SUMMER PROJECT

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Abstract.

1. Introduction

We will focus on Research Project 11 in [GK20]. The following are more specific directions that we plan to pursue.

Goal (8/1/2022)**.**

(1) Prove/disprove: for an oriented graph G, one always has $\mathrm{Pic}(G) = \mathbb{Z} \times \mathrm{Jac}(G)$, i.e., as a finitely generated abelian group, the rank of $\mathrm{Pic}(G)$ is 1.
(2) Prove/disprove: for C_n, and $0 \leq m \leq n$, one can always find an orientation of C_n so that $\mathrm{Jac}(C_n) = \mathbb{Z}_m$ (with the orientation).
(3) Prove/disprove: for an oriented graph G, if $v_0 \in V(G)$ is a sink (or a source) and G' is the graph obtained by reserving the direction for all arrows adjacent to v_0 from G, then $\mathrm{Jac}(G) = \mathrm{Jac}(G')$. (Note: we believe that this should be true for at least some classes of graphs such as cyclic graphs.)
(4) Prove/disprove: for an oriented planar graph G and its planar dual (should be defined) $\hat{G}$, one has $\mathrm{Jac}(G) = \mathrm{Jac}(\hat{G})$.
(5) Prove/disprove: for oriented graphs G_1, G_2, let G be the graph obtained by gluing G_1 and G_2 along one vertex. Then $\mathrm{Jac}(G) = \mathrm{Jac}(G_1) \times \mathrm{Jac}(G_2)$.

2. Background

2.1. Chip Firing. The game at the heart of this paper is the Chip-Firing game. When a game is started, each vertex on a graph is assigned a certain number of chips. During play, chips can be lent or borrowed at each node where one or more chips are either sent or received along each outgoing edge equally. In the case of a directed graph, vertices can only interact with another along an outgoing or bidirectional edge. The game is won once every vertex has a positive number of chips (i.e., this vertex is not in debt).

2.2. Divisors and Equivalence Relations. In the study of this game a **Divisor** of a graph ($\mathrm{Div}(G)$) is an integer vector $v \in \mathbb{Z}^n$ where n is the number of vertices in the graph. The i^{th} of element of the vector v is the number of chips on the i^{th} vertex of the graph. Two divisors have an **Equivalence Relation** ($\sim$) if one divisor can be gotten from the other by a finite series of lending or borrowing moves $D_1 \sim D_2 \leftrightarrow (D_1 \xleftrightarrow{\text{moves}} D_2)$. An **Equivalence Class** *[D]* is the set of all divisors that are equivalent to each other, $[D] = \{D_i \mid D_i \sim D\}$.

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2.3. The Picard Group and The Jacobian. The **Picard Group** of a graph $\mathrm{Pic}(G)$ is the set of all equivalence classes that the divisors of that graph can be a part of. The **Jacobian** of a graph $\mathrm{Jac}(G)$ is a subset of $\mathrm{Pic}(G)$ such that every divisor in each equivalency class has a degree of 0 where the degree of a divisor $\deg(D)$ is the sum of each of the divisor's elements. If a divisor is in one of the Jacobian's classes, it can be made winning after a finite series of moves.

3. Preliminaries

4. Propositions

References

[GK20] Darren Glass and Nathan Kaplan. Chip-firing games and critical groups. In *A Project-Based Guide to Undergraduate Research in Mathematics*, pages 107–152. Springer, 2020.

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