**(13.1)** If $a$ and $b$ are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$, $0 \leq x \leq 1$.
To find critical points, we will take the derivative of the function $f(x) = x^a(1-x)^b$, and solve for 0.

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^a(1-x)^b\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^a\right)(1-x)^b + \frac{\mathrm{d}}{\mathrm{d}x}\left((1-x)^b\right)x^a \\
&= (ax^{a-1})(1-x)^b + b(1-x)^{b-1}\frac{\mathrm{d}}{\mathrm{d}x}(1-x)x^a \\
&= -x^a b(1-x)^{b-1} + a(x-1)^b x^{a-1} \\
&= x^{a-1}(1-x)^{b-1}[(1-x)a - bx] \\
&= x^{a-1}(1-x)^{b-1}[a - x(a+b)]
\end{aligned}$$

Now to find the critical points, we must solve for $f' = 0$.

$$x^{a-1}(1-x)^{b-1}[a - x(a+b)] = 0$$

From here we can see that x must either equal 0, 1 , or $\dfrac{a}{a+b}$. Because we are trying to find the maximum on the interval [0,1], the critical points must be on the open interval (0,1). Thus, we can eliminate $x = 0$ and $x = 1$ from the pool of possible critical points, leaving $x = \dfrac{a}{a+b}$ as the only possible critical point on the interval.
Now that we have our critical point, we can just solve $f(\dfrac{a}{a+b})$ to find the possible maximum of the function in the interval $0 \leq x \leq 1$.

$$\begin{aligned}
f\left(\frac{a}{a+b}\right) &= \left(\frac{a}{a+b}\right)^a\left(1 - \frac{a}{a+b}\right)^b \\
&= \left(\frac{a}{a+b}\right)^a\left(\frac{b}{a+b}\right)^b \\
&= \frac{a^a}{(a+b)^a}\cdot\frac{b^b}{(a+b)^b} \\
&= \frac{a^a b^b}{(a+b)^{a+b}}
\end{aligned}$$

To determine whether $\dfrac{a}{a+b}$ is a maximum, we must check if $f(0)$ and $f(1)$ are greater than $f\left(\dfrac{a}{a+b}\right)$.

$$\begin{aligned}
f(0) &= 0^a(1-0)^b \\
&= 0 \cdot 1^b \\
&= 0 \\
f(1) &= 1^a(1-1)^b \\
&= 1^a \cdot 0 \qquad\qquad = 0
\end{aligned}$$

Now we know that $\dfrac{a}{a+b}$ is the maximum point because $a$ and $b$ are positive numbers and thus are greater than 0.

$$\boxed{\text{The maximum value of } f(x) = x^a(1-x)^b,\ 0 \leq x \leq 1 \text{ is } \frac{a^a b^b}{(a+b)^{a+b}}}$$

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**(13.2)** Sketch a function $f$ that satifisfies all of the following conditions (simultaneously):

(a) $f'(x) > 0$ if $|x| < 2, f'(x) < 0$ if $|x| > 2$

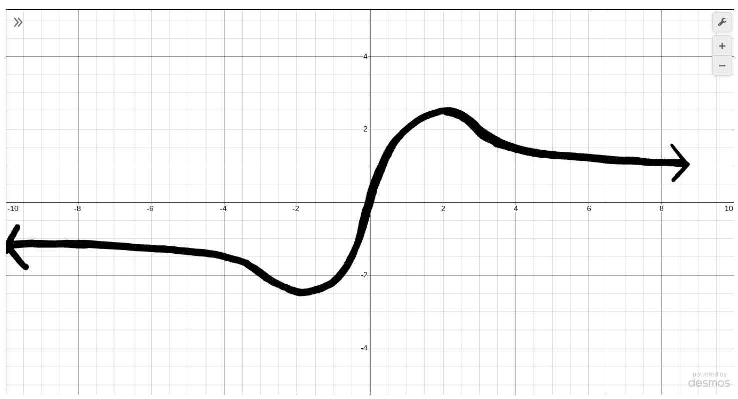
(b) $x = 2$ is a critical number

(c) $\lim_{x\to\infty} f(x) = 1$

(d) $f(-x) = -f(x)$

(e) $f''(x) < 0$ if $0 < x < 3$, $f''(x) > 0$ if $x > 3$

Now we will interpret what each of the conditions mean to help us sketch our graph.
The condition $a$) means that the function is increasing from (-2,2) and decreasing everywhere else. When we take into account condition ($b$), 2 is a maximum and then it goes down to the line $y =$ because there is a horizontal asymptote there from condition ($c$). The function $f$ will also be reflected over the origin because of ($d$), so everything that we said before now also holds true for the 3rd quadrant. And finally, ($e$) means that the function will be concave down for (0,3) and concave up elsewhere. This is a graph of the function that satfisfies all of those conditions:



We can see that both local and global maximums are the same, and there is only one local maximum. The local and global maximum is $(2, 2.5)$ and the local and global minimum is $(-2, -2.5)$.

**(13.3)** For the function $f(x) = \dfrac{x^2}{(x-4)^2}$

a Find the vertical and horizontal asymptotes

b Find the open intervals of increase and decrease

c Find local maximums and minimums

d Find open intervals of concavity and inflection points

e Use the information from the parts above to sketch $f$.

(a) HA:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2}{(x-4)^2} &= \left(\lim_{x\to\infty} \frac{x}{x-4}\right)^2 \\
&= \left(\lim_{x\to\infty} \frac{1}{1-\frac{4}{x}}\right)^2 \\
&= \left(\frac{1}{1}\right)^2 \\
&= 1
\end{aligned}$$

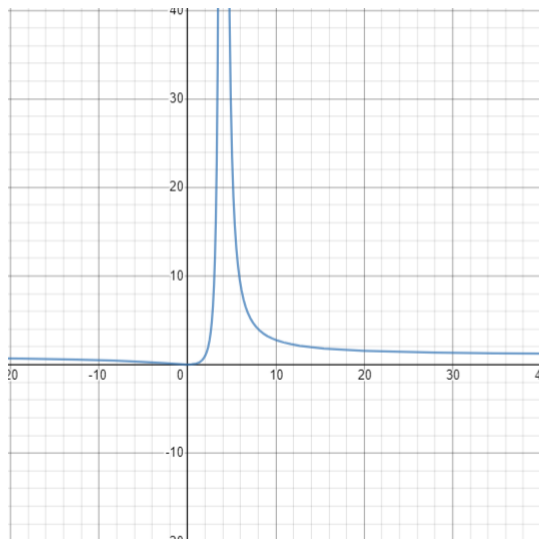
VA: Vertical asymptote is at $x = 4$ because the denominator of $f$ is undefined at 4.

(b) $f'(x) = \frac{-8x}{(x-4)^3}$. $f(x)$ is negative when $-\infty < x < 0$ and $4 < x < \infty$. $x$ is increasing on the intervals $(-\infty, 0)$U$(4, \infty)$. $x$ is increasing on the interval $(0, 4)$.

(c) $f'(x) = \frac{-8x}{(x-4)^3}$. $f'(x)$ is equal to 0 at $x = 0$. $f(x)$ has a global and local minimum at $(0, 0)$ and no maximum.

(d) $f''(x) = \frac{16(x+2)}{(x-4)^4}$. Intervals are: Concave down: $(-\infty$,-2) and Concave up: (-2,$\infty$). Thus the inflection point is $-2$.

(e) My figure of the equation is:



**(13.4) Professional Problem:** A cubic function is a polynomial of degree 3. It has the form $f(x) = ax^3 + bx^2 + cx + d$, where $a$, $b$, $c$, and $d$ are real numbers and $a \neq 0$.

(a) Prove that a cubic function can have two, one, or no critical numbers by providing explicit examples. Sketch your examples.

(b) How many local extrema can a cubic function have? Provide explicit examples that exhibit each possibility; sketch your examples. Prove that you have found all of the possibilities.

(a) For two solutions, we can take the derivative

$$\begin{aligned} f'(x) &= 3x^2 + 4x + 1 \\ &= (3x+1)(x+1) \end{aligned}$$

For the function with one solution we can take the derivative

$$f'(x) = (x+1)^2$$

For the function with no solutions, we take the derivative

$$f'(x) = 3x^2 + 2x^2 + 1$$

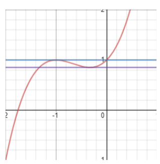


Figure 1: 2 extrema

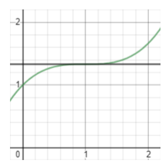


Figure 2: 1 extrema

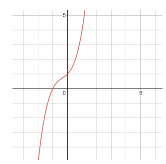


Figure 3: No extrema

(b) A cubic function can have 0, 1, or 2 extrema. My examples are:

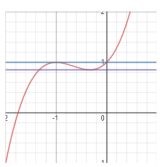


Figure 4: 2 extrema

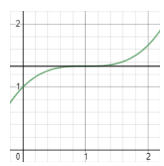


Figure 5: 1 extrema

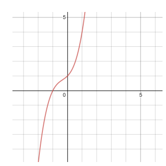


Figure 6: No extrema

A function's extrema are critical points, or points whose tangent lines have a slope of 0. The critical points then, are points for which the derivative of the function solves for 0. The derivative of a cubic function

$$f(x) = ax^3 + bx^2 + cx + d,$$

is

$$f'(x) = 3ax^2 + 2bx + c$$

From here we can see that the derivative of a cubic polynomial is a quadratic polynomial. Now, from the discriminant, we can tell that a quadratic, at most, can have 2 solutions and at least, 0. Therefore, a cubic function can have either 0, 1, or 2 extrema.