(9.1) Let $a_n = \ln(4n+2) - \ln(n+1)$. Write the first three numbers in the sequence a_n and calculate its limit. If the limit doesn't exist, explain why.

Firstly, let's condense the sequence a_n.

$$ln(4n+2) - \ln(n+1) = \ln\left(\frac{4n+2}{n+1}\right)$$

Then we substitute 1, 2, and 3.

$$\begin{aligned} a_1, a_2, a_3 &= \ln\left(\frac{4+2}{1+1}\right), \ln\left(\frac{8+2}{2+1}\right), \ln\left(\frac{12+2}{3+1}\right) \\ &= \ln 3, \ln\frac{10}{3}, \ln\frac{14}{4} \end{aligned}$$

Next, let us take the limit of $\ln\left(\frac{4n+2}{n+1}\right)$ as n approaches ∞. First we must change a_n into $f(x)$ and take the limit

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \ln\left(\frac{4x+2}{x+1}\right)$$

We can use the limit law that states that "If $\lim_{n\to\infty} a_n = L$ and $f(x)$ is continuous at L, $\lim_{n\to\infty} f(a_n) = f(L)$" to rearrange our limit and make it easier to evaluate.

$$\lim_{x\to\infty} \ln\left(\frac{4x+2}{x+1}\right) = \ln\left(\lim_{x\to\infty} \frac{4x+2}{x+1}\right)$$

This is relatively simple to evaluate as we simply divide everything within the limit by x.

$$\begin{aligned} \ln\left(\lim_{x\to\infty} \frac{4x+2}{x+1}\right) &= \ln\left(\lim_{x\to\infty} \frac{4+\frac{2}{x}}{1+\frac{1}{x}}\right) \\ &= \ln\left(\frac{4+0}{1+0}\right) \\ &= \ln 4 \end{aligned}$$

Thus, the limit of the sequence a_n is $\ln 4$.

(9.2) A sequence b_n is given by $b_1 = 2$, $b_{n+1} = \sqrt{6 + b_n}$.

(a) Use induction to prove b_n is increasing.

(b) Use induction to prove b_n is bounded.

(c) Explain why Monotonic Sequence Theorem applies to b_n.

(d) Find $\lim_{n\to\infty} b_n$.

(a) First let us do the Base Case.

$$\begin{aligned} b_2 &= \sqrt{6 + b_1} \\ &= \sqrt{6+2} \\ &= \sqrt{8} \\ &= 2\sqrt{2} > b_1 \end{aligned}$$

Thus we have seen that for $n = 1$, the sequence is indeed increasing. For our assumption, let us assume that the sequence is increasing for $n = k$ and that

$$b_k \geq b_{k-1} \tag{1}$$

Then, we shall show that this is true for $n = k + 1$ and that:

$$b_{k+1} \geq b_k$$

Now substituting our b_k in we can do the math:

$$\begin{aligned} b_{k+1} &\geq b_k \\ \sqrt{6 + b_k} &\geq \sqrt{6 + b_{k-1}} \\ 6 + b_k &\geq 6 + b_{k-1} \\ b_k &\geq b_{k-1} \end{aligned}$$

Looking back at (1), note that we assumed this inequality to be true, and therefore, b_n is increasing.

(b) Let the bound of b_n be 3. Then, our base case is

$$\begin{aligned} b_1 &= 2 \\ &\leq 3 \end{aligned}$$

Thus, we shall assume that $b_k \leq 3$ for $n = k$, and prove that $b_{k+1} \leq 3$. We know that

$$b_{k+1} = \sqrt{6 + b_k}$$

Using our assumption that $b_k \leq 3$, we can say that

$$\sqrt{6 + b_k} \leq \sqrt{6+3} \qquad = \sqrt{6 + b_k} \leq 33 \qquad \leq 3$$

Thus we have proved that b_n is bounded.

(c) The Monotonic Sequence Theorem applies to b_n since we have proved that b_n is both monotonic and bounded, which the two requirements for the Monotonic Sequence theorem and thus b_n converges.

(d) First of all note that for our sequence

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{n+1} \tag{2}$$

Let

$$B = \lim_{n\to\infty} b_n$$

Then, we can do the algebra using (1) that

$$\begin{aligned} B &= \lim_{n\to\infty} b_{n+1} \\ &= \lim_{n\to\infty} \sqrt{6 + b_n} \end{aligned}$$

Then, using limit laws, we can change our limit above to:

$$\begin{aligned} \lim_{n\to\infty} \sqrt{6 + b_n} &= \sqrt{6 + \lim_{n\to\infty} b_n} \\ &= \sqrt{6 + B} \\ B &= \sqrt{6 + B} \\ B^2 &= 6 + B \\ 0 &= B^2 - B - 6 \\ &= (B - 3)(B + 2) \end{aligned}$$

From this we can see that $B = 3, -2$, however since b_n is increasing and $-2 < b_1$, $B = 3$. Therefore $\lim_{n\to\infty} b_n = 3$

(9.3) Let $c_n = \dfrac{3-2n}{n}$.

(a) Prove that c_n is decreasing w/ algebra

(b) Use algebra to prove that c_n is bounded below by -2 and above by 1.

(a) If c_n is decreasing, then

$$\begin{aligned}\frac{3-2(n+1)}{n+1} &\leq \frac{3-2n}{n} \\ 0 &\leq \frac{3-2n}{n} - \frac{3-2(n+1)}{n+1} \\ &\leq (n+1)(3-2n) - n(1-2n) \\ &\leq 3n - 2n^2 + 3 - 2n - n + 2n^2 \\ &\leq 3\end{aligned}$$

Thus, we have proved that since the value of n has no bearing on the relationship between c_n and c_{n+1} and our proposed inequality was true, c_n is decreasing.

(b) Since $c_1 = 1$ and the sequence is decreasing, we know that it is bounded above by 1. To prove that the sequence is bounded below by -2, we can find the range of y like so:

$$\begin{aligned}\frac{3-2x}{x} &= y \\ 3-2x &= xy \\ 3 &= xy + 2x \\ &= x(2+y) \\ \frac{3}{2+y} &= x\end{aligned}$$

From this, we can clearly see that the range of c_n is $1, -2$ since the domain is $1, \infty$.

(9.4) Professional Problem: Prove or provide a specific counterexample to the following statements;

(a) If $|a_n|$ is convergent, a_n is convergent.
(b) If a_n and b_n are divergent, then $b_n + a_n$ diverges.
(c) If a_n and b_n are divergent, then $a_n \cdot b_n$ diverges.
(d) If a_n and $a_n \cdot b_n$ are convergent sequences, then b_n converges.

(a) **This statement is false.** The sequence $a_n = (-1)^n$ is a counterexample to the statement

"If $|a_n|$ is convergent, a_n is convergent."

This is because $(-1)^n$ is divergent by the theorem in the textbook which states,

"The sequence r^n is convergent if $-1 < r \leq 1$ and divergent for all other values of r."

However, $|a_n| = 1$ because of the limit law

$$\lim_{x\to\infty} c = c \tag{3}$$

Therefore, the statement is false.

(b) **This statement is false.** The sequences

$$a_n = n$$

and

$$b_n = -n$$

are counterexamples to the statement

"If a_n and b_n are divergent, then $b_n + a_n$ diverges."

This is because they are both divergent on their own, which can be found by simply evaluating the limits of both, but $a_n + b_n = 0$.
According to the limit law referenced in equation (1), $a_n + b_n$ is therefore divergent since the sequence is a constant.

(c) **This statement is false.** The sequences of

$$a_n = \cos n$$

and

$$b_n = \sec n$$

are counterexamples to the statement

"If a_n and b_n are divergent, then $a_n \cdot b_n$ diverges."

They diverge because evaluation of them using the theorem

"If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

requires the evaluation of $\cos\infty$ and $\sec\infty$. Therefore, a_n and b_n are divergent since they require the evaluation of ∞, $a_n \cdot b_n = 1$ and again using equation (1), is convergent.

(d) **This statement is false.** The combination of the sequences $a_n = 0$ and $b_n = n$ is a counterexample of the statement

"If a_n and $a_n \cdot b_n$ are convergent sequences, then b_n converges."

According to (1), then the limit is convergent. However, $b_n = n$ isn't convergent on its own which can be found from basic evaluation of the limit.