RAMSEY NUMBERS WITH PRESCRIBED RATE OF GROWTH

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Abstract. Let $R(G)$ be the two-colour Ramsey number of a graph G. In this note, we prove that for any non-decreasing function $n \leqslant f(n) \leqslant R(K_n)$, there exists a sequence of connected graphs $(G_n)_{n\in\mathbb{N}}$, with $|V(G_n)| = n$ for all $n \geqslant 1$, such that $R(G_n) = \Theta(f(n))$. In contrast, we also show that an analogous statement does not hold for hypergraphs of uniformity at least 5.

We also use our techniques to answer a question posed by DeBiasio about the existence of sequences of graphs whose 2-colour Ramsey number is linear whereas their 3-colour Ramsey number has superlinear growth.

1. Introduction

For a graph G and $r \geqslant 2$, the *r-colour Ramsey number* $R_r(G)$ of G is the smallest number n such that every r-edge-colouring of the edges of the complete graph K_n contains a monochromatic copy of G, that is, a copy of G with all its edges in the same colour. For $r = 2$ we will simply write $R_2(G) = R(G)$ and refer to this as *the* Ramsey number of G. The most notorious open problem here is to determine the Ramsey number of cliques. The classical bounds on $R(K_n)$ by Erdős [Erd47] and Erdős and Szekeres [ES35] imply that $\sqrt{2}^n \leqslant R(K_n) \leqslant 4^n$, so $R(K_n)$ is exponential in n, but despite tremendous efforts its exact behaviour remains unknown.

In general, if a graph G on n vertices has m edges, then $2^{\Omega(m/n)} \leqslant R(G) \leqslant 2^{O(\sqrt{m})}$, where the lower bound follows from a probabilistic construction and the upper bound was shown by Sudakov [Sud11]. Given additional structure on G, there are many cases where we can even obtain $R(H) = O(n)$. This holds, for instance, for graphs with bounded maximum degree [Chv+83], bounded arrangeability [CS93], and bounded degeneracy [Lee17]. We recommend [CFS15] for a survey in the area.

As we have seen, the Ramsey number of an n-vertex graph can vary between linear and exponential in n. A natural question is thus to ask which values (between n and $R(K_n)$) can be attained as the Ramsey number of some n-vertex graph. The aim of this note is to study this question, and, in particular, to determine which functions $f : \mathbb{N} \to \mathbb{N}$, with $n \leqslant f(n) \leqslant R(K_n)$ for all $n \in \mathbb{N}$, are the rate of growth of the Ramsey numbers of some sequence of n-vertex graphs.

It is natural here to restrict our analysis to connected graphs. Note that after adding $n-r$ isolated vertices to an r-vertex graph H, we end with an n-vertex graph H satisfying $R(H') = \max\{n, R(H)\}$. This means that we can obtain values for the Ramsey number of n-vertex graphs which in essence correspond to the Ramsey number of r-vertex graphs; restricting to connected graphs rules out such constructions. Our first result is that every function can be attained as the rate of growth of some sequence of graphs, up to a multiplicative factor.

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Theorem 1. *There exists a positive constant C such that for every non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, with $n \leqslant f(n) \leqslant R(K_n)$, there exists a sequence of connected graphs $(G_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $|V(G_n)| = n$ and $f(n) \leqslant R(G_n) \leqslant Cf(n)$.*

In other words, we have $R(G_n) = \Theta(f(n))$, where the implicit constants do not depend on the function f. We remark that by a result of Burr and Erdős [BE76] on the Ramsey number of trees, it is known that every n-vertex connected graph G satisfies $R(G) \geqslant \lceil \frac{4}{3}n \rceil - 1$; thus taking the function $f(n) = \alpha n$ for any $1 \leqslant \alpha < 4/3$ shows that the conclusion of Theorem 1 cannot hold with $R(G_n) = (1 + o(1))f(n)$ instead. We discuss the structure of these 'gaps' further in Section 5.

Our second result concerns k-uniform hypergraphs. A *k-graph* H is a pair $H=(V,E)$ where V is the set of vertices of H and every edge $e \in E$ is a k-element subset of V. For $n \in \mathbb{N}$, the k-uniform clique on n vertices $K_n^{(k)}$ is the hypergraph consisting of n vertices such that every k-element subset of vertices is an edge. Given a k-graph H, the Ramsey number $R(H)$ of H is the smallest number n such that every red-blue colouring of the edges of $K_n^{(k)}$ yields a monochromatic copy of H.

We prove that an analogue of Theorem 1 fails for k-graphs if $k \geqslant 5$ (even without any kind of connectivity restrictions).

Theorem 2. *Let $k \geqslant 5$. There exists a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ with $n \leqslant f(n) \leqslant R(K_n^{(k)})$, such that for all $c, C > 0$ and any n_0, there is an $n > n_0$ such that*

$$R(H) \leqslant cf(n) \quad \text{or} \quad R(H) \geqslant Cf(n)$$

for every n-vertex k-graph H.

Using our techniques we can also answer a question posed by DeBiasio [DeB], who asked about the existence of a sequence G_n of graphs where $R_2(G_n)$ is linear whilst $R_3(G_n)$ is superlinear. Similar differences in behaviour depending on the number of colours have been observed before in infinite graphs (see [CDM20, Section 10.1]) and in k-graphs with $k \geqslant 3$ (see [CFR17]). We answer DeBiasio's question in the affirmative.

Theorem 3. *There exists a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs such that $|V(G_n)| = n$, $R_2(G_n) = O(n)$ and $R_3(G_n) = \Omega(n \log n)$.*

The graphs we construct for Theorem 3 have isolated vertices. If we insist on sequences of connected graphs, we can get the following.

Theorem 4. *There is a sequence $(G_n)_{n\in\mathbb{N}}$ of connected graphs such that $|V(G_n)| = n$, $R_2(G_n) = O(n \log n)$ and $R_3(G_n) = \Omega(n \log^2 n)$.*

2. Proof of Theorem 1

Conlon, Fox and Sudakov [CFS20, Lemma 5.5] proved that the Ramsey number of a dense graph cannot decrease by much under the deletion of one vertex. Recently, Wigderson [Wig22] investigated this phenomenon in sparser graphs. A graph on n vertices has density d if it has $d\binom{n}{2}$ edges.

Lemma 5 ([CFS20])**.** *There exists a function $g : [0,1] \to \mathbb{R}$ and a universal constant $c > 0$ such that for every graph H of density at least d and any graph H' obtained by deleting a single vertex from H, we have $R(H) \leqslant cg(d)R(H')$.*

In fact, in [CFS20, Lemma 5.5] it is claimed that the statement is true with $g(d) = \log(1/d)/d$, but such a function works for d bounded away from 1 only. Their proof can be trivially changed to obtain Lemma 5, which is all we need to show the following corollary.

Lemma 6. *There exist $c_1, c_2 > 0$ so that for any $n \geqslant 1$,*
- (i) $R(K_{n+1}) \leqslant c_1 R(K_n)$,
- (ii) $R(K_{n+1,n+1}) \leqslant c_2 R(K_{n,n})$.

We also need the Ramsey number of a path P_n with n edges, determined by Gerencsér and Gyarfás [GG67].

Lemma 7. *For every $n \geqslant 1$, $R(P_n) = \lceil (3n+1)/2 \rceil$.*

We shall also use a lower bound on the Ramsey number of complete bipartite graphs, which follows from a standard probabilistic construction.

Lemma 8. *For $t \geqslant 1$, $R(K_{t,t}) \geqslant 2^{t/2}$.*

The proof of Theorem 1 will be a direct consequence of the following two results.

Lemma 9. *Suppose $1 \leqslant t \leqslant n/2$. Let $H_{n,t}$ be the graph formed by taking a copy of $K_{t,t}$ and attaching a path on $n-2t$ new vertices. Then*

$$\max\{\lceil 3n/2 \rceil - 1, R(K_{t,t})\} \leqslant R(H_{n,t}) \leqslant 3R(K_{t,t})/2 + 3n\,.$$

Proof. Let $H = H_{n,t}$. Note that H contains both $K_{t,t}$ and a path P_{n-1} on $n-1$ edges. Since $R(P_{n-1}) = \lceil 3n/2 \rceil - 1$ by Theorem 7, the bound $\max\{R(K_{t,t}), R(P_{n-1})\} \leqslant R(H)$ is immediate. The goal is thus to prove the upper bound.

Let $N = 3R(K_{t,t})/2 + 3n$ and consider an arbitrary red-blue edge-colouring of K_N, we shall show that it contains a monochromatic copy of H. For a contradiction, assume it does not. By Lemma 7, there exists a monochromatic path P' in K_N of length at least $2N/3 \geqslant R(K_{t,t}) + 2n$, and we assume without loss of generality that P' is red. Let $P \subseteq P'$ be obtained after removing n vertices at one extreme of the path P'. Thus P has at least $R(K_{t,t}) + 2n - n \geqslant R(K_{t,t}) + n$ vertices. Let $S = V(P)$.

If S contains a red copy of $K_{t,t}$, then together with P' we can find in K_N a red path of length at least n joined to one of its vertices, a contradiction. Since $|S| \geqslant R(K_{t,t}) + n$, we can find a monochromatic copy of $K_{t,t}$ in S, which must be blue. In fact, we can greedily find vertex-disjoint blue copies of $K_{t,t}$ until less than $R(K_{t,t})$ vertices remain uncovered. Let $K^1, \ldots, K^s$ be the copies that we have found. Note that these copies together cover more than $|S| - R(K_{t,t}) \geqslant n$ vertices.

For all $1 \leqslant i \leqslant s$, let A_i, B_i be the two classes of K^i. Given $1 \leqslant i < s$, note that not all edges between B_i and A_{i+1} can be red, as that would yield a red monochromatic copy of $K_{t,t}$ in S. Therefore, there are blue edges $e_1, \ldots, e_{s-1}$ where each e_i has one endpoint $b_i \in B_i$ and other endpoint $a_{i+1} \in A_{i+1}$. Let $a_1 \in A_1$ be arbitrary. For all $1 \leqslant i < s$, take a blue path $P_i \subseteq K^i$ which spans $V(K^i)$ and has endpoints a_i and b_i. Thus, the concatenation $P_1 + e_1 + \cdots + P_{s-1} + e_{s-1}$, together with K^s, forms a blue copy of H, a final contradiction. □

Lemma 10. *Suppose $2 \leqslant t \leqslant n$. Let $J_{n,t}$ be the graph formed by taking a copy of K_t and attaching a path on $n-t$ new vertices. Then*

$$\max\{\lceil 3n/2 \rceil - 1, R(K_t)\} \leqslant R(J_{n,t}) \leqslant 3(R(K_t) + (t+1)n)/2\,.$$

Proof. Let $J = J_{n,t}$. Again, the bound $\max\{R(K_t), R(P_{n-1})\} \leqslant R(H)$ is trivial and thus we need to show the upper bound only.

Let $N = 3(R(K_t) + (t+1)n)/2$ and consider an arbitrary red-blue edge-colouring of K_N, we shall show that it contains a monochromatic copy of J. For a contradiction, assume it does not. By Lemma 7, there exists a monochromatic path P' in K_N of length at least $2N/3 \geqslant R(K_t) + (t+1)n$, and we assume without loss of generality that P' is red. Let $P \subseteq P'$ be obtained after removing n vertices at one extreme of the path P'. Thus P has at least $R(K_t) + nt$ vertices. Let $S = V(P)$.

If S contains a red copy of K_t, then together with P' we can find in K_N a red path of length at least n joined to one of its vertices, a contradiction. Since $|S| \geqslant R(K_t) + nt$, we can find a monochromatic copy of K_t in S, which must be blue. In fact, we can greedily find vertex-disjoint blue copies of K_t until less than $R(K_t)$ vertices remain uncovered. Let $Q^1, \dots, Q^s$ be the copies that were found. These copies, together, cover at least $|S| - R(K_t) \geqslant nt$ vertices of S, and thus we have $s \geqslant n$.

Define a *clique-path* P to be a sequence of vertex-disjoint blue cliques $Q^1, \dots, Q^l$ such that for each $1 \leqslant i < l$ there is a blue edge e_i between Q^i and Q^{i+1}, and the edges e_i are vertex-disjoint for all $1 \leqslant i < l$.

Claim 11. *There is a set of at most $t-1$ clique-paths that together cover all cliques $Q^1, \dots, Q^r$ exactly once.*

Proof. Suppose otherwise and let $P_1, \dots, P_{t-1}$ be $t-1$ clique-paths which use pairwise-disjoint sets of cliques and together use the maximum possible number of cliques. Let $P_0 = Q^0$ be a clique-path consisting of any clique not used by any P_i, and for each $1 \leqslant i \leqslant t-1$, let Q^i be an "end-clique" of each P_i. In each $Q^0, \dots, Q^{t-1}$, we select a vertex q_i which is not in any of the edges of the clique-paths (here we use $t \geqslant 2$). Since S contains no red K_t, there must be a blue edge between some pair $q_i q_j$. But then we can merge P_i and P_j into a longer clique-path using the blue edge $q_i q_j$, and thus we have found a set of $t-1$ clique-paths covering one more clique, a contradiction. □

Therefore, there is a clique-path which uses at least $s/(t-1) \geqslant s/t$ cliques. In such a clique-path, we can easily find a blue clique K_t together with a blue path which together use at least $t \cdot (s/t) = s \geqslant n$ vertices, as required. □

Now we are ready to prove the main result of this section.

Proof of Theorem 1. Let $f : \mathbb{N} \to \mathbb{N}$ be non-decreasing with $n \leqslant f(n) \leqslant R(K_n)$ for all $n \in \mathbb{N}$. Let c_1 and c_2 be the constants from Lemma 6 such that $R(K_t) \leqslant c_1 R(K_{t-1})$ and $R(K_{t,t}) \leqslant c_2 R(K_{t-1,t-1})$ for all $t \in \mathbb{N}$, and let c be a sufficiently large constant.

We will split the proof into two cases, depending on how large $f(n)$ is. In fact, the two ranges we consider are not disjoint, but they are enough to cover all possibilities between n and $R(K_n)$.

Case 1: $n \leqslant f(n) \leqslant 2^{n/8}$. Let t be the minimal number such that $R(K_{t,t}) > f(n)$. We note that by the choice of t, we have $R(K_{t-1,t-1}) \leqslant f(n) < R(K_{t,t})$. By Lemma 8, we have $2^{(t-1)/2} \leqslant f(n)$ and thus $t \leqslant 2\log_2(f(n)) + 1$. Since $f(n) \leqslant 2^{n/8}$, we certainly have $2t \leqslant n$. Let $G_n = H_{n,t}$ be the graph as in Lemma 9. Since $K_{t,t} \subseteq G_n$, we have $R(G_n) \geqslant R(K_{t,t}) > f(n)$. For the upper-bound, using Lemmas 6 and 9 we deduce that

$$R(G_n) \leqslant \frac{3}{2} R(K_{t,t}) + 3n \leqslant \frac{3}{2} c_2 f(n) + 3n \leqslant c f(n).$$

Case 2: $n^2 \leqslant f(n) \leqslant R(K_n)$. Take t minimal subject to $R(K_t) \geqslant f(n)$. Clearly, such t always exists and is at most n. Thus we have $R(K_{t-1}) < f(n) \leqslant R(K_t)$. Moreover, since $R(K_r) \geqslant 2^{r/2}$ holds for all r, we know that $t \leqslant \min\{n, 2\log_2 f(n)\}$. Let $G_n = J_{n,t}$ be as in Lemma 10 and note that, since $K_t \subseteq G_n$, we have $R(G_n) \geqslant R(K_t) > f(n)$. For the upper-bound, using Lemmas 10 and 6 we have

$$\begin{aligned} R(G_n) \leqslant \tfrac{3}{2}(R(K_t) + (t+1)n) &\leqslant \tfrac{3}{2}(c_1 f(n) + (t+1)n) \\ &\leqslant c f(n), \end{aligned}$$

where the last inequality follows from $(t+1)n \leqslant 3n \log_2 f(n) \leqslant 6n^2 \leqslant 6 f(n)$. □

3. Proof of Theorem 2

For k-graphs, the so-called 'stepping-up lemma' by Erdős, Hajnal, and Rado [EHR65] allows us to deduce a tower-type lower bound for the Ramsey number $R(K_n^{(k)})$ for every $k \geqslant 3$, namely

$$2^{an^2} \leqslant \log^{(k-2)}(R(K_n^{(k)})), \tag{1}$$

where $a > 0$ is a constant depending only on k and $\log^{(i)}(\cdot)$ denotes the ith iterated logarithm.

Proof of Theorem 2. Let $k \geqslant 5$. We find a function $g \colon \mathbb{N} \to \mathbb{N}$, with $n \leqslant g(n) \leqslant R(K_n^{(k)})$ as follows. For every $n \in \mathbb{N}$, let $I_n = [\log n, \log R(K_n^{(k)})]$ be an interval in $\mathbb{R}$. Note that, since $k \geqslant 5$, inequality (1) implies that

$$\log R(K_n^{(k)}) - \log n \geqslant 2^{2^{an}} - \log n.$$

Since the number of k-graphs on n vertices is at most 2^{n^k}, by averaging we find a sub-interval $I'_n \subseteq I_n$ which does not contain $\log R(H)$ for any n-vertex k-graph H, and such that I'_n has length at least

$$\frac{2^{2^{an}} - \log n}{2^{n^k}} \geqslant 2n + 1,$$

where we used that n is sufficiently large. By passing to a sub-interval we might assume that I'_n has exactly $2n+1$ elements. Let $m_n \in I'_n$ be the middle point of I'_n. Then, for large n and every n-vertex k-graph H, we have

$$\log R(H) \leqslant m_n - n \qquad \text{or} \qquad \log R(H) \geqslant m_n + n. \tag{2}$$

Let $g : \mathbb{N} \to \mathbb{N}$ be defined by $g(n) = 2^{m_n}$. Since $m_n \in I_n$ we have $n \leqslant g(n) \leqslant R(K_n^{(k)})$. Then, due to (2) we deduce that for every n and every n-vertex k-graph H,

$$R(H) \leqslant 2^{-n} g(n) \qquad \text{or} \qquad R(H) \geqslant 2^n g(n)\,.$$

In particular, for every two positive constants $c, C > 0$ and for every sufficiently large n, we have $R(H) < cg(n)$ or $R(H) > Cg(n)$, as required.

Note that g might decrease. To overcome this, we define $f : \mathbb{N} \to \mathbb{N}$ by setting $f(1) = g(1)$ and, for $n \geqslant 2$,

$$f(n) = \begin{cases} g(n) & \text{if } g(n) \geqslant f(n-1), \\ f(n-1) & \text{if } g(n) < f(n-1)\,. \end{cases}$$

Thus, it is straightforward to check that f is non-decreasing and satisfies the desired conditions. □

Notice that the proof of Theorem 2 relies on the fact that $\log R(K_n^k) = \omega(2^{n^k})$ for every $k \geqslant 5$. Erdős, Hajnal, and Rado [EHR65] conjectured that the lower bound in inequality (1) can be improved to $\log^{(k-1)}(R(K_n^{(k)}))$ for every $k \geqslant 3$, in which case our proof of Theorem 2 works for 4-uniform hypergraphs as well. The situation for 3-uniform hypergraphs is not clear, even if this conjecture were true.

4. Proof of Theorems 3 and 4

We shall use the following simple lemma. We remark that similar statements were obtained before, e.g., by Lefmann [Lef87].

Lemma 12. *For every graph G and connected $H \subseteq G$, we have*

$$R_3(G) \geqslant (\chi(H) - 1)(R_2(H) - 1) + 1\,.$$

Proof. Let $N = (\chi(H) - 1)(R_2(H) - 1)$. We construct a red-blue-green colouring of K_N as follows: partition $V(K_N)$ into $\chi(H) - 1$ sets $V_1, \dots, V_{\chi(H)-1}$ of size $R_2(H) - 1$ each. Inside each V_i use colours red and blue in such a way that the colouring does not contain a red-blue copy of H; and colour every other edge green.

This colouring does not contain a monochromatic copy of G. Indeed, a hypothetical such copy cannot be red or blue, as otherwise there must exist a red or blue copy of H. Since H is connected, such a copy of H must lie inside one of the sets V_i, but we have chosen the red-blue edges so that this does not happen. Also, there are no green copies of G_n, since the graph formed by the green edges is $(\chi(H) - 1)$-partite but $\chi(G) \geqslant \chi(H)$. We conclude that $R_3(G) > N$. □

Proof of Theorem 3. Given n, let t be the least integer such that $n \leqslant R_2(K_t)$. By choice, we have $R_2(K_{t-1}) < n$ and, by Lemma 6, we have $R_2(K_t) \leqslant c_1 R_2(K_{t-1}) < c_1 n$. Let G_n be the graph obtained from K_t by adding $n - t$ isolated vertices. Therefore, $|V(G_n)| = n$ and $R_2(G_n) = \max\{n, R_2(K_t)\} = R_2(K_t) < c_1 n = O(n)$. On the other hand, since $n \leqslant R_2(K_t) \leqslant 4^t$; by the choice of t we know that $t \geqslant \frac{1}{2}\log_2 n$ and therefore by Lemma 12 we have $R_3(G_n) = \Omega(n \log n)$. □

Proof of Theorem 4. Let $t = \log_2(n)/2$ and let $G_n = J_{n,t}$. Applying Lemma 10 we obtain a constant $C > 0$ such that

$$R(G_n) \leqslant \frac{3}{2}(R(K_t) + (t+1)n) \leqslant Cn\log(n)\,,$$

where we use that $R(K_t) \leqslant 4^t$. Since $\chi(G_n) = \Omega(\log n)$, then by Lemma 12 we have $R_3(G_n) > (\chi(G_n) - 1)(R_2(G_n) - 1) = \Omega(n \log^2 n)$, as required. □

5. Concluding remarks

For $n \in \mathbb{N}$, let us consider the sets

$$\begin{aligned}
\mathcal{R}_n &= \{R(G)\colon |V(G)| = n\},\\
\mathcal{R}_n^\circ &= \{R(G)\colon G \text{ does not contain isolated vertices and } |V(G)| = n\}, \quad \text{and}\\
\mathcal{R}_n^{\mathfrak{c}} &= \{R(G)\colon G \text{ is connected and } |V(G)| = n\}.
\end{aligned}$$

It is clear that $\mathcal{R}_n^{\mathfrak{c}} \subseteq \mathcal{R}_n^\circ \subseteq \mathcal{R}_n \subseteq [n, R(K_n)]$. Observe that $n \in \mathcal{R}_n$ since $R(\overline{K_n}) = n$, where $\overline{K_n}$ corresponds to an independent set on n vertices. Furthermore, consider a disjoint union of two stars $\Sigma_{a,b} = K_{1,a} \cup K_{1,b}$. A result due to Grossman [Gro79] implies that $R(\Sigma_{a,a-i}) = 3a - 2i$ for $i \in \{0, 1, 2\}$. Thus, by adding $n - (2a - i + 2)$ extra isolated vertices to $\Sigma_{a,a-i}$ and letting the value of a vary from $\lfloor n/3 \rfloor$ to $\lfloor (n-2)/2 \rfloor$, we can deduce that $[n, \lfloor \frac{3(n-2)}{2} \rfloor - 3] \subseteq \mathcal{R}_n$. Other families of sparse graphs can also be used to show other inclusions of this kind.

As mentioned in the introduction, $R(G) \geqslant \lceil \frac{4}{3}n \rceil - 1$ holds for every connected graph G on n vertices, and this bound is tight. In particular, it implies that

$$\mathcal{R}_n^{\mathfrak{c}} \subseteq \left[\left\lceil \tfrac{4}{3}n \right\rceil - 1, R(K_n)\right].$$

In a similar fashion, Burr and Erdős [BE76] showed that $R(G) \geqslant n + \log n - O(\log\log n)$ holds for every $G \in \mathcal{R}_n^\circ$, which is almost tight as shown by Csákány and Komlós [CK99]. It would be interesting to get a better understanding of the structures of $\mathcal{R}_n$, $\mathcal{R}_n^\circ$, and $\mathcal{R}_n^{\mathfrak{c}}$.

Given a constant $c > 1$, we say that $a \in [n, R(K_n)]$ is a *c-gap for $\mathcal{R}_n^{\mathfrak{c}}$* if $[a, ca] \cap \mathcal{R}_n^{\mathfrak{c}} = \emptyset$. It is not difficult to see that Theorem 1 is equivalent to the existence of a constant $c \geqslant 1$ for which $\mathcal{R}_n^{\mathfrak{c}}$ has no c-gaps for every sufficiently large n. In this direction a proper (but non-empty) subset of the authors of this paper believe that the answer to the following question should be affirmative.

Question 13. Does there exist an $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$

$$\mathcal{R}_n = [n, R(K_n)] \quad \text{and} \quad \mathcal{R}_n^{\mathfrak{c}} = \left[\lceil \tfrac{4}{3}n \rceil - 1, R(K_n)\right]\,?$$

Observe that the first equality would imply that for every function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ with $n \leqslant f(n) \leqslant R(K_n)$ there is a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ such that $f(n) = R(G_n)$. An analogous statement would hold for connected graphs if the second identity was true.

Finally, observe that the proof of the first case of Theorem 1 can be modified by replacing the rôle of $K_{t,t}$ with a complete k-partite graph $K_{t,\dots,t}$. For this, we need of course that $n \geqslant k$. In this way, we may ensure that the graphs in the sequence $(G_n)_{n\in\mathbb{N}}$ have a large chromatic number, at least for sufficiently large n.

Theorem 14. *For every $k \geqslant 2$, there are positive constants c, C, and $n_0 \in \mathbb{N}$ such that for every non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, with $n \leqslant f(n) \leqslant R(K_n)$, there is a sequence of connected graphs $(G_n)_{n\in\mathbb{N}}$ with $|V(G_n)| = n$ such that $cf(n) \leqslant R(G_n) \leqslant Cf(n)$ for all $n \in \mathbb{N}$. Moreover, $\chi(G_n) \geqslant k$ for every $n \geqslant n_0$.*

It would be interesting to ensure other properties for the graphs in this sequence. In particular, we believe the graphs can also be taken to have large connectivity.

Conjecture 15. For every $k \geqslant 2$ and for every non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ with $n \leqslant f(n) \leqslant R(K_n)$ there is a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ with $|V(G_n)| = n$ such that $R(G_n) = \Theta(f(n))$, and G_n is k-connected for all n sufficiently large.

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References

[BE76] S. A. Burr and P. Erdős. "Extremal Ramsey theory for graphs". *Utilitas Math.* 9 (1976), 247–258. URL: `https://www.renyi.hu/~p_erdos/1976-13.pdf`.

[CDM20] Jan Corsten, Louis DeBiasio, and Paul McKenney. "Density of monochromatic infinite subgraphs II". Preprint (2020). DOI: `10.48550/ARXIV.2007.14277`.

[CFR17] David Conlon, Jacob Fox, and Vojtěch Rödl. "Hedgehogs are not colour blind". *J. Comb.* 8.3 (2017), 475–485. DOI: `10.4310/JOC.2017.v8.n3.a4`.

[CFS15] David Conlon, Jacob Fox, and Benny Sudakov. "Recent developments in graph Ramsey theory". *Surveys in Combinatorics 2015.* Vol. 424. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2015, 49–118. URL: `https://www.cambridge.org/core/books/abs/surveys-in-combinatorics-2015/recent-developments-in-graph-ramsey-theory/82997361A922AF213D13AEFFB46AF519`.

[CFS20] David Conlon, Jacob Fox, and Benny Sudakov. "Short proofs of some extremal results III". *Random Structures Algorithms* 57.4 (2020), 958–982. DOI: `10.1002/rsa.20953`.

[Chv+83] C. Chvatál, V. Rödl, E. Szemerédi, and W.T. Trotter. "The Ramsey number of a graph with bounded maximum degree". *Journal of Combinatorial Theory, Series B* 34.3 (1983), 239–243. DOI: `https://doi.org/10.1016/0095-8956(83)90037-0`.

[CK99] Rita Csákány and János Komlós. "The smallest Ramsey numbers". *Discrete Math.* 199.1-3 (1999), 193–199. DOI: `10.1016/S0012-365X(98)00300-8`.

[CS93] Guantao Chen and R. H. Schelp. "Graphs with linearly bounded Ramsey numbers". *J. Combin. Theory Ser. B* 57.1 (1993), 138–149. DOI: `10.1006/jctb.1993.1012`.

[DeB] Louis DeBiasio. "Graphs with linear Ramsey number for two colors, but super-linear Ramsey number for three colors?" MathOverflow. URL: `https://mathoverflow.net/q/385498`.

[EHR65] P. Erdős, A. Hajnal, and R. Rado. "Partition relations for cardinal numbers". *Acta Math. Acad. Sci. Hungar.* 16 (1965), 93–196. DOI: `10.1007/BF01886396`.

[Erd47] P. Erdös. "Some remarks on the theory of graphs". *Bull. Amer. Math. Soc.* 53 (1947), 292–294. DOI: `10.1090/S0002-9904-1947-08785-1`.

[ES35] P. Erdös and G. Szekeres. "A combinatorial problem in geometry". *Compositio Math.* 2 (1935), 463–470. URL: `http://www.numdam.org/item?id=CM_1935__2__463_0`.

[GG67] L. Gerencsér and A. Gyárfás. "On Ramsey-type problems". *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* 10 (1967), 167–170. URL: https://users.renyi.hu/~gyarfas/Cikkek/01_GerencserGyarfas_OnRamseyTypeProblems.pdf.

[Gro79] Jerrold W. Grossman. "The Ramsey numbers of the union of two stars". *Utilitas Math.* 16 (1979), 271–279.

[Lee17] Choongbum Lee. "Ramsey numbers of degenerate graphs". *Ann. of Math. (2)* 185.3 (2017), 791–829. DOI: 10.4007/annals.2017.185.3.2.

[Lef87] H. Lefmann. "A note on Ramsey numbers". *Studia Sci. Math. Hungar.* 22.1-4 (1987), 445–446. URL: http://real-j.mtak.hu/5464/.

[Sud11] Benny Sudakov. "A conjecture of Erdős on graph Ramsey numbers". *Adv. Math.* 227.1 (2011), 601–609. DOI: 10.1016/j.aim.2011.02.004.

[Wig22] Yuval Wigderson. "Ramsey numbers upon vertex deletion". Preprint (2022). DOI: 10.48550/ARXIV.2208.11181.

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