(3.1) Give the general solution for the following differential equations:

(a) $4y'' - 12y' + 9y = 0$

(b) $4y'' - 4y' + 5y = 0$

(c) What is the particular solution to part (b) if $y(\pi) = e^{\pi/2}$ and $y'(\pi) = 0$.

(a) This is homogeneous, first order DE, so the auxiliary equation is

$$4r^2 - 12r + 9$$

The solution to this is clearly

$$r = \frac{3}{2}$$

Because the auxiliary equation has a repeated root, the general solution is

$$\alpha e^{\frac{3}{2}x} + \beta x e^{\frac{3}{2}x}$$

(b) The auxiliary equation for this differential equation is

$$4r^2 - 4r + 5 = 0$$

The solution to this is

$$r = \frac{1}{2} \pm i$$

Because the solution is in the form of a complex conjugate, we have to express the solution using the form

$$y = e^{\frac{1}{2}x}[c_1 \cos(x) + c_2 \sin(x)]$$

(c) Using the first initial condition, we can do the following arithmetic:

$$\begin{aligned} e^{\pi/2} &= e^{\frac{1}{2}\pi}[c_1 \cos(\pi) + c_2 \sin(\pi)] \\ &= e^{\pi/2}[-c_1] \\ c_1 &= -1 \end{aligned}$$

We now have the new equation $y = e^{\frac{x}{2}}[c_2 \sin(x) - \cos(x)]$. Computing, the derivative with the product rule, we yield

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(e^{x/2})[c_2 \sin(x) - \cos(x)] + \frac{\mathrm{d}}{\mathrm{d}x}[c_2 \sin(x) - \cos(x)]e^{x/2} \\ &= \frac{1}{2}e^{x/2}[c_2 \sin(x) - \cos(x)] + e^{x/2}[\sin(x) + c_2 \cos(x)] \end{aligned}$$

Substitution solves for c_2 like so

$$\begin{aligned} 0 &= \frac{1}{2}e^{\pi/2}[c_2 \sin(\pi) - \cos(\pi)] + e^{\pi/2}[\sin(\pi) + c_2 \cos(\pi)] \\ &= e^{\pi/2}(\frac{1}{2} - c_2) \\ c_2 &= \frac{1}{2} \end{aligned}$$

Now, we know c_2 so we can determine that the specific solution is

$$y = e^{x/2}[\frac{1}{2}\sin(x) - \cos(x)]$$

(3.2) Solve the initial-value problem $x^2\frac{\mathrm{d}y}{\mathrm{d}x} - y = 2e^{1/x}$, $y(1) = -e$.

First, we must put this first-order linear differential equation into standard form. The standard form is

$$y' - \frac{y}{x^2} = \frac{2e^{1/x}}{x^2}$$

To find the integrating factor, we must find the value of $e^{\int -\frac{1}{x^2}\,dx}$. This is integration is simple to complete, and yields $\frac{1}{x}$. Thus, the function is transformed into

$$\int \frac{\mathrm{d}}{\mathrm{d}x}(e^{1/x}y)\,dx = \int \frac{2e^{2/x}}{x^2}\,dx$$

To integrate the left hand side, use the u-substitution of $\frac{2}{x}$. The value of du is $\frac{-2}{x^2}$. The left hand side is transformed into

$$\int -e^u\,du$$

which is equal to $-e^{2/x}$. The equation is now

$$e^{1/x}y = -e^{2/x} + C$$

Before isolating y, it is easier to solve for the value of C like so

$$\begin{aligned} e^{1/1}(-e) &= -e^{2/1} + C \\ -e^2 &= -e^2 + C \\ 0 &= C \end{aligned}$$

Isolating y is simply done.

$$\begin{aligned} y &= \frac{-e^{2/x}}{e^{1/x}} \\ &= -(e^{\frac{2}{x}-\frac{1}{x}}) \\ &= -e^{1/x} \end{aligned}$$

The particular solution for this initial-value problem is

$$y = -e^{1/x}$$

(3.3) Professional Problem: Chocolate factory tank contains 100 L of pure chocolate. The mixture of crushed Oreos and chocolate is added at 5 L/min. The mixed tank is drained at a rate of 3 L/min. Find and solve the initial value problem. Also, if after 20 minutes, the tank contains 32 kg of Oreo, find the concentration M of Oreos.

We know that the rate of change of the amount Oreos in the tank is $\frac{\mathrm{d}y}{\mathrm{d}t} = [\text{rate in}] - [\text{rate out}]$. The amount of Oreos entering the tank is the concentration of Oreos, M, multiplied by the rate at which the mixture enters the tank. The rate out is the concentration of Oreos in the tank multiplied by the rate out. This looks like

$$\frac{\mathrm{d}y}{\mathrm{d}t} = [M \cdot 5] - \left[\frac{y}{100+2t} \cdot 3\right]$$

The problem in standard form looks like

$$y' + \frac{3y}{100+2t} = 5M$$

To solve, this we need to solve $e^{\int \frac{3}{100+2t}\,dt}$ which is $(100+2t)^{3/2}$. Our new equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}((100+2t)^{3/2}y) = 5(100+2t)^{3/2}M$$

Solving this by integrating both sides yields

$$(100+2t)^{3/2}y = (100+2t)^{5/2}M + C$$

Using the initial condition $y(0) = 0$, we can isolate C. After substitution, we find that

$$C = -(100)^{5/2}M$$

Isolating y provides the solved equation

$$\begin{aligned} y &= \frac{\frac{2}{5}(100+2t)^{5/2}M - (100)^{5/2}M}{(100+2t)^{3/2}} \\ &= \frac{\frac{3}{5}(100+2t)^{5/2}M}{(100+2t)^{3/2}} \end{aligned}$$

To solve the initial value problem of $y(20) = 32$, we should isolate M. The isolation of M is that

$$\frac{(100+2t)^{3/2}y}{\frac{3}{5}(100+2t)^{5/2}} = M$$

This is a bit of a hairy equation, so plugging these answers into WolframAlpha is probably the right way to go. Plugging these in provides

$$C = \frac{(100+2(20))^{3/2}(32)}{\frac{3}{5}(100+2(20)^{5/2}} = \frac{8}{21}$$

The concentration of Oreos to melted chocolate is $\frac{8}{21}$ kg/L.