

# ON PICARD GROUPS OF DIRECTED GRAPHS

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**ABSTRACT.** The Picard group  $\text{Pic}(G)$  of a graph  $G$  is a finitely generated abelian group, and the Jacobian  $\text{Jac}(G)$  is the torsion subgroup of  $\text{Pic}(G)$ . These groups can be computed by using the Smith normal form of the Laplacian matrix  $L_G$  or by using chip-firing games. One may consider its generalization to directed graphs by using Laplacian matrices. In this paper, we investigate Picard groups and Jacobians for several classes of directed graphs, including directed trees and directed cycles. Different from the undirected case, even for trees and cycles, one can find very interesting computational results. Our investigation is based on experimental mathematics; we compute a large number of examples, find some patterns from them, and prove them.

## 1. Introduction

A chip-firing game is a combinatorial game that one can play on finite graphs. To play a chip-firing game, one first puts chips (possibly negative as “debt”) at each vertex of a graph  $G$ . At each turn, a vertex can borrow / lend chips from / to all neighbors simultaneously. The goal of the game is to find a finite sequence of borrowing / lending moves so that every vertex is debt-free. Hence, a natural question to ask is whether or not one can determine there is a winning moves for a given configuration of chips.

To study the game more systemically, one can write any chip configuration as an element of the free abelian group generated by the set of vertices  $V(G)$  of a graph  $G$  which is denoted by  $\text{Div}(G)$ . Each element of  $\text{Div}(G)$  is called a divisor, and a divisor  $D$  is effective if  $D = \sum_{v \in V(G)} a_v v$ , with  $a_v \geq 0$ . Then, one defines an equivalence relation on  $\text{Div}(G)$  as follows:  $D \sim D'$  if and only if  $D'$  can be obtained from  $D$  by a finite sequence of borrowing / lending movers. This defines a group

$$\text{Pic}(G) := \text{Div}(G) / \sim,$$

called the Picard group of  $G$ . The torsion subgroup of  $\text{Pic}(G)$ , denoted by  $\text{Jac}(G)$ , is called the Jacobian of  $G$ .<sup>1</sup>

The combinatorial theory of chip-firing games has connections to various areas in mathematics. For instance, from a perspective motivated by algebraic geometry, one may view finite graphs as a discrete model for Riemann surfaces. In this case, chip configurations play the role of divisors on curves. In fact, Baker and Norine [BN07] formulated and proved a version of Riemann-Roch theorem for finite graphs as follows:

$$r(D) - r(K - D) = \deg(D) - g + 1,$$

where  $g = |E(G)| - |V(G)| + 1$ ,  $\deg(D)$  for  $D \in \text{Div}(G)$  is the total number of chips,  $K = \sum_{v \in V(G)} (\deg(v) - 2)v \in \text{Div}(G)$  (the canonical divisor of  $G$ ), and the rank  $r(D)$  is one less than the minimum number of chips which need to be removed so that  $D$  is no longer equivalent to an effective divisor. When  $G$  is a dual graph of a strongly semistable model for an algebraic curve  $X$  (smooth, proper, geometrically connected) over a valued field, one also has a (degree-preserving) group homomorphism  $\rho : \text{Pic}(X) \rightarrow \text{Pic}(G)$ .

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2020 *Mathematics Subject Classification.* 05C50, 05C76.

*Key words and phrases.* Jacobian of a graph, sandpile group, critical group, chip-firing game, directed graphs, cycle graph, wheel graphs, Laplacian of graph, trees, Smith normal form.

<sup>1</sup>Depending on literature,  $\text{Jac}(G)$  is also called as a critical group.

For a graph  $G$  with the vertices  $\{v_1, v_2, \dots, v_n\}$ , one defines an  $n \times n$  matrix  $L_G$ , called the Laplacian matrix of  $G$ , as follows:

$$(L_G)_{ij} = \begin{cases} \text{valency of } v_i, & i = j \\ -(\# \text{ of edges between } v_i \text{ and } v_j), & i \neq j. \end{cases}$$

To compute  $\text{Pic}(G)$ , one can use the Laplacian matrix  $L_G$  of a graph  $G$ . One can easily check that any configuration of chips can be considered as a vector  $v \in \mathbb{Z}^{V(G)}$ , and the chip configuration  $v'$  by making a lending move (resp. borrowing move) at a vertex  $i$  from  $v$  is the following:

$$v' = v - (L_G^t e_i) \quad (\text{resp. } v' = v + (L_G^t e_i)). \quad (1)$$

With this, by considering  $L_G^t : \mathbb{Z}^{V(G)} \rightarrow \mathbb{Z}^{V(G)}$ , one can see that  $\text{Pic}(G) = \text{coker}(L_G^t)$ , and  $\text{Pic}(G) = \mathbb{Z} \times \text{Jac}(G)$ .

From (1), one is naturally led to think that one can play a chip-firing game with a given matrix  $M$  (replacing  $L_G$ ), namely firing at a site  $i$  is:

$$v' = v - (M^t e_i). \quad (2)$$

Now we specialize the case when a matrix  $A$  is a directed Laplacian. Perhaps we need to cite some results in Section 6 of [?]

**1.1. to be edited.** The main things that we have done so far:

- (1)  $\text{Jac}(T) = \{0\}$  (Proposition 3.4).
- (2)  $\text{Jac}(C_n)$  can be anything (Conjecture 4.1).
- (3) An explicit construction of an orientation of  $C_n$  so that  $\text{Jac}(C_n) = \mathbb{Z}_m$  (Conjecture 4.6).

The remaining things that we want to do:

- (1) Add experimental data visualization for number of paths needed and sizes of graphs.
- (2) Anything else??

Things that we might want to do (or perhaps leave them as conjectures):

- (1) Proof of wheel graph conjecture (should be rewritten with Lucas numbers). [Big99] should be relevant.
- (2) Proof of multipartite graphs. [?] could be relevant.

**Theorem A.** *Let  $T$  be a tree with any orientation. Then  $\text{Jac}(T) = \{0\}$ .*

**Theorem B.** *Let  $C_n$  be a cycle graph with  $n$  vertices. For any  $0 \leq m \leq n$ , there exists an orientation of  $C_n$  such that  $\text{Jac}(C_n) = \mathbb{Z}_m$  with the orientation. Furthermore, we explicitly describe how to find an orientation of  $C_n$  to obtain  $\mathbb{Z}_m$ .*

For all bidirectional edges, Biggs computed the Jacobian of a wheel graph in [Big99]. For the directional cases, we compute two special cases. Let  $W_n$  be the wheel graph on  $n$ -vertices with bidirectional edges, and  $W_n'$  with the edges of the rim are bidirectional and all its spoke edges point to the axel. Let  $W_n''$  be the wheel graph on  $n$ -vertices where the edges of the rim are bidirectional and all its spoke edges point away from the axel.

**Theorem C.** *With the same notation as above, we have the following.*

- (1) *The Laplacian matrices of  $W_n$  and  $W_n'$  are row equivalent. In particular, one has  $\text{Pic}(W_n) \cong \text{Pic}(W_n')$ .*
- (2) *The smith normal form of the Laplacian of  $W_n''$  is of the form*

$$\left[ \begin{array}{c|ccc} I_{n-3} & 0 & 0 & 0 \\ \hline 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

$$\text{where } (a, b) = \begin{cases} (n-1, n-1) & \text{if } n \text{ is even;} \\ (\frac{n-1}{2}, 2(n-1)) & \text{if } n \text{ is odd} \end{cases}.$$

**Acknowledgment** This research was supported by Research and Creative Activities (RSCA) at SUNY New Paltz. We would like to thank RSCA for their support. [Add here thanks to CSUSB HPC.](#)

## 2. Preliminaries

**Definition 2.1.** *Jaiung:* Recall terminal strong components

**Theorem 2.2.** *Jaiung:* Here we recall [Wag00] the torsion-free part theorem.

**Remark 2.3.** Let  $M \in M_{m \times n}(\mathbb{Z})$ , and let  $I_k(M)$  denote the ideal generated by  $k \times k$  minors of  $M$ , where  $I_k(M) = 0$  if  $k > \min\{m, n\}$  and  $I_k = (1)$  if  $k \leq 0$ . For a matrix  $N = \left[ \begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array} \right]$  and for any  $k$ ,  $I_k(M) = I_{k+1}(N)$ , and the cokernels of  $M$  and  $N$  are isomorphic.

The following is well-known. For instance, see [?, Theorem 2.4].

**Theorem 2.4.** Let  $R$  be a unique factorization domain such that any two elements have a greatest common divisor (gcd). Suppose that  $M \in \text{Mat}_{m \times n}(R)$  has a Smith normal form  $L = (x_1, \dots, x_m)$ . Then, for  $1 \leq k \leq m$ , the product  $x_1 \cdots x_k$  is equal to the gcd of all  $k \times k$  minors of  $M$ , with the convention that if all  $k \times k$  minors are 0 then their gcd is 0.

## 3. Picard groups of oriented trees

**Lemma 3.1.** Let  $G$  be a directed graph. If we attach either an incoming arrow or a two-sided arrow to create another directed graph  $G'$ , then  $\text{Pic}(G) = \text{Pic}(G')$ .

*Proof.* Let  $\alpha$  be an arrow which is glued to  $G$ . Let  $|V(G)| = n$ . We label the vertexes of  $G$  as  $1, 2, \dots, n$ . Suppose first that  $\alpha$  is incoming and  $\alpha$  is glued at the vertex  $n$ . Let  $L_G = (a_{ij})$  (resp.  $L_{G'}$ ) be the Laplacian matrix of  $G$  (resp.  $G'$ ). Then the matrix  $L_{G'}$  is of the following form.

$$L_{G'} = \left[ \begin{array}{ccc|cc} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 0 \\ \hline 0 & 0 & \cdots & -1 & 1 \end{array} \right] \quad (3)$$

By a column operation between the last two columns, we obtain the following matrix:

$$\left[ \begin{array}{ccc|cc} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array} \right] \quad (4)$$

This shows that  $\text{Pic}(G) = \text{Pic}(G')$ .

Next, suppose that  $\alpha$  is a two-sided arrow. Then similar to the above, we obtain the following Laplacian matrix for  $G'$ :

$$L_{G'} = \left[ \begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n + 1 & -1 \\ \hline 0 & 0 & \cdots & -1 & 1 \end{array} \right] \quad (5)$$

By a column operation, the matrix (5) becomes the matrix (4). This shows that  $\text{Pic}(G) = \text{Pic}(G')$ .  $\square$

The following example shows that if we change the direction of arrows, Picard groups can change drastically.

**Example 3.2.** Consider the following directed tree:

$$T = \left( \begin{array}{c} 1 \\ \downarrow \\ 2 \longrightarrow 3 \longleftarrow 4 \\ \uparrow \\ 5 \end{array} \right) \quad (6)$$

We have

$$L_T = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \implies \text{SNF}(L_T) = \left[ \begin{array}{c|c} I_4 & 0 \\ \hline 0 & 0 \end{array} \right]$$

Hence,  $\text{Pic}(T) = \mathbb{Z}$ . On the other hand, consider the following

$$T' = \left( \begin{array}{c} 1 \\ \uparrow \\ 2 \longleftarrow 3 \longrightarrow 4 \\ \downarrow \\ 5 \end{array} \right) \quad (7)$$

We have

$$L_{T'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \implies \text{Pic}(T') = \mathbb{Z}^4.$$

**Remark 3.3.** For the undirected case, when one glues two graphs  $G_1$  and  $G_2$  along one vertex to obtain  $G$ , then  $\text{Pic}(G) = \text{Pic}(G_1) \times \text{Pic}(G_2)$ . But, this is no longer true for directed graphs. For instance, the tree  $T$  in (6) can be considered as a directed graph obtained by gluing the following two

directed graphs  $G_1$  and  $G_2$  along the vertices 3 and 3''

$$G_1 = \left( \begin{array}{ccc} & 1 & \\ & \downarrow & \\ 2 & \longrightarrow & 3 \end{array} \right), \quad G_2 = \left( \begin{array}{ccc} & 3' & \longleftarrow 4 \\ & \uparrow & \\ & 5 & \end{array} \right) \quad (8)$$

But, we have

$$L_{G_1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies \text{Pic}(G_1) = \mathbb{Z}, \quad L_{G_2} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \implies \text{Pic}(G_2) = \mathbb{Z}$$

It follows that  $\text{Pic}(T) \neq \text{Pic}(G_1) \times \text{Pic}(G_2)$ .

For the undirected trees  $T$ ,  $\text{Pic}(T) = \mathbb{Z}$ . But, for directed trees, the rank of  $\text{Pic}(T)$  can be arbitrarily large depending the number of strong terminal component of  $T$ . Nonetheless, we prove that  $\text{Jac}(T) = \{0\}$  in the following.

**Proposition 3.4.** *Let  $T$  be a tree with any orientation. Then  $\text{Jac}(T) = \{0\}$ , i.e.,  $\text{Pic}(T)$  is torsion-free.*

*Proof.* We inductively prove this. When  $T_0$  is a tree with one arrow, one can easily check that  $\text{Pic}(T_0) = \mathbb{Z}$  or  $\{0\}$  (depending on the number of strong terminal components).

Suppose that  $T_k$  is an oriented tree with  $k$  arrows. When we add one arrow  $\alpha$  to  $T_k$  to construct  $T_{k+1}$ , there are three cases;  $\alpha$  is (1) incoming, (2) outgoing, and (3) two-sided. When  $\alpha$  is either incoming or two-sided arrow, then it follows from Lemma 3.1 that  $\text{Pic}(T_k) = \text{Pic}(T_{k+1}) = \mathbb{Z}^r$ , where  $r$  is the number of the terminal strong components of  $T_k$  and  $T_{k+1}$ , since in this case it does not increase the number of the terminal strong components.

Next, suppose that  $\alpha$  is an outgoing arrow. Let's label the vertexes of  $G$  as  $v_1, v_2, \dots, v_n$ . Suppose that the arrow  $\alpha$  is attached to the vertex  $v_j$ . Let  $L_k = (a_{ij})$  be the Laplacian matrix of  $T_k$ . Then we have the following:

$$L_{k+1} = \left[ \begin{array}{ccccc|c|c} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jj} + 1 & \cdots & a_{jn} & -1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} & 0 \\ \hline 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \end{array} \right] \quad (9)$$

To compute the Smith normal form, by relabeling vertices, we may assume  $L_{k+1}$  is the following matrix:

$$\left[ \begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 1 \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array} \right] \quad (10)$$

Since  $\text{Pic}(T_k) = \mathbb{Z}^r$ , there exist  $P, Q \in \text{GL}_n(\mathbb{Z})$  such that

$$PL_kQ = \left[ \begin{array}{c|c} I_{n-r} & 0 \\ \hline 0 & 0_r \end{array} \right] \quad (11)$$

where  $0_r$  is an  $r \times r$  zero matrix. Consider the following block matrices of size  $(n+1) \times (n+1)$ :

$$P' = \left[ \begin{array}{c|c} P & 0 \\ \hline 0 & 1 \end{array} \right], \quad Q' = \left[ \begin{array}{c|c} Q & 0 \\ \hline 0 & 1 \end{array} \right] \quad (12)$$

Then, we have

$$P'L_{k+1}Q' = \left[ \begin{array}{c|c} P & 0 \\ \hline 0 & 1 \end{array} \right] \left[ \begin{array}{c|c} L_k & e_n \\ \hline 0 & 0 \end{array} \right] \left[ \begin{array}{c|c} Q & 0 \\ \hline 0 & 1 \end{array} \right] = \left[ \begin{array}{c|c} PL_kQ & Pe_n \\ \hline 0 & 0 \end{array} \right] \quad (13)$$

We first consider the case when  $v_n$  is a sink. In particular,  $T_{k+1}$  and  $T_k$  have the same number of terminal strong components. In this case, the  $n^{\text{th}}$  row of  $L_{k+1}$  in (10) is the zero row. In particular, we can take  $P$  so that

$$Pe_n = e_n \quad (14)$$

Therefore, we have

$$P'L_{k+1}Q' = \left[ \begin{array}{c|c} PL_kQ & Pe_n \\ \hline 0 & 0 \end{array} \right] = \left[ \begin{array}{c|c} PL_kQ & e_n \\ \hline 0 & 0 \end{array} \right] = \left[ \begin{array}{c|c|c|c} I_{n-r} & & 0 & 0 \\ \hline & 0_{r-1} & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array} \right] \quad (15)$$

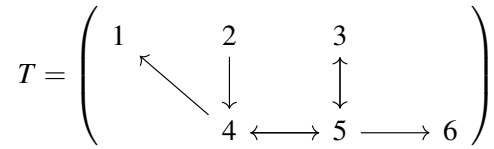
It follows that  $\text{Pic}(T_{k+1}) = \text{Pic}(T_k)$ .

Now, suppose that  $v_n$  is not a sink. Let  $Pe_n = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ . There are two cases:

Case 1: Suppose that  $x_{n-r+1} = x_{n-r+2} = \cdots = x_n = 0$ . In this case, one can easily observe that after some column operations,  $P'L_{k+1}Q'$  becomes the Smith normal form of  $L_{k+1}$ . In particular,  $\text{Jac}(T_{k+1}) = \text{Jac}(T_k)$ , and hence  $\text{Pic}(T_{k+1}) = \mathbb{Z} \times \text{Pic}(T_k)$ .

Case 2: Suppose that at least one of  $x_{n-r+1}, x_{n-r+2}, \dots, x_n$  is not equal to zero. Then, the Smith normal form of  $L_{k+1}$  becomes the last matrix in (15). In particular,  $\text{Jac}(T_{k+1}) = \text{Jac}(T_k)$ , and hence  $\text{Pic}(T_{k+1}) = \mathbb{Z} \times \text{Pic}(T_k)$ .  $\square$

**Example 3.5.** Consider the following oriented tree:



The Laplacian matrix of  $T$  is the following:

$$L_T = D_T - A_T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Smith normal form of  $L_T$  is the following  $6 \times 6$  matrix:

$$\left[ \begin{array}{c|c} I_4 & 0 \\ \hline 0 & 0_2 \end{array} \right]$$

This shows that  $\text{Pic}(T) = \mathbb{Z}^2$ .

**Example 3.6.** Consider the following oriented tree obtain from Example 3.5 by gluing an outgoing arrow  $\alpha$ :

$$T' = \left( \begin{array}{ccccc} 1 & 2 & 3 & & \\ & \swarrow & \downarrow & \updownarrow & \\ 7 & \xleftarrow{\alpha} & 4 & \longleftrightarrow & 5 \longrightarrow 6 \end{array} \right)$$

Now, we have

$$L_{T'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Smith normal form of  $L_{T'}$  is the following  $7 \times 7$  matrix:

$$\left[ \begin{array}{c|c} I_4 & 0 \\ \hline 0 & 0_3 \end{array} \right]$$

This shows that  $\text{Pic}(T) = \mathbb{Z}^3$ .

**Example 3.7.** Consider the following oriented tree obtain from Example 3.5 by gluing an outgoing arrow  $\alpha$ :

$$T'' = \left( \begin{array}{cccc} 1 & 2 & 3 & 7 \\ & \swarrow & \downarrow & \uparrow \alpha \\ & & 4 & \longleftrightarrow 5 \longrightarrow 6 \end{array} \right)$$

Now, we have

$$L_{T''} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Smith normal form of  $L_{T''}$  is the following  $7 \times 7$  matrix:

$$\left[ \begin{array}{c|c} I_5 & 0 \\ \hline 0 & 0_2 \end{array} \right]$$

This shows that  $\text{Pic}(T'') = \mathbb{Z}^2$ .

**Remark 3.8.** Jaiung: Here we add a version of directed matrix-tree theorem. I think it does not prove the above theorem that  $\text{Jac}(T)$  is trivial.

#### 4. Picard groups of oriented cycles

We computed a large set of examples (from  $C_3$  to  $C_{80}$ ) with all possible orientations. By doing so, we found some patterns which lead us to two conjectures. In what follows, we let  $C_n$  be an undirected cycle graph on  $n$  vertices.

**4.1. Oriented cycles with cyclic Jacobians.** In this subsection, we prove the following conjecture.

**Conjecture 4.1.** Let  $n \geq 3$ . For each  $k \leq n$ , there exists an orientation of  $C_n$  such that  $\text{Jac}(C_n)$  (with that orientation) is  $\mathbb{Z}_k$ .

The following examples confirms the conjecture for the class of  $C_3$ .

**Example 4.2.** With the following orientations of  $C_3$

$$G_1 = \left( \begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \\ \uparrow & & \swarrow \\ \bullet & & \bullet \end{array} \right), \quad G_2 = \left( \begin{array}{ccc} \bullet & \xleftarrow{\quad} & \bullet \\ \uparrow & & \swarrow \\ \bullet & & \bullet \end{array} \right), \quad G_3 = \left( \begin{array}{ccc} \bullet & \xleftrightarrow{\quad} & \bullet \\ \updownarrow & & \swarrow \\ \bullet & & \bullet \end{array} \right)$$

we have  $\text{Jac}(G_1) = 0$ ,  $\text{Jac}(G_2) = \mathbb{Z}_2$ , and  $\text{Jac}(G_3) = \mathbb{Z}_3$ .

**Lemma 4.3.** Fix an orientation of  $C_n$ . If not every edge of  $C_n$  (with a fixed orientation) is bidirectional, there is an orientation of  $C_{n+1}$  such that  $\text{Pic}(C_n) \cong \text{Pic}(C_{n+1})$ .

*Proof.* Let  $V(C_n) = \{v_1, \dots, v_n\}$  and  $D_{C_n} = (d_{ij})$  be the diagonal matrix of  $C_n$  (with a given orientation). Since not every edge is bidirectional, there exists  $i$  such that  $d_{ii} = 0$  or  $1$ . Without loss of generality, we may assume  $i = n$  so that the adjacent vertices are  $v_{n-1}$  and  $v_1$ .

Case 1: Suppose that  $d_{nn} = 0$ . In this case, the Laplacian matrix of  $C_n$  is the following.

$$L_{C_n} = \begin{bmatrix} l+1 & -l & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & 0 & -k & k+1 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \quad (16)$$

where  $l, k \in \{0, 1\}$ . We define  $C_{n+1}$  extending  $C_n$  as follows. Add a vertex  $v_{n+1}$  to  $C_n$ , replace the edge  $e_{1 \rightarrow n}$  by  $e_{1 \rightarrow n+1}$ , and add a new edge  $e_{n \rightarrow n+1}$ . Pictorially, we have the following

$$C_n : (\cdots v_{n-1} \rightarrow v_n \leftarrow v_1 \cdots) \longrightarrow C_{n+1} : (\cdots v_{n-1} \rightarrow v_n \rightarrow v_{n+1} \leftarrow v_1 \cdots). \quad (17)$$

Now, one has the following equivalence of matrices.

$$\begin{aligned} L_{C_{n+1}} &= \begin{bmatrix} l+1 & -l & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 \rightarrow r_1 - r_n} \begin{bmatrix} l+1 & -l & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{c_n \rightarrow c_n + c_{n+1}} \begin{bmatrix} l+1 & -l & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} l+1 & -l & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \left[ \begin{array}{c|c} L_C & 0 \\ \hline 0 & 1 \end{array} \right]. \end{aligned}$$

Now our claim follows from Remark 2.3.

Case 2: Now suppose  $d_{nn} = 1$ . Without loss of generality, we may assume the vertex  $v_n$  has one outgoing edge to  $v_1$ . There are two cases, depending on the existence of  $e_{1 \rightarrow n}$ . The Laplacian of these



two graphs are equivalent.

$$L_{C_n} = \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & 0 & -k & k+1 & -1 \\ -1 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \rightarrow r_1 - r_n} \begin{bmatrix} l+1 & -l & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \cdots & & \\ 0 & \cdots & 0 & -k & k+1 & -1 \\ -1 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (18)$$

where  $l, k \in \{0, 1\}$ . Therefore, we may assume that  $L_{C_n}$  is the first matrix in (18). We define  $C_{n+1}$  extending  $C_n$  as follows. Add a vertex  $v_{n+1}$  to  $C_n$ , replace the edge  $e_{n \rightarrow 1}$  by  $e_{n+1 \rightarrow 1}$ , and add a new edge  $e_{n \rightarrow n+1}$ . Pictorially, we have the following

$$C_n : (\cdots v_{n-1} \rightarrow v_n \rightarrow v_1 \cdots) \longrightarrow C_{n+1} : (\cdots v_{n-1} \rightarrow v_n \rightarrow v_{n+1} \rightarrow v_1 \cdots).$$

Then we have the following equivalence of matrices.

$$\begin{aligned} L_{C_{n+1}} &= \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & & \\ 0 & \cdots & 0 & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -1 \\ -1 & \cdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_{n+1} \leftrightarrow r_n} \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & & \\ 0 & \cdots & 0 & -k & k+1 & -1 & 0 \\ -1 & \cdots & 0 & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\ &\xrightarrow{r_n \rightarrow r_{n+1} + r_n} \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & & \\ 0 & \cdots & 0 & -k & k+1 & -1 & 0 \\ -1 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{c_n \rightarrow c_n - c_{n+1}} \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & & \\ 0 & \cdots & 0 & -k & k+1 & -1 & 0 \\ -1 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \\ &\longrightarrow \left[ \begin{array}{c|c} L_C & 0 \\ \hline 0 & 1 \end{array} \right]. \end{aligned}$$

Our claim follows from Remark 2.3. □

**Lemma 4.4.** *For any  $n \geq 4$  and any  $m \in \{n-1, n\}$ , there exists an orientation of  $C_n$  such that  $\text{Jac}(C_n) = \mathbb{Z}_m$ .*

*Proof.* If all edges of  $C_n$  are bidirectional, then  $\text{Pic}(C_n) \cong \mathbb{Z}_n$ . We claim that an orientation of  $C_n$  such that

$$C_n : (\cdots v_n \rightarrow v_1 \leftarrow v_2 \leftarrow v_3 \cdots)$$

and all other edges are bidirectional has the Picard group  $\mathbb{Z}_{n-1}$ . In fact, we have the following equivalence of matrices:

$$\begin{aligned}
L_{C_n} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ -1 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \xrightarrow{r_n \rightarrow r_n - r_2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \\
&\xrightarrow{c_2 \rightarrow c_2 + c_1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \quad (\text{See the remark below for the last equivalence})
\end{aligned}$$

Now the  $(n-1) \times (n-1)$  bottom right submatrix is that of  $C_{n-1}$  with all bidirectional edges and Remark 2.3 proves the claim.  $\square$

**Remark 4.5.** For  $n \geq 3$ , the Laplacian of  $C_n$  each of its edge is bidirectional, the Laplacian of  $C_n$  is of the form

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

Since  $[1 \cdots 1]L = 0$ , the first row is a  $\mathbb{Z}$ -linear combination of the next  $(n-1)$  rows. This justifies the last equivalence in the proof above.

*Proof of the Conjecture 4.1.* We induct on the number of edges. The base case is when  $n = 3$  which is Example 4.2. For  $n > 3$ , by Lemma 4.4, we have orientations of  $C_n$  with  $\text{Jac}(C_n) = \mathbb{Z}_n$  and  $\mathbb{Z}_{n-1}$ . Now the rest follows from Lemma 4.3.  $\square$

**4.2. Construction of cyclic Jacobians of  $C_n$ .** By a *path* of  $C_n$ , we mean a connected subgraph of  $C_n$  in which all arrows are oriented in a single direction or are bidirectional. In other words, a path is a subgraph of  $C_n$  with one strong terminal component.

In graphs with exactly two of these paths, a consistent pattern arises that allows for the rapid calculation of the Jacobian in only  $O(n)$  time compared to the  $O(n^4)$  time that calculating the Smith normal form takes. This pattern relies on calculating the number of bidirectional arrows between these two paths on one side. Our experimental computation suggests the following.

**Conjecture 4.6.** Let  $C_n$  be a cycle graph with a fixed orientation. Then, we have  $\text{Jac}(G) = \mathbb{Z}_{x+2}$ , where  $x$  is the number of bidirectional edges clockwise of the counter-clockwise path and counter-clockwise of the clockwise path.

**Lemma 4.7.** For  $n \geq 2$ , consider the following square matrix  $M_n$

$$M_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}. \quad (19)$$

This is a matrix whose diagonal entries are 2 and sub-diagonals are  $-1$ . Then  $\det(M_n) = n + 1$ .

*Proof.* We induct on  $n$ . When  $n = 2$ , a direct computation shows that  $\det(M_2) = 3$ . Suppose the statement is true for all  $k \leq n - 1$ . To compute  $\det M_n$ , we use the Laplace expansion along the first row. That is,  $\det M_n = 2 \det(M_{n-1}) + \det(N)$ , where  $N$  is

$$N = \left[ \begin{array}{c|cccc} -1 & -1 & 0 & \cdots & 0 \\ \hline 0 & & M_{n-2} & & \end{array} \right]. \quad (20)$$

(Thus,  $\det N = -\det(M_{n-2})$ . [Youngsu: to be removed](#)) By induction,  $\det M_n = 2 \det(M_{n-1}) - \det(M_{n-2}) = 2(n - 1 + 1) - (n - 2 + 1) = 2n - n + 1 = n + 1$ .  $\square$

**Remark 4.8.** Note that the  $(n - 1)$  by  $(n - 1)$  minor of  $M_n$  after deleting the last row and last column is  $(-1)^{n-1}$ . Hence the Smith normal form of  $M_n$  is

$$\left[ \begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & n + 1 \end{array} \right], \quad (21)$$

where  $I_{n-1}$  is the identity matrix of size  $n - 1$ .

**Lemma 4.9.** Consider a directed cycle graph  $C_n$  on  $n$  vertices. Assume that the vertex  $v_1$  does not have any outgoing edge, and all other vertices have two outgoing edges. Then  $\text{Jac}(C_n) = \mathbb{Z}_n$ .

*Proof.* The Laplacian of  $C_n$  is

$$\begin{bmatrix} 0 & 0 & \cdots & & & \\ -1 & 2 & -1 & \cdots & & \\ 0 & -1 & 2 & -1 & \cdots & \\ \vdots & & & & & \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}. \quad (22)$$

By the same argument as in Remark 4.5, [Youngsu: we need to state Remark 4.5 for the general case](#) This matrix is equivalent to

$$\left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 & M_{n-1} \end{array} \right], \quad (23)$$

where  $M_{n-1}$  is the matrix in Lemma 4.7. Thus, by the remark above,  $\text{Jac}(C_n) = \mathbb{Z}_n$ . (One can also use the proof of Lemma 4.4 with Lemma 4.7.)  $\square$

The general case follows from this lemma combined with the first and the second parts of the proof of Lemma 4.3 for  $\mathbb{Z}_{n-1}$ . (I believe Matt's observation about straightning intermediate birational edges corresponds to the reduction in the 2nd case of  $d_{n,n} = 1$ .)

**Jaiung:** here we study how Picard groups change when we glue two directed graphs along one vertex or connect two graphs along one arrow. If we are lucky this should provide a way to compute Picard groups for pseudotrees by using the tree case and the cycle case in the previous sections.

## 5. Picard groups of oriented wheel graphs

**Jaiung:** the following conjecture should generalize the results in [Big99].

For wheel graphs, we looked for patterns that arose within the invariant factors of the Jacobian as a general formula was not immediately obvious. For this strategy, we broke the edges of the wheel graph into their two most obvious groups, those belonging to the rim of the wheel and those of the spokes. By orienting all the edges of either group the same way and trying all nine combinations, we noticed a well-defined pattern for each as the size of the wheel graph changed. These patterns fell into four distinct groups.

- (1) When the spoke edges all pointed towards the axel and the rim was not bidirectional, or when the spoke edges were bidirectional, but the rim was not then a graph of size  $n$  had a Jacobian of  $\mathbb{Z}_{2^{(n-1)}-1}$ .
- (2) When the spoke edges all pointed away from the axel and the rim was not bidirectional, a graph of size  $n$  had a Jacobian of  $\mathbb{Z}_{n-1}$ .
- (3) When the spoke edges point away from the axel and the rim was bidirectional and a graph of size  $n$  had a Jacobian of  $\mathbb{Z}_{n-1} \times \mathbb{Z}_{n-1}$  when  $n$  was even and  $\mathbb{Z}_{\frac{n-1}{2}} \times \mathbb{Z}_{(n-1) \times 2}$  when  $n$  was odd.
- (4) When all edges were bidirectional or when the spoke direction was towards the axel and the rim was bidirectional, a graph of size  $n$  had a Jacobian of  $\mathbb{Z}_{\alpha\phi^n} \times \mathbb{Z}_{5\alpha\phi^n}$  when the size was odd where  $\alpha \cong 0.27555$  and  $\mathbb{Z}_{\beta\phi^n} \times \mathbb{Z}_{\beta\phi^n}$  when the size was even where  $\beta \cong 0.618035$ . In both of these patterns,  $\phi$  represents the golden ratio.

**Definition 5.1** (Wheel graph). By a wheel graph  $W_n$ , we mean a graph obtained by connecting a single universal vertex to all vertices of a cycle  $C_{n-1}$ .

**Proposition 5.2.** Let  $W_n$  be the wheel graph with bidirectional edges, and let  $W'_n$  be the wheel graph such that the edges of the rim are bidirectional and all its spoke edges point to the axel. Then  $L'_{W_n}$  and  $L_{W_n}$  are row equivalent. In particular, one has  $\text{Pic}(W_n) \cong \text{Pic}(W'_n)$ .

*Proof.* We use the labeling convention that  $v_1$  is the axel and the vertices on the rim are  $v_2, \dots, v_n$ . One can directly see that the Laplacian matrices of  $W_n$  and  $W'_n$  are row equivalent. To be precise, from the Laplacian  $L_{W'_n}$ , one can obtain the Laplacian  $L_{W_n}$  by subtracting all other rows from the first row by using the fact that the  $\mathbf{1}_n \cdot L_{W_n} = 0$ :

$$L_{W'_n} = \left[ \begin{array}{c|ccccccc} 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \hline -1 & 3 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & -1 & 3 & -1 & \cdots & \cdots & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & \cdots & 0 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 & \cdots & 0 \\ \hline -1 & \vdots & \vdots & & & & & \\ -1 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ -1 & -1 & 0 & 0 & \cdots & 0 & -1 & 3 \end{array} \right]$$

$$L_{W_n} = \left[ \begin{array}{c|ccccccc} n-1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 \\ \hline -1 & 3 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & -1 & 3 & -1 & \cdots & \cdots & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & \cdots & 0 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 & \cdots & 0 \\ \hline -1 & \vdots & \vdots & & & & & \\ -1 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ -1 & -1 & 0 & 0 & \cdots & 0 & -1 & 3 \end{array} \right]$$

□

Yongsu: This approach probably works for complete graphs where all edges are bidirectional except for one vertex.

**Proposition 5.3.** *Let  $W_n$  be the wheel graph where the edges of the rim are bidirectional and all its spoke edges point away from the axel. Then the Smith normal form of the Laplacian of  $W_n$  is of the following form*

$$\left[ \begin{array}{c|ccc} I_{n-3} & 0 & 0 & 0 \\ \hline 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{array} \right], \quad \text{where } (a, b) = \begin{cases} (n-1, n-1) & \text{if } n \text{ is even;} \\ (\frac{n-1}{2}, 2(n-1)) & \text{if } n \text{ is odd} \end{cases}$$

*Proof.* We use the labeling convention that  $v_1$  is the axel and the vertices on the rim are  $v_2, \dots, v_n$ . Let  $L_{W_n}$  be the Laplacian matrix of  $W_n$ . It is clear that  $\det(L_{W_n}) = 0$ . Hence, from Theorem 2.4, it suffices to show that

$$I_{n-1}(L_{W_n}) = \langle (n-1)^2 \rangle \text{ and } I_{n-2}(L_{W_n}) = \begin{cases} \langle n-1 \rangle & \text{if } n \text{ is even} \\ \langle (n-1)/2 \rangle & \text{if } n \text{ is odd.} \end{cases}$$

We reduce its Laplacian matrix  $L_{W_n}$  as follows:

$$\begin{aligned} L_{W_n} &= \left[ \begin{array}{c|cccccccc} n-1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 \\ \hline 0 & 2 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & \vdots & \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{array} \right] \xrightarrow{c_1 \rightarrow c_1 + \cdots + c_n} \left[ \begin{array}{c|cccccccc} 0 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 \\ \hline 0 & 2 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & \vdots & \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{array} \right] \\ &\xrightarrow{r_n \rightarrow r_2 + \cdots + r_n} \left[ \begin{array}{c|cccccccc} 0 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \hline 0 & 2 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & \vdots & \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array} \right] \xrightarrow{c_n \rightarrow c_2 + \cdots + c_n} \left[ \begin{array}{c|cccccccc} 0 & 1 & 1 & \cdots & \cdots & \cdots & 1 & n-1 \\ \hline 0 & 2 & -1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & \vdots & \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

Thus, to compute the Smith normal form of  $L_{W_n}$ , it suffices to find the Smith normal form of the following matrix.

$$N = \left[ \begin{array}{c|c} \mathbf{1}_{n-2} & n-1 \\ \hline M_{n-2} & \mathbf{0}_{n-1} \end{array} \right],$$

where  $M_{n-2}$  is the matrix in Lemma 4.7. Note that  $\det(M_{n-2}) = n-1$  and this shows that  $I_{n-1}(L_{W_n}) = I_{n-1}(N) = \langle \det(N) \rangle = \langle (n-1)^2 \rangle$ .

To compute the  $(n-2)$  minors, first note that if the last column is a part of a minor, then it is divisible by  $n-1$ . Since  $\det(M_{n-2}) = n-1 \in I_{n-2}(N)$ , we only need to consider the  $n-2$  minors of the first  $n-2$  columns of  $N$ .

In other words,  $I_{n-2}(L_{W_n})$  is generated by  $n-1$  and the  $(n-2)$ -determinants of the matrix which is obtained from  $M_{n-2}$  by replacing the  $i^{\text{th}}$  row by  $\mathbf{1}_{n-2}$ .

Note that  $M_{n-2}$  is symmetric. By Cramer's rule, these  $(n-2)$ -minors are the entries of the solution matrix  $\mathbf{x}$  of the matrix equation

$$M_{n-2}\mathbf{x} = \det(M_{n-2}) \cdot \mathbf{1}_{n-2}^T$$

up to sign. This is done in the following lemma, and it completes the proof.  $\square$

**Lemma 5.4.** Consider the matrix equation

$$M_n \mathbf{x} = \det(M_n) \mathbf{1}_n^T = (n+1) \mathbf{1}_n^T,$$

where  $M_n$  is as in Lemma 4.7. Let  $x_k$  denote the  $k^{\text{th}}$  entry of  $\mathbf{x}$ . Then  $x_k = ak^2 + bk$ , where

$$a = -\frac{(n+1)}{2}, \quad b = \frac{(n+1)^2}{2}.$$

Furthermore,  $\gcd(x_1, \dots, x_n)$  is  $n+1$  if  $n$  is even and  $(n+1)/2$  if  $n$  is odd.

*Proof.* Note that  $\det(M_n) = n+1$  (Lemma 4.7). Consider a sequence  $f(k) = f_k = ak^2 + bk$ . For any  $k$ , we have

$$-f_{k-1} + 2f_k - f_{k+1} = -2a.$$

First, we set  $a = -(n+1)/2$ . In addition, the equations  $2x_1 - x_2 = n+1$  and  $-x_{n-1} + 2x_n = n+1$  are equivalent to the conditions  $f_0 = f_{n+1} = 0$ . These conditions imply  $b = (n+1)^2/2$  since  $0, -b/a$  are the roots of  $f(k)$ .

We compute the gcd of  $f_1, \dots, f_n$ . First, we claim that  $\gcd(f_1, \dots, f_n) = \gcd(f_1, n+1)$ . Since  $\gcd(f_1, \dots, f_n)$  divides  $2f_1 - f_2 = n+1$ , it suffices to show that  $\gcd(f_1, n+1)$  divides  $f_2, \dots, f_n$ . This follows by induction since for  $k = 2, \dots, n$ ,  $f_k = 2f_{k-1} - f_{k-2} - (n+1)$ . Here we use the fact that  $f_0 = 0$ . Since  $f_1 = -(n+1)/2 + (n+1)^2/2 = \frac{n(n+1)}{2}$  is an integer, the expressions  $f_k = -2f_{k-1} + f_{k-2}$  also proves that  $f_1, \dots, f_n$  are integers.

Finally, we show that  $\gcd(f_1, n+1)$  is  $n+1$  when  $n$  is even and  $(n+1)/2$  when  $n$  is odd. Recall that  $f_1 = \frac{n(n+1)}{2}$ . If  $n$  is even, then  $n+1$  divides  $f_1$ . Thus,  $\gcd(f_1, n+1) = n+1$ . If  $n$  is odd, then

$$\gcd(f_1, n+1) = \gcd\left(\frac{n+1}{2} \cdot n, n+1\right) = \gcd\left(\frac{n+1}{2}, n+1\right) = \frac{n+1}{2}.$$

The 2<sup>nd</sup> last equality follows from the fact that  $\frac{n+1}{2}$  is an integer and for integers  $a, b, c$ ,  $\gcd(ab, c) = \gcd(a, c)$  if  $b, c$  are relatively prime.

Thus  $\mathbf{x} = [f_1 \ \cdots \ f_n]^T$  is a solution having the asserted gcd. This completes the proof.  $\square$

## 6. Picard groups of oriented multipartite graphs

**Jaiung:** this paper [?] could be relevant here. For our purposes, a multipartite graph is a graph whos vertices can be partitioned between several independent groups, arranged in a linear order. Vertices have no connections to members of their own group, but are strongly connected to all vertices of their two adjacent groups.

The structure of the graphs that we investigate are intentionally designed to resemble artificial neural networks. To further facilitate this comparison, we direct all edges *forward* such that, after numbering the groupings of these vertices in some order, edges always point towards the next highest numbered grouping.

We were able to find notable patterns in both a *Perceptron* style model with two layers and a *Hidden Layer* model with three layers.

**6.1. Picard groups for two layers.** For two layers in the form of  $f \rightarrow s$  where  $f$  and  $s$  are the number of nodes in the first and second layers, respectively,  $\text{Pic}(G) = \mathbb{Z}_s^{f-1} \times \mathbb{Z}^s$ .

**6.2. Picard groups for three layers.** For three layers in the form of  $f \rightarrow s \rightarrow t$  where  $f$ ,  $s$ , and  $t$  are the number of nodes in the first, second, and third layers, respectively the Picard group is significantly more complex.

- (1) When ( $s$  is odd,  $s$  is not a factor of  $t$  and  $f \leq s$ ) or ( $t$  is odd,  $s$  is even, and  $f \leq s$ ) or  $\text{Pic}(G) = \mathbb{Z}_t^{s-f-1} \times \mathbb{Z}_{s \times t}^f \times \mathbb{Z}^t$
- (2) When ( $s$  is odd,  $s$  is not a factor of  $t$  and  $f > s$ ) or ( $t$  is odd,  $s$  is even and  $f > s$ ),  $\text{Pic}(G) = \mathbb{Z}_s^{f-s+1} \times \mathbb{Z}_{s \times t}^{s-1} \times \mathbb{Z}^t$

- (3) When ( $s$  is odd and  $s$  is a factor of  $t$ ) or ( $t$  is even,  $s$  is even, and  $s$  is a factor of  $t$ ),  $\text{Pic}(G) = Z_s^{f-1} \times Z_t \times Z_{s \times t} \times Z^t$
- (4) When ( $t$  is even,  $s$  is even, and  $s$  is a not factor of  $t$ ),  $\text{Pic}(G) = Z_2^{f-1} \times Z_t^{s-f-1} \times Z_{s \times t/2}^{f-1} \times Z_{s \times t} \times Z^t$

## 7. Experimental Results

### 8. Conjectures and Future directions

**Conjecture 8.1.** For any cycle graph  $C_n$ , the orientation with no paths always has a Jacobian of  $\mathbb{Z}_n$ . Either of the single path orientations have a trivial Jacobian. The set of all orientations with two paths always contains all single invariant factors  $\mathbb{Z}_2 \dots \mathbb{Z}_n$ . For all graphs at least up to  $C_{10}$  and likely well beyond that point, the sets that contain all other paths do not contain all of the single invariant factors.

**Jaiung:** I think this should not be difficult....

It should be noted that for four paths and upward, the sets that contain these paths often also contain Jacobians of  $\mathbb{Z}_3$  and  $\mathbb{Z}_4$ . The number of each of these increases with the size of the graph, so it is possible that these sets will contain all of the single invariant factors for very large cycle graphs.

## References

- [Big99] Norman L Biggs. Chip-firing and the critical group of a graph. *Journal of Algebraic Combinatorics*, 9(1):25–45, 1999.
- [BN07] Matthew Baker and Serguei Norine. Riemann–Roch and Abel–Jacobi theory on a finite graph. *Advances in Mathematics*, 215(2):766–788, 2007.
- [Wag00] David G Wagner. The critical group of a directed graph. *arXiv preprint math/0010241*, 2000.

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