What If?

The Comparison Test:

1. If $\sum a_n$ and $\sum b_n$ are series with positive terms, $\sum b_n$ is convergent, and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
2. If $\sum a_n$ and $\sum b_n$ are series with positive terms, $\sum b_n$ is divergent, and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

1. Show that if $\sum a_n$ is allowed to have negative terms, then part 1 of the Comparison Test is false. Give a specific counterexample.

2. Show that if $\sum b_n$ is allowed to have negative terms, then part 2 of the Comparison Test is false. Give a specific counterexample.

The Integral Test:

1. If f is a continuous, positive, and decreasing function on $[1, \infty)$, $a_n = f(n)$, and $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
2. If f is a continuous, positive, and decreasing function on $[1, \infty)$, $a_n = f(n)$, and $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

1. Let $f(x) = \sin(\pi x)$ and let $a_n = \sin(\pi n)$.

   (a) Is $\int_1^{\infty} f(x)\, dx$ convergent?

   (b) Is $\sum_{n=1}^{\infty} a_n$ convergent?

   (c) Which hypotheses of the Integral Test does $f(x)$ satisfy? Which doesn't it satisfy? What does this tell you about the hypotheses of the Integral Test?

2. Let $g(x) = |\sin(\pi x)| + \frac{1}{x^2}$ and let $b_n = g(n)$.

   (a) Is $\int_1^{\infty} g(x)\, dx$ convergent?

   (b) Is $\sum_{n=1}^{\infty} b_n$ convergent?

   (c) Which hypotheses of the Integral Test does $g(x)$ satisfy? Which doesn't it satisfy? What does this tell you about the hypotheses of the Integral Test?