STAT 588

Fun with Bayesian Estimators

Name: Solutions

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Theorem 10.6 says that the conjugate prior for the mean of a normal random variable is a normal distribution, as shown below.

THEOREM 6. If $\overline{X}$ is the mean of a random sample of size n from a normal population with the known variance σ^2 and the prior distribution of M (capital Greek *mu*) is a normal distribution with the mean μ_0 and the variance σ_0^2, then the posterior distribution of M given $\overline{X} = \bar{x}$ is a normal distribution with the mean μ_1 and the variance σ_1^2, where

$$\mu_1 = \frac{n\bar{x}\sigma_0^2 + \mu_0\sigma^2}{n\sigma_0^2 + \sigma^2} \quad \text{and} \quad \frac{1}{\sigma_1^2} = \frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}$$

Consider the following real-life scenario:

The authorities are alerted to the scene of a home burglary where a window was broken in order to gain entrance to the home. As evidence, the crime scene unit collects some fragments of broken window glass and analyze it via refractive index. Later that evening, authorities apprehend a suspect and notice that his clothes have fragments of glass on it. The suspect claims that the glass fragments on his clothes are there because he broke a glass flower vase earlier that day and the fragments must have gotten on his clothes while he was cleaning it up. How can we tell if the glass on the suspect's clothes could have come from the broken window at the crime scene using a Bayesian analysis?

1. We need to determine a distribution for the refractive indices of glass from windows. Suppose that it has been shown that the refractive index of window glass has a normal distribution with a known variance $\sigma^2 = 16$. However, the mean μ can shift around based on changing recipes for how window glass is made, so the mean is unknown. $X_i \overset{iid}{\sim} N(\mu, \sigma^2 = 16)$
2. Since the mean is unknown, we need to determine a prior distribution for it. Let's choose the conjugate prior and use a normal distribution. After talking to some experts, we decide that the mean of this normal prior distribution should be $\mu_0 = 1.58$ and the variance should be $\sigma_0^2 = 4$.

$$\mu \sim N(\mu_0 = 1.58, \sigma_0^2 = 4)$$

3. Now, we need to gather some data to update our prior distribution. So, we look at the evidence that was collected at the crime scene, and we find that $n = 15$ pieces of broken glass were taken from the window and that the average refractive index of those pieces is $\bar{x} = 1.6$.
4. Use the result from Theorem 10.6 above to find the Bayesian estimator of μ.

$$\hat{\mu} = \mu_1 = \frac{n\bar{x}\sigma_0^2 + \mu_0\sigma^2}{n\sigma_0^2 + \sigma^2} = \frac{15(1.6)(4) + 1.58(16)}{15(4) + 16} = 1.596$$

5. Finally, to determine whether the glass on the suspect could have come from broken window at the crime scene, we consider a normal distribution with mean $\hat{\mu} = \mu_1$ and variance σ_1^2, where μ_1 and σ_1^2 are given by Theorem 6.
   a. If the refractive index from the glass on the suspect is near the mean μ_1, then the glass probably came from the broken window at the crime scene.
   b. If the refractive index from the glass on the suspect is far from the mean μ_1, then the glass probably came from the broken flower vase.

$$\frac{1}{\sigma_1^2} = \frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}$$

$$\sigma_1^2 = \frac{1}{\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right)} = \frac{1}{\left(\frac{n\sigma_0^2 + \sigma^2}{\sigma_0^2\sigma^2}\right)} = \frac{\sigma_0^2\sigma^2}{n\sigma_0^2 + \sigma^2} = \frac{4(16)}{15(4) + 16} = 0.842$$