(13.1) If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$, $0 \le x \le 1$.
To find critical points, we will take the derivative of the function $f(x) = x^a(1-x)^b$, and solve for 0.

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^a(1-x)^b\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^a\right)(1-x)^b + \frac{\mathrm{d}}{\mathrm{d}x}\left((1-x)^b\right)x^a \\
&= (ax^{a-1})(1-x)^b + b(1-x)^{b-1}\frac{\mathrm{d}}{\mathrm{d}x}(1-x)x^a \\
&= -x^a b(1-x)^{b-1} + a(x-1)^b x^{a-1} \\
&= x^{a-1}(1-x)^{b-1}[(1-x)a - bx] \\
&= x^{a-1}(1-x)^{b-1}[a - x(a+b)]
\end{aligned}$$

Now to find the critical points, we must solve for $f' = 0$.

$$x^{a-1}(1-x)^{b-1}[a - x(a+b)] = 0$$

From here we can see that x must either equal 0, 1 , or $\dfrac{a}{a+b}$. Because we are trying to find the maximum on the interval [0,1], the critical points must be on the open interval (0,1). Thus, we can eliminate $x = 0$ and $x = 1$ from the pool of possible critical points, leaving $x = \dfrac{a}{a+b}$ as the only possible critical point on the interval.
Now that we have our critical point, we can just solve $f(\frac{a}{a+b})$ to find the possible maximum of the function in the interval $0 \le x \le 1$.

$$\begin{aligned}
f\left(\frac{a}{a+b}\right) &= \left(\frac{a}{a+b}\right)^a \left(1 - \frac{a}{a+b}\right)^b \\
&= \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b \\
&= \frac{a^a}{(a+b)^a} \cdot \frac{b^b}{(a+b)^b} \\
&= \frac{a^a b^b}{(a+b)^{a+b}}
\end{aligned}$$

To determine whether $\dfrac{a}{a+b}$ is a maximum, we must check if $f(0)$ and $f(1)$ are greater than $f\left(\dfrac{a}{a+b}\right)$.

$$\begin{aligned}
f(0) &= 0^a(1-0)^b \\
&= 0 \cdot 1^b \\
&= 0 \\
f(1) &= 1^a(1-1)^b \\
&= 1^a \cdot 0 \qquad\qquad = 0
\end{aligned}$$

Now we know that $\dfrac{a}{a+b}$ is the maximum point because a and b are positive numbers and thus are greater than 0.

$$\boxed{\text{The maximum value of } f(x) = x^a(1-x)^b,\ 0 \le x \le 1 \text{ is } \frac{a^a b^b}{(a+b)^{a+b}}}$$

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