

(13.1) If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$, $0 \leq x \leq 1$.

To find critical points, we will take the derivative of the function $f(x) = x^a(1-x)^b$, and solve for 0.

$$\begin{aligned}
 f'(x) &= \frac{d}{dx} (x^a(1-x)^b) \\
 &= \frac{d}{dx} (x^a) (1-x)^b + \frac{d}{dx} ((1-x)^b) x^a \\
 &= (ax^{a-1})(1-x)^b + b(1-x)^{b-1} \frac{d}{dx} (1-x)x^a \\
 &= -x^a b(1-x)^{b-1} + a(x-1)^b x^{a-1} \\
 &= x^{a-1}(1-x)^{b-1}[(1-x)a - bx] \\
 &= x^{a-1}(1-x)^{b-1}[a - x(a+b)]
 \end{aligned}$$

Now to find the critical points, we must solve for $f' = 0$.

$$x^{a-1}(1-x)^{b-1}[a - x(a+b)] = 0$$

From here we can see that x must either equal 0, 1, or $\frac{a}{a+b}$. Because we are trying to find the maximum on the interval $[0,1]$, the critical points must be on the open interval $(0,1)$. Thus, we can eliminate $x = 0$ and $x = 1$ from the pool of possible critical points, leaving $x = \frac{a}{a+b}$ as the only possible critical point on the interval.

Now that we have our critical point, we can just solve $f(\frac{a}{a+b})$ to find the possible maximum of the function in the interval $0 \leq x \leq 1$.

$$\begin{aligned}
 f\left(\frac{a}{a+b}\right) &= \left(\frac{a}{a+b}\right)^a \left(1 - \frac{a}{a+b}\right)^b \\
 &= \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b \\
 &= \frac{a^a}{(a+b)^a} \cdot \frac{b^b}{(a+b)^b} \\
 &= \frac{a^a b^b}{(a+b)^{a+b}}
 \end{aligned}$$

To determine whether $\frac{a}{a+b}$ is a maximum, we must check if $f(0)$ and $f(1)$ are greater than $f\left(\frac{a}{a+b}\right)$.

$$\begin{aligned}
 f(0) &= 0^a(1-0)^b \\
 &= 0 \cdot 1^b \\
 &= 0 \\
 f(1) &= 1^a(1-1)^b \\
 &= 1^a \cdot 0 = 0
 \end{aligned}$$

Now we know that $\frac{a}{a+b}$ is the maximum point because a and b are positive numbers and thus are greater than 0.

The maximum value of $f(x) = x^a(1-x)^b$, $0 \leq x \leq 1$ is $\frac{a^a b^b}{(a+b)^{a+b}}$

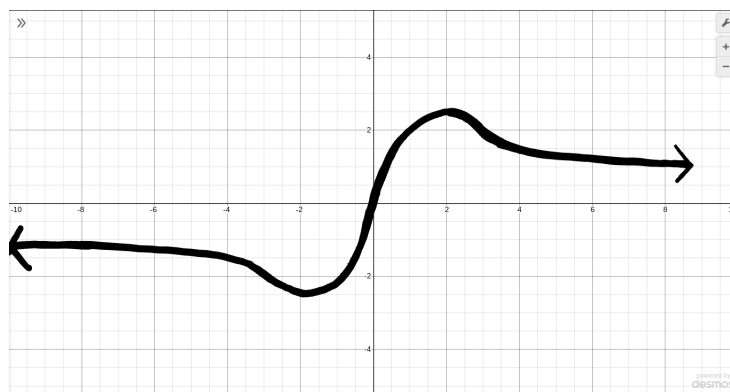
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(13.2) Sketch a function f that satisfies all of the following conditions (simultaneously):

- (a) $f'(x) > 0$ if $|x| < 2$, $f'(x) < 0$ if $|x| > 2$
- (b) $x = 2$ is a critical number
- (c) $\lim_{x \rightarrow \infty} f(x) = 1$
- (d) $f(-x) = -f(x)$
- (e) $f''(x) < 0$ if $0 < x < 3$, $f''(x) > 0$ if $x > 3$

Now we will interpret what each of the conditions mean to help us sketch our graph.

The condition $a)$ means that the function is increasing from $(-2, 2)$ and decreasing everywhere else. When we take into account condition (b) , 2 is a maximum and then it goes down to the line $y = 1$ because there is a horizontal asymptote there from condition (c) . The function f will also be reflected over the origin because of (d) , so everything that we said before now also holds true for the 3rd quadrant. And finally, (e) means that the function will be concave down for $(0, 3)$ and concave up elsewhere. This is a graph of the function that satisfies all of those conditions:



We can see that both local and global maximums are the same, and there is only one local maximum. The local and global maximum is $(2, 2.5)$ and the local and global minimum is $(-2, -2.5)$.

(13.3) For the function $f(x) = \frac{x^2}{(x-4)^2}$

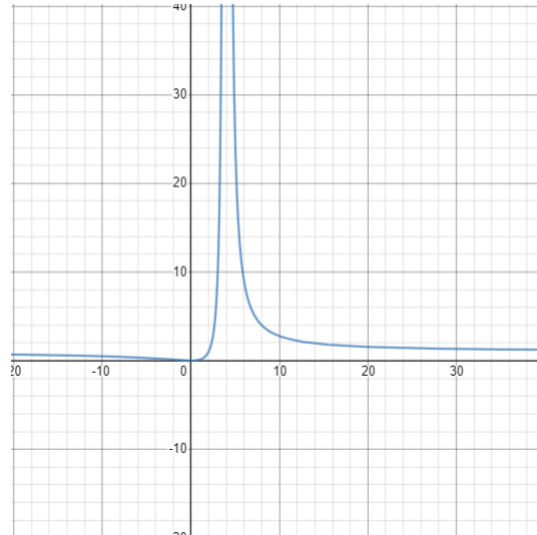
- a Find the vertical and horizontal asymptotes
- b Find the open intervals of increase and decrease
- c Find local maximums and minimums
- d Find open intervals of concavity and inflection points
- e Use the information from the parts above to sketch f .

(a) HA:

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{x^2}{(x-4)^2} &= \left(\lim_{x \rightarrow \infty} \frac{x}{x-4} \right)^2 \\
 &= \left(\lim_{x \rightarrow \infty} \frac{1}{1 - \frac{4}{x}} \right)^2 \\
 &= \left(\frac{1}{1} \right)^2 \\
 &= 1
 \end{aligned}$$

VA: Vertical asymptote is at $x = 4$ because the denominator of f is undefined at 4.

- (b) $f'(x) = \frac{-8x}{(x-4)^3}$. $f(x)$ is negative when $-\infty < x < 0$ and $4 < x < \infty$. x is increasing on the intervals $(-\infty, 0) \cup (4, \infty)$. x is increasing on the interval $(0, 4)$.
- (c) $f'(x) = \frac{-8x}{(x-4)^3}$. $f'(x)$ is equal to 0 at $x = 0$. $f(x)$ has a global and local minimum at $(0, 0)$ and no maximum.
- (d) $f''(x) = \frac{16(x+2)}{(x-4)^4}$. Intervals are: Concave down: $(-\infty, -2)$ and Concave up: $(-2, \infty)$. Thus the inflection point is -2 .
- (e) My figure of the equation is:



(13.4) Professional Problem: A cubic function is a polynomial of degree 3. It has the form $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c , and d are real numbers and $a \neq 0$.

- Prove that a cubic function can have two, one, or no critical numbers by providing explicit examples. Sketch your examples.
- How many local extrema can a cubic function have? Provide explicit examples that exhibit each possibility; sketch your examples. Prove that you have found all of the possibilities.

(a) For two solutions, we can take the derivative

$$\begin{aligned} f'(x) &= 3x^2 + 4x + 1 \\ &= (3x + 1)(x + 1) \end{aligned}$$

For the function with one solution we can take the derivative

$$f'(x) = (x + 1)^2$$

For the function with no solutions, we take the derivative

$$f'(x) = 3x^2 + 2x^2 + 1$$

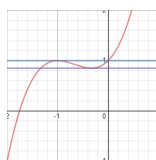


Figure 1: 2 extrema

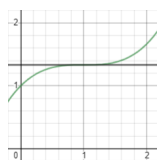


Figure 2: 1 extrema

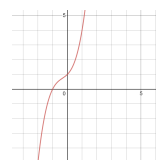


Figure 3: No extrema

- (b) A cubic function can have 0, 1, or 2 extrema. My examples are:

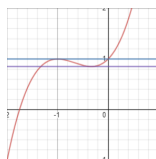


Figure 4: 2 extrema

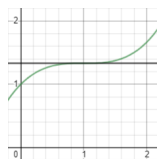


Figure 5: 1 extrema

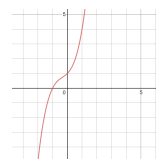


Figure 6: No extrema

A function's extrema are critical points, or points whose tangent lines have a slope of 0. The critical points then, are points for which the derivative of the function solves for 0. The derivative of a cubic function

$$f(x) = ax^3 + bx^2 + cx + d,$$

is

$$f'(x) = 3ax^2 + 2bx + c$$

From here we can see that the derivative of a cubic polynomial is a quadratic polynomial. Now, from the discriminant, we can tell that a quadratic, at most, can have 2 solutions and at least, 0. Therefore, a cubic function can have either 0, 1, or 2 extrema.