(3.1) Evaluate the definite integral: $\int_0^2 t^3\sqrt{9+t^4}\,dt$.

For this problem we can use u substitution. Let $u = 9 + t^4$ to simplify the radicand in the integrand. Subsequently we have to find the value of du

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 4t^3, \text{ therefore } du = 4t^3 dx$$

Now we must manipulate the integrand to contain the value $4t^3$. This can be achieved by multiplying by 1, or $\frac{4}{4}$.

$$\begin{aligned}\int_0^2 t^3\sqrt{9+t^4}\,dt &= \int_0^2 \frac{4}{4}\cdot t^3\sqrt{9+t^4}\,dt\\ &= \int_0^2 \frac{1}{4}\cdot 4t^3\sqrt{9+t^4}\,dt\\ &= \frac{1}{4}\int_0^2 \sqrt{u}\,du\end{aligned}$$

We have done this via the Substitution rule which states that $\int f\left(g\left(x\right)\right)g'(x)dx = \int f(u)du$ if $u = g(x)$. Now we must adjust the bounds of the integral which is accomplished by applying the function u to the bounds.

$$\begin{aligned}x = 0 : 9 + 0^4 = 9\\ x = 2 : 9 + 2^4 = 25\end{aligned}$$

Now we are done with the setup to simplifying the process of integrating this, we can see that we have an entirely new, simple integral left: $\int_9^{25}\sqrt{u}\,du$. All that is left is to evaluate the integral.

$$\begin{aligned}\frac{1}{4}\int_9^{25}\sqrt{u}\,du &= \frac{1}{4}\left[\frac{2u^{\frac{3}{2}}}{3}\right]_9^{25}\\ &= \frac{1}{4}\left(\frac{2\cdot 5^3}{3} - \frac{2\cdot 3^3}{3}\right)\\ &= \frac{1}{4}\left(\frac{250}{3} - \frac{54}{3}\right)\\ &= \frac{1}{4}\left(\frac{196}{3}\right)\\ &= \frac{49}{3}\end{aligned}$$

Thus, we can see the value of the definite integral is $\frac{49}{3}$.

Problem 2 on next page. →

(3.2) Evaluate the indefinite integral: $\displaystyle\int \frac{e^{2x}}{(e^x+1)^2}\,dx$.

Using the same method as last time, we define u as e^x+1. Then we find the value of du which is $e^x dx$. Then we simplify

$$\begin{aligned}\frac{e^{2x}}{(e^x+1)^2}dx &= \frac{e^x\cdot e^x}{(e^x+1)^2}\,dx\\ &= \frac{e^x}{(e^x+1)^2}e^x\,dx\\ &= \frac{e^x}{u^2}\,du\end{aligned}$$

To get rid of the e^x we can use our equation from before, $u=e^x+1$, to get $u-1=e^x$.

$$\int \frac{e^x}{u^2}\,du = \int \frac{u-1}{u^2}\,du$$

Now that we have the u form of the integrand, next we split the numerator into two separate integrals.

$$\begin{aligned}\int \frac{u-1}{u^2}\,du &= \int \frac{u}{u^2}-\frac{1}{u^2}\,du\\ &= \int \frac{1}{u}-u^{-2}\,du\\ &= \int \frac{1}{u}\,du - \int u^{-2}\,du\end{aligned}$$

The final step is to evaluate the antiderivative of both integrals and combine them. We can use the power rule for u^{-2} and we already know that the antiderivative of $\frac{1}{u}$ is $\ln|u|$.

$$\begin{aligned}\int \frac{1}{u}\,du - \int u^{-2}\,du &= \ln|u| + C - \left(\frac{u^{-1}}{-1}\right) + C\\ &= \ln|u| + \frac{1}{u} + C\\ &= \ln\left(|e^x+1|\right) + \frac{1}{e^x+1} + C\end{aligned}$$

Now we have simplified and added a constant C to our equation, so the final answer is

$$\int \frac{e^{2x}}{(e^x+1)^2}\,dx = \ln\left(|e^x+1|\right) + \frac{1}{e^x+1} + C$$

Problem 3 on next page. →

(3.3) Prove the following two statements

(a) Prove $\displaystyle\int \frac{1}{x\ln(x)}\,dx = \ln(|\ln(x)|) + C$.

(b) Prove $\displaystyle\int_a^{a^2} \frac{1}{x\ln(x)}\,dx = \ln(2)$ for any $a > 1$.

(a) To prove that $\displaystyle\int \frac{1}{x\ln(x)}\,dx = \ln(|\ln(x)|) + C$ we need to turn $\displaystyle\int \frac{1}{x\ln(x)}\,dx$ into $\ln(|\ln(x)|)$. To do that let us use u-substitution to calculate the value of the indefinite integral.
Let $u = \ln(x)$. Then du is $\dfrac{1}{x}\,dx$. This is really simple because both u and du are in the integrand so there is no algebra required. The new integral is

$$\int \frac{1}{u}\,du$$

Now we take the antiderivative of $\dfrac{1}{u}$ which is simply $\ln(|u|)$ and add C to get the expression

$$\ln(|u|) + C$$

Finally, we substitute the value for u back in to get a final value of

$$\ln(|\ln(x)|) + C$$

Thus, we have proved that $\displaystyle\int \frac{1}{x\ln(x)}\,dx = \ln(|\ln(x)|) + C$.

(b) We found the antiderivative to the integrand of the definite integral in part (a), so we can skip straight to the Evaluation Theorem. First, however we must split the integrals into two terms

$$\int_a^{a^2} \frac{1}{x\ln(x)}\,dx = \int_0^{a^2} \frac{1}{x\ln(x)}\,dx - \int_0^{a} \frac{1}{x\ln(x)}\,dx$$

Next apply the Evaluation Theorem to the integrals to get the expression using the antiderivative that we found in part (a).

$$\left(\ln\left|\ln\left(a^2\right)\right| - \ln|\ln(0)|\right) - \left(\ln|\ln(a)| - \ln|\ln(0)|\right)$$

This simplifies to

$$\ln|2\ln(a)| - \ln|\ln(a)|$$

Using the Product rule we can separate the logarithms leaving us with

$$\ln|2\ln(a)| - \ln|\ln(a)| = \ln(2) + (\ln|\ln(a)| - \ln|\ln(a)|) = \ln(2)$$

However, a cannot be 1 because if $a = 1$ then $a^2 = a$, therefore the integral is equal to 0 due to the bounds being equal. Thus, $\displaystyle\int_a^{a^2} \frac{1}{x\ln(x)}\,dx = \ln(2)$ for $a > 1$. This can also be expressed as $a \neq 1$. Otherwise, a can be any value.

Problem 4 on next page. →

(3.4) Professional Problem: Suppose f is continuous and $\displaystyle\int_1^4 f(x)\,dx = 7$.

(a) Calculate $\int_{-1}^{2} xf(x^2)\,dx$.

(b) Calculate $\int_{1}^{2} xf(x^2)\,dx$.

(c) Explain why your answers are the same.

(a) We can use u substitution for this problem again. Let $u = x^2$. Then, we find du which is

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

Utilizing some more algebra, $du = 2x\,dx$. To substitute du into the integral using the Substitution Rule, we must use algebra to force a $2x$ into the integrand. To do this, we multiply the integral by $\frac{2}{2}$ which creates the expression

$$\begin{aligned}\int_{-1}^{2} \frac{2}{2} \cdot xf\left(x^2\right)\,dx &= \int_{-1}^{2} \frac{1}{2} \cdot 2xf\left(x^2\right)\,dx \\ &= \frac{1}{2}\int_{-1}^{2} f\left(u\right)\,du\end{aligned}$$

Now that the integral is in respect to u, we must adjust the bounds of the integral, (a, b), by applying $g(x)$ so the new bounds are $(g(a), g(b))$. Squaring the values because $g(x) = x^2$, our new bounds are $(4, 1)$. Our integral is $\frac{1}{2}\int_1^4 f(u)\,du$. Using the definition of the integral, it evaluates to

$$\frac{1}{2} \cdot 7 = 3.5$$

Thus the integral is equal to 3.5.

(b) As the integrand of the integral is same as the prior, we know that the integral after u substitution is

$$\frac{1}{2}\int_1^2 f(u)\,du$$

Now we adjust the bounds by squaring them to get the integral

$$\frac{1}{2}\int_1^4 f(u)\,du$$

This is the same expression as in part (a), and the same integral from the problem statement. Therefore, we already know the value of this expression, which is 3.5.

(c) Let

$$\int_{-1}^{2} xf\left(x^2\right)\,dx = \int_{1}^{2} xf\left(x^2\right)\,dx + \int_{-1}^{1} xf\left(x^2\right)\,dx$$

Using u substitution, our equation becomes

$$\int_{1}^{4} f\left(u\right)\,du = \int_{1}^{4} f\left(u\right)\,du + \int_{1}^{1} f\left(u\right)\,du$$

Notice that the bounds of the third integral have become equal. Therefore, the value of that integral is 0. Restating, the equation becomes

$$\int_{1}^{4} f\left(u\right)\,du = \int_{1}^{4} f\left(u\right)\,du$$

Our answers for (a) and (b) are the same because the integral of the integrand is equal to 0 with the bound $(-1, 1)$. Thus, $\int_{-1}^{2} f(u)\,du = 0 + \int_1^2 f(u)\,du$ and (a) and (b) are equivalent.