THE CODEGREE TURÁN DENSITY OF TIGHT CYCLES MINUS ONE EDGE

SIMÓN PIGA, MARCELO SALES, AND BJARNE SCHÜLKE

ABSTRACT. Given $\alpha > 0$ and an integer $\ell \geqslant 5$, we prove that every sufficiently large 3-uniform hypergraph H on n vertices in which every two vertices are contained in at least αn edges contains a copy of C_ℓ^-, a tight cycle on ℓ vertices minus one edge. This improves a previous result by Balogh, Clemen, and Lidický.

§1. INTRODUCTION

A k-uniform hypergraph H consists of a vertex set $V(H)$ together with a set of edges $E(H) \subseteq V(H)^{(k)} = \{S \subseteq V(H) : |S| = k\}$. Throughout this note, if not stated otherwise, by *hypergraph* we always mean a 3-uniform hypergraph. Given a hypergraph F, the extremal number of F for n vertices, $\mathrm{ex}(n, F)$, is the maximum number of edges an n-vertex hypergraph can have without containing a copy of F. Determining the value of $\mathrm{ex}(n, F)$, or the Turán density $\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,F)}{\binom{n}{3}}$, is one of the core problems in combinatorics. In particular, the problem of determining the Turán density of the complete 3-uniform hypergraph on four vertices, i.e., $\pi(K_4^{(3)})$, was asked by Turán in 1941 [12] and Erdős [4] offered 1000$ for its resolution. Despite receiving a lot of attention (see for instance the survey by Keevash [7]) this problem, and even the seemingly simpler problem of determining $\pi(K_4^{(3)-})$, where $K_4^{(3)-}$ is the $K_4^{(3)}$ minus one edge, remain open.

Several variations of this type of problem have been considered, see for instance [1, 6, 11] and the references therein. The one that we are concerned with in this note asks how large the minimum codegree of an F-free hypergraph can be. Given a hypergraph H and $S \subseteq V$ we define the degree $d(S)$ of S (in H) as the number of edges containing S, i.e., $d(S) = |\{e \in E(H) : S \subseteq e\}|$. If $S = \{v\}$ or $S = \{u, v\}$ (and H is 3-uniform), we omit the parentheses and speak of $d(v)$ or $d(uv)$ as the degree of v or codegree of u and v, respectively. We further

---

2020 *Mathematics Subject Classification.* Primary: 05D99. Secondary: 05C65.

Key words and phrases. Codegree density, Hypergraphs.

S. Piga, is supported by EPSRC grant EP/V002279/1. The second author is partially suported by NSF grant DMS 1764385. There are no additional data beyond that contained within the main manuscript.

write $\delta(H) = \delta_1(H) = \min_{v\in V(H)} d(v)$ and $\delta_2(H) = \min_{uv\in V(H)^{(2)}} d(uv)$ for the minimum degree and the minimum codegree of H, respectively.

Given a hypergraph F and $n \in \mathbb{N}$, Mubayi and Zhao [10] introduced the *codegree Turán number* $\mathrm{ex}_2(n, F)$ of n and F as the maximum d such that there is an F-free hypergraph H on n vertices with $\delta_2(H) \geqslant d$. Moreover, they defined the *codegree Turán density of* F as

$$\gamma(F) := \lim_{n\to\infty} \frac{ex_2(n, F)}{n}$$

and proved that this limit always exists. It is not hard to see that

$$\gamma(F) \leqslant \pi(F)\,.$$

The codegree Turán density is known only for a few (non-trivial) hypergraphs (and blow-ups of these), see the table in [1]. The first result that determined $\gamma(F)$ exactly is due to Mubayi [8] who showed that $\gamma(\mathbb{F}) = 1/2$, where $\mathbb{F}$ denotes the 'Fano plane'. Later, using a computer assisted proof, Falgas-Ravry, Pikhurko, Vaughan, and Volec [5] proved that $\gamma(K_4^{(3)-}) = 1/4$. As far as we know, the only other hypergraph for which the codegree Turán density is known is $F_{3,2}$, a hypergraph with vertex set $[5]$ and edges 123, 124, 125, and 345. The problem of determining the codegree Turán density of $K_4^{(3)}$ remains open, and Czygrinow and Nagle [2] conjectured that $\gamma(K_4^{(3)}) = 1/2$. For more results concerning $\pi(F)$, $\gamma(F)$, and other variations of the Turán density see [1].

Given an integer $\ell \geqslant 3$, a *tight cycle* C_ℓ is a hypergraph with vertex set $\{v_1, \dots, v_\ell\}$ and edge set $\{v_i v_{i+1} v_{i+2} : i \in \mathbb{Z}/\ell\mathbb{Z}\}$. Moreover, we define C_ℓ^- as C_ℓ minus one edge. In this note we prove that the Turán codegree density of C_ℓ^- is zero for every $\ell \geqslant 5$.

Theorem 1.1. *Let $\ell \geqslant 5$ be an integer. Then $\gamma(C_\ell^-) = 0$.*

The previously known best upper bound was given by Balogh, Clemen, and Lidický [1] who used flag algebras to prove that $\gamma(C_\ell^-) \leqslant 0.136$.

§2. Proof of Theorem 1.1

For singletons, pairs, and triples we may omit the set parentheses and commas. For a hypergraph $H = (V, E)$ and $v \in V$, the *link of* v (in H) is the graph $L_v = (V \smallsetminus v, \{e \smallsetminus v : v \in e \in E\})$. For $x, y \in V$, the neighbourhood of x and y (in H) is the set $N(xy) = \{z \in V : xyz \in E\}$. For positive integers ℓ, k and a hypergraph F on k vertices, denote the *ℓ-blow-up of F* by $F(\ell)$. This is the

k-partite hypergraph $F(\ell) = (V, E)$ with $V = V_1 \dot\cup \ldots \dot\cup V_k$, $|V_i| = \ell$ for $1 \leqslant i \leqslant k$, and $E = \{v_{i_1}v_{i_2}v_{i_3} : v_{i_j} \in V_{i_j} \text{ and } i_1i_2i_3 \in E(F)\}$.

In their seminal paper, Mubayi and Zhao [10] proved the following supersaturation result for the codegree Turán density.

Proposition 2.1 (Mubayi and Zhao [10])**.** *For every hypergraph F and $\varepsilon > 0$, there are n_0 and $\delta > 0$ such that every hypergraph H on $n \geqslant n_0$ vertices with $\delta_2(H) \geqslant (\gamma(F) + \varepsilon)n$ contains at least $\delta n^{v(F)}$ copies of F. Consequently, for every positive integer ℓ, $\gamma(F) = \gamma(F(\ell))$.* □

Proof of Theorem 1.1. We begin by noting that it is enough to show that $\gamma(C_5^-) = 0$. Indeed, we shall prove by induction that $\gamma(C_\ell^-) = 0$ for every $\ell \geqslant 5$. For $\ell = 6$, the result follows since C_6^- is a subgraph of $C_3(2)$. Hence, by Proposition 2.1, we have $\gamma(C_6^-) \leqslant \gamma(C_3(2)) = \gamma(C_3) = 0$. For $\ell = 7$, note that C_7^- is a subgraph of $C_5^-(2)$. To see that, let $v_1, \ldots, v_5$ be the vertices of a C_5^- with edge set $\{v_iv_{i+1}v_{i+2} : i \neq 4\}$, where the indices are taken modulo 5. Now add one copy v_2' of v_2 and one copy v_3' of v_3. Then $v_1v_3v_2v_4v_3'v_5v_2'$ is the cyclic ordering of a C_7^- with the missing edge being $v_3'v_5v_2'$. Therefore, if $\gamma(C_5^-) = 0$, then, by Proposition 2.1, we have $\gamma(C_7^-) = 0$. Finally, for $\ell \geqslant 8$, $\gamma(C_\ell^-) = 0$ follows by induction using the same argument and observing that C_ℓ^- is a subgraph of $C_{\ell-3}^-(2)$.

Given $\varepsilon \in (0, 1)$, consider a hypergraph $H = (V, E)$ on $n \geqslant \left(\frac{2}{\varepsilon}\right)^{5/\varepsilon^2+2}$ vertices with $\delta_2(H) \geqslant \varepsilon n$. We claim that H contains a copy of a C_5^-.

Given $v, b \in V$, $S \subseteq V$, and $P \subseteq (V \smallsetminus S)^2$, we say that (v, S, b, P) is a *nice picture* if it satisfies the following:

(i) $S \subseteq N_{L_v}(b)$, where $N_{L_v}(b)$ is the neighborhood of b in the link L_v.
(ii) For every vertex $u \in S$ and ordered pair $(x, y) \in P$, the sequence $ubxy$ is a path of length 3 in L_v.

Note that if (v, S, b, P) is a nice picture and there exists $u \in S$ and $(x, y) \in P$ such that $uxy \in E$, then $ubvxy$ is a copy of C_5^- (with the missing edge being yub)

To find such a copy of C_5^- in H, we are going to construct a sequence of nested sets $S_t \subseteq S_{t-1} \subseteq \ldots \subseteq S_0$, where $t = 5/\varepsilon^2 + 1$, and nice pictures (v_i, S_i, b_i, P_i) satisfying $v_i \in S_{i-1}$, $|S_i| \geqslant \left(\frac{\varepsilon}{2}\right)^{i+1} n \geqslant 1$ and $|P_i| \geqslant \varepsilon^2n^2/5$ for $1 \leqslant i \leqslant t$. Suppose that such a sequence exists. Then by the pigeonhole principle, there exist two indices $i, j \in [t]$ such that $P_i \cap P_j \neq \varnothing$ and $i < j$. Let (x, y) be an element of $P_i \cap P_j$. Hence, we obtain a nice picture (v_i, S_i, b_i, P_i), $v_j \in S_i$ and $(x, y) \in P_i$ such that $v_jxy \in E$ (since xy is an edge in L_{v_j}). Consequently, $v_jb_iv_ixy$ is a copy of C_5^- in H.

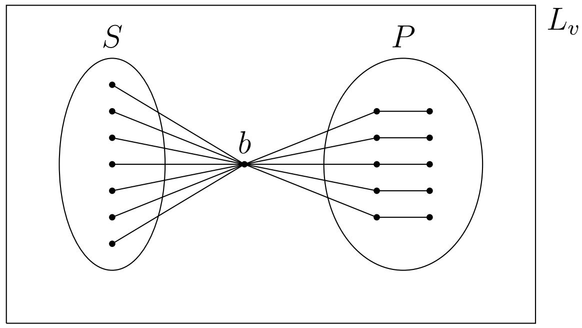

FIGURE 2.1. A nice picture (v, S, b, P)

It remains to prove that the sequence described above always exists. We construct it recursively. Let $S_0 \subseteq V$ be an arbitrary subset of size $\varepsilon n/2$. Suppose that we already constructed nice pictures (v_i, S_i, b_i, P_i) for $1 \leqslant i < k \leqslant t$ and now we want to construct (v_k, S_k, b_k, P_k). Pick $v_k \in S_{k-1}$ arbitrarily. The minimum codegree of H implies that $\delta(L_{v_k}) \geqslant \varepsilon n$ and thus for every $u \in S_{k-1}$, we have that $d_{L_{v_k}}(u) \geqslant \varepsilon n$. Observe that

$$\sum_{b \in V \smallsetminus v_k} |N_{L_{v_k}}(b) \cap S_{k-1}| = \sum_{u \in S_{k-1} \smallsetminus v_k} d_{L_{v_k}}(u) \geqslant \varepsilon n\big(|S_{k-1}| - 1\big) \geqslant \Big(\frac{\varepsilon}{2}\Big)^{k+1} n^2$$

and therefore, by an averaging argument there is a vertex $b_k \in V \smallsetminus v_k$ such that the subset $S_k := N_{L_{v_k}}(b_k) \cap S_{k-1} \subseteq S_{k-1}$ is of size at least $|S_k| \geqslant \big(\frac{\varepsilon}{2}\big)^{k+1} n$. Let P_k be all the pairs $(x, y) \in (V \smallsetminus S_k)^2$ such that for every vertex $v \in S_k$, the sequence v, b_k, x, y forms a path of length 3 in L_{v_k}. Since $|S_k| \leqslant \varepsilon n/2$ and $\delta(L_{v_k}) \geqslant \varepsilon n$, it is easy to see that $|P_k| \geqslant \varepsilon^2 n^2/5$. That is to say (v_k, S_k, b_k, P_k) is a nice picture satisfying the desired conditions. □

§3. CONCLUDING REMARKS

A famous result by Erdős [3] asserts that a hypergraph F satisfies $\pi(F) = 0$ if F is tripartite (i.e., $V(F) = X_1 \dot\cup X_2 \dot\cup X_3$ and for every $e \in E(F)$ we have $|e \cap X_i| = 1$ for every $i \in [3]$). Note that if H is tripartite, then every subgraph of H is tripartite as well and there are tripartite hypergraphs H with $|E(H)| = \frac{2}{9}\binom{|V(H)|}{3}$. Therefore, if F is not tripartite, then $\pi(F) \geqslant 2/9$. In other words, Erdős' result implies that there are no Turán densities in the interval $(0, 2/9)$. It would be interesting to understand the behaviour of the codegree Turán density in the range close to zero.

Question 3.1. Is it true that for every $\xi \in (0, 1]$, there exists a hypergraph F such that

$$0 < \gamma(F) \leqslant \xi \ ?$$

Mubayi and Zhao [10] answered this question affirmatively if we consider the codegree Turán density of a family of hypergraphs instead of a single hypergraph.

Since C_5^- is not tripartite, we have that $\pi(C_5^-) \geqslant 2/9$. The following construction attributed to Mubayi and Rödl (see e.g. [1]) provides a better lower bound. Let $H = (V, E)$ be a C_5^--free hypergraph on n vertices. Define a hypergraph $\widetilde{H}$ on $3n$ vertices with $V(\widetilde{H}) = V_1 \dot\cup V_2 \dot\cup V_3$ such that $\widetilde{H}[V_i] = H$ for every $i \in [3]$ plus all edges of the form $e = \{v_1, v_2, v_3\}$ with $v_i \in V_i$. Then, it is easy to check that $\widetilde{H}$ is also C_5^--free. We may recursively repeat this construction starting with H being a single edge and obtain an arbitrarily large C_5^--free hypergraph with density $1/4 - o(1)$. In fact, those hypergraphs are C_ℓ^--free for every ℓ not divisible by three. The following is a generalisation of a conjecture in [9].

Conjecture 3.2. *If $\ell \geqslant 5$ is not divisible by three, then $\pi(C_\ell^-) = \frac{1}{4}$.*

References

[1] J. Balogh, F. C. Clemen, and B. Lidickỳ, *Hypergraph Turán Problems in ℓ_2-Norm*, arXiv preprint arXiv:2108.10406 (2021). ↑1, 1, 3

[2] A. Czygrinow and B. Nagle, *A note on codegree problems for hypergraphs*, Bulletin of the Institute of Combinatorics and its Applications (2001), 63 - 69. ↑1

[3] P. Erdős, *On extremal problems of graphs and generalized graphs*, Israel J. Math. **2** (1964), 183–190, DOI 10.1007/BF02759942. ↑3

[4] P. Erdős, *Paul Turán, 1910–1976: his work in graph theory*, J. Graph Theory **1** (1977), no. 2, 97–101, DOI 10.1002/jgt.3190010204. MR441657 ↑1

[5] V. Falgas-Ravry, O. Pikhurko, E. Vaughan, and J. Volec, *The codegree threshold of K4-*, Electronic Notes in Discrete Mathematics **61** (2017), 407–413. ↑1

[6] R. Glebov, D. Král', and J. Volec, *A problem of Erdős and Sós on 3-graphs*, Israel J. Math. **211** (2016), no. 1, 349–366, DOI 10.1007/s11856-015-1267-4. MR3474967 ↑1

[7] P. Keevash, *Hypergraph Turán problems*, Surveys in combinatorics 2011, London Math. Soc. Lecture Note Ser., vol. 392, Cambridge Univ. Press, Cambridge, 2011, pp. 83–139. MR2866732 ↑1

[8] D. Mubayi, *The co-degree density of the Fano plane*, J. Combin. Theory Ser. B **95** (2005), no. 2, 333–337, DOI 10.1016/j.jctb.2005.06.001. MR2171370 ↑1

[9] D. Mubayi, O. Pikhurko, and Benny Sudakov, *Hypergraph Turán Problem: Some Open Questions*, https://homepages.warwick.ac.uk/∼maskat/Papers/TuranQuestions.pdf (2011). ↑3

[10] D. Mubayi and Y. Zhao, *Co-degree density of hypergraphs*, Journal of Combinatorial Theory, Series A **114** (2007), no. 6, 1118–1132. ↑1, 2, 2.1, 3

[11] Chr. Reiher, V. Rödl, and M. Schacht, *On a Turán problem in weakly quasirandom 3-uniform hypergraphs*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 5, 1139–1159, DOI 10.4171/JEMS/784. MR3790065 ↑1

[12] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452 (Hungarian, with German summary). MR18405 ↑1

Mathematics Department, University of Birmingham, UK
Email address: `s.piga@bham.ac.uk`

Mathematics Department, Emory University, USA
Email address: `mtsales@emory.edu`

Mathematics Department, California Institute of Technology, USA
Email address: `schuelke@caltech.edu`