ON PICARD GROUPS OF DIRECTED GRAPHS

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Abstract. We explore a combinatorial game on finite graphs, called Chip-Firing Games, which has various connections to other areas, such as algebraic geometry, number theory and economics. To play the game, one first puts an integer amount of chips at each vertex. Then, each vertex is allowed to borrow or lend chips from its neighbors equally as the game progresses. One can study chip-firing games on a graph G through a finitely generated abelian group $\mathrm{Pic}(G)$ (Picard group) and its torsion subgroup $\mathrm{Jac}(G)$ (Jacobian) which can be computed by using the Laplacian matrix of G.

When a graph G is directed, one may define $\mathrm{Pic}(G)$ and $\mathrm{Jac}(G)$ as in the case of undirected graphs by using Laplacian matrices, but computations become much more complicated in this case. For example, $\mathrm{Pic}(G) = \mathbb{Z}$, when G is a tree, from the matrix-tree theorem, which tells us that $|\mathrm{Jac}(G)|$ is the number of spanning trees of G in the undirected case. In the case of directed trees, even the rank of $\mathrm{Pic}(T)$ can be arbitrarily large. For example, for any natural number n we can construct a tree T_n such that the rank of $\mathrm{Pic}(T_n)$ is n.

In our ongoing project, we study Picard groups and Jacobians for directed trees, cycles, and pseudotrees. Even in these seemingly simple cases, we find some new phenomenon. For instance, for the undirected cycle C_n, $\mathrm{Jac}(C_n) = \mathbb{Z}_n$, however, we prove that in the directed case, for any given $m \leq n$, one can always find an orientation of C_n in such a way that $\mathrm{Jac}(C_n)$ is $\mathbb{Z}_m$. By closely examining trees and cycles, and how Picard groups and Jacobians change with (suitably defined) vertex and edge gluing, we obtain several results for pseudotrees.

1. Introduction

We will focus on Research Project 11 in [GK20]. The following are more specific directions that we plan to pursue.

Goal (8/1/2022).

(1) Prove/disprove: for an oriented graph G, one always has $\mathrm{Pic}(G) = \mathbb{Z} \times \mathrm{Jac}(G)$, i.e., as a finitely generated abelian group, the rank of $\mathrm{Pic}(G)$ is 1. Jaiung: we disproved this by using [Wag00].

(2) Prove/disprove: for C_n, and $0 \leq m \leq n$, one can always find an orientation of C_n so that $\mathrm{Jac}(C_n) = \mathbb{Z}_m$ (with the orientation).

(3) Prove/disprove: for an oriented graph G, if $v_0 \in V(G)$ is a sink (or a source) and G' is the graph obtained by reserving the direction for all arrows adjacent to v_0 from G, then $\mathrm{Jac}(G) = \mathrm{Jac}(G')$. (Note: we believe that this should be true for at least some classes of graphs such as cyclic graphs.)

(4) Prove/disprove: for an oriented planar graph G and its planar dual (should be defined) $\hat{G}$, one has $\mathrm{Jac}(G) = \mathrm{Jac}(\hat{G})$.

(5) Prove/disprove: for oriented graphs G_1, G_2, let G be the graph obtained by gluing G_1 and G_2 along one vertex. Then $\mathrm{Jac}(G) = \mathrm{Jac}(G_1) \times \mathrm{Jac}(G_2)$. Jaiung: we are currently working on this (8/31)

2. Preliminaries

2.1. Chip Firing. The game at the heart of this paper is the Chip-Firing game. When a game is started, each vertex on a graph is assigned a certain number of chips. During play, chips can be lent or borrowed at each node where one or more chips are either sent or received along each outgoing edge equally. In the case of a directed graph, vertices can only interact with another along an outgoing or bidirectional edge. The game is won once every vertex has a positive number of chips (i.e., this vertex is not in debt).

2.2. Divisors and Equivalence Relations. In the study of this game a **Divisor** of a graph ($\mathrm{Div}(G)$) is an integer vector $v \in \mathbb{Z}^n$ where n is the number of vertices in the graph. The ith element of the vector v is the number of chips on the ith vertex of the graph. Two divisors have an **Equivalence Relation** ($\sim$) if one divisor can be gotten from the other by a finite series of lending or borrowing moves $D_1 \sim D_2 \leftrightarrow (D_1 \xleftrightarrow{\text{moves}} D_2)$. An **Equivalence Class** *[D]* is the set of all divisors that are equivalent to each other, $[D] = \{D_i \mid D_i \sim D\}$.

2.3. The Picard Group and The Jacobian. The **Picard Group** of a graph $\mathrm{Pic}(G)$ is the set of all equivalence classes that the divisors of that graph can be a part of. The **Jacobian** of a graph $\mathrm{Jac}(G)$ is a subset of $\mathrm{Pic}(G)$ such that every divisor in each equivalency class has a degree of 0 where the degree of a divisor $\deg(D)$ is the sum of each of the divisor's elements. If a divisor is in one of the Jacobian's classes, it can be made winning after a finite series of moves. Here, we represent the Picard Group as $\mathrm{Pic}(G) = \mathrm{Jac}(G) \times \mathbb{Z}^n$.

3. Describing The Picard Group of a Tree

Conjecture 3.1. The Jacobian of a tree graph is always $\mathbb{Z}$ as follows from the matrix-tree theorem as a tree graph only has one possible spanning tree. Prove and cite

Lemma 3.2. *Let G be a graph with any orientation. If we attach either an incoming arrow or a two-sided arrow to create a directed graph G', then* $\mathrm{Pic}(G) = \mathrm{Pic}(G')$.

Proof. Let α be an arrow which is glued to G. Let $|V(G)| = n$. We label the vertexes of G as $1, 2, \ldots, n$. Suppose first that α is incoming and α is glued at the vertex n. Let $A_G = (a_{ij})$ (resp. $A_{G'}$) be the Laplacian of G (resp. G'). In this case, one can easily observe that the matrix $A_{G'}$ is of the following form:

$$A_{G'} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{array} \right] \tag{1}$$

By a column operation between the last two columns, we obtain the following matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{array} \right] \tag{2}$$

This shows that $\mathrm{Pic}(G) = \mathrm{Pic}(G')$.

Next, suppose that α is a two-sided arrow. Then similar to the above, we obtain the following matroid for G':

$$A_{G'} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n+1 & -1 \\ 0 & 0 & \cdots & -1 & 1 \end{array} \right] \tag{3}$$

By a column operation, the matrix (3) becomes the matrix (2). This shows that $\mathrm{Pic}(G) = \mathrm{Pic}(G')$. $\square$

Proposition 3.3. *Let T be a tree with any orientation. Then $\mathrm{Pic}(T) = \mathbb{Z}^n$, where n is the number of sinks of T.*

Proof. Note that from [Wag00, Corollary 3.5] that the rank of $\mathrm{Pic}(T)$ is the number of sinks of T.

Now, we can inductively prove this. When T_0 is a tree with one arrow, one can easily check that $\mathrm{Pic}(T_0) = \mathbb{Z}$ or $\{0\}$ (depending on the number of sinks). Suppose that T_k is a directed tree with k arrows. When we add one arrow α to T_k to construct T_{k+1}, there are three cases; (1) incoming, (2) outgoing, and (3) two-sided. When α is either incoming or two-sided arrow, then it follows from Lemma 3.2 that $\mathrm{Pic}(T_k) = \mathrm{Pic}(T_{k+1}) = \mathbb{Z}^n$, where n is the number of sinks of T_k and T_{k+1}, since in this case it does not increase the number of sinks.

Next, suppose that α is an outgoing arrow. In this case, T_{k+1} has one more sink than T_k. We claim that $\mathrm{Pic}(T_{k+1}) = \mathrm{Pic}(T_k) \times \mathbb{Z}$. We label the vertexes of G as $1, 2, \ldots, n$ and suppose that the arrow α is attached to the vertex n. Let $A_{T_k} = (a_{ij})$ be the Laplacian of T_k. Then we have the following:

$$A_{T_{k+1}} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n+1 & -1 \\ 0 & 0 & \cdots & 0 & 0 \end{array} \right] \tag{4}$$

Jaiung: I will finish writing down the proof before our next meeting. $\square$

4. Describing The Picard Group for a Cycle

4.1. Representing Single Term Jacobians.

Conjecture 4.1. Let C_n be the cycle graph with n vertices. For each $k \leq n$, there exists an orientation of C_n such that $\mathrm{Jac}(C_n)$ (with that orientation) is $\mathbb{Z}_k$.

Proof. We have proven this experimentally for cycles using two methods. For the first, we proved this for C_{330} by testing out all possible orientations for each size of cycle graph until at least one of each $\mathrm{Jac}(C_n) = \mathbb{Z}_k$ has been found. By using our second method, we proved this for C_{380} by only looking as specific orientations of two path cycle graphs of size n. $\square$

The following examples shows the conjecture for C_3.

Example 4.2.

The number of paths in the cycle graph also plays a role in if it can represent all single term Jacobians up to $\mathbb{Z}_n$.

Conjecture 4.3. For any cycle graph C_n, the orientation with no paths always has a Jacobian of C_n. Either of the single path orientations have a trivial Jacobian. The set of all orientations with two paths always contains all single invariant factors $\mathbb{Z}_2 \ldots \mathbb{Z}_n$. For all graphs at least up to C_{10} and likely well beyond that point, the sets that contain all other paths do not contain all of the single invariant factors.

It should be noted that for four paths and upward, the sets that contain these paths often also contain Jacobians of $\mathbb{Z}_3$ and $\mathbb{Z}_4$. The number of each of these increases with the size of the graph, so it is possible that these sets will contain all of the single invariant factors for very large cycle graphs.

4.2. Describing Two Path Cycle Graphs.
Here, a *path* represents a region of a cycle graph in which all arrows are oriented in a single direction or are bidirectional. In graphs with exactly two of these paths, a consistent pattern arises that allows for the rapid calculation of the Jacobian in only $O(n)$ time compared to the $O(n^4)$ time that calculating the smith normal form takes. This pattern relies on calculating the number of bidirectional arrows between these two paths on one side.

Conjecture 4.4. For any given cycle graph, $\mathrm{Jac}(G) = \mathbb{Z}_{x-2}$, where x is the number of bidirectional edges clockwise of the counter-clockwise path (or counter-clockwise of the clockwise path).

Proof. We have proven this completely for cycle graphs up to C_{80} when using two continuous paths without any bidirectional arrows separating them on the other size. This has also held true for many other arbitrary orientations that still meet the two-path definition. $\square$

This holds true for nearly every graph tested, with very few exceptions.

4.3. Representing The Rank of the Picard Group for Cycle Graphs.
The following example shows that the rank of $\mathrm{Pic}(C_n)$ (with an orientation) does not have to be 1; this directly follows from [Wag00, Corollary 3.5].

Example 4.5.

$\mathrm{Pic}(G)$ is often in the form $\mathbb{Z}_1 \times \cdots \times \mathbb{Z}_n \times \mathbb{Z}^m$ where m is the rank of the picard group. A terminal strong component describes itself well. It is a terminal component, meaning that its only connections are edges directed into it. It is also strongly connected, meaning that every vertex in the component has at least one path to all other components. For cycle graphs and trees, the number of these terminal strong components is exactly equal to $Rank(\mathrm{Pic}(G))$.

Proof. The $Rank(\mathrm{Pic}(G))$ comes directly from the number of all zero rows of the Smith Normal Form of the laplacian of G. Terminal strong components are either already zero rows (in the case of a single sink) or there exists a series of row and column operations that, when preformed on the laplacian result in a row of all zeroes. $\square$

Lemma 4.6. *Let C_n be the cycle graph with n vertices. Consider the equi-orientation of C_n. Then* $\mathrm{Pic}(C_n) = \mathbb{Z}$.

Proof. In this case C_n will always have a laplacian with a diagonal of 1 and a diagonal of -1 above or below it depending on the orientation, with another -1 in the top right or bottom left corner where V_0 connects V_{n-1}. Turing this laplacian into a smith normal form of all 1s is as simple as cascading additions from the first row in a counter-clockwise case and the last row in the clockwise case. This will result in a diagonal of ones with the first or last column made up of mostly -1. From here all that needs to be done is to eliminate these with the first or last row. Since each equi-oriented C_n will follow this pattern, this algorithm will prove $\mathrm{Pic}(C_n) = \mathbb{Z}$ for all C_n. □

5. Example: Pseudotrees

Jaiung: here we study how Picard groups change when we glue two directed graphs along one vertex or connect two graphs along one arrow. If we are lucky this should provide a way to compute Picard groups for pseudotrees by using the tree case and the cycle case in the previous sections.

6. Strongly connected directed graphs

References

[GK20] Darren Glass and Nathan Kaplan. Chip-firing games and critical groups. In *A Project-Based Guide to Undergraduate Research in Mathematics*, pages 107–152. Springer, 2020.
[Wag00] David G Wagner. The critical group of a directed graph. *arXiv preprint math/0010241*, 2000.

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