UMTYMP Calculus I Name:

Sequences and Series Toolbox

During the sections on Sequences and Series you learned *a lot* of definitions and *a lot* of theorems. This workbook will help you to keep track of your growing toolbox! Fill this out as a first step towards reviewing for the exam. Answers can all be found in your textbook. Keep it nearby and use it during your exam.

Due: on Gradescope by 4pm on 5/10/21. This will be counted as your 8 point professional problem for the Section 11.9 assignment.

Definitions (from §11.1 through §11.6)

Sequence (p.690) A sequence can be thought of as a list of written in a : $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$

Convergent Sequence (p.692) A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.
If $\lim_{n\to\infty} a_n$ exists, we say the sequence (or is). Otherwise, we say the sequence (or is **divergent**).

Diverges to infinity (p.): $\lim_{n\to} a_n = \infty$ means that for every positive number M there is an integer N such that

 whenever

Increasing sequence (p.) A sequence $\{a_n\}$ is called **increasing** if for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$.

*What if the inequality is only true for $n \geq 10$? We can call that "**eventually increasing.**" Think about why that is still a useful property.*

It is called **decreasing** if for all $n \geq 1$.

*What if the inequality is only true for $n \geq 10$? We can call that " **eventually decreasing.**" Think about why that is still a useful property.*

It is called **monotonic** if it is either or .
How can the ratio $\dfrac{a_n}{a_{n+1}}$ help you determine if the sequence is monotonic?

Warning: The book's definitions of "increasing/decreasing" are really "*strictly* increasing/decreasing." In our class, "increasing/decreasing" means "monotonically increasing/decreasing," where $a_n \leq a_{n+1}$ or $a_n \geq a_{n+1}$ for all n. See the lecture videos/notes for details. If we mean *strictly*, we'll say so.

Factorial For any integer $n \geq 1$, the factorial $n! =$.

Bounded sequence (p.697) A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

for all $n \geq 1$

It is **bounded below** if there is a number m such that

for all $n \geq 1$

If it is bounded above *and* below, then $\{a_n\}$ is a .

Infinite series (p.704) If we try to add the terms of an infinite sequence $\{a_n\}$ we get an expression of the form

$$a_1 + a_2 + \cdots + a_n + \cdots$$

which is called an (or just a) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

Partial sum (p. 705) Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i =$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = \quad .$$

The number s is called the of the series. Otherwise, the series is called .

Geometric series (p.706) The geometric series is $\sum_{n=1}^{\infty}$. It is convergent only when .

Harmonic series (p.708) $\sum_{n=1}^{\infty}$ =

What is the harmonic series famous for?

***p*-series** (p.) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and divergent if .

Alternating series (p.727) An **alternating series** is a series whose terms are alternately positive and negative. We write it as

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = \qquad (b_n > 0).$$

How is the notation b_n in an alternating series different from a_n in other series?

Absolute convergence, conditional convergence (pp.732–733) A series $\sum a_n$ is called **absolutely convergent** if the series of $\sum$ is convergent.
It is called **conditionally convergent** if $\sum a_n$ is convergent but not .

Additional review: Give an example of...

- A series which converges: .
- A series which diverges: .
- A series which converges absolutely: .
- A series which converges conditionally: .
- A sequence $\{a_n\}$ and a function $f(x)$ for which $a_n = f(n)$ for every $n \geq 1$, such that $\lim_{n\to\infty} a_n$ converges but $\lim_{x\to\infty} f(x)$ diverges: $a_n =$, $f(x) =$.
- A sequence $\{a_n\}$ which converges, but $\sum a_n$ diverges: .

More on the next pages!

Theorems for Sequences

These theorems are useful for determining if a *sequence* converges. **Do not use them on series!!**

(p.693) If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is any integer, then $\lim_{n\to\infty} a_n = L$.

Limit Laws for Sequences (p.693) If $\{a_n\}$ and $\{b_n\}$ are ______ sequences and c is a constant, then

1. $\lim_{n\to\infty}(a_n + b_n) =$ ______ $+$ ______
2. $\lim_{n\to\infty}(a_n - b_n) =$ ______ $-$ ______
3. $\lim_{n\to\infty}(c\, a_n) =$ ______ $\qquad \lim_{n\to\infty} c = c$
4. $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to} a_n) \cdot$ ______
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if ______
6. $\lim_{n\to\infty} a_n^p = [$ ______ $]^p$ if $p > 0$ and ______.

The ______ **Theorem** (p.694) If $a_n \le b_n \le c_n$ for $n \ge n_0$ and

$$\_\_\_\_\_\_ = \_\_\_\_\_\_ = L$$

then $\lim\limits_{n\to\infty} b_n = L$.

(p. ______) If $\lim_{n\to\infty} |a_n| =$ ______, then $\lim_{n\to\infty} a_n =$ ______.
Is this true for any limit?

r^n (p.696) The sequence $\{r^n\}$ is ______ if $-1 < r \le 1$ and ______ for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1 \\ 1 & \text{if} \quad \text{r=1} \end{cases}$$

Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.
If $\{a_n\}$ is increasing, it is bounded below by ______ *; if it is decreasing, it is bounded above by* ______ .
If an increasing sequence is bounded below, must it necessarily diverge?

Theorems for Series

The following theorems are useful for determining if a series converges. **Don't use them on sequences or integrals or anything other than a series!!** They are listed from easiest to use to hardest, not in the order they appear in the book. Always make sure your series satisfies all of the hypotheses, and that you know what the conclusion really says! §11.7 has some useful advice about when to use which theorem.

Constant Multiple / Sum / Difference (Limit) Laws for Series (p.709) If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) =$ ______ + ______

(iii) ______ $= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Don't use these if they don't apply. If $\sum a_n$ converges and $\sum b_n$ diverges, don't write $\sum(a_n + b_n) = \sum a_n + \sum b_n$.

(p. 708) If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n =$ ______.

The Test for Divergence (p.709) If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq$ ______, then the series $\sum_{n=1}^{\infty} a_n$ is ______.
If $\lim_{n\to\infty} a_n = 0$, *can the series diverge?*

Hypotheses: $\lim_{n\to\infty} a_n \neq 0$ or does not exist.

Conclusion: $\sum_{k=1}^{\infty} a_k$ **diverges**

*What can you **never** conclude from the test for divergence?*

Alternating series test (p.727) If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies
(i) $b_{n+1} \leq b_n$ for all n
(ii) $\lim_{n\to\infty} b_n = 0$
then the series is convergent.
Hypotheses:

______

Conclusion:

______

Don't get carried away and try to use this test on a non-alternating series!

Integral test(p.) Suppose f is a continuous, positive, and decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is . In other words:
(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is .
(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is .

Hypotheses:

Conclusion:

Come up with some examples to help you remember when you cannot use this test.

The Comparison Test (p.722) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is .
2. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is .

Hypotheses:

Conclusion:

Think about why we require $0 \leq a_n$ *and* $0 \leq b_n$.

Limit Comparison Test (p.) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c,$$

where c is a and $c > 0$, then either both series converge or both series diverge.

Hypotheses:

Conclusion:

If your ***comparison test*** *fails, try a* ***limit comparison test*** *instead with the same* b_n.

Absolute convergence (p.733) if a series $\sum a_n$ is absolutely convergent, then it is convergent.

Hypotheses:

Conclusion:

This lets you use the Comparison Tests because $|a_n| \geq 0$ automatically! But it does not help you if the series is only conditionally convergent.

Ratio test (p.734)

(i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is _____ _____ (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is _____ .

(iii) If If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Hypotheses:

Conclusion:

Warning: sometimes this test gives you no information at all! This sort of test is easiest to do when a_n involves things that cancel nicely in the ratio, such as 3^n and $n!$.

Root test(p.)

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is (and therefore).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is .

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Hypotheses:

Conclusion:

Warning: sometimes this test gives you no information at all! This test is useful when every part of a_n is raised to the power n.

The **Ratio Test** and **Root Test** give the same information about a series, because

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Write a short reminder to yourself about when it is worth choosing the Ratio Test, and when you should use the Root Test instead:

Power Series. Explain how to find the **radius of convergence** of a power series $\sum c_n(x-a)^n$ using the Ratio Test or Root test.

After finding the radius of convergence, how do you find the **interval of convergence**?