1. Show that $y = \frac{2}{3}e^x + e^{-2x}$ is a solution of the differential equation $y' + 2y = 2e^x$.

First, let us define $y = \frac{2}{3}e^x + e^{-2x}$ as Equation 1, or (1), $y' + 2y = 2e^x$ as Equation 2, or (2). Next, the value of y' needs to be found. Because both terms in the function are very similar to the natural exponential function e^x, this function is relatively simple to differentiate.

$$\begin{aligned} y &= \frac{2}{3}e^x + e^{-2x} \\ y' &= \frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}x}(x)e^x + \frac{\mathrm{d}}{\mathrm{d}x}(-2x)e^{-2x} \\ &= \frac{2}{3}e^x - 2e^{-2x} \end{aligned}$$

Substituting the values for y' and y that we found in the first step into Equation 2 provides the new equation:

$$\frac{2}{3}e^x - 2e^{-2x} + 2\left(\frac{2}{3}e^x + e^{-2x}\right) = 2e^x$$

To verify this equation, all that is needed is the simplification of the left hand side, which is done below.

$$\begin{aligned} \frac{2}{3}e^x - 2e^{-2x} + 2\left(\frac{2}{3}e^x + e^{-2x}\right) &= \frac{2}{3}e^x - 2e^{-2x} + \frac{4}{3}e^x + 2e^{-2x} \\ &= \frac{6}{3}e^x \\ &= 2e^x \end{aligned}$$

Thus, it has been shown that $y = \frac{2}{3}e^x + e^{-2x}$ is a solution of the differential equation $y' + 2y = 2e^x$.

2. Verify that $y = -t\cos t - t$ is a solution of the initial-value problem

$$t\frac{\mathrm{d}y}{\mathrm{d}t} = y + t^2 \sin t \qquad\qquad y(\pi) = 0$$

Again, the first step needs to be finding the value of y'. But before that, the initial value requirement needs to be addressed. This can be accomplished as follows:

$$\begin{aligned} y(\pi) &= 0 \\ -\pi\cos\pi - \pi &= 0 \\ &= \pi - \pi \\ &= 0 \end{aligned}$$

Therefore, the initial value is true and the function can be differentiated as follows.

$$\begin{aligned} y' &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(t\cos t + t\right) \\ &= -\left(\frac{\mathrm{d}}{\mathrm{d}t}(t)\cos t + \frac{\mathrm{d}}{\mathrm{d}t}(\cos t)t\right) - 1 \\ &= -\cos t + t\sin t - 1 \end{aligned}$$

Since the value of y' is now known, simple substitution is all that remains to verify the solution of the problem.

$$\begin{aligned} y + t^2\sin t &= t\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= t\left(-\cos t + t\sin t - 1\right) \\ &= -t\cos t + t^2\sin t - t \\ &= (-t\cos t - t) + t^2\sin t \\ &= y + t^2\sin t \end{aligned}$$

With this, $y = -t\cos t - t$ has been verified as a solution of the provided initial-value problem.

3. A solution is modeled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 1.2P\left(1 - \frac{P}{4200}\right)$$

(a) For what values is the population increasing?

(b) For what values is the population decreasing?

(c) What are the equilibrium solutions?

(a) This is a very simple problem. The population is increasing when the derivative is positive and vice versa. Therefore, all that needs to be found for this question are the values for which $\frac{\mathrm{d}P}{\mathrm{d}t}$ is greater than zero. The eye immediately goes to the expression inside the parentheses, $1 - \frac{P}{4200}$. It is easily seen that for the derivative to be positive, $P < 4200$. The other P in the differential equation creates the lower bound of 0. The values for which the population is increasing are

$$(0, 4200)$$

(b) Using a similar approach to the one used in (a), we just need to find the values of P for which the derivative is negative. First, the P outside of the parentheses ensures that a population below 0 will continually keep decreasing. The expression inside of the parentheses can be used to infer that a population above 4200 will start decreasing because at that point, $1 - \frac{P}{4200}$ will become negative. The values for which the population is decreasing are

$$(-\infty, 0) \cup (4200, \infty)$$

(c) We can take the phrase to equilibrium solutions to mean where the value of the derivative is a 0. This only occurs when one of the terms of the differential equation are equal to 0. In this case, the terms are P and $\left(1 - \frac{P}{4200}\right)$. Some quick mental math results in the values of 0 and 4200 and population values where the population is neither increasing nor decreasing.

4. A function $y(t)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^4 - 6y^3 + 5y^2$$

(a) What are the constant solutions of the equation?

(b) For what values of y is y increasing?

(c) For what values of y is y decreasing?

(a) Using the same approach as the last problem, we simply solve the right hand side for when derivative is in a certain range. For this part, we need to find the solutions for which $\frac{\mathrm{d}y}{\mathrm{d}t}$ is 0. This can be solved with some general algebra to yield

$$\begin{aligned} y^4 - 6y^3 + 5y^2 &= y^2(y^2 - 6y + 5) \\ &= y^2(y-5)(y-1) \end{aligned}$$

Now we can see that the constant solutions are at $y = 0, 5, 1$.

(b) To find when y is increasing, the factored form of the differential equation is useful in determining when the derivative is positive. If $y^2(y^2 - 6y + 5)$ is the standard form of the differential equation, y^2 can be disregarded because it is always positive. Thus, the only question remaining is when is the y value of the parabola above 0. Because the parabola is positive and the zeroes are at 5 and 1, it is inferred that all value that are not in the interval $(1, 5)$ are positive. Thus, y is increasing for the values of $(-\infty, 0) \cup (0, 1) \cup (5, \infty)$.

(c) Using the results from part (b) it is clear that the only values of y for which y is decreasing are $(1, 5)$.

5. Match the differential equations with the solution graphs labeled I-IV. Give reasons for your choices.