(11.1) Use the Comparison Test or the Limit Comparison Test to determine the convergence of the following series:

$$(a)\sum_{n=1}^{\infty}\frac{4^n}{3^n+2^n} \qquad (b)\sum_{n=3}^{\infty}\frac{n}{n^5-2}$$

(a) Let $a_n = \dfrac{4^n}{3^n+3^n}$ and $b_n = \dfrac{4^n}{3^n+2^n}$. Because n is only a power, we know that both sequences are positive for all $n > 0$. Therefore we can use the Comparison Test. Because $2^n < 3^n$, we can make the comparison:

$$0 < \frac{4^n}{2(3^n)} < \frac{4^n}{3^n+2^n}$$

Because b_n is a geometric sequence of the form ar^n, we can easily determine its convergence. In our case of $b_n = \dfrac{1}{2}\left(\dfrac{4}{3}\right)^n$, $r > 1$, therefore b_n diverges. Since $b_n < a_n$, a_n also converges.

(b) Let $a_n = \dfrac{1}{n^4}$ and $b_n = \dfrac{n}{n^5-2}$. To use the Limit Comparison Test, let us verify that the sequences are positive. The first sequence is positive because $n^4 > 0$ for all n and b_n is positive because for $n > 3$, $n^5 - 2 > 0$. Then we construct the limit

$$\begin{aligned}\lim_{x\to\infty}\frac{\frac{1}{n^4}}{\frac{n}{n^5-2}} &= \lim_{x\to\infty}\frac{n^5-2}{n^5}\\ &= \lim_{x\to\infty}\frac{1-\frac{2}{n^5}}{1}\\ &= 1\end{aligned}$$

Since the limit of $\dfrac{a_n}{b_n} > 0$, they both either converge or diverge. Through the p-test, we know that a_n converges, therefore b_n also converges.

(11.2) Use the Integral Test to determine whether $\sum_{n=1}^{\infty} \frac{n^2}{n^3+7}$ converges.

Before using the integral test to determine the series' convergence, first we must verify the conditions that the sequence is continuous, decreasing, and positive. First, let $a_n = \frac{n^2}{n^3+7}$.

Positive: The numerator of $a_n = n^2$, which is positive for all n. The denominator is $n^3 + 7$, which is also positive for all $n > 0$.

Decreasing: Applying the Quotient Rule for derivatives, we find that the derivative of a_n is $\frac{-n^4 + 14n}{(n^3+7)^2}$ which is negative for $n > \sqrt[3]{14}$. Although the sequence is not decreasing for all n, it is only increasing for a finite interval, so the bounds of the integral can be adjusted to fit.

Continuous: The functions n and $n^3 + 7$ do not have any points where they are undefined , therefore the sequence is continuous on all n.

To use the Integral Test, we simply construct an integral from the terms in the series. Doing this results in the integral

$$\int_{\sqrt[3]{14}}^{\infty} \frac{n^2}{n^3+7}\, dn$$

Note that the lower bound is not 1 to satisfy the decreasing condition of the Integral Test. To compute this we first turn it into a proper integral and then use a u-substitution of $n^3 + 7$.

$$\begin{aligned}\int_{\sqrt[3]{14}}^{\infty} \frac{n^2}{n^3+7}\, dn &= \lim_{t\to\infty} \int_{\sqrt[3]{14}}^{t} \frac{n^2}{n^3+7}\, dn \\ &= \lim_{t\to\infty} \int_{\sqrt[3]{14}}^{t} \frac{1}{3u}\, du\end{aligned}$$

Normally it would be necessary to complete evaluating the integral, but in this case we know that the series is divergent. This is because the integrand is of the form $\frac{1}{n}$ and according to the p-test, we know that integrals of this form diverge. Thus, the series $\sum_{n=1}^{\infty} \frac{n^2}{n^3+7}$ diverges.

(11.3) Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series with positive terms.

(a) Explain why, eventually $0 \leq a_n < 1$.

(b) Use the comparison test to explain why $\sum_{n=1}^{\infty} a_n b_n$ is also convergent.

(a) If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. Therefore, at some point, the values of a_n must be less than 1 as it approaches 0. Also, because the a_n has positive terms, the values of it cannot be less that 0.
Thus,

$$0 \leq a_n < 1$$

(b) Since we know from (a) that there is some N that $a_N < 1$ and $b_N < 1$, we know that the sum is a fraction multiplied by a fraction. On the other hand, $\sum_{n=N}^{\infty} a_n$ is simply a singular fraction. Thus, we can make the comparison that

$$0 \leq \sum_{n=N}^{\infty} a_n b_n < \sum_{n=N} a_n$$

And since $\sum_{n=N}^{\infty} a_n$ converges, we know that $\sum_{n=1}^{\infty} a_n b_n$ also converges because $[1, N)$ is a finite interval.

(11.4) Professional Problem: Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series and $\sum_{n=1}^{\infty} b_n$ is a divergent series who both have positive terms.

(a) Prove $\sum_{n=1}^{\infty} \sin(a_n)$ converges
(b) Provide a specific counterexample to the notion $\sum_{n=1}^{\infty} \cos(a_n)$ always converges
(c) Provide a specific counterexample to the notion $\sum_{n=1}^{\infty} \sin(b_n)$ always diverges.

(a) We shall use the LCT with the sequences $\sin(a_n)$ and a_n. Although $\sin(a_n)$ is not always positive, as $n \to \infty$ $a_k \to 0$. Thus, after some N $\sin(a_n)$ is always positive. Then we can create the limit

$$\lim_{n\to\infty} \frac{\sin(a_n)}{a_n}$$

Because $\lim\limits_{n\to\infty} a_n = 0$, we can perform the substitution

$$\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = \lim_{u\to\infty} \frac{\sin u}{u}$$

Using the identity $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 1$, we find that

$$\lim_{u\to\infty} \frac{\sin u}{u} = 1$$

Thus, if $\sum\limits_{n=1}^{\infty} a_n$ is convergent, $\sum\limits_{n=1}^{\infty} \sin(a_n)$ converges.

(b) A specific counterexample is the sequence $a_n = 0$. The series

$$\sum_{n=1}^{\infty} a_n = 0$$

However,

$$\begin{aligned}\sum_{n=1}^{\infty} \cos(a_n) &= \sum_{n=1}^{\infty} \cos(0) \\ &= \sum_{n=1}^{\infty} 1 \\ &= \infty\end{aligned}$$

Thus, if $\sum\limits_{n=1}^{\infty} a_n$ converges, $\sum\limits_{n=1}^{\infty} \cos(a_n)$ does not necessarily converge.

(c) A specific counterexample to this notion is the sequence $b_n = \pi$. The series

$$\sum_{n=1}^{\infty} b_n = \infty$$

However,

$$\begin{aligned}\sum_{n=1}^{\infty} \sin(b_n) &= \sum_{n=1}^{\infty} \sin(\pi) \\ &= \sum_{n=1}^{\infty} 0 \\ &= 0\end{aligned}$$

Thus, if $\sum\limits_{n=1}^{\infty} b_n$ diverges, $\sum\limits_{n=1}^{\infty} \sin(b_n)$ does not necessarily diverge.