

ON PICARD GROUPS AND JACOBIANS OF DIRECTED GRAPHS

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Abstract. We explore a combinatorial game on finite graphs, called Chip-Firing Games, which has various connections to other areas, such as algebraic geometry, number theory and economics. To play the game, one first puts an integer amount of chips at each vertex. Then, each vertex is allowed to borrow or lend chips from its neighbors equally as the game progresses.

A configuration of chips on the vertices of a graph is called a divisor, a vector in \mathbb{Z}^n , where n is the number of vertices. The degree of a divisor is the total number of chips combined on the vertices. The collection of all divisors on a graph defines a free abelian group $\text{Div}(G)$, the divisor group of G . Under the equivalence relation \sim generated by borrowing and lending moves, one obtains the Picard group $\text{Pic}(G) = \text{Div}(G)/\sim$. The Jacobian $\text{Jac}(G)$ of G is the subgroup of $\text{Pic}(G)$ consisting of degree-zero divisors. These groups can be computed from the Laplacian matrix of a graph G . Through the structure of these groups, we gain a better understanding of how any given game can be played and evolve.

When a graph G is directed, one may define $\text{Pic}(G)$ and $\text{Jac}(G)$ as in the case of undirected graphs by using Laplacian matrices, but computations become much more complicated in this case. For example, $\text{Pic}(G) = \mathbb{Z}$, when G is a tree, from the matrix-tree theorem, which tells us that $|\text{Jac}(G)|$ is the number of spanning trees of G in the undirected case. In the case of directed trees, even the rank of $\text{Pic}(T)$ can be arbitrarily large. For example, for any natural number n we can construct a tree T_n such that the rank of $\text{Pic}(T_n)$ is n .

In our ongoing project, we study Picard groups and Jacobians for directed trees, cycles, and pseudotrees. Even in these seemingly simple cases, we find some new phenomenon. For instance, for the undirected cycle C_n , $\text{Jac}(C_n) = \mathbb{Z}_n$, however, we prove that in the directed case, for any given $m \leq n$, one can always find an orientation of C_n in such a way that $\text{Jac}(C_n)$ is \mathbb{Z}_m . By closely examining trees and cycles, and how Picard groups and Jacobians change with (suitably defined) vertex and edge gluing, we obtain several results for pseudotrees.

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