(6.1) Determine whether $\int_0^1 \frac{1}{3y-2}\,dy$ is convergent or divergent. If it is convergent evaluate it.

To find the integral of $\int_0^1 \frac{1}{3y-2}\,dy$, we can use u-substitution with a u of $3y-2$ and a $du = 3$. Thus,

$$\int_0^1 \frac{1}{3y-2}\,dy = \int_0^1 \frac{1}{3u}\,du$$

Computing the antiderivative gives us

$$\begin{aligned}
\int_0^1 \frac{1}{3u}\,du &= \left[\frac{1}{3}\ln u\right]_0^1 \\
&= \left[\frac{\ln(3y-2)}{3}\right]_0^1 \\
&= \frac{\ln 1}{3} - \frac{\ln(-2)}{3} \\
&= -\frac{\ln(-2)}{3}
\end{aligned}$$

Because our expression for the antiderivative contains $\ln(-2)$ and the natural logarithm is undefined for negative numbers, the integral $\int_0^1 \frac{1}{3y-2}\,dy$ is divergent.

(6.2) Use the Comparison Theorem to determine whether $\displaystyle\int_4^\infty \frac{5+e^{-x}}{x}\,dx$ is convergent or divergent.

Let $\displaystyle\int_4^\infty \frac{5+e^{-x}}{x}\,dx$ be called (1). Currently (1) is not in a form the Comparison Theorem is able to be used on. To fix that, we can split the integral into two.

$$\int_4^\infty \frac{5+e^{-x}}{x}\,dx = \int_1^\infty \frac{5+e^{-x}}{x}\,dx - \int_1^4 \frac{5+e^{-x}}{x}\,dx$$

Now we can use the Comparison Theorem on the first integral,and compute a finite value for the second integral. We shall compare $\displaystyle\int_1^\infty \frac{5+e^{-x}}{x}\,dx$ to $\displaystyle\frac{1}{x}$, and $1 < x < \infty$.

$$\begin{aligned} e^{-x} &> 1 \\ 5+e^{-x} &> 1 \\ \frac{5+e^{-x}}{x} &> \frac{1}{x} \end{aligned}$$

Now if we use the p-test on $\frac{1}{x}$, we find that $1 \not< 1$ and therefore it is divergent.
Now since

$$\frac{5+e^{-x}}{x} > \frac{1}{x}$$

the integral $\displaystyle\frac{5+e^{-x}}{x}$ is also divergent.
Therefore,

$$\int_1^\infty \frac{5+e^{-x}}{x}\,dx - \int_1^4 \frac{5+e^{-x}}{x}\,dx$$

is divergent which means that

$$\int_4^\infty \frac{5+e^{-x}}{x}\,dx$$

is divergent.

(6.3) Use the Comparison Theorem to determine whether $\int_0^{\pi/2} \sec^3 x\,dx$ is convergent or divergent.

Let us compare our integral to the integral of $\sec x$ with the same bounds. Remember that $\int \sec x\,dx = \ln|\tan x + \sec x| + C$. Therefore, using our methods for evaluating improper integrals,

$$\begin{aligned}\int_0^{\pi/2} \sec x\,dx &= \int_0^{t} \sec x\,dx \\ &= [\ln|\tan x + \sec x|]_0^t \\ &= \ln|\tan t + \sec t| - \ln|\tan 0 + \sec 0| \\ &= \ln|\tan t + \sec t| - \ln 1 \\ &= \ln|\tan t + \sec t|\end{aligned}$$

Plugging this in to our limit gives us

$$\begin{aligned}\lim_{t\to\infty} \ln|\tan t + \sec t| &= \infty \\ &= \text{Diverges}\end{aligned}$$

Now we know that $\int_0^{\pi/2} \sec x\,dx$ diverges, we can say that for $0 < x < \frac{\pi}{2}$

$$\begin{gathered}\sec x > 1 \\ \sec^3 x > \sec x \\ \int_0^{\pi/2} \sec^3 x\,dx > \int_0^{\pi/2} \sec x\,dx\end{gathered}$$

And since we know that $\int_0^{\pi/2} \sec x\,dx$ diverges and $\int_0^{\pi/2} \sec^3 x\,dx > \int_0^{\pi/2} \sec x\,dx$, we can say using the Comparison Theorem that $\int_0^{\pi/2} \sec^3 x\,dx$ is divergent.

(6.4) Professional Problem: Decide whether the following statements are true or false. If a statement is true, prove it and if it's false, provide a specific counterexample.

(a) If $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ both converge, then $\int_1^\infty f(x)+g(x)\,dx$ also converges.

(b) If $\int_1^\infty f(x)+g(x)\,dx$ converges, then $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ also converge.

(c) If $\int_1^\infty f(x)\,dx$ diverges, then $\int_1^\infty (f(x))^2\,dx$ diverges.

(a) Let the statement "If $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ both converge, then $\int_1^\infty f(x)+g(x)\,dx$ also converges" be true. To prove this, first using basic integration we can establish that

$$\int_1^\infty f(x)\,dx = \lim_{x\to\infty}(F(x)-F(1)) \tag{1}$$

and that (1) converges. Now if we evaluate the integral $\int_1^\infty f(x)+g(x)\,dx$, we can complete the math

$$\begin{aligned}\int_1^\infty f(x)+g(x)\,dx &= \lim_{t\to\infty}[F(x)+G(x)]_1^t \\ &= \lim_{t\to\infty}(F(t)+G(t)-F(1)-G(1))\end{aligned}$$

Rearranging the limt produces

$$\lim_{t\to\infty}(F(t)+G(t)-F(1)-G(1)) = \lim_{t\to\infty}(F(t)-F(1)) + \lim_{t\to\infty}(G(t)-G(1)) \tag{2}$$

which using the property in (1) we can clearly see is equal to $\int_1^\infty f(x)\,dx + \int_1^\infty g(x)\,dx$ and since we know both of those integral converge, we can conclude that $\int_1^\infty f(x)+g(x)\,dx$ also converges.

(b) To prove that if $\int_1^\infty f(x)+g(x)\,dx$ converges, then $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ also converge, we can take the fact that

$$\int_1^\infty f(x)+g(x)\,dx = \int_1^\infty f(x)\,dx + \int_1^\infty g(x)\,dx$$

which was shown in (2) if we convert the limits to integrals and use it to prove that $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ are convergent since $\int_1^\infty f(x)+g(x)\,dx$ converges, each of it's parts must converge to produce a finite sum.

(c) A simple counterexample to the statement "If $\int_1^\infty f(x)\,dx$ diverges, then $\int_1^\infty (f(x))^2\,dx$ diverges" is the function $f(x) = \dfrac{1}{x}$. If we apply the p-test to $f(x)$ because it is in the form $\dfrac{1}{x^p}$, we find that since $f(x)$ has a p of 1, it fails the p-test and therefore $\int_1^\infty f(x)\,dx$ diverges.
However,

$$\int_1^\infty (f(x))^2\,dx = \int_1^\infty \frac{1}{x^2}\,dx$$

which means that $f(x)^2$ has a p of 2 which is greater than 1 and $\int_1^\infty (f(x))^2\,dx$ iss convergent. We have just disproved the statement "If $\int_1^\infty f(x)\,dx$ diverges, then $\int_1^\infty (f(x))^2\,dx$ diverges."