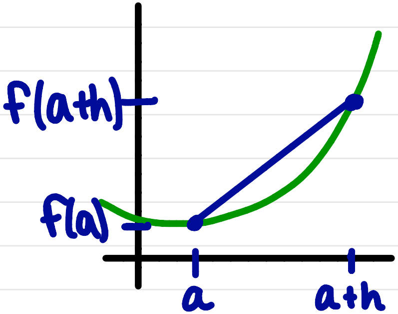
Linearization

A long time ago... in a Calculus class far, far away...

We defined the derivative of a function $f(x)$ at a point $x=a$:



Question: What is the slope of the secant line through the points $(a, f(a))$ and $(a+h, f(a+h))$?

WS 1

Recall The idea was that the average rate of change given by the slope of the secant line approximates the instantaneous rate of change, which is the slope of the tangent line.

Question What is the slope of the tangent line to the graph of $f(x)$ at the point $x=a$?

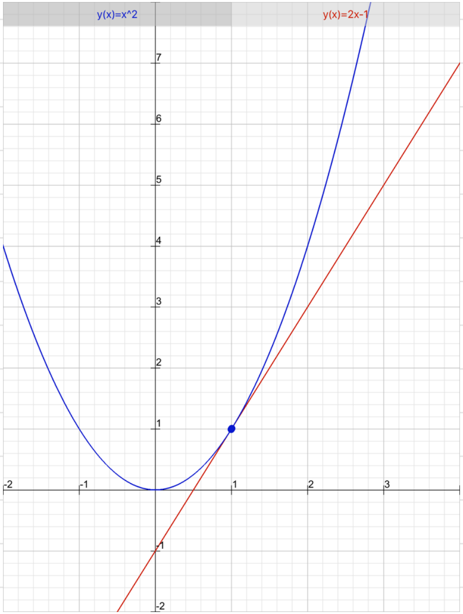
$$f'(a) = \boxed{\text{WS 2}}$$

The idea of linear approximation is to flip this:

- We used average rates of change to approximate instantaneous rates of change.
- We can also use instantaneous rates of change (i.e., the tangent line) to approximate average rates of change (i.e., the original function).

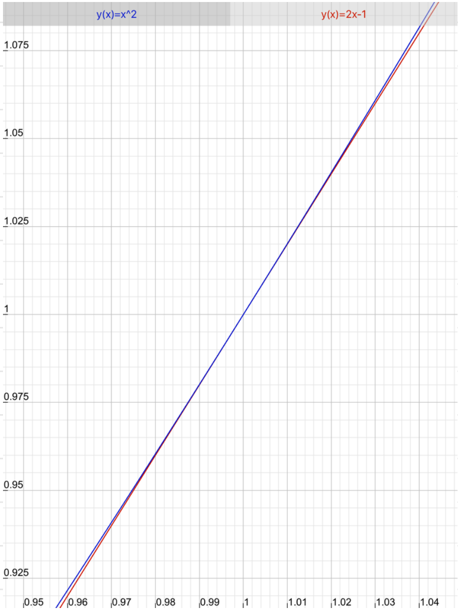
The tangent line gives a linear approximation to the original function near that point.

Another way to say this is that if we graph our function and the tangent line at a point, and then zoom in on that point, the graphs will be very close together.



graph of $y=x^2$ and tangent line at $x=1$.

zoomed in



zoomed in more

So, the tangent line gives us a good linear approximation of our function near that point:

$$f(x) \approx f(a) + f'(a)(x-a) \qquad \text{for } x \text{ near } a$$

The linear function $L(x)$ whose graph is the tangent line is called the linearization of f at a:

$$L(x) = f(a) + f'(a)(x-a)$$

So, $f(x) \approx L(x)$ for x near a.

What is this good for?

Suppose we want an approximate value for $\sqrt{4.1}$, but we don't have a calculator.

Idea We can use the linearization (or tangent line) to $f(x)=\sqrt{x}$ at the point $x=4$. This works well because we can easily compute $f(4)$ and $f'(4)$.

Solution Let $f(x)=\sqrt{x}$. Then $f'(x)=\frac{1}{2\sqrt{x}}$, so $f'(4)=\frac{1}{4}$ and $f(4)=2$.

So the linearization of f at a is:

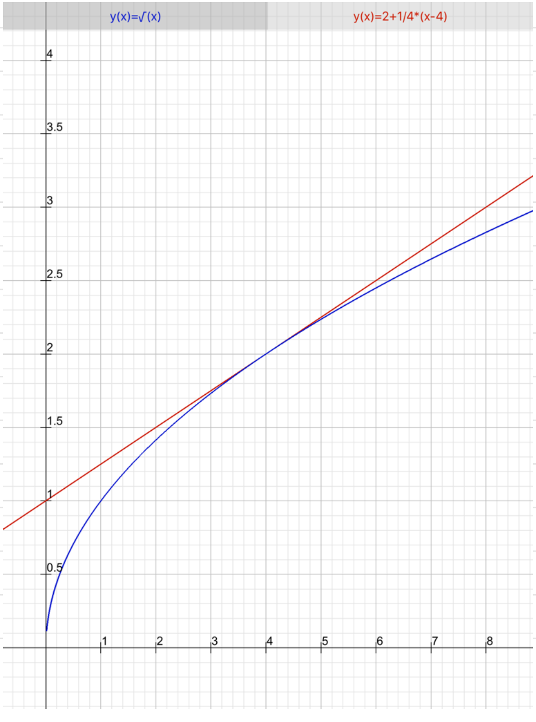
$$\begin{aligned} L(x) &= f(a)+f'(a)(x-a) \\ &= 2+\tfrac{1}{4}(x-4) \end{aligned}$$

Then we have:

$$\begin{aligned} \sqrt{4.1} &= f(4.1) \\ &\approx L(4.1) \\ &= 2+\tfrac{1}{4}(4.1-4) \\ &= 2+\tfrac{1}{4}\cdot 0.1 \\ &= 2.025 \end{aligned}$$

So linear approximation gives us $\sqrt{4.1} \approx 2.025$

Question Use a calculator to compute $\sqrt{4.1}$. How does this compare to our approximation? Was our approximation an overestimate or an underestimate?



WS 3

Question Looking at the graph of $f(x)=\sqrt{x}$ and the tangent line at $x=4$, explain how the concavity relates to this being an overestimate or underestimate.

WS 4

Newton's Method

We can also use tangent lines to approximate zeros of equations, although this process is a bit more complicated.

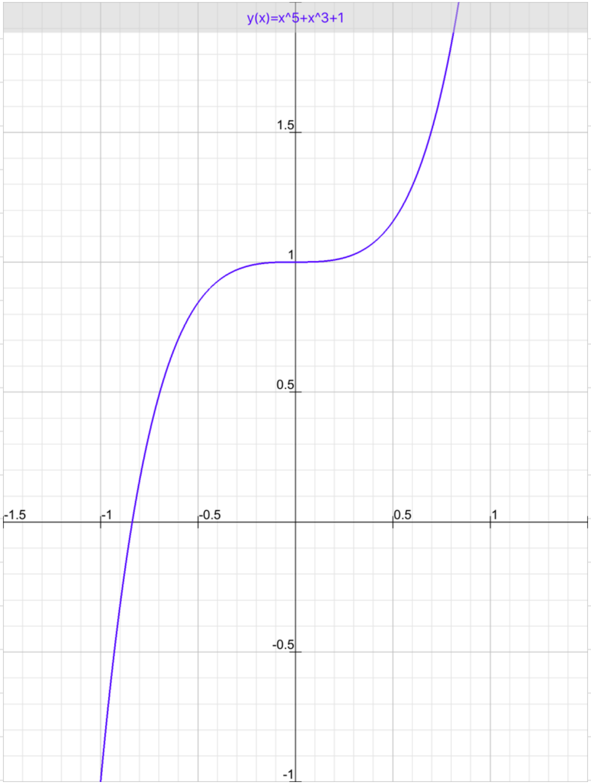
It would be strange to do Newton's method by hand (although it is possible), so use a calculator as much as you like for these problems (I certainly will).

It's easiest to see how this works with an example:

Problem Approximate x such that $x^5+x^3+1=0$.

You could try to solve this algebraically, but you would not succeed. It turns out that this is not solvable using algebra.

Problem Approximate x such that $x^5+x^3+1=0$.

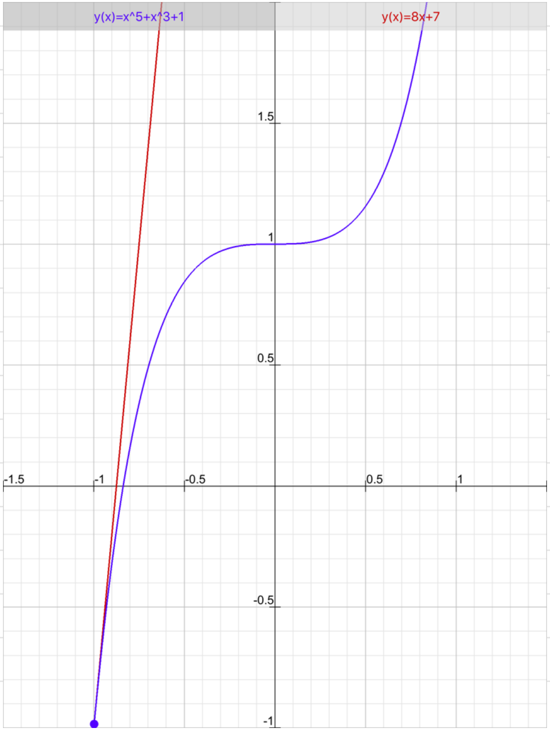


By looking at the graph, we can see that there is a solution, between -1 an $-\frac{1}{2}$. (You could also show this using the Intermediate Value Theorem)

Let's choose -1 as our initial guess for the root:

$$x_0 = -1$$

Idea The tangent line to $f(x)=x^5+x^3+1$ at $x=-1$ gives an approximation of $f(x)$ near $x=-1$. So, we can approximate where $f(x)=0$ by finding the x-intercept of the tangent line.



$f'(x)=5x^4+3x^2$, so $f'(-1)=8$ and an equation for the tangent line at $x=-1$ is:

$$y=8x+7$$

Solving for when $y=0$, we get

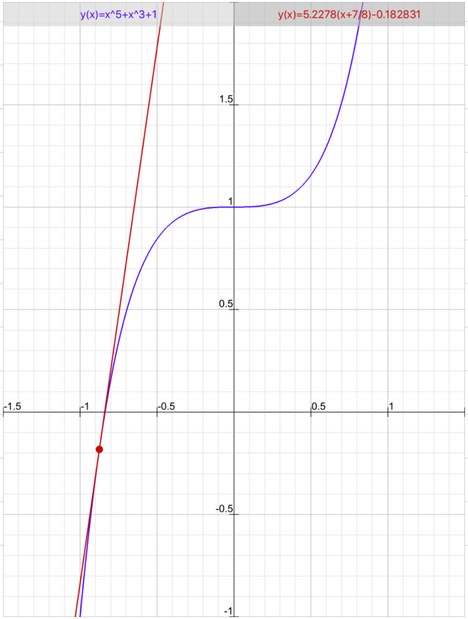
$$x=-\frac{7}{8}.$$

So our approximation is $x_1=-\frac{7}{8}$.

But, we can see from the graph that this isn't a great approximation.

It is, however, an improvement over our initial guess of -1.

<u>Idea</u> Let's repeat this, now finding the x-intercept of the tangent line at $x=-\frac{7}{8}$!



$$f'(-\tfrac{7}{8})(x-{-\tfrac{7}{8}})+f(-\tfrac{7}{8})=0$$

$$f'(-\tfrac{7}{8})(x-{-\tfrac{7}{8}})=-f(-\tfrac{7}{8})$$

$$(x-{-\tfrac{7}{8}})=\frac{-f(-7/8)}{f'(-7/8)}$$

$$x=\frac{-f(-7/8)}{f'(-7/8)}+-\frac{7}{8}$$

$$\approx -0.8400$$

$$x_2=-0.8400$$

We can see from the graph that this is a big improvement.

We can repeat this as many times as we like to get a more accurate approximation.

In general, we can get the next approximation x_n from the previous one x_{n-1} with the formula:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

So we can continue to approximate the root of $f(x)=x^5+x^3+1$ with initial guess $x_0=-1$:

$$x_0 = -1$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{f(-1)}{f'(-1)} = -\frac{7}{8} = -0.875$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -(-0.875) - \frac{f(-0.875)}{f'(-0.875)} \approx -0.8400$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx -0.8376$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx -0.8376$$

These become very accurate very quickly.

You try!

Question Use Newton's method to approximate the zero of $f(x)=e^x-2$ using an initial guess $x_0=1$.
Give your as a decimal.

$$x_0=1$$

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}\approx \_\_\_\_\_\_$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}\approx \_\_\_\_\_\_$$

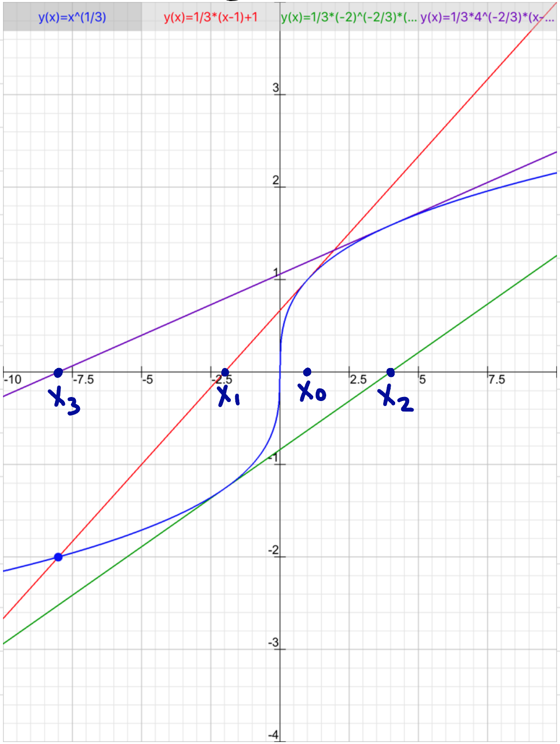
$$x_3=x_2-\frac{f(x_2)}{f'(x_2)}\approx \_\_\_\_\_\_$$

$$x_4=x_3-\frac{f(x_3)}{f'(x_3)}\approx \_\_\_\_\_\_$$

WS 5

What could go wrong?

Unfortunately, sometimes strange things happen and this doesn't work.



For example, if we approximate the zero of $f(x)=\sqrt[3]{x}$ with $x_0=1$, we get:

$$x_0 = 1$$

$$x_1 = 1 - \frac{\sqrt[3]{1}}{\frac{1}{3}(1)^{-2/3}} = -2$$

$$x_2 = -2 - \frac{\sqrt[3]{-2}}{\frac{1}{3}(-2)^{-2/3}} = 4$$

$$x_3 = 4 - \frac{\sqrt[3]{4}}{\frac{1}{3}(4)^{-2/3}} = -8$$

Although we can see from the graph that the zero is $x=0$, the "approximations" do not approach 0.