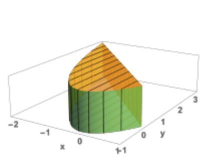
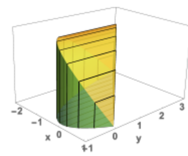
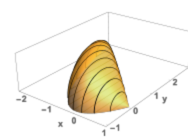
(8.1) Set up integrals for the volume of each of the solids below. The base of each solid is the region bounded by $y = 1 - x$ and $y = x^2 - 1$. The cross sections perpendicular to the x-axis are describe below



(a) Rectangles of height 2



(b) Squares



(c) Semicircles

(a) First we need to find the area function, $A(x)$ of a cross section of the solid. We know that the height of the cross sectional shape is 2, therefore

$$A(x) = 2 \cdot S$$

where S is the length of the base. Notice that the lines of the base are parallel to the y-axis. Therefore,

$$\begin{aligned} S &= (1 - x) - (x^2 - 1) && (1) \\ &= 2 - x - x^2 && (2) \end{aligned}$$

Thus our area function is

$$A(x) = 2(2 - x - x^2)$$

Next, we have to find the bounds of our integral which can be accomplished by finding the intersections between $y = 1 - x$ and $y = x^2 - 1$. Subtracting them gives the equation

$$\begin{aligned} 0 &= 2 - x - x^2 \\ &= -\left(x^2 + x - 2\right) \\ &= -(x + 2)(x - 1) \\ &= (x + 2)(x - 1) \end{aligned}$$

Thus, we get the x-values for intersection as $x = 1$, $x = -2$. Finally setting up the integral using the x-values as bounds, we get the integral

$$\int_{-2}^{1} \left(2x - 2x^2\right)\, dx$$

(b) The area function for a square is

$$A(x) = S^2$$

Using the our definition of S from (1) and our bounds from part (a), we can set up the integral

$$\int_{-2}^{1} \left(2 - x - x^2\right)^2\, dx$$

(c) Using the same method as for the last two parts, we determine that the area function for this is

$$A(x) = \frac{\pi S^2}{2}$$

Once more, we use the mathematics from part (a) to construct the integral

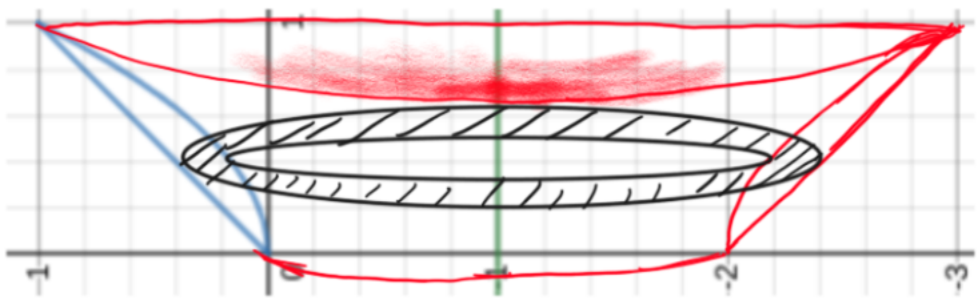
$$\int_{-2}^{1} \frac{\pi \left(2 - x - x^2\right)^2}{2}\, dx$$

(8.2) Let S be the solid generated by rotating the region bounded by $y = x$ and $y = x^2$ around the line $y = -1$.

(a) Sketch S

(b) Use the method of washers to set up and evaluate an integral to find the value of S. Sketch a representative cross section on your drawing of S.

(a) Figure Below



(b) We will use the general formula and find the variable values for them. The general equation is

$$\int_a^b \pi(f(x)^2 - g(x)^2)\,dx$$

Since S was rotated around the line $y = -1$ and not the z-axis, we must add one to our functions before writing them into the integrand. Letting $f(x)$ be the top function and $g(X)$ be the bottom, we can conclude that

$$f(x) = x + 1$$
$$g(x) = x^2 + 1$$

The next step is to find the bounds of the integral. This is an relatively simple step, as all it requires is to find the intersection points of $y = x$ and $y = x^2$. Subtracting the two equations gives us

$$\begin{aligned} x^2 - x &= 0 \\ &= x(x-1) \end{aligned}$$

Giving us the bounds of $[a, b]$ as $[0, 1]$. Thus we have our integral

$$\int_0^1 \pi\left((x+1)^2 - \left(x^2+1\right)^2\right)\,dx$$

To compute this integral, first we must simplify which creates the simplified result of

$$\pi \int_0^1 -x^4 - x^2 + 2x\,dx$$

Then, we use the Evaluation Theorem to evaluate the integral.

$$\begin{aligned}
\pi \int_0^1 -x^4 - x^2 + 2x \, dx &= \pi \left[-\frac{x^5}{5} - \frac{x^3}{3} + x^2 \right]_0^1 \\
&= \pi \left(-\frac{1^5}{5} - \frac{1^3}{3} + 1^2 \right) - \pi \left(-\frac{0^5}{5} - \frac{0^3}{3} + 0^2 \right) \\
&= \pi \left(1 - \frac{1}{3} - \frac{1}{5} \right) \\
&= \frac{7\pi}{15}
\end{aligned}$$

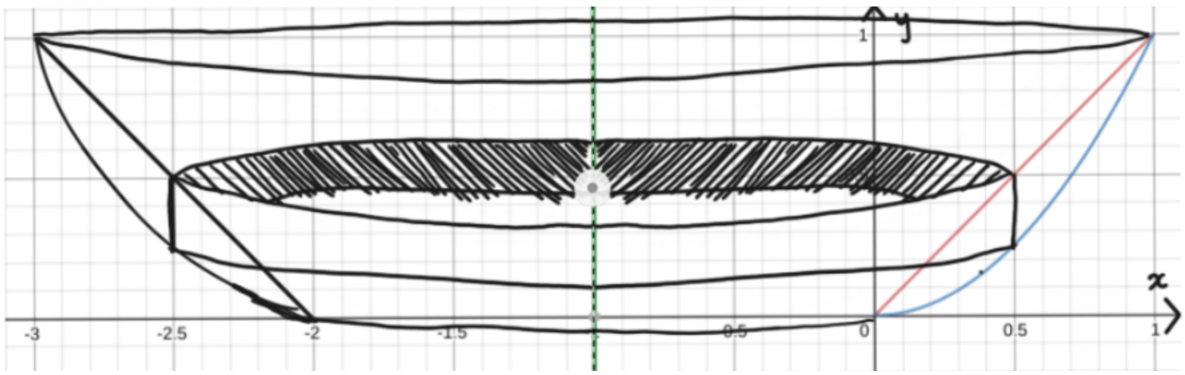
Thus the volume of S is $\frac{7\pi}{15}$.

(8.3) Let S be the solid generated by rotating the region bounded by $y = x$ and $y = x^2$ around the line $x = -1$.

(a) Sketch S

(b) Use the method of cylindrical shells to set up and evaluate an integral to find the value of S. Sketch a representative cross section on your drawing of S.

(a) Figure Below



(b) To find the area of S, we need to use the integral,

$$\int_a^b 2\pi x f(x)\, dx$$

We can find $f(x)$ by finding the vertical distance of the area bound by $y = x$ and $y = x^2$ which is $x - x^2$. Thus, our integral becomes

$$\int_a^b 2\pi x(x - x^2)\, dx$$

Next, we replace x with $1 + x$, since the radius of the function will be one away from the x of $f(x)$ since the solid is rotated around $x = -1$. Our bounds become 0 and 1 since the functions $y = x$ and $y = x^2$ intersect at o and 1. Thus, the final integral is

$$2\pi \int_0^1 (1 + x)(x - x^2)\, dx$$

The first step to evaluating this is to simplify. After simplification, the integral becomes

$$2\pi \int_0^1 x - x^3\, dx$$

This becomes fairly simple to integrate, with the integration being

$$\begin{aligned} 2\pi \int_0^1 x - x^3\, dx &= 2\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 \\ &= 2\pi \left(\frac{1^2}{2} - \frac{1^4}{4}\right) - 2\pi \left(\frac{0^2}{2} - \frac{0^4}{4}\right) \\ &= 2\pi \left(\frac{1}{4}\right) \\ &= \frac{\pi}{2} \end{aligned}$$

The volume of S is $\dfrac{\pi}{2}$.

(8.4) Professional Problem:

(a) Find the volume of the solid of revolution created by rotating the graph of $\sqrt{x}$, $0 \leq x \leq 9$, about the $x-$axis.

(b) A different solid of revolution S was created by rotating the same graph in part (a) around a line $y = k$ where $k > 0$. The solid has the same volume as the solid in part (a). Find k.

(a) We shall use the cylindrical shell method to find the volume of the solid of revolution created by rotating the graph of $\sqrt{x}$, $0 \leq x \leq 9$, about the $x-$axis. To do this, first we must find the area of one of these shells which is equal to the surface of a hollowed out cylinder. The surface area of a holow cylinder is $2\pi rh$ where r is the radius and h is the height. Since our cylinder lying down, $r = y$ and $h = 9 - x$, which means that if $y = \sqrt{x}$, then $h = 9 - y^2$. Thus, our shell surface area is

$$S(x) = 2\pi y(9 - y^2) \tag{3}$$

To find the volume we sum up and infinite number of the these cylinders from the x values for which the function is defined. In this case, the integral from 0 to 3 of (3) since our function is in respect to y. Thus, the integral is

$$\begin{aligned} 2\pi \int_0^3 9y - y^3 \, dy &= 2\pi \left[\frac{9y^2}{2} - \frac{y^4}{4}\right]_0^3 \\ &= 2\pi \left(9\frac{3^2}{2} - \frac{3^4}{4}\right) \\ &= 2\pi(\frac{405}{4}) \\ &= \frac{405\pi}{2} \end{aligned}$$

Thus, we have determined volume of the solid of revolution is $\frac{405\pi}{2}$.

(b) To find k we can define the integral

$$2\pi \int_0^3 (k - y)(9 - y^2) \, dy$$

is the volume of S. We convert y to $k - y$ because the height is just the distance between $y = k$ and $y = \sqrt{x}$. Since we already know the volume of S, we can just plug it in and solve like so

$$\begin{aligned} \frac{405\pi}{2} &= 2\pi \int_0^3 (k - y)(9 - y^2) \, dy \\ \frac{405}{4} &= \int_0^3 9k - ky^2 + 9y + y^3 \, dy \\ &= \left[9ky - k\frac{y^3}{3} + 9\frac{y^2}{2} + \frac{y^4}{4}\right]_0^3 \\ &= 18k + \frac{243}{4} \\ \frac{162}{4} &= 18k \\ \frac{9}{4} &= k \end{aligned}$$

Using the evaluation theorem, we have found that $k = \frac{9}{4}$.