** Class to handle points in 3D space. T can be e.g. double or long long.

template<class T> struct Point3D {     typedef Point3D P;     typedef const P& R;     T x, y, z;     explicit Point3D(T x=0, T y=0, T z=0) : x(x), y(y), z(z) {}     bool operator<(R p) const {         return tie(x, y, z) < tie(p.x, p.y, p.z); }     bool operator==(R p) const {         return tie(x, y, z) == tie(p.x, p.y, p.z); }     P operator+(R p) const { return P(x+p.x, y+p.y, z+p.z); }     P operator-(R p) const { return P(x-p.x, y-p.y, z-p.z); }     P operator*(T d) const { return P(x*d, y*d, z*d); }     P operator/(T d) const { return P(x/d, y/d, z/d); }     T dot(R p) const { return x*p.x + y*p.y + z*p.z; }     P cross(R p) const {	8058ae, 32 lines
--	------------------

```
    return P(y*p.z - z*p.y, z*p.x - x*p.z, x*p.y - y*p.x);
}
T dist2() const { return x*x + y*y + z*z; }
double dist() const { return sqrt((double)dist2()); }
//Azimuthal angle (longitude) to x-axis in interval [-pi, pi]
double phi() const { return atan2(y, x); }
//Zenith angle (latitude) to the z-axis in interval [0, pi]
double theta() const { return atan2(sqrt(x*x+y*y),z); }
P unit() const { return *this/(T)dist(); } //makes dist()==1
//returns unit vector normal to *this and p
P normal(P p) const { return cross(p).unit(); }
//returns point rotated 'angle' radians ccw around axis
P rotate(double angle, P axis) const {
    double s = sin(angle), c = cos(angle); P u = axis.unit();
    return u*dot(u)*(1-c) + (*this)*c - cross(u)*s;
}
};
```

3dHull.h

**Description:** Computes all faces of the 3-dimension hull of a point set. \*No four points must be coplanar\*, or else random results will be returned. All faces will point outwards.

**Time:**  $\mathcal{O}(n^2)$

"Point3D.h"	c172e9, 49 lines
-------------	------------------

```
typedef Point3D<double> P3;

struct PR {
    void ins(int x) { (a == -1 ? a : b) = x; }
    void rem(int x) { (a == x ? a : b) = -1; }
    int cnt() { return (a != -1) + (b != -1); }
    int a, b;
};

struct F { P3 q; int a, b, c; };
```

```
vector<F> hull3d(const vector<P3>& A) {
    assert(sz(A) >= 4);
    vector<vector<PR>> E(sz(A), vector<PR>(sz(A), {-1, -1}));
    #define E(x,y) E[f.x][f.y]
    vector<F> FS;
    auto mf = [&](int i, int j, int k, int l) {
        P3 q = (A[j] - A[i]).cross((A[k] - A[i]));
        if (q.dot(A[l]) > q.dot(A[i]))
            q = q * -1;
        F f{q, i, j, k};
        E(a,b).ins(k); E(a,c).ins(j); E(b,c).ins(i);
        FS.push_back(f);
    };
    rep(i,0,4) rep(j,i+1,4) rep(k,j+1,4)
        mf(i, j, k, 6 - i - j - k);
```

```
    rep(i,4,sz(A)) {
        rep(j,0,sz(FS)) {
            F f = FS[j];
            if(f.q.dot(A[i]) > f.q.dot(A[f.a])) {
                E(a,b).rem(f.c);
                E(a,c).rem(f.b);
                E(b,c).rem(f.a);
                swap(FS[j-], FS.back());
                FS.pop_back();
            }
            int nw = sz(FS);
            rep(j,0,nw) {
                F f = FS[j];
            }
        }
        #define C(a, b, c) if (E(a,b).cnt() != 2) mf(f.a, f.b, i, f.c);
        C(a, b, c); C(a, c, b); C(b, c, a);
    }
}
```

```
    }
    trav(it, FS) if ((A[it.b] - A[it.a]).cross(
        A[it.c] - A[it.a]).dot(it.q) <= 0) swap(it.c, it.b);
    return FS;
};
```

### sphericalDistance.h

**Description:** Returns the shortest distance on the sphere with radius `radius` between the points with azimuthal angles (longitude) `f1` ( $\phi_1$ ) and `f2` ( $\phi_2$ ) from `x` axis and zenith angles (latitude) `t1` ( $\theta_1$ ) and `t2` ( $\theta_2$ ) from `z` axis. All angles measured in radians. The algorithm starts by converting the spherical coordinates to cartesian coordinates so if that is what you have you can use only the two last rows. `dx*radius` is then the difference between the two points in the `x` direction and `d*radius` is the total distance between the points.

```
double sphericalDistance(double f1, double t1,
    double f2, double t2, double radius) {
    double dx = sin(t2)*cos(f2) - sin(t1)*cos(f1);
    double dy = sin(t2)*sin(f2) - sin(t1)*sin(f1);
    double dz = cos(t2) - cos(t1);
    double d = sqrt(dx*dx + dy*dy + dz*dz);
    return radius*2*asin(d/2);
}
```

## Strings (9)

### KMP.h

**Description:** `pi[x]` computes the length of the longest prefix of `s` that ends at `x`, other than `s[0...x]` itself (abacaba -> 0010123). Can be used to find all occurrences of a string.

**Time:**  $\mathcal{O}(n)$

```
vi pi(const string& s) {
    vi p(sz(s));
    rep(i,1,sz(s)) {
        int g = p[i-1];
        while (g && s[i] != s[g]) g = p[g-1];
        p[i] = g + (s[i] == s[g]);
    }
    return p;
}
```

```
vi match(const string& s, const string& pat) {
    vi p = pi(pat + '\0' + s), res;
    rep(i,sz(p)-sz(s),sz(p))
        if (p[i] == sz(pat)) res.push_back(i - 2 * sz(pat));
    return res;
}
```

### Zfunc.h

**Description:** `z[x]` computes the length of the longest common prefix of `s[i:]` and `s`, except `z[0] = 0`. (abacaba -> 0010301)

**Time:**  $\mathcal{O}(n)$

```
vi Z(string S) {
    vi z(sz(S));
    int l = -1, r = -1;
    rep(i,1,sz(S)) {
        z[i] = i >= r ? 0 : min(r - i, z[i - l]);
        while (i + z[i] < sz(S) && S[i + z[i]] == S[z[i]])
            z[i]++;
        if (i + z[i] > r)
            l = i, r = i + z[i];
    }
    return z;
}
```

### Manacher.h

**Description:** For each position in a string, computes `p[0][i] = half length of longest even palindrome around pos i`, `p[1][i] = longest odd (half rounded down)`.

**Time:**  $\mathcal{O}(N)$

```
array<vi, 2> manacher(const string& s) {
    int n = sz(s);
    array<vi,2> p = {vi(n+1), vi(n)};
    rep(z,0,2) for (int i=0,l=0,r=0; i < n; i++) {
        int t = r-i+!z;
        if (i<r) p[z][i] = min(t, p[z][l+t]);
        int L = i-p[z][i], R = i+p[z][i]-!z;
        while (L>=1 && R+1<n && s[L-1] == s[R+1])
            p[z][i]++, L--, R++;
        if (R>r) l=L, r=R;
    }
    return p;
}
```

### MinRotation.h

**Description:** Finds the lexicographically smallest rotation of a string.

**Usage:** `rotate(v.begin(), v.begin()+min_rotation(v), v.end());`

**Time:**  $\mathcal{O}(N)$

```
int min_rotation(string s) {
    int a=0, N=sz(s); s += s;
    rep(b,0,N) rep(k,0,N) {
        if (a+k == b || s[a+k] < s[b+k]) {b += max(0, k-1); break;}
        if (s[a+k] > s[b+k]) {a = b; break;}
    }
    return a;
}
```

### SuffixArray.h

**Description:** Builds suffix array for a string. `sa[i]` is the starting index of the suffix which is `i`'th in the sorted suffix array. The returned vector is of size `n + 1`, and `sa[0] = n`. The `lcp` array contains longest common prefixes for neighbouring strings in the suffix array: `lcp[i] = lcp(sa[i], sa[i-1])`, `lcp[0] = 0`. The input string must not contain any zero bytes.

**Time:**  $\mathcal{O}(n \log n)$

```
struct SuffixArray {
    vi sa, lcp;
    SuffixArray(string& s, int lim=256) { // or basic_string<int>
        int n = sz(s) + 1, k = 0, a, b;
        vi x(all(s)+1), y(n), ws(max(n, lim)), rank(n);
        sa = lcp = y, iota(all(sa), 0);
        for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim = p) {
            p = j, iota(all(y), n - j);
            rep(i,0,n) if (sa[i] >= j) y[p++] = sa[i] - j;
            fill(all(ws), 0);
            rep(i,0,n) ws[x[i]]++;
            rep(i,1,lim) ws[i] += ws[i - 1];
            for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
            swap(x, y), p = 1, x[sa[0]] = 0;
            rep(i,1,n) a = sa[i - 1], b = sa[i], x[b] =
                (y[a] == y[b] && y[a + j] == y[b + j]) ? p - 1 : p++;
        }
        rep(i,1,n) rank[sa[i]] = i;
        for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
            for (k && k--, j = sa[rank[i] - 1];
                s[i + k] == s[j + k]; k++);
    }
};
```

### SuffixTree.h

**Description:** Ukkonen's algorithm for online suffix tree construction. Each node contains indices `[l, r]` into the string, and a list of child nodes. Suffixes are given by traversals of this tree, joining `[l, r]` substrings. The root is 0 (has `l = -1, r = 0`), non-existent children are -1. To get a complete tree, append a dummy symbol – otherwise it may contain an incomplete path (still useful for substring matching, though).

**Time:**  $\mathcal{O}(26N)$

```
struct SuffixTree {
    enum { N = 200010, ALPHA = 26 }; // N ~ 2*maxlen+10
    int toi(char c) { return c - 'a'; }
    string a; // v = cur node, q = cur position
    int t[N][ALPHA], l[N], r[N], p[N], s[N], v=0, q=0, m=2;

    void ukkadd(int i, int c) { suff:
        if (r[v]<=q) {
            if (t[v][c]==-1) { t[v][c]=m; l[m]=i;
                p[m++]=v; v=s[v]; q=r[v]; goto suff; }
            v=t[v][c]; q=l[v];
        }
        if (q==-1 || c==toi(a[q])) q++; else {
            l[m+1]=i; p[m+1]=m; l[m]=l[v]; r[m]=q;
            p[m]=p[v]; t[m][c]=m+1; t[m][toi(a[q])]=v;
            l[v]=q; p[v]=m; t[p[m]][toi(a[l[m]])]=m;
            v=s[p[m]]; q=l[m];
            while (q<r[m]) { v=t[v][toi(a[q])]; q+=r[v]-l[v]; }
            if (q==r[m]) s[m]=v; else s[m]=m+2;
            q=r[v]-(q-r[m]); m+=2; goto suff;
        }
    }

    SuffixTree(string a) : a(a) {
        fill(r,r+N,sz(a));
        memset(s, 0, sizeof s);
        memset(t, -1, sizeof t);
        fill(t[1],t[1]+ALPHA,0);
        s[0] = 1; l[0] = l[1] = -1; r[0] = r[1] = p[0] = p[1] = 0;
        rep(i,0,sz(a)) ukkadd(i, toi(a[i]));
    }

    // example: find longest common substring (uses ALPHA = 28)
    pii best;
    int lcs(int node, int i1, int i2, int olen) {
        if (l[node] <= i1 && i1 < r[node]) return 1;
        if (l[node] <= i2 && i2 < r[node]) return 2;
        int mask = 0, len = node ? olen + (r[node] - l[node]) : 0;
        rep(c,0,ALPHA) if (t[node][c] != -1)
            mask |= lcs(t[node][c], i1, i2, len);
        if (mask == 3)
            best = max(best, {len, r[node] - len});
        return mask;
    }
    static pii LCS(string s, string t) {
        SuffixTree st(s + (char)('z' + 1) + t + (char)('z' + 2));
        st.lcs(0, sz(s), sz(s) + 1 + sz(t), 0);
        return st.best;
    }
};
```

### Hashing.h

**Description:** Self-explanatory methods for string hashing.

```
// Arithmetic mod 2^64-1. 2x slower than mod 2^64 and more
// code, but works on evil test data (e.g. Thue-Morse, where
// ABBA... and BAAB... of length 2^10 hash the same mod 2^64).
// "typedef ull H;" instead if you think test data is random,
// or work mod 10^9+7 if the Birthday paradox is not a problem.
struct H {
```

```
typedef uint64_t ull;
ull x; H(ull x=0) : x(x) {}
#define OP(O,A,B) H operator O(H o) { ull r = x; asm \
(A "addq %%rdx, %0\n adcq $0,%0" : "+a"(r) : B); return r; }
OP(+,,"d"(o.x)) OP(*,"mul %1\n", "r"(o.x) : "rdx")
H operator~(H o) { return *this + ~o.x; }
ull get() const { return x + !~x; }
bool operator==(H o) const { return get() == o.get(); }
bool operator<(H o) const { return get() < o.get(); }
};
static const H C = (1l)1e1l+3; // (order ~ 3e9; random also ok)

struct HashInterval {
    vector<H> ha, pw;
    HashInterval(string& str) : ha(sz(str)+1), pw(ha) {
        pw[0] = 1;
        rep(i,0,sz(str))
            ha[i+1] = ha[i] * C + str[i],
            pw[i+1] = pw[i] * C;
    }
    H hashInterval(int a, int b) { // hash [a, b)
        return ha[b] - ha[a] * pw[b - a];
    }
};

vector<H> getHashes(string& str, int length) {
    if (sz(str) < length) return {};
    H h = 0, pw = 1;
    rep(i,0,length)
        h = h * C + str[i], pw = pw * C;
    vector<H> ret = {h};
    rep(i,length,sz(str)) {
        ret.push_back(h = h * C + str[i] - pw * str[i-length]);
    }
    return ret;
}

H hashString(string& s) { H h{}; trav(c,s) h=h*C+c; return h; }
```

AhoCorasick.h

**Description:** Aho-Corasick tree is used for multiple pattern matching. Initialize the tree with create(patterns). find(word) returns for each position the index of the longest word that ends there, or -1 if none. findAll(., word) finds all words (up to  $N\sqrt{N}$  many if no duplicate patterns) that start at each position (shortest first). Duplicate patterns are allowed; empty patterns are not. To find the longest words that start at each position, reverse all input. **Time:** create is  $\mathcal{O}(26N)$  where  $N$  is the sum of length of patterns. find is  $\mathcal{O}(M)$  where  $M$  is the length of the word. findAll is  $\mathcal{O}(NM)$ .

```
struct AhoCorasick {
    enum {alpha = 26, first = 'A'};
    struct Node {
        // (nmatches is optional)
        int back, next[alpha], start = -1, end = -1, nmatches = 0;
        Node(int v) { memset(next, v, sizeof(next)); }
    };
    vector<Node> N;
    vector<int> backp;
    void insert(string& s, int j) {
        assert(!s.empty());
        int n = 0;
        trav(c, s) {
            int& m = N[n].next[c - first];
            if (m == -1) { n = m = sz(N); N.emplace_back(-1); }
            else n = m;
        }
        if (N[n].end == -1) N[n].start = j;
        backp.push_back(N[n].end);
        N[n].end = j;
    }
};
```

```
    N[n].nmatches++;
}
AhoCorasick(vector<string>& pat) {
    N.emplace_back(-1);
    rep(i,0,sz(pat)) insert(pat[i], i);
    N[0].back = sz(N);
    N.emplace_back(0);

    queue<int> q;
    for (q.push(0); !q.empty(); q.pop()) {
        int n = q.front(), prev = N[n].back;
        rep(i,0,alpha) {
            int &ed = N[n].next[i], y = N[prev].next[i];
            if (ed == -1) ed = y;
            else {
                N[ed].back = y;
                (N[ed].end == -1 ? N[ed].end : backp[N[ed].start])
                    = N[y].end;
                N[ed].nmatches += N[y].nmatches;
                q.push(ed);
            }
        }
    }
}

vi find(string word) {
    int n = 0;
    vi res; // ll count = 0;
    trav(c, word) {
        n = N[n].next[c - first];
        res.push_back(N[n].end);
        // count += N[n].nmatches;
    }
    return res;
}

vector<vi> findAll(vector<string>& pat, string word) {
    vi r = find(word);
    vector<vi> res(sz(word));
    rep(i,0,sz(word)) {
        int ind = r[i];
        while (ind != -1) {
            res[i - sz(pat[ind]) + 1].push_back(ind);
            ind = backp[ind];
        }
    }
    return res;
}
};
```

Various (10)

10.1 Intervals

IntervalContainer.h

**Description:** Add and remove intervals from a set of disjoint intervals. Will merge the added interval with any overlapping intervals in the set when adding. Intervals are [inclusive, exclusive). **Time:**  $\mathcal{O}(\log N)$

```
set<pii>::iterator addInterval(set<pii>& is, int L, int R) {
    if (L == R) return is.end();
    auto it = is.lower_bound({L, R}), before = it;
    while (it != is.end() && it->first <= R) {
        R = max(R, it->second);
        before = it = is.erase(it);
    }
    if (it != is.begin() && (--it)->second >= L) {
        L = min(L, it->first);
    }
}
```

```
    R = max(R, it->second);
    is.erase(it);
}
return is.insert(before, {L,R});
}

void removeInterval(set<pii>& is, int L, int R) {
    if (L == R) return;
    auto it = addInterval(is, L, R);
    auto r2 = it->second;
    if (it->first == L) is.erase(it);
    else (int&)it->second = L;
    if (R != r2) is.emplace(R, r2);
}
```

IntervalCover.h

**Description:** Compute indices of smallest set of intervals covering another interval. Intervals should be [inclusive, exclusive). To support [inclusive, inclusive], change (A) to add || R.empty(). Returns empty set on failure (or if G is empty). **Time:**  $\mathcal{O}(N \log N)$

```
template<class T>
vi cover(pair<T, T> G, vector<pair<T, T>> I) {
    vi S(sz(I)), R;
    iota(all(S), 0);
    sort(all(S), [&](int a, int b) { return I[a] < I[b]; });
    T cur = G.first;
    int at = 0;
    while (cur < G.second) { // (A)
        pair<T, int> mx = make_pair(cur, -1);
        while (at < sz(I) && I[S[at]].first <= cur) {
            mx = max(mx, make_pair(I[S[at]].second, S[at]));
            at++;
        }
        if (mx.second == -1) return {};
        cur = mx.first;
        R.push_back(mx.second);
    }
    return R;
}
```

ConstantIntervals.h

**Description:** Split a monotone function on [from, to) into a minimal set of half-open intervals on which it has the same value. Runs a callback g for each such interval. **Usage:** constantIntervals(0, sz(v), [&](int x){return v[x];}, [&](int lo, int hi, T val){...}); **Time:**  $\mathcal{O}(k \log \frac{n}{k})$

```
template<class F, class G, class T>
void rec(int from, int to, F& f, G& g, int& i, T& p, T q) {
    if (p == q) return;
    if (from == to) {
        g(i, to, p);
        i = to; p = q;
    } else {
        int mid = (from + to) >> 1;
        rec(from, mid, f, g, i, p, f(mid));
        rec(mid+1, to, f, g, i, p, q);
    }
}

template<class F, class G>
void constantIntervals(int from, int to, F f, G g) {
    if (to <= from) return;
    int i = from; auto p = f(i), q = f(to-1);
    rec(from, to-1, f, g, i, p, q);
    g(i, to, q);
}
```



## 10.2 Misc. algorithms

### TernarySearch.h

**Description:** Find the smallest  $i$  in  $[a, b]$  that maximizes  $f(i)$ , assuming that  $f(a) < \dots < f(i) \geq \dots \geq f(b)$ . To reverse which of the sides allows non-strict inequalities, change the  $<$  marked with (A) to  $\leq$ , and reverse the loop at (B). To minimize  $f$ , change it to  $>$ , also at (B).  
**Usage:** `int ind = ternSearch(0, n-1, [&](int i){return a[i];});`  
**Time:**  $\mathcal{O}(\log(b-a))$

9155b4, 11 lines

```
template<class F>
int ternSearch(int a, int b, F f) {
    assert(a <= b);
    while (b - a >= 5) {
        int mid = (a + b) / 2;
        if (f(mid) < f(mid+1)) a = mid; // (A)
        else b = mid+1;
    }
    rep(i, a+1, b+1) if (f(a) < f(i)) a = i; // (B)
    return a;
}
```

### LIS.h

**Description:** Compute indices for the longest increasing subsequence.  
**Time:**  $\mathcal{O}(N \log N)$

2932a0, 17 lines

```
template<class I> vi lis(const vector<I>& S) {
    if (S.empty()) return {};
    vi prev(sz(S));
    typedef pair<I, int> p;
    vector<p> res;
    rep(i, 0, sz(S)) {
        // change 0 -> i for longest non-decreasing subsequence
        auto it = lower_bound(all(res), p{S[i], 0});
        if (it == res.end()) res.emplace_back(), it = res.end()-1;
        *it = {S[i], i};
        prev[i] = it == res.begin() ? 0 : (it-1)->second;
    }
    int L = sz(res), cur = res.back().second;
    vi ans(L);
    while (L--) ans[L] = cur, cur = prev[cur];
    return ans;
}
```

## 10.3 Dynamic programming

### KnuthDP.h

**Description:** When doing DP on intervals:  $a[i][j] = \min_{i < k < j} (a[i][k] + a[k][j]) + f(i, j)$ , where the (minimal) optimal  $k$  increases with both  $i$  and  $j$ , one can solve intervals in increasing order of length, and search  $k = p[i][j]$  for  $a[i][j]$  only between  $p[i][j-1]$  and  $p[i+1][j]$ . This is known as Knuth DP. Sufficient criteria for this are if  $f(b, c) \leq f(a, d)$  and  $f(a, c) + f(b, d) \leq f(a, d) + f(b, c)$  for all  $a \leq b \leq c \leq d$ . Consider also: LineContainer (ch. Data structures), monotone queues, ternary search.  
**Time:**  $\mathcal{O}(N^2)$

d38d2b, 18 lines

### DivideAndConquerDP.h

**Description:** Given  $a[i] = \min_{l \circ(i) \leq k < h i(i)} (f(i, k))$  where the (minimal) optimal  $k$  increases with  $i$ , computes  $a[i]$  for  $i = L..R-1$ .  
**Time:**  $\mathcal{O}((N + (hi - lo)) \log N)$

```
struct DP { // Modify at will:
    int lo(int ind) { return 0; }
    int hi(int ind) { return ind; }
    ll f(int ind, int k) { return dp[ind][k]; }
    void store(int ind, int k, ll v) { res[ind] = pii(k, v); }
```

```
void rec(int L, int R, int LO, int HI) {
    if (L >= R) return;
    int mid = (L + R) >> 1;
    pair<ll, int> best(LLONG_MAX, LO);
    rep(k, max(LO, lo(mid)), min(HI, hi(mid)))
        best = min(best, make_pair(f(mid, k), k));
    store(mid, best.second, best.first);
    rec(L, mid, LO, best.second+1);
    rec(mid+1, R, best.second, HI);
}
void solve(int L, int R) { rec(L, R, INT_MIN, INT_MAX); }
```

## 10.4 Debugging tricks

- `signal(SIGSEGV, [](int) { _Exit(0); });` converts segfaults into Wrong Answers. Similarly one can catch SIGABRT (assertion failures) and SIGFPE (zero divisions). `_GLIBCXX_DEBUG` failures generate SIGABRT (or SIGSEGV on gcc 5.4.0 apparently).
- `feenableexcept(29);` kills the program on NaNs (1), 0-divs (4), infinities (8) and denormals (16).

## 10.5 Optimization tricks

### 10.5.1 Bit hacks

- `x & -x` is the least bit in `x`.
- `for (int x = m; x; ) { --x &= m; ... }` loops over all subset masks of `m` (except `m` itself).
- `c = x&-x, r = x+c; ((r^x) >> 2)/c | r` is the next number after `x` with the same number of bits set.
- `rep(b, 0, K) rep(i, 0, (1 << K))`  
    `if (i & 1 << b) D[i] += D[i^(1 << b)];`  
computes all sums of subsets.

### 10.5.2 Pragas

- `#pragma GCC optimize ("Ofast")` will make GCC auto-vectorize for loops and optimizes floating points better (assumes associativity and turns off denormals).
- `#pragma GCC target ("avx,avx2")` can double performance of vectorized code, but causes crashes on old machines.
- `#pragma GCC optimize ("trapv")` kills the program on integer overflows (but is really slow).

### FastMod.h

**Description:** Compute  $a\%b$  about 4 times faster than usual, where  $b$  is constant but not known at compile time. Fails for  $b = 1$ .  
c977c5, 10 lines

```
typedef unsigned long long ull;
typedef __uint128_t L;
struct FastMod {
    ull b, m;
    FastMod(ull b) : b(b), m(ull((L(1) << 64) / b)) {}
    ull reduce(ull a) {
        ull q = (ull)((L(m) * a) >> 64), r = a - q * b;
        return r >= b ? r - b : r;
    }
};
```

### BumpAllocator.h

**Description:** When you need to dynamically allocate many objects and don't care about freeing them. "new X" otherwise has an overhead of something like 0.05us + 16 bytes per allocation.  
745db2, 8 lines

```
// Either globally or in a single class:
static char buf[450 << 20];
void* operator new(size_t s) {
    static size_t i = sizeof buf;
    assert(s < i);
    return (void*)&buf[i -= s];
}
void operator delete(void*) {}
```

### SmallPtr.h

**Description:** A 32-bit pointer that points into BumpAllocator memory.  
"BumpAllocator.h" 2dd6c9, 10 lines

```
template<class T> struct ptr {
    unsigned ind;
    ptr(T* p = 0) : ind(p ? unsigned((char*)p - buf) : 0) {
        assert(ind < sizeof buf);
    }
    T& operator*() const { return *(T*)(buf + ind); }
    T* operator->() const { return &***this; }
    T& operator[](int a) const { return (&***this)[a]; }
    explicit operator bool() const { return ind; }
};
```

### BumpAllocatorSTL.h

**Description:** BumpAllocator for STL containers.  
**Usage:** `vector<vector<int, small<int>>> ed(N);`  
bb66d4, 14 lines

```
char buf[450 << 20] alignas(16);
size_t buf_ind = sizeof buf;
```

```
template<class T> struct small {
    typedef T value_type;
    small() {}
    template<class U> small(const U&) {}
    T* allocate(size_t n) {
        buf_ind -= n * sizeof(T);
        buf_ind &= 0 - alignof(T);
        return (T*)(buf + buf_ind);
    }
    void deallocate(T*, size_t) {}
};
```

### Unrolling.h

520e76, 4 lines

```
#define F {...; ++i;}
int i = from;
while (i&3 && i < to) F // for alignment, if needed
while (i + 4 <= to) { F F F F } while (i < to) F
```



# Techniques (A)

techniques.txt	159 lines
Recursion	
Divide and conquer	
Finding interesting points in N log N	
Algorithm analysis	
Master theorem	
Amortized time complexity	
Greedy algorithm	
Scheduling	
Max contiguous subvector sum	
Invariants	
Huffman encoding	
Graph theory	
Dynamic graphs (extra book-keeping)	
Breadth first search	
Depth first search	
* Normal trees / DFS trees	
Dijkstra's algorithm	
MST: Prim's algorithm	
Bellman-Ford	
Konig's theorem and vertex cover	
Min-cost max flow	
Lovasz toggle	
Matrix tree theorem	
Maximal matching, general graphs	
Hopcroft-Karp	
Hall's marriage theorem	
Graphical sequences	
Floyd-Warshall	
Euler cycles	
Flow networks	
* Augmenting paths	
* Edmonds-Karp	
Bipartite matching	
Min. path cover	
Topological sorting	
Strongly connected components	
2-SAT	
Cut vertices, cut-edges and biconnected components	
Edge coloring	
* Trees	
Vertex coloring	
* Bipartite graphs (=> trees)	
* 3^n (special case of set cover)	
Diameter and centroid	
K'th shortest path	
Shortest cycle	
Dynamic programming	
Knapsack	
Coin change	
Longest common subsequence	
Longest increasing subsequence	
Number of paths in a dag	
Shortest path in a dag	
Dynprog over intervals	
Dynprog over subsets	
Dynprog over probabilities	
Dynprog over trees	
3^n set cover	
Divide and conquer	
Knuth optimization	
Convex hull optimizations	
RMQ (sparse table a.k.a 2^k-jumps)	
Bitonic cycle	
Log partitioning (loop over most restricted)	
Combinatorics	

Computation of binomial coefficients
Pigeon-hole principle
Inclusion/exclusion
Catalan number
Pick's theorem
Number theory
Integer parts
Divisibility
Euclidean algorithm
Modular arithmetic
* Modular multiplication
* Modular inverses
* Modular exponentiation by squaring
Chinese remainder theorem
Fermat's little theorem
Euler's theorem
Phi function
Frobenius number
Quadratic reciprocity
Pollard-Rho
Miller-Rabin
Hensel lifting
Vieta root jumping
Game theory
Combinatorial games
Game trees
Mini-max
Nim
Games on graphs
Games on graphs with loops
Grundy numbers
Bipartite games without repetition
General games without repetition
Alpha-beta pruning
Probability theory
Optimization
Binary search
Ternary search
Unimodality and convex functions
Binary search on derivative
Numerical methods
Numeric integration
Newton's method
Root-finding with binary/ternary search
Golden section search
Matrices
Gaussian elimination
Exponentiation by squaring
Sorting
Radix sort
Geometry
Coordinates and vectors
* Cross product
* Scalar product
Convex hull
Polygon cut
Closest pair
Coordinate-compression
Quadtrees
KD-trees
All segment-segment intersection
Sweeping
Discretization (convert to events and sweep)
Angle sweeping
Line sweeping
Discrete second derivatives
Strings
Longest common substring
Palindrome subsequences

Knuth-Morris-Pratt
Tries
Rolling polynomial hashes
Suffix array
Suffix tree
Aho-Corasick
Manacher's algorithm
Letter position lists
Combinatorial search
Meet in the middle
Brute-force with pruning
Best-first (A*)
Bidirectional search
Iterative deepening DFS / A*
Data structures
LCA (2^k-jumps in trees in general)
Pull/push-technique on trees
Heavy-light decomposition
Centroid decomposition
Lazy propagation
Self-balancing trees
Convex hull trick (wcipeg.com/wiki/Convex_hull_trick)
Monotone queues / monotone stacks / sliding queues
Sliding queue using 2 stacks
Persistent segment tree