(4.1) Integrate $\int p^5 \ln(p)\, dp$

We can use the formula

$$\int u\, dv = uv - \int v\, du$$

otherwise know as IBP. Using the acronym L.I.A.T.E. to pick our u and v, our values for them are

$$u = \ln(p)$$
$$dv = p^5\, dp$$

Then, we substitute our values into the IBP formula.

$$\begin{aligned}\int u\, dv &= uv - \int v\, du \\ &= \ln(p)\frac{p^{5+1}}{5+1} - \int \frac{p^{5+1}}{5+1} \cdot \frac{1}{p}\, dp \\ &= \frac{\ln(p)p^6}{6} - \int \frac{p^5}{6}\, dp \\ &= \frac{\ln(p)p^6}{6} - \frac{1}{6}\int p^5\, dp\end{aligned}$$

Now we should evaluate the indefinite integral with basic antiderivatives.

$$\begin{aligned}\int p^5\, dp &= \frac{p^{5+1}}{5+1} + C \\ &= \frac{p^6}{6} + C\end{aligned}$$

Substitution back into our previous expression sums as

$$\frac{\ln(p)p^6}{6} - \frac{1}{6} \cdot \frac{p^6}{6} = \frac{\ln(p)p^6}{6} - \frac{p^6}{36}$$

Thus, our final answer is $\frac{\ln(p)p^6}{6} - \frac{p^6}{36}$.

Problem 2 on next page. →

(4.2) Use substitution on the integral $\int \frac{1}{\sqrt{x+2}+x}$ to express the integrand as a rational function, then evaluate the integral.

Let $u = \sqrt{x+2}$. Thus, $du = \frac{1}{2\sqrt{x+2}}\, dx$. Now, we can use algebra to apply the Substitution Rule.

$$\begin{aligned}
\int \frac{1}{\sqrt{x+2}+x} &= \int \frac{2\sqrt{x+2}}{2\sqrt{x+2}} \cdot \frac{1}{\sqrt{x+2}+x} \\
&= \int \frac{2\sqrt{x+2}}{\sqrt{x+2}+x} \cdot \frac{dx}{2\sqrt{x+2}} \\
&= \int \frac{2\sqrt{x+2}}{\sqrt{x+2}+x}\, du \\
&= 2\int \frac{u}{u+x}\, du
\end{aligned}$$

To get rid of x in the denominator we can manipulate the equation of u

$$\begin{aligned}
u &= \sqrt{x+2} \\
u^2 &= x+2 \\
u^2 - 2 &= x
\end{aligned}$$

Therefore,

$$2\int \frac{u}{u+x}\, du = 2\int \frac{u}{u^2+u-2}\, du$$

Factoring the denominator leaves $(x+2)(x-1)$. Notice that the integrand now can be used for partial fraction decomposition.

$$\begin{aligned}
\frac{u}{(u+2)(u-1)} &= \frac{A}{u+2} + \frac{B}{u-1} \\
&= A(u-1) + B(u+2) \\
&= (A+B)u + (2B-A) = (1)u + 0
\end{aligned}$$

Thus we must solve the system of equations

$$\begin{cases} A + B = 1 \\ 2B - A = 0 \end{cases}$$

Thus we use use substitution to solve our system of equations.

$$\begin{aligned}
A &= 2B \\
A + B &= 2B + B \\
3B &= 1 \\
B &= \frac{1}{3}
\end{aligned}$$

We can easily solve to get $A = \frac{2}{3}$. Substituting this decomposition into the integrand changes the integral into one that can be easily manipulated.

$$\begin{aligned}
2\int \frac{u}{u^2+u-2}\,du &= 2\int \frac{2}{3(u+2)} + \frac{1}{3(u-1)}\,du \\
&= \frac{4}{3}\int \frac{1}{u+2} + \frac{2}{3}\int \frac{1}{u-1} \\
&= \frac{4}{3}\ln(|u|) + \frac{2}{3}\ln(|u|) + C \\
&= \frac{4}{3}\ln(|\sqrt{x+2}|) + \frac{2}{3}\ln(|\sqrt{x+2}|) + C
\end{aligned}$$

Thus we have reached our final answer.

$$\int \frac{1}{\sqrt{x+2}+x} = \frac{4}{3}\ln(|\sqrt{x+2}|) + \frac{2}{3}\ln(|\sqrt{x+2}|) + C$$

Problem 3 on Page 4. $\rightarrow$

(4.3) Let f be a continuous, increasing function, and let g be the inverse of f.

(a) Use IBP to show $\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$

(b) Show that $\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy$.

(c) Suppose $f(x) > 0$ and $0 < a < b$, as shown in the diagram below. Reproduce the diagram, the shade/label it to give a geometric interpretation of (b).

(a) Using IBP in the integral allows us to set our $u = f(x)$ and $dv = 1\,dx$. Now if we substitute the left hand side into the IBP formula we get the equation

$$\begin{aligned}\int f(x)\,dx &= f(x)\cdot\frac{x^{0+1}}{0+1} - \int \frac{x^{0+1}}{0+1} f'(x)\,dx \\ &= xf(x) - \int xf'(x)\,dx\end{aligned}$$

Thus we have successfully converted LHS to equal RHS and have shown that $\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$.

(b) In the previous part, we calculated the indefinite integral of $f(x)$ using IBP. We can now use that and then apply the Evaluation Theorem to the indefinite integral.

$$\begin{aligned}\left[xf(x) - \int xf'(x)\,dx\right]_a^b &= [xf(x)]_a^b - \left[\int xf'(x)\,dx\right]_a^b \\ &= (bf(b) - af(a)) - \int_a^b xf'(x)\,dx\end{aligned}$$

Now, we substitute in y as $y = f(x)$. This makes it so $dy = f'(x)dx$. Thus the last integral in our equation is now.

$$\int x\,dy$$

We now have to remove the x from the integrand and adjust the bounds. Since $y = f(x)$, we can derive that $x = f^{-1}(y)$. To adjust the bounds by applying y to them. Thus, the integral is now

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,dy$$

Looking back at the problem, it states that g is the inverse of f, so doing one more simplification, the integral is now,

$$\int_{f(a)}^{f(b)} g(y)\,dy$$

Substituting the new integral back into the original equation yields

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,d$$

Thus we have shown that the equation on (b) is true.

(c)

(4.4) Professional Problem:

(a)

(b)

(c)