(5.1) Use the substitution $u - \sec x$ to evaluate the integral $\displaystyle\int \tan^7 x \sec^3 x\, dx$.

If we let $u = \sec x$, then $du = \sec x \tan x$.

$$\begin{aligned}\int \tan^7 x \sec^3 x\, dx &= \int \tan^6 x \sec^2 x \cdot \tan x \sec x\, dx \\ &= \int \tan^6 x\, u^2\, du\end{aligned}$$

To remove the $\tan^6 x$ in the integrand, we can utilize the identity $\tan^2 x + 1 = \sec^2 x$ to derive the equation $\tan^6 x = \left(\sec^2 x - 1\right)^3$.

$$\begin{aligned}\int \tan^6 x\, u^2\, du &= \int \left(\sec^2 x - 1\right)^3 u^2\, du \\ &= \int \left(u^2 - 1\right)^3 u^2\, du \\ &= \int \left(u^8 - 3u^6 + 3u^4 - u^2\right)\end{aligned}$$

Now the integrand is in a form which is easily integrated with the power rule

$$\begin{aligned}\int \left(u^8 - 3u^6 + 3u^4 - u^2\right) &= \frac{u^9}{9} - 3\frac{u^7}{7} + 3\frac{u^5}{5} - \frac{u^3}{3} + C \\ &= \frac{\sec^9 x}{9} - 3\frac{\sec^7 x}{7} + 3\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C\end{aligned}$$

Thus we have found out final answer which is

$$\boxed{\int \tan^7 x \sec^3 x\, dx = \frac{\sec^9 x}{9} - 3\frac{\sec^7 x}{7} + 3\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$$

Problem 2 on next page. →

(5.2) Evaluate the integral $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$ using the following two methods.

(a) Use the substitution $u = x^2 + 1$.

(b) Use the substitution $x = \tan\theta$.

(a) Let $u = x^2 + 1$. Therefore, $du = 2x$. Substitution into the integral looks like

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+1}}\,dx &= \int \frac{2}{2}\frac{x^2}{\sqrt{x^2+1}}\,dx \\ &= \int \frac{1}{2}\frac{x^2 \cdot 2x}{\sqrt{x^2+1}}\,dx \\ &= \frac{1}{2}\int \frac{x^2}{\sqrt{u}}\,du\end{aligned}$$

Removal of the x^2 in the integrand can be accomplished using the equation $x^2 = u - 1$, which is a manipulation of $u = x^2 + 1$.

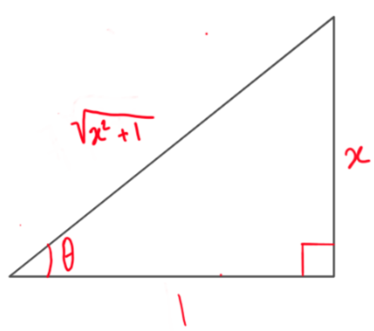
$$\begin{aligned}\frac{1}{2}\int \frac{x^2}{\sqrt{u}}\,du &= \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du \\ &= \frac{1}{2}\int u^{1/2} - u^{-1/2}\,du\end{aligned}$$

Now the integral is in a form which is easily integratable using the power rule.

$$\begin{aligned}\frac{1}{2}\int u^{1/2} - u^{-1/2}\,du &= \frac{u^{3/2}}{3} - \sqrt{u} + C \\ &= \frac{\left(x^2+1\right)^{3/2}}{3} - \sqrt{x^2+1} + C\end{aligned}$$

Thus we have our answer that $\boxed{\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \frac{\left(x^2+1\right)^{3/2}}{3} - \sqrt{x^2+1} + C}$

(b) If we set $x = \tan\theta$ and $dx = \sec^2\theta\,d\theta$, we can draw the following triangle using the the trigonometric identity $\tan\theta = \frac{opp}{adj}$.



Now we can substitute in our values for x and dx.

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+1}}\,dx &= \int \frac{x^3\sec^2\theta}{\sqrt{\tan^2\theta+1}}\,d\theta \\ &= \int \frac{\tan^3\theta\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta \\ &= \int \tan^3\theta\sec\theta\,d\theta\end{aligned}$$

Now we can use u substitution to solve this integral. Let $u = \sec\theta$ and $du = \sec\theta\tan\theta$. First we use the trig identity $\tan^2\theta = \sec^2\theta - 1$ and then u substitution.

$$\begin{aligned}\int \tan^3\theta\sec\theta\,d\theta &= \int \left(\sec^2 - 1\right)\tan\theta\sec\theta\,d\theta \\ &= \int u^2 - 1\,du\end{aligned}$$

Now we can evaluate the indefinite integral.

$$\begin{aligned}intu^2 - 1\,du &= \frac{u^3}{3} - u + C \\ &= \frac{\sec^3\theta}{3} - \sec\theta + C\end{aligned}$$

We can find the value of $\sec\theta$ because $\sec\theta = \frac{hyp}{opp}$ which is equal to $\sqrt{x^2+1}$ from the triangle above. Therefore,

$$\frac{\sec^3\theta}{3} - \sec\theta + C = \frac{\left(\sqrt{x^2+1}\right)^3}{3} - \sqrt{x^2+1} + C$$

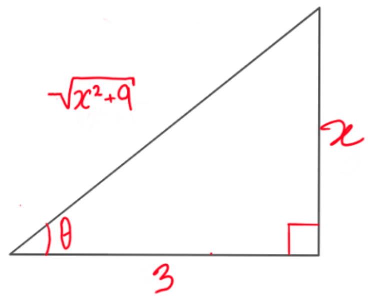
Since parts (a) and (b) are equal, we can safely conclude that

$$\boxed{\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \frac{\left(x^2+1\right)^{3/2}}{3} - \sqrt{x^2+1} + C}$$

Problem 3 on next page. →

(5.3) Evaluate the integral $\int \frac{x^2}{x^2+9}\,dx$ and draw a triangle according to your trig substitution.

To start, let us determine what side length we are looking for. The integral has a denominator of x^2+9 so we look for a triangle which has that to some power on a side. The triangle that fulfills that requirement is



Thus we can tell that our substitution is $x = 3\tan\theta$ and $du = 3\sec^2\theta\,d\theta$.

$$\begin{aligned}\int \frac{x^2}{x^2+9}\,dx &= \int \frac{9\tan^2\theta 3\sec^2\theta}{9\tan^2\theta+9}\,d\theta \\ &= \int \frac{3\tan^2\theta\sec^2\theta}{\tan^2\theta+1}\,d\theta \\ &= \int \frac{3\tan^2\theta\sec^2\theta}{\sec^2\theta}\,d\theta \\ &= 3\int \tan^2\theta\,d\theta\end{aligned}$$

Now we can evaluate the integral because of the identity $\tan^2\theta = \sec^2\theta - 1$.

$$\begin{aligned}3\int \tan^2\theta\,d\theta &= 3\int \sec^2\theta - 1\,d\theta \\ &= 3\left(\int \sec^2\theta\,d\theta - \int 1\,d\theta\right) \\ &= 3\left(\tan\theta - \theta + C\right)\end{aligned}$$

Now we have to re-substitute in the values for θ. Using our value for x we find that $\theta = \arctan\left(\frac{x}{3}\right)$.

$$\begin{aligned}3\left(\tan\theta - \theta + C\right) &= 3\left(\frac{x}{3} - \arctan\left(\frac{x}{3}\right) + C\right) \\ &= x - 3\arctan\left(\frac{x}{3}\right)\end{aligned}$$

That is the final answer to our problem

$$\boxed{\int \frac{x^2}{x^2+9}\,dx = x - 3\arctan\left(\frac{x}{3}\right)}$$

Problem 4 on next page. →

(5.4) Professional Problem: Define $G(n) = \int_0^\infty x^n e^{-x}\, dx$.

(a) Show $G(n) = nG(n-1)$ for $n \geq 1$.

(b) Let n be a positive integer. Explain why $G(n) = n!$.

If we use IBP on the indefinite integral form of $G(n)$ with a $u = x^n$ and a $dv = e^{-x}\, dx$, we create the equation

$$\int x^n e^{-x}\, dx = -x^n e^{-x} + n \int x^{n-1} e^{-x}\, dx + C. \tag{1}$$

Notice that $\int x^{n-1} e^{-x}\, dx = G(n-1)$. We can see from this that the integral will keep repeating until we reach $G(1)$. Recalling the groupwork Alternative Factorials, we know that $G(1) = 1$ Thus, the integral is actually

$$\int x^n e^{-x}\, dx = \left(-x^n e^{-x} - nx^{n-1} e^{-x} \ldots - e^{-x}\right) + n! + C. \tag{2}$$

Now we have the complete antiderivative we can evaluate $G(n)$ by finding the limit as x approaches ∞ of the integral from 0 to t. This yields the equations

$$\left[\left(-x^n e^{-x} - nx^{n-1} e^{-x} \ldots - e^{-x}\right) + n!\right]_0^t = \left(-t^n e^{-t} - nt^{n-1} e^{-t} \ldots - e^{-t}\right) + n! \tag{3}$$

Now to take the limit of this expression we can again utilize the groupwork, where we showed that $\lim_{x\to\infty} \dfrac{x^n}{e^x} = 0$. By this logic, everything within parentheses should equal 0. Applying the limit gives us

$$\begin{aligned}
\lim_{t\to\infty} \left(-t^n e^{-t} - nt^{n-1} e^{-t} \ldots - e^{-t}\right) + n! &= \lim_{x\to\infty} \left(-t^n e^{-t} - nt^{n-1} e^{-t} \ldots - e^{-t}\right) + \lim_{x\to\infty} n! \\
&= 0 + n! \lim_{x\to\infty} 1 \\
&= n!.
\end{aligned}$$

Thus we can see that $G(n) = n!$ for a positive n. We can also conclude that $G(n) = nG(n-1)$ for $n \geq 1$ because

$$\begin{aligned}
G(n) &= nG(n-1) \\
&= n \cdot (n-1)! \\
&= n! \\
&= G(n).
\end{aligned}$$

We have concluded that $G(n) = nG(n-1)$ for $n \geq 1$ and $G(n) = n!$ when n is a positive integer.