(4.1) Integrate $\int p^5 \ln(p)\, dp$

We can use the formula

$$\int u\, dv = uv - \int v\, du$$

otherwise know as IBP. Using the acronym L.I.A.T.E. to pick our u and v, our values for them are

$$u = \ln(p)$$
$$dv = p^5\, dp$$

Then, we substitute our values into the IBP formula.

$$\begin{aligned}\int u\, dv &= uv - \int v\, du \\ &= \ln(p)\frac{p^{5+1}}{5+1} - \int \frac{p^{5+1}}{5+1} \cdot \frac{1}{p}\, dp \\ &= \frac{\ln(p)p^6}{6} - \int \frac{p^5}{6}\, dp \\ &= \frac{\ln(p)p^6}{6} - \frac{1}{6}\int p^5\, dp\end{aligned}$$

Now we should evaluate the indefinite integral with basic antiderivatives.

$$\begin{aligned}\int p^5\, dp &= \frac{p^{5+1}}{5+1} + C \\ &= \frac{p^6}{6} + C\end{aligned}$$

Substitution back into our previous expression sums as

$$\frac{\ln(p)p^6}{6} - \frac{1}{6} \cdot \frac{p^6}{6} = \frac{\ln(p)p^6}{6} - \frac{p^6}{36}$$

Thus, our final answer is $\frac{\ln(p)p^6}{6} - \frac{p^6}{36}$.

Problem 2 on next page. →

(4.2) Use substitution on the integral $\int \frac{1}{\sqrt{x+2}+x}$ to express the integrand as a rational function, then evaluate the integral.

Let $u = \sqrt{x+2}$. Thus, $du = \frac{1}{2\sqrt{x+2}}\,dx$. Now, we can use algebra to apply the Substitution Rule.

$$\begin{aligned}
\int \frac{1}{\sqrt{x+2}+x} &= \int \frac{2\sqrt{x+2}}{2\sqrt{x+2}} \cdot \frac{1}{\sqrt{x+2}+x} \\
&= \int \frac{2\sqrt{x+2}}{\sqrt{x+2}+x} \cdot \frac{dx}{2\sqrt{x+2}} \\
&= \int \frac{2\sqrt{x+2}}{\sqrt{x+2}+x}\,du \\
&= 2\int \frac{u}{u+x}\,du
\end{aligned}$$

To get rid of x in the denominator we can manipulate the equation of u

$$\begin{aligned}
u &= \sqrt{x+2} \\
u^2 &= x+2 \\
u^2 - 2 &= x
\end{aligned}$$

Therefore,

$$2\int \frac{u}{u+x}\,du = 2\int \frac{u}{u^2+u-2}\,du$$

Factoring the denominator leaves $(x+2)(x-1)$. Notice that the integrand now can be used for partial fraction decomposition.

$$\begin{aligned}
\frac{u}{(u+2)(u-1)} &= \frac{A}{u+2} + \frac{B}{u-1} \\
&= A(u-1) + B(u+2) \\
&= (A+B)u + (2B-A) = (1)u + 0
\end{aligned}$$

Thus we must solve the system of equations

$$\begin{cases} A+B=1 \\ 2B-A=0 \end{cases}$$

Thus we use use substitution to solve our system of equations.

$$\begin{aligned}
A &= 2B \\
A+B &= 2B+B \\
3B &= 1 \\
B &= \frac{1}{3}
\end{aligned}$$

We can easily solve to get $A = \frac{2}{3}$. Substituting this decomposition into the integrand changes the integral into one that can be easily manipulated.

$$\begin{aligned}
2\int \frac{u}{u^2+u-2}\,du &= 2\int \frac{2}{3(u+2)} + \frac{1}{3(u-1)}\,du \\
t &= \frac{4}{3}\int \frac{1}{u+2} + \frac{2}{3}\int \frac{1}{u-1} \\
&= \frac{4}{3}\ln(|u|) + \frac{2}{3}\ln(|u|) + C \\
&= \frac{4}{3}\ln(|\sqrt{x+2}|) + \frac{2}{3}\ln(|\sqrt{x+2}|) + C
\end{aligned}$$

Thus we have reached our final answer.

$$\int \frac{1}{\sqrt{x+2}+x} = \frac{4}{3}\ln(|\sqrt{x+2}|) + \frac{2}{3}\ln(|\sqrt{x+2}|) + C$$

Problem 3 on Page 4. →

(4.3) Let f be a continuous, increasing function, and let g be the inverse of f.

(a) Use IBP to show $\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$

(b) Show that $\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy.$

(c) Suppose $f(x) > 0$ and $0 < a < b$, as shown in the diagram below. Reproduce the diagram, the shade/label it to give a geometric interpretation of (b).

(a) Using IBP in the integral allows us to set our $u = f(x)$ and $dv = 1\,dx$. Now if we substitute the left hand side into the IBP formula we get the equation

$$\begin{aligned}\int f(x)\,dx &= f(x)\cdot\frac{x^{0+1}}{0+1} - \int \frac{x^{0+1}}{0+1}f'(x)\,dx\\ &= xf(x) - \int xf'(x)\,dx\end{aligned}$$

Thus we have successfully converted LHS to equal RHS and have shown that $\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$.

(b) In the previous part, we calculated the indefinite integral of $f(x)$ using IBP. We can now use that and then apply the Evaluation Theorem to the indefinite integral.

$$\begin{aligned}\left[xf(x) - \int xf'(x)\,dx\right]_a^b &= [xf(x)]_a^b - \left[\int xf'(x)\,dx\right]_a^b\\ &= (bf(b) - af(a)) - \int_a^b xf'(x)\,dx\end{aligned}$$

Now, we substitute in y as $y = f(x)$. This makes it so $dy = f'(x)dx$. Thus the last integral in our equation is now.

$$\int x\,dy$$

We now have to remove the x from the integrand and adjust the bounds. Since $y = f(x)$, we can derive that $x = f^{-1}(y)$. To adjust the bounds by applying y to them. Thus, the integral is now

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,dy$$

Looking back at the problem, it states that g is the inverse of f, so doing one more simplification, the integral is now,

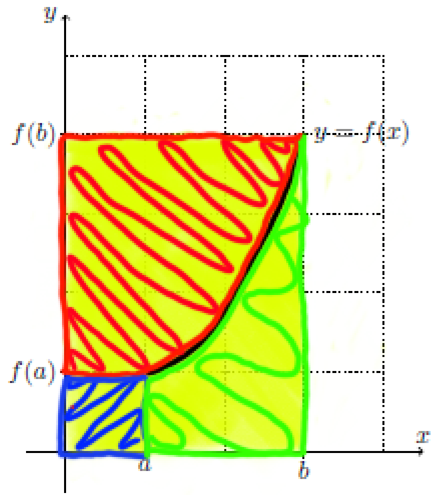
$$\int_{f(a)}^{f(b)} g(y)\,dy$$

Substituting the new integral back into the original equation yields

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy$$

Thus we have shown that the equation on (b) is true.

(c) My graph is below.



Now I will talk about the different aspects of the graph and why they are shaded a certain way. The yellow highlighted area is the area that returned with $bf(b)$. This is because it is a rectangle with sides of length b and $f(b)$. The next term is $af(a)$. This term is the red shaded area. The area in the blue is the integral $\displaystyle\int_{f(a)}^{f(b)} g(y)\,dy$. The area in the green is the remaining area after subtracting the blue and the red from the yellow area. This area is equivalent to $\displaystyle\int_a^b f(x)\,dx$ which shows that (b) is correct geometrically.

Professional Problem on Page 6. →

(4.4) Professional Problem:

(a) Use the reduction formula to show that $\int_{-\pi}^{\pi} \sin^n x\,dx = \dfrac{n-1}{n} \int_{-\pi}^{\pi} \sin^{n-2} x\,dx$.

(b) Use part (a) to evaluate $\int_{-\pi}^{\pi} \sin^2 x\,dx$.

(c) Use induction to prove $\int_{-\pi}^{\pi} \sin^{2n} x\,dx = \frac{(2n-1)\ldots 5\cdot 3\cdot 1}{(2n)\ldots 6\cdot 4\cdot 2}(2\pi)$.

(a) The reduction formula is

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

Using the Evaluation Theorem, we can use it on the reduction formula and simplify. This results in an expression like

$$\begin{aligned}\left[-\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx\right]_{-\pi}^{\pi} &= \left[-\frac{1}{n}\cos x\sin^{n-1} x\right]_{\pi}^{\pi} + \left[\frac{n-1}{n}\int \sin^{n-2} x\,dx\right]_{-\pi}^{\pi} \\ &= -\frac{\cos(-\pi)\sin^{n-1}(-\pi)}{n} + \frac{\cos\pi \sin^{n-1}\pi}{n} i + \frac{n-1}{n}\int_{-\pi}^{\pi} \sin^{n-2} x\,dx \\ &= \frac{1}{n}\cdot 0 + \frac{1}{n}\cdot 0 + \frac{n-1}{n}\int_{-\pi}^{\pi}\sin^{n-2} x\,dx \\ &= \frac{n-1}{n}\int_{-\pi}^{\pi}\sin^{n-2} x\,dx\end{aligned}$$

We have shown that $\int_{-\pi}^{\pi} \sin^n x\,dx = \dfrac{n-1}{n}\int_{-\pi}^{\pi}\sin^{n-2} x\,dx$.

(b) Part (a) states that the power of the sine in the original integrand is equal to n. Therefore, a $\sin^2 x$ in the integrand means $n = 2$. We can now plug n into the simplified reduction formula now to form the integral

$$\frac{2-1}{2}\int_{-\pi}^{\pi}\sin^{2-2} x\,dx$$

Simplifying this will form the expressions

$$\begin{aligned}\frac{1}{2}\int_{-\pi}^{\pi}\sin^0 x\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} 1\,dx \\ &= \frac{1}{2}\cdot(\pi - (-\pi)) \\ &= \pi\end{aligned}$$

Simplifying has given us the answer π. The integral $\int_{-\pi}^{\pi}\sin^2 x\,dx = \pi$.

(c) Let us start our proof by induction by setting out the base case where $n = 1$. Solving this using the formula provides the solution,

$$\frac{1}{2}(2\pi) = \pi$$

We know this is true since we solved for $n = 1$ in part (b). For our induction hypothesis let us assume that the prior formula holds true for all $n = k$. If the equation is true for all $k+1$, then

$$int_{-\pi}^{\pi}\sin^{2k+2} x\,dx = \int_{-\pi}^{\pi}\sin^{2k} x\,dx \cdot \frac{2k+1}{2k+2} \tag{1}$$

If we use the equation we proved was true in part (a), we can change the left hand side to

$$\frac{2k+2-1}{2k+2}\int_{-\pi}^{\pi}\sin^{2k+2-2} x\,dx = \frac{2k+1}{2k+2}\int_{-\pi}^{\pi}\sin^{2k} x\,dx$$

Now that we have shown that both sides of equation (1) are equal, we have proved that $\int_{-\pi}^{\pi}\sin^{2n} x\,dx = \frac{(2n-1)\ldots 5\cdot 3\cdot 1}{(2n)\ldots 6\cdot 4\cdot 2}(2\pi)$.