ON PICARD GROUPS OF DIRECTED GRAPHS

JAIUNG JUN AND MATTHEW PISANO

Abstract. We explore a combinatorial game on finite graphs, called Chip-Firing Games, which has various connections to other areas, such as algebraic geometry, number theory and economics. Roughly speaking, to play the game, one first puts an integer amount of chips at each vertex. Then, each vertex is allowed to borrow or lend chips from all of its neighbors equally the game progresses.

A configuration of chips on the vertices of a graph is called a divisor, a vector in $\mathbb{Z}^n$ for a graph of size n. The degree of divisor is the total number of chips combined on the vertices. The collection of all divisors on a graph defines a free abelian group $\mathrm{Div}(G)$, the divisor group of G. Under the equivalence relation $\sim$ generated by borrowing and lending moves, one obtain the Picard group $\mathrm{Pic}(G) = \mathrm{Div}(G)/\sim$. The Jacobian $\mathrm{Jac}(G)$ of G is the subgroup of $\mathrm{Pic}(G)$ consisting of the degree-zero divisors. These groups can be computed from the Laplacian matrix of a graph G. Through the structure of these groups, we gain a better understanding of how any given game can be played and evolve.

When a graph G is directed, one may define $\mathrm{Pic}(G)$ and $\mathrm{Jac}(G)$ as in the case of undirected graphs by using Laplacian matrices, but computations become much more complicated in this case. For example, $\mathrm{Pic}(G) = \mathbb{Z}$, when G is a tree, from the matrix-tree theorem, which tells us that $|\mathrm{Jac}(G)|$ is the number of spanning trees of G in the undirected case. For the case of directed trees T, even the rank of $\mathrm{Pic}(T)$ can be arbitrarily large depending on the size of T.

In our ongoing project, we study Picard groups and Jacobians for trees, cycles, and pseudotrees. Even in these seemingly simple cases, we find some new phenomenon. For instance, for the undirected cycle C_n, $\mathrm{Jac}(C_n) = \mathbb{Z}_n$, however, we prove that in the directed case, for any given $m \leq n$, one can always find an orientation of C_n in such a way that $\mathrm{Jac}(C_n)$ (with this orientation) is $\mathbb{Z}_m$. By closely examining trees and cycles, and how Picard groups and Jacobians change with (suitable defined) vertex and edge gluing, we obtain several results for pseudotrees.

1. Introduction

We will focus on Research Project 11 in [GK20]. The following are more specific directions that we plan to pursue.

Goal (8/1/2022)**.**

(1) Prove/disprove: for an oriented graph G, one always has $\mathrm{Pic}(G) = \mathbb{Z} \times \mathrm{Jac}(G)$, i.e., as a finitely generated abelian group, the rank of $\mathrm{Pic}(G)$ is 1. Jaiung: we disproved this by using [Wag00].

(2) Prove/disprove: for C_n, and $0 \leq m \leq n$, one can always find an orientation of C_n so that $\mathrm{Jac}(C_n) = \mathbb{Z}_m$ (with the orientation).

(3) Prove/disprove: for an oriented graph G, if $v_0 \in V(G)$ is a sink (or a source) and G' is the graph obtained by reserving the direction for all arrows adjacent to v_0 from G, then $\mathrm{Jac}(G) = \mathrm{Jac}(G')$. (Note: we believe that this should be true for at least some classes of graphs such as cyclic graphs.)

(4) Prove/disprove: for an oriented planar graph G and its planar dual (should be defined) $\hat{G}$, one has $\mathrm{Jac}(G) = \mathrm{Jac}(\hat{G})$.

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(5) Prove/disprove: for oriented graphs G_1, G_2, let G be the graph obtained by gluing G_1 and G_2 along one vertex. Then $\mathrm{Jac}(G) = \mathrm{Jac}(G_1) \times \mathrm{Jac}(G_2)$. Jaiung: we are currently working on this (8/31)

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2. Preliminaries

2.1. Chip Firing. The game at the heart of this paper is the Chip-Firing game. When a game is started, each vertex on a graph is assigned a certain number of chips. During play, chips can be lent or borrowed at each node where one or more chips are either sent or received along each outgoing edge equally. In the case of a directed graph, vertices can only interact with another along an outgoing or bidirectional edge. The game is won once every vertex has a positive number of chips (i.e., this vertex is not in debt).

2.2. Divisors and Equivalence Relations. In the study of this game a **Divisor** of a graph ($\mathrm{Div}(G)$) is an integer vector $v \in \mathbb{Z}^n$ where n is the number of vertices in the graph. The i^{th} element of the vector v is the number of chips on the i^{th} vertex of the graph. Two divisors have an **Equivalence Relation** ($\sim$) if one divisor can be gotten from the other by a finite series of lending or borrowing moves $D_1 \sim D_2 \leftrightarrow (D_1 \xleftrightarrow{\text{moves}} D_2)$. An **Equivalence Class** *[D]* is the set of all divisors that are equivalent to each other, $[D] = \{D_i \mid D_i \sim D\}$.

2.3. The Picard Group and The Jacobian. The **Picard Group** of a graph $\mathrm{Pic}(G)$ is the set of all equivalence classes that the divisors of that graph can be a part of. The **Jacobian** of a graph $\mathrm{Jac}(G)$ is a subset of $\mathrm{Pic}(G)$ such that every divisor in each equivalency class has a degree of 0 where the degree of a divisor $\deg(D)$ is the sum of each of the divisor's elements. If a divisor is in one of the Jacobian's classes, it can be made winning after a finite series of moves.

3. Propositions

3.1. Calculating Sinks and Sources. Vertices on a directed graph can be classified in three ways, as a sink, source, or neither depending on the direction of the edges connected to it. A sink is a vertex where all edges are directed into that vertex, a source is a vertex where all edges are directed away from that vertex and a vertex is neither when it has a mixture of the two. The number of sinks and sources can be calculated inductively by reconstructing the original graph from a single vertex.

Beginning with any two vertices that are adjacent on the original graph, observe their connecting edge. If the edge is directed, the count of sinks and sources begins at 1 for both. If the edge is undirected (bidirectional), the count begins at zero. Next, pick a vertex adjacent to one of the original edges and observe its connection. From this point on, when a new edge is observed the number of sinks and sources changes on a series of rules. For these rules, let the vertex being added to be V_1 and the vertex being added be V_2.

(1) If V_1 is neither a sink nor a source.
  (a) Adding an edge directed towards V_2 adds a sink.
  (b) Adding an edge directed away from V_2 adds a source.
  (c) Adding a bidirectional edge between V_1 and V_2 adds nothing.
(2) If V_1 is a source.
  (a) Adding an edge directed towards V_2 adds a sink.
  (b) Adding an edge directed away from V_2 adds nothing (Here the number of sources stays the same as V_1 is no longer a source but V_2 now is).
  (c) Adding a bidirectional edge between V_1 and V_2 removes a source.
(3) If V_1 is a sink.

(a) Adding an edge directed towards V_2 adds nothing (See 2b).
(b) Adding an edge directed away from V_2 adds a source.
(c) Adding a bidirectional edge between V_1 and V_2 removes a sink.

When accounting for a vertex that has multiple connections, apply these rules for every V_{1i} that V_2 is connected to, taking into account all the edges of V_2 when evaluating its class. After every vertex has been accounted for, the number of sinks and sources will have been calculated in polynomial time between $O(n)$ at the average case and $O(n^2)$ for the case of a connected graph.

4. Example: Trees

Lemma 4.1. *Let G be a graph with any orientation. If we attach either an incoming arrow or a two-sided arrow to create a directed graph G', then* $\mathrm{Pic}(G) = \mathrm{Pic}(G')$.

Proof. Let α be an arrow which is glued to G. Let $|V(G)| = n$. We label the vertexes of G as $1, 2, \ldots, n$. Suppose first that α is incoming and α is glued at the vertex n. Let $A_G = (a_{ij})$ (resp. $A_{G'}$) be the Laplacian of G (resp. G'). In this case, one can easily observe that the matrix $A_{G'}$ is of the following form:

$$A_{G'} = \left[\begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 0 \\ \hline 0 & 0 & \cdots & -1 & 1 \end{array}\right] \tag{1}$$

By a column operation between the last two columns, we obtain the following matrix:

$$\left[\begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right] \tag{2}$$

This shows that $\mathrm{Pic}(G) = \mathrm{Pic}(G')$.

Next, suppose that α is a two-sided arrow. Then similar to the above, we obtain the following matroid for G':

$$A_{G'} = \left[\begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n+1 & -1 \\ \hline 0 & 0 & \cdots & -1 & 1 \end{array}\right] \tag{3}$$

By a column operation, the matrix (3) becomes the matrix (2). This shows that $\mathrm{Pic}(G) = \mathrm{Pic}(G')$. $\square$

Proposition 4.2. *Let T be a tree with any orientation. Then* $\mathrm{Pic}(T) = \mathbb{Z}^n$*, where n is the number of sinks of T.*

Proof. Note that from [Wag00, Corollary 3.5] that the rank of $\mathrm{Pic}(T)$ is the number of sinks of T.

Now, we can inductively prove this. When T_0 is a tree with one arrow, one can easily check that $\mathrm{Pic}(T_0) = \mathbb{Z}$ or $\{0\}$ (depending on the number of sinks). Suppose that T_k is a directed tree with k arrows. When we add one arrow α to T_k to construct T_{k+1}, there are three cases; (1) incoming, (2) outgoing, and (3) two-sided. When α is either incoming or two-sided arrow, then it follows from Lemma 4.1 that $\mathrm{Pic}(T_k) = \mathrm{Pic}(T_{k+1}) = \mathbb{Z}^n$, where n is the number of sinks of T_k and T_{k+1}, since in this case it does not increase the number of sinks.

Next, suppose that α is an outgoing arrow. In this case, T_{k+1} has one more sink than T_k. We claim that $\text{Pic}(T_{k+1}) = \text{Pic}(T_k) \times \mathbb{Z}$. We label the vertexes of G as $1, 2, \ldots, n$ and suppose that the arrow α is attached to the vertex n. Let $A_{T_k} = (a_{ij})$ be the Laplacian of T_k. Then we have the following:

$$A_{T_{k+1}} = \left[\begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n + 1 & -1 \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array}\right] \tag{4}$$

Jaiung: I will finish writing down the proof before our next meeting. □

5. Example: Cycles

Here's our conjecture:

Conjecture 5.1. Let C_n be the cycle graph with n vertices. For each $k \leq n$, there exists an orientation of C_n such that $\text{Jac}(C_n)$ (with that orientation) is $\mathbb{Z}_k$.

The following examples shows the conjecture for C_3.

Example 5.2.

Lemma 5.3. *Let C_n be the cycle graph with n vertices. Consider the orientation of C_n which provides precisely two directed paths in opposite direction. All invariant factors of Jac(G), where $G = C_n$, can be represented in the Jacobian of that graph using two directed paths.*

Proof. Beginning with a non-oriented cyclic graph, $\text{Jac}(G) = \mathbb{Z}_n$. Orienting one edge $(x, x+1)$ in the counter-clockwise direction, another edge $(x-1, x)$ to clockwise, and $(x+1, x+2)$ to counter-clockwise yields a Jacobian of $\mathbb{Z}_{n-1}$. Orienting either the edge before $(x-1, x)$ to clockwise or the edge after $(x+1, x+2)$ to counter-clockwise yields $\text{Jac}(G) = \mathbb{Z}_{n-2}$. Continuing this pattern until the two paths meet yields all invariant factors down to $\mathbb{Z}_2$. Going further to orient the entire graph to one direction results in a trivial Jacobian of 0.This pattern should hold for all cycle graphs $C_3 \ldots C_x$, and had been proven for graphs $C_3 \ldots C_{80}$. □

The following example shows that the rank of $\text{Pic}(C_n)$ (with an orientation) does not have to be 1; this directly follows from [Wag00, Corollary 3.5].

Example 5.4.

5.1. Calculating Rank((Pic(G)) Using Terminal Strong Components.

Pic(G) is often in the form $\mathbb{Z}_1 \times \cdots \times \mathbb{Z}_n \times \mathbb{Z}^m$ where m is the rank of the picard group. A terminal strong component describes itself well. It is a terminal component, meaning that its only connections are edges directed into it. It is also strongly connected, meaning that every vertex in the component has at least one path to all other components. For cycle graphs and trees, the number of these terminal strong components is exactly equal to $Rank(\text{Pic}(G))$.

Proof. The $Rank(\text{Pic}(G))$ comes directly from the number of all zero rows of the Smith Normal Form of the laplacian of G. Terminal strong components are either already zero rows (in the case of a single sink) or there exists a series of row and column operations that, when preformed on the laplacian result in a row of all zeroes. □

Lemma 5.5. *Let C_n be the cycle graph with n vertices. Consider the equi-orientation of C_n. Then* $\text{Pic}(C_n) = \mathbb{Z}$.

Proof. In this case C_n will always have a laplacian with a diagonal of 1 and a diagonal of -1 above or below it depending on the orientation, with another -1 in the top right or bottom left corner where V_0 connects V_{n-1}. Turing this laplacian into a smith normal form of all 1s is as simple as cascading additions from the first row in a counter-clockwise case and the last row in the clockwise case. This will result in a diagonal of ones with the first or last column made up of mostly -1. From here all that needs to be done is to eliminate these with the first or last row. Since each equi-oriented C_n will follow this pattern, this algorithm will prove $\mathrm{Pic}(C_n) = \mathbb{Z}$ for all C_n. □

6. Example: Pseudotrees

Jaiung: here we study how Picard groups change when we glue two directed graphs along one vertex or connect two graphs along one arrow. If we are lucky this should provide a way to compute Picard groups for pseudotrees by using the tree case and the cycle case in the previous sections.

7. Strongly connected directed graphs

References

[GK20] Darren Glass and Nathan Kaplan. Chip-firing games and critical groups. In *A Project-Based Guide to Undergraduate Research in Mathematics*, pages 107–152. Springer, 2020.

[Wag00] David G Wagner. The critical group of a directed graph. *arXiv preprint math/0010241*, 2000.

Department of Mathematics, State University of New York at New Paltz, NY 12561, USA
Email address: junj@newpaltz.edu

Department of Mathematics, State University of New York at New Paltz, NY 12561, USA
Email address: pisanom1@newpaltz.edu