ON PICARD GROUPS OF DIRECTED GRAPHS

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ABSTRACT. The Picard group $\mathrm{Pic}(G)$ of a graph G is a finitely generated abelian group, and the Jacobian $\mathrm{Jac}(G)$ is the torsion subgroup of $\mathrm{Pic}(G)$. These groups can be computed by using the Smith normal from of the Laplacian matrix L_G or by using chip-firing games. One may consider its generalization to directed graphs by using Laplacian matrices. In this paper, we investigate Picard groups and Jacobians for several classes of directed graphs, including directed trees and directed cycles. Different from the undirected case, even for trees and cycles, one can find very interesting computational results. Our investigation is based on experimental mathematics; we compute a large number of examples, find some patterns from them, and prove them.

1. Introduction

The main things that we have done so far:

(1) $\mathrm{Jac}(T) = \{0\}$ (Proposition 3.4).
(2) $\mathrm{Jac}(C_n)$ can be anything (Conjecture 4.1).
(3) An explicit construction of an orientation of C_n so that $\mathrm{Jac}(C_n) = \mathbb{Z}_m$ (Conjecture 4.7).

The remaining things that we want to do:

(1) Add experimental data visualization for number of paths needed and sizes of graphs.
(2) Anything else??

Things that we might want to do (or perhaps leave them as conjectures):

(1) Proof of wheel graph conjecture (should be rewritten with Lucas numbers). [Big99] should be relevant.
(2) Proof of multipartite graphs. [**?**] could be relevant.

Acknowledgment This research was supported by Research and Creative Activities (RSCA) at SUNY New Paltz. We would like to thank RSCA for their support.

2. Preliminaries

Definition 2.1. Jaiung: Recall terminal strong components

Theorem 2.2. *Jaiung: Here we recall* [Wag00] *the torsion-free part theorem.*

3. Picard groups of oriented trees

Lemma 3.1. *Let G be a directed graph. If we attach either an incoming arrow or a two-sided arrow to create another directed graph G', then* $\mathrm{Pic}(G) = \mathrm{Pic}(G')$.

2020 *Mathematics Subject Classification.* 05C50, 05C76.

Key words and phrases. Jacobian of a graph, sandpile group, critical group, chip-firing game, gluing graphs, cycle graph, directed graphs, Laplacian of graph.

Proof. Let α be an arrow which is glued to G. Let $|V(G)| = n$. We label the vertexes of G as $1, 2, \ldots, n$. Suppose first that α is incoming and α is glued at the vertex n. Let $L_G = (a_{ij})$ (resp. $L_{G'}$) be the Laplacian matrix of G (resp. G'). Then the matrix $L_{G'}$ is of the following form.

$$L_{G'} = \left[\begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 0 \\ \hline 0 & 0 & \cdots & -1 & 1 \end{array}\right] \tag{1}$$

By a column operation between the last two columns, we obtain the following matrix:

$$\left[\begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right] \tag{2}$$

This shows that $\mathrm{Pic}(G) = \mathrm{Pic}(G')$.

Next, suppose that α is a two-sided arrow. Then similar to the above, we obtain the following Laplacian matrix for G':

$$L_{G'} = \left[\begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n + 1 & -1 \\ \hline 0 & 0 & \cdots & -1 & 1 \end{array}\right] \tag{3}$$

By a column operation, the matrix (3) becomes the matrix (2). This shows that $\mathrm{Pic}(G) = \mathrm{Pic}(G')$. $\square$

The following example shows that if we change the direction of arrows, Picard groups can change drastically.

Example 3.2. Consider the following directed tree:

$$T = \left(\begin{tikzcd} & 1 \arrow[d] & \\ 2 \arrow[r] & 3 & 4 \arrow[l] \\ & 5 \arrow[u] & \end{tikzcd} \right) \tag{4}$$

We have

$$L_T = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \implies \mathrm{SNF}(L_T) = \left[\begin{array}{c|c} I_4 & 0 \\ \hline 0 & 0 \end{array}\right]$$

Hence, $\mathrm{Pic}(T) = \mathbb{Z}$. On the other hand, consider the following

$$T' = \left(\begin{array}{ccccc} & & 1 & & \\ & & \uparrow & & \\ 2 & \longleftarrow & 3 & \longrightarrow & 4 \\ & & \downarrow & & \\ & & 5 & & \end{array} \right) \tag{5}$$

We have

$$L_T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \implies \mathrm{Pic}(T') = \mathbb{Z}^4.$$

Remark 3.3. For the undirected case, when one glues two graphs G_1 and G_2 along one vertex to obtain G, then $\mathrm{Pic}(G) = \mathrm{Pic}(G_1) \times \mathrm{Pic}(G_2)$. But, this is no longer true for directed graphs. For instance, the tree T in (4) can be considered as a directed graph obtained by gluing the following two directed graphs G_1 and G_2 along the vertices 3 and 3′"

$$G_1 = \left(\begin{array}{ccc} & & 1 \\ & & \downarrow \\ 2 & \longrightarrow & 3 \end{array} \right), \quad G_2 = \left(\begin{array}{ccc} 3' & \longleftarrow & 4 \\ \uparrow & & \\ 5 & & \end{array} \right) \tag{6}$$

But, we have

$$L_{G_1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathrm{Pic}(G_1) = \mathbb{Z}, \quad L_{G_2} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \implies \mathrm{Pic}(G_2) = \mathbb{Z}$$

It follows that $\mathrm{Pic}(T) \neq \mathrm{Pic}(G_1) \times \mathrm{Pic}(G_2)$.

For the undirected trees T, $\mathrm{Pic}(T) = \mathbb{Z}$. But, for directed trees, the rank of $\mathrm{Pic}(T)$ can be arbitrarily large depending the number of strong terminal component of T. Nonetheless, we prove that $\mathrm{Jac}(T) = \{0\}$ in the following.

Proposition 3.4. *Let T be a tree with any orientation. Then* $\mathrm{Jac}(T) = \{0\}$*, i.e.,* $\mathrm{Pic}(T)$ *is torsion-free.*

Proof. We inductively prove this. When T_0 is a tree with one arrow, one can easily check that $\mathrm{Pic}(T_0) = \mathbb{Z}$ or $\{0\}$ (depending on the number of strong terminal components).

Suppose that T_k is an oriented tree with k arrows. When we add one arrow α to T_k to construct T_{k+1}, there are three cases; α is (1) incoming, (2) outgoing, and (3) two-sided. When α is either incoming or two-sided arrow, then it follows from Lemma 3.1 that $\mathrm{Pic}(T_k) = \mathrm{Pic}(T_{k+1}) = \mathbb{Z}^r$, where r is the number of the terminal strong components of T_k and T_{k+1}, since in this case it does not increase the number of the terminal strong components.

Next, suppose that α is an outgoing arrow. Let's label the vertexes of G as $v_1, v_2, \ldots, v_n$. Suppose that the arrow α is attached to the vertex v_j. Let $L_k = (a_{ij})$ be the Laplacian matrix of T_k. Then we have the following:

$$L_{k+1} = \left[\begin{array}{cccccc|c|c} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jj}+1 & \cdots & a_{jn} & -1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} & 0 \\ \hline 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \end{array}\right] \tag{7}$$

To compute the Smith normal form, by relabeling vertices, we may assume L_{k+1} is the following matrix:

$$\left[\begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 1 \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array}\right] \tag{8}$$

Since $\mathrm{Pic}(T_k) = \mathbb{Z}^r$, there exist $P, Q \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$PL_kQ = \left[\begin{array}{c|c} I_{n-r} & 0 \\ \hline 0 & 0_r \end{array}\right] \tag{9}$$

where 0_r is an $r \times r$ zero matrix. Consider the following block matrices of size $(n+1) \times (n+1)$:

$$P' = \left[\begin{array}{c|c} P & 0 \\ \hline 0 & 1 \end{array}\right], \quad Q' = \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 1 \end{array}\right] \tag{10}$$

Then, we have

$$P'L_{k+1}Q' = \left[\begin{array}{c|c} P & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} L_k & e_n \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} PL_kQ & Pe_n \\ \hline 0 & 0 \end{array}\right] \tag{11}$$

We first consider the case when v_n is a sink. In particular, T_{k+1} and T_k have the same number of terminal strong components. In this case, the n^{th} row of L_{k+1} in (8) is the zero row. In particular, we can take P so that

$$Pe_n = e_n \tag{12}$$

Therefore, we have

$$P'L_{k+1}Q' = \left[\begin{array}{c|c} PL_kQ & Pe_n \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} PL_kQ & e_n \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c|c|c} I_{n-r} & & 0 & 0 \\ \hline & 0_{r-1} & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \tag{13}$$

It follows that $\mathrm{Pic}(T_{k+1}) = \mathrm{Pic}(T_k)$.

Now, suppose that v_n is not a sink. Let $Pe_n = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. There are two cases:

<u>Case 1:</u> Suppose that $x_{n-r+1} = x_{n-r+2} = \cdots = x_n = 0$. In this case, one can easily observe that after some column operations, $P'L_{k+1}Q'$ becomes the Smith normal form of L_{k+1}. In particular, $\mathrm{Jac}(T_{k+1}) = \mathrm{Jac}(T_k)$, and hence $\mathrm{Pic}(T_{k+1}) = \mathbb{Z} \times \mathrm{Pic}(T_k)$.

<u>Case 2:</u> Suppose that at least one of $x_{n-r+1}, x_{n-r+2}, \cdots, x_n$ is not equal to zero. Then, the Smith normal form of L_{k+1} becomes the last matrix in (13). In particular, $\mathrm{Jac}(T_{k+1}) = \mathrm{Jac}(T_k)$, and hence $\mathrm{Pic}(T_{k+1}) = \mathbb{Z} \times \mathrm{Pic}(T_k)$. □

Example 3.5. Consider the following oriented tree:

$$T = \left(\begin{tikzcd} 1 & 2 \arrow[d] & 3 \arrow[d, leftrightarrow] & \\ & 4 \arrow[ul] \arrow[r, leftrightarrow] & 5 \arrow[r] & 6 \end{tikzcd} \right)$$

The Laplacian matrix of T is the following:

$$L_T = D_T - A_T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Smith normal form of L_T is the following 6×6 matrix:

$$\left[\begin{array}{c|c} I_4 & 0 \\ \hline 0 & 0_2 \end{array} \right]$$

This shows that $\mathrm{Pic}(T) = \mathbb{Z}^2$.

Example 3.6. Consider the following oriented tree obtain from Example 3.5 by gluing an outgoing arrow α:

$$T' = \left(\begin{tikzcd} 1 & 2 \arrow[d] & 3 \arrow[d, leftrightarrow] & \\ 7 & 4 \arrow[ul] \arrow[l, "\alpha"'] \arrow[r, leftrightarrow] & 5 \arrow[r] & 6 \end{tikzcd} \right)$$

Now, we have

$$L_{T'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Smith normal form of $L_{T'}$ is the following 7×7 matrix:

$$\left[\begin{array}{c|c} I_4 & 0 \\ \hline 0 & 0_3 \end{array} \right]$$

This shows that $\mathrm{Pic}(T) = \mathbb{Z}^3$.

Example 3.7. Consider the following oriented tree obtain from Example 3.5 by gluing an outgoing arrow α:

$$T'' = \left(\begin{tikzcd} 1 & 2 \arrow[d] & 3 \arrow[d, leftrightarrow] & 7 \\ & 4 \arrow[ul] \arrow[r, leftrightarrow] & 5 \arrow[r] & 6 \arrow[u, "\alpha"'] \end{tikzcd} \right)$$

Now, we have

$$L_{T''} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Smith normal form of $L_{T''}$ is the following 7×7 matrix:

$$\left[\begin{array}{c|c} I_5 & 0 \\ \hline 0 & 0_2 \end{array}\right]$$

This shows that $\mathrm{Pic}(T'') = \mathbb{Z}^2$.

Remark 3.8. Jaiung: Here we add a version of directed matrix-tree theorem. I think it does not prove the above theorem that $\mathrm{Jac}(T)$ is trivial.

4. Picard groups of oriented cycles

We computed a large set of examples (from C_3 to C_{80}) with all possible orientations. By doing so, we found some patterns which lead us to two conjectures. In what follows, we let C_n be an undirected cycle graph on n vertices.

4.1. Oriented cycles with cyclic Jacobians.

In this subsection, we prove the following conjecture.

Conjecture 4.1. Let $n \geq 3$. For each $k \leq n$, there exists an orientation of C_n such that $\mathrm{Jac}(C_n)$ (with that orientation) is $\mathbb{Z}_k$.

The following examples confirms the conjecture for the class of C_3.

Example 4.2. With the following orientations of C_3

$$G_1 = \left(\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \uparrow & \swarrow & \\ \bullet & & \end{array}\right), \quad G_2 = \left(\begin{array}{ccc} \bullet & \longleftarrow & \bullet \\ \uparrow & \swarrow & \\ \bullet & & \end{array}\right), \quad G_3 = \left(\begin{array}{ccc} \bullet & \longleftrightarrow & \bullet \\ \updownarrow & \swarrow\!\!\!\!\nearrow & \\ \bullet & & \end{array}\right)$$

we have $\mathrm{Jac}(G_1) = 0$, $\mathrm{Jac}(G_2) = \mathbb{Z}_2$, and $\mathrm{Jac}(G_3) = \mathbb{Z}_3$.

Remark 4.3. Let $M \in M_{m\times n}(\mathbb{Z})$, and let $I_k(M)$ denote the ideal generated by $k \times k$ minors of M, where $I_k(M) = 0$ if $k > \min\{m,n\}$ and $I_k = (1)$ if $k \leq 0$. For a matrix $N = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array}\right]$ and for any k, $I_k(M) = I_{k+1}(N)$, and the cokernels of M and N are isomorphic.

Lemma 4.4. *Fix an orientation of C_n. If not every edge of C_n (with a fixed orientation) is bidirectional, there is an orientation of C_{n+1} such that* $\mathrm{Pic}(C_n) \cong \mathrm{Pic}(C_{n+1})$.

Proof. Let $V(C_n) = \{v_1, \dots, v_n\}$ and $D_{C_n} = (d_{ij})$ be the diagonal matrix of C_n (with a given orientation). Since not every edge is bidirectional, there exists i such that $d_{ii} = 0$ or 1. Without loss of generality, we may assume $i = n$ so that the adjacent vertices are v_{n-1} and v_1.

<u>Case 1:</u> Suppose that $d_{nn} = 0$. In this case, the Laplacian matrix of C_n is the following.

$$L_{C_n} = \begin{bmatrix} l+1 & -l & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & 0 & -k & k+1 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

where $l,k \in \{0,1\}$. We define C_{n+1} extending C_n as follows. Add a vertex v_{n+1} to C_n, replace the edge $e_{1\to n}$ by $e_{1\to n+1}$, and add a new edge $e_{n\to n+1}$. Pictorially, we have the following

$$C_n : \Big(\cdots\ v_{n-1} \to v_n \leftarrow v_1\ \cdots\Big) \longrightarrow C_{n+1} : \Big(\cdots\ v_{n-1} \to v_n \to v_{n+1} \leftarrow v_1\ \cdots\Big). \tag{15}$$

Now, one has the following equivalence of matrices.

$$L_{C_{n+1}} = \begin{bmatrix} l+1 & -l & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 \to r_1 - r_n} \begin{bmatrix} l+1 & -l & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

$$\xrightarrow{c_n \to c_n + c_{n+1}} \begin{bmatrix} l+1 & -l & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} l+1 & -l & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

$$= \left[\begin{array}{c|c} L_C & 0 \\ \hline 0 & 1 \end{array}\right]. \tag{18}$$

Now our claim follows from Remark 4.3.

<u>Case 2:</u> Now suppose $d_{nn} = 1$. Without loss of generality, we may assume the vertex v_n has one outgoing edge to v_1. There are two cases, depending on the existence of $e_{1\to n}$. The Laplacian of these two graphs are equivalent.

$$L_{C_n} = \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & 0 & -k & k+1 & -1 \\ -1 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \to r_1 - r_n} \begin{bmatrix} l+1 & -l & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & 0 & -k & k+1 & -1 \\ -1 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

where $l,k \in \{0,1\}$. Therefore, we may assume that L_{C_n} is the first matrix in (19). We define C_{n+1} extending C_n as follows. Add a vertex v_{n+1} to C_n, replace the edge $e_{n\to 1}$ by $e_{n+1\to 1}$, and add a new edge $e_{n\to n+1}$. Pictorially, we have the following

$$C_n : \Big(\cdots\ v_{n-1} \to v_n \to v_1\ \cdots\Big) \longrightarrow C_{n+1} : \Big(\cdots\ v_{n-1} \to v_n \to v_{n+1} \to v_1\ \cdots\Big).$$

Then we have the following equivalence of matrices.

$$L_{C_{n+1}} = \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & & \\ 0 & \cdots & 0 & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -1 \\ -1 & \cdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_{n+1} \leftrightarrow r_n} \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & & \\ 0 & \cdots & 0 & -k & k+1 & -1 & 0 \\ -1 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{20}$$

$$\xrightarrow{r_n \to r_{n+1}+r_n} \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & & \\ 0 & \cdots & 0 & -k & k+1 & -1 & 0 \\ -1 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{c_n \to c_n - c_{n+1}} \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & & \\ 0 & \cdots & 0 & -k & k+1 & -1 & 0 \\ -1 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{21}$$

$$\longrightarrow \left[\begin{array}{c|c} L_C & 0 \\ \hline 0 & 1 \end{array}\right]. \tag{22}$$

Our claim follows from Remark 4.3. ☐

Lemma 4.5. *For any $n \geq 4$ and any $m \in \{n-1, n\}$, there exists an orientation of C_n such that* $\mathrm{Jac}(C_n) = \mathbb{Z}_m$.

Proof. If all edges of C_n are bidirectional, then $\mathrm{Pic}(C_n) \cong \mathbb{Z}_n$. We claim that an orientation of C_n such that

$$C_n : \Big(\cdots \ v_n \ \rightarrow \ v_1 \ \leftarrow \ v_2 \ \leftarrow \ v_3 \ \cdots \Big)$$

and all other edges are bidirectional has the Picard group $\mathbb{Z}_{n-1}$. In fact, we have the following equivalence of matrices:

$$L_{C_n} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ -1 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \xrightarrow{r_n \to r_n - r_2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \tag{23}$$

$$\xrightarrow{c_2 \to c_2 + c_1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \tag{24}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \quad \text{(See the remark below for the last equivalance)}$$

Now the $(n-1)\times(n-1)$ bottom right submatrix is that of C_{n-1} with all bidirecional edges and Remark 4.3 proves the claim. $\square$

Remark 4.6. For $n \geq 3$, the Lapacian of C_n each of its eage is bidirecional, the Laplician of C_n is of the form

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

Since $[1\cdots 1]L = 0$, the first row is a $\mathbb{Z}$-linear combination of the next $(n-1)$ rows. This justifies the last equivalence in the proof above.

Proof of the Conjecture 4.1. We induct on the number of edges. The base case is when $n = 3$ which is Example 4.2. For $n > 3$, by Lemma 4.5, we have orientations of C_n with $\mathrm{Jac}(C_n) = \mathbb{Z}_n$ and $\mathbb{Z}_{n-1}$. Now the rest follows from Lemma 4.4. $\square$

4.2. Construction of cyclic Jacobians of C_n. By a *path* of C_n, we mean a connected subgraph of C_n in which all arrows are oriented in a single direction or are bidirectional. In other words, a path is a subgraph of C_n with one strong terminal component.

In graphs with exactly two of these paths, a consistent pattern arises that allows for the rapid calculation of the Jacobian in only $O(n)$ time compared to the $O(n^4)$ time that calculating the Smith normal form takes. This pattern relies on calculating the number of bidirectional arrows between these two paths on one side. Our experimental computation suggests the following.

Conjecture 4.7. Let C_n be a cycle graph with a fixed orientation. Then, we have $\mathrm{Jac}(G) = \mathbb{Z}_{x-2}$, where x is the number of bidirectional edges clockwise of the counter-clockwise path and counter-clockwise of the clockwise path.

Lemma 4.8. *For $n \geq 2$, consider the following square matrix M_n*

$$M_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}. \tag{25}$$

This is a matrix whose digonal entries are 2 and sub-diagonals are -1. Then $\det(M_n) = n+1$.

Proof. We induct on n. When $n = 2$, a direct computation shows that $\det(M_2) = 3$. Suppose the statement is true for all $k \leq n-1$. To compute $\det M_n$, we use the Laplace expantion alone the first row. That is, $\det M_n = 2\det(M_{n-1}) + \det(N)$, where N is

$$N = \left[\begin{array}{c|cccc} -1 & -1 & 0 & \cdots & 0 \\ \hline 0 & & M_{n-2} & & \end{array}\right]. \tag{26}$$

(Thus, $\det N = -\det(M_{n-2})$. Youngsu: to be removed) By induction, $\det M_n = 2\det(M_{n-1}) - \det(M_{n-2}) = 2(n-1+1) - (n-2+1) = 2n - n + 1 = n+1$. $\square$

Remark 4.9. Note that the $(n-1)$ by $(n-1)$ minor of M_n after deleting the last row and last column is $(-1)^{n-1}$. Hence the Smith normal form of M_n is

$$\left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & n+1 \end{array}\right], \tag{27}$$

where I_{n-1} is the identity matrix of size $n-1$.

Lemma 4.10. *Consider a directed cycle graph C_n on n vertices. Assume that the vertex v_1 does not have any outgoing edge, and all other vertices have two outgoing edges. Then* $\mathrm{Jac}(C_n) = \mathbb{Z}_n$.

Proof. The Laplacian of C_n is

$$\begin{bmatrix} 0 & 0 & \cdots & & & \\ -1 & 2 & -1 & \cdots & & \\ 0 & -1 & 2 & -1 & \cdots & \\ \vdots & & & & & \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}. \tag{28}$$

By the same argument as in Remark 4.6, Youngsu: we need to state Remark 4.6 for the general case This matrix is equivalent to

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & M_{n-1} \end{array}\right], \tag{29}$$

where M_{n-1} is the matrix in Lemma 4.8. Thus, by the remark above, $\mathrm{Jac}(C_n) = \mathbb{Z}_n$. (One can also use the proof of Lemma 4.5 with Lemma 4.8.) $\square$

The general case follows from this lemma combined with the first and the second parts of the proof of Lemma 4.4 for $\mathbb{Z}_{n-1}$. (I believe Matt's observation about straightning intermediate birational edges corresponses to the reduction in the 2nd case of $d_{n,n} = 1$.)

4.3. Representing The Rank of the Picard Group for Cycle Graphs. The following example shows that the rank of $\mathrm{Pic}(C_n)$ (with an orientation) does not have to be 1; this directly follows from [Wag00, Corollary 3.5].

Example 4.11.

$\mathrm{Pic}(G)$ is often in the form $\mathbb{Z}_1 \times \cdots \times \mathbb{Z}_n \times \mathbb{Z}^m$ where m is the rank of the picard group. A terminal strong component describes itself well. It is a terminal component, meaning that its only connections are edges directed into it. It is also strongly connected, meaning that every vertex in the component has at least one path to all other components. For cycle graphs and trees, the number of these terminal strong components is exactly equal to $Rank(\mathrm{Pic}(G))$.

Proof. The $Rank(\mathrm{Pic}(G))$ comes directly from the number of all zero rows of the Smith Normal Form of the laplacian of G. Terminal strong components are either already zero rows (in the case of a single sink) or there exists a series of row and column operations that, when preformed on the laplacian result in a row of all zeroes. $\square$

Lemma 4.12. *Let C_n be the cycle graph with n vertices. Consider the equi-orientation of C_n. Then* $\mathrm{Pic}(C_n) = \mathbb{Z}$.

Proof. In this case C_n will always have a laplacian with a diagonal of 1 and a diagonal of -1 above or below it depending on the orientation, with another -1 in the top right or bottom left corner where V_0 connects V_{n-1}. Turing this laplacian into a smith normal form of all 1s is as simple as cascading additions from the first row in a counter-clockwise case and the last row in the clockwise case. This will result in a diagonal of ones with the first or last column made up of mostly -1. From here all that needs to be done is to eliminate these with the first or last row. Since each equi-oriented C_n will follow this pattern, this algorithm will prove $\mathrm{Pic}(C_n) = \mathbb{Z}$ for all C_n. $\square$

5. Picard groups of oriented pseudotrees

Jaiung: here we study how Picard groups change when we glue two directed graphs along one vertex or connect two graphs along one arrow. If we are lucky this should provide a way to compute Picard groups for pseudotrees by using the tree case and the cycle case in the previous sections.

6. Picard groups of oriented wheel graphs

Jaiung: the following conjecture should generalize the results in [Big99].

6.1. Patterns in The Jacobian. For wheel graphs, we looked for patterns that arose within the invariant factors of the Jacobian as a general formula was not immediately obvious. For this strategy, we broke the edges of the wheel graph into their two most obvious groups, those belonging to the rim of the wheel and those of the spokes. By orienting all the edges of either group the same way and trying all nine combinations, we noticed a well-defined pattern for each as the size of the wheel graph changed. These patterns fell into four distinct groups.

(1) When the spoke edges all pointed towards the axel and the rim was not bidirectional, or when the spoke edges were bidirectional, but the rim was not then a graph of size n had a Jacobian of $\mathbb{Z}_{2(n-1)-1}$.
(2) When the spoke edges all pointed away from the axel and the rim was not bidirectional, a graph of size n had a Jacobian of $\mathbb{Z}_{n-1}$.
(3) When the spoke edges point away from the axel and the rim was bidirectional and a graph of size n had a Jacobian of $\mathbb{Z}_{n-1} \times \mathbb{Z}_{n-1}$ when n was even and $Z_{\frac{n-1}{2}} x Z_{(n-1)\times 2}$ when n was odd.
(4) When all edges were bidirectional or when the spoke direction was towards the axel and the rim was bidirectional, a graph of size n had a Jacobian of $\mathbb{Z}_{\alpha\phi^n} \times \mathbb{Z}_{5\alpha\phi^n}$ when the size was odd where $\alpha \approxeq 0.27555$ and $\mathbb{Z}_{\beta\phi^n} \times \mathbb{Z}_{\beta\phi^n}$ when the size was even where $\beta \approxeq 0.618035$. In both of these patterns, ϕ represents the golden ratio.

7. Picard groups of oriented multipartite graphs

Jaiung: this paper [**?**] could be relevant here. For our purposes, a multipartite graph is a graph whos vertices can be partitioned between several independent groups, arranged in a linear order. Vertices have no connections to members of their own group, but are strongly connected to all vertices of their two adjacent groups.

The structure of the graphs that we investigate are intentionally designed to resemble artificial neural networks. To further facilitate this comparison, we direct all edges *forward* such that, after numbering the groupings of these vertices in some order, edges always point towards the next highest numbered grouping.

We were able to find notable patterns in both a *Perceptron* style model with two layers and a *Hidden Layer* model with three layers.

7.1. Picard groups for two layers. For two layers in the form of $f \to s$ where f and s are the number of nodes in the first and second layers, respectively, $\text{Pic}(G) = \mathbb{Z}_s^{f-1} \times \mathbb{Z}^s$.

7.2. Picard groups for three layers. For three layers in the form of $f \to s \to t$ where f, s, and t are the number of nodes in the first, second, and third layers, respectively the Picard group is significantly more complex.

(1) When s is odd, s is not a factor of t and $f <= s$, $\text{Pic}(G) = Z_t^{s-f-1} \times Z_{s\times t}^f \times Z^t$

8. Experimental Results

9. Conjectures and Future directions

Conjecture 9.1. For any cycle graph C_n, the orientation with no paths always has a Jacobian of $\mathbb{Z}_n$. Either of the single path orientations have a trivial Jacobian. The set of all orientations with two paths always contains all single invariant factors $\mathbb{Z}_2 \ldots \mathbb{Z}_n$. For all graphs at least up to C_{10} and likely well beyond that point, the sets that contain all other paths do not contain all of the single invariant factors. Jaiung: I think this should not be difficult....

It should be noted that for four paths and upward, the sets that contain these paths often also contain Jacobians of $\mathbb{Z}_3$ and $\mathbb{Z}_4$. The number of each of these increases with the size of the graph, so it is possible that these sets will contain all of the single invariant factors for very large cycle graphs.

References

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