Oscillatory Motion

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Block-Spring System

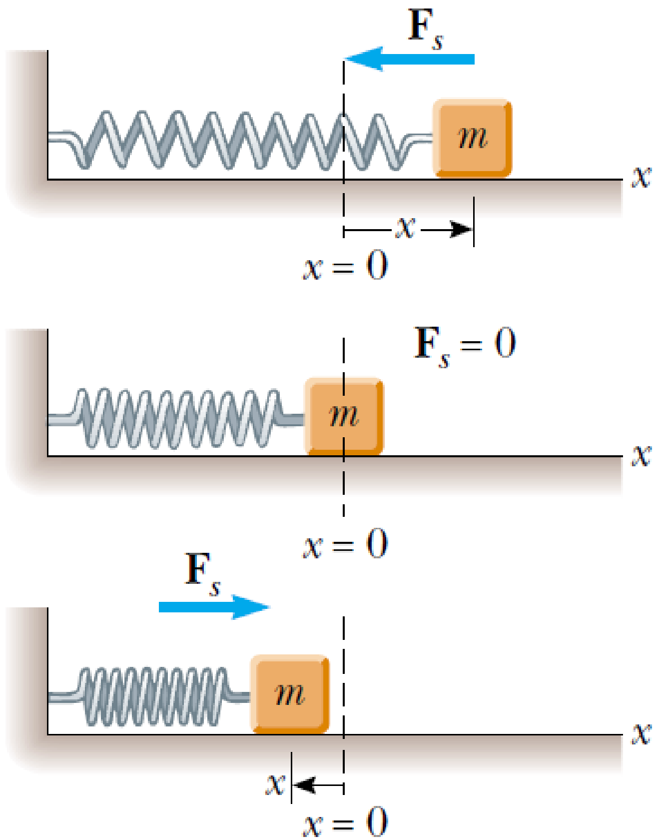
Hooke's Law:

$$F_s = -kx$$

F_s is a restoring force (ergo, the negative sign)

Simple harmonic motion – when an object's acceleration is proportional to its position and is oppositely directed to the displacement from equilibrium

$$a_x = -\frac{k}{m}x$$



If the block on the end of the spring is pulled to position A and released. Over one full cycle of motion, what will its total distance travelled be? What is its total displacement?

Simple Harmonic Motion

Let’s model the previous example analytically:

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Setting $\omega^2 = k/m$,

$$\omega = \sqrt{\frac{k}{m}}$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Remember, x is a function (it can be more accurately written as x(t)), so it is a function that, when differentiated twice, yields itself multiplied by a factor of $-\omega^2$:

$$x(t) = A\cos(\omega t + \phi)$$

Where A is the amplitude of motion, ω the angular frequency in rad/s, and ϕ

The phase constant – determined by the position and velocity of the object at $t = 0$.

Simple Harmonic Motion

Period of motion – time interval of one full cycle (in s)

$$T = \frac{2\pi}{\omega}$$

Frequency of motion – number of oscillations per unit time interval (in Hz)

$$f = \frac{1}{T}$$

Note: frequency f and angular frequency ω are not the same, but they are related

Relating T and f to mass and the spring constant:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

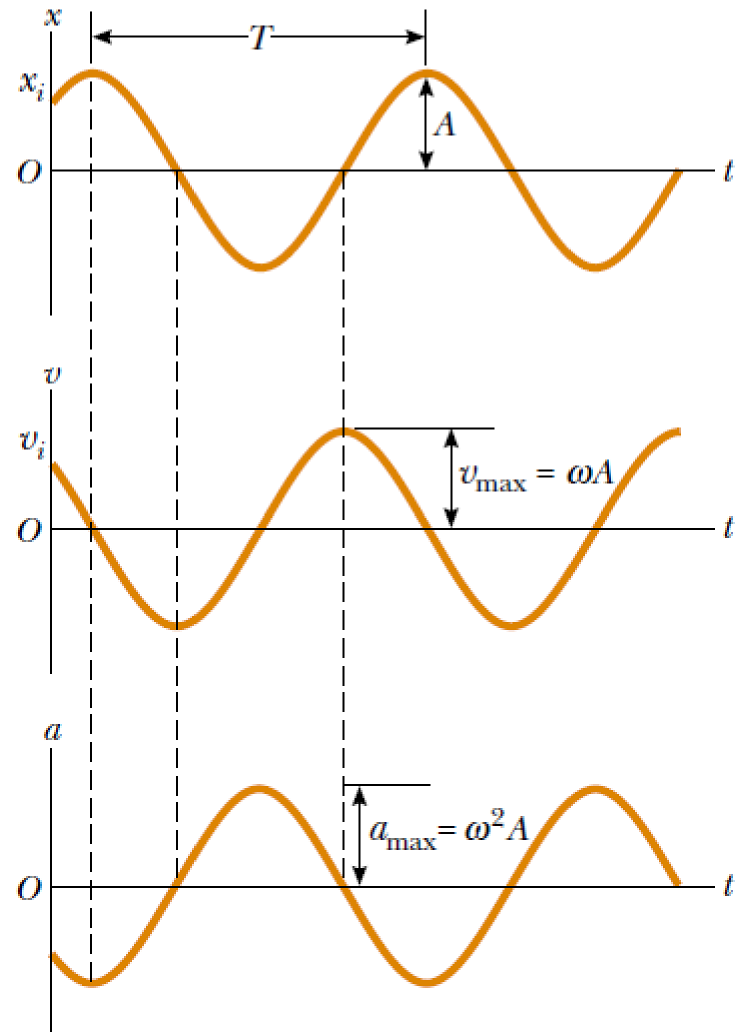
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Kinematics of SHO

$$x(t) = A\cos(\omega t + \phi)$$

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi)$$



Example: Oscillating Object

An object oscillates with simple harmonic motion along the x axis. Its position varies with time according to the equation $x = 4.00m\ \cos\left(\pi t + \frac{\pi}{4}\right)$, where time is in seconds and all angles are in radians.

a) Find the amplitude, frequency, and period of the motion
b) Find the velocity and acceleration of the object at any time t
c) Find the position, velocity, and acceleration of the object at 1.00 s.
d) Find the maximum speed and maximum acceleration of the object
e) Find the displacement of the object between 0 and 1 second

Energy of the Simple Harmonic Oscillator

For a frictionless surface in our block-spring system, the total mechanical energy is constant, with kinetic and potential energies given by:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin(\omega t + \phi)$$

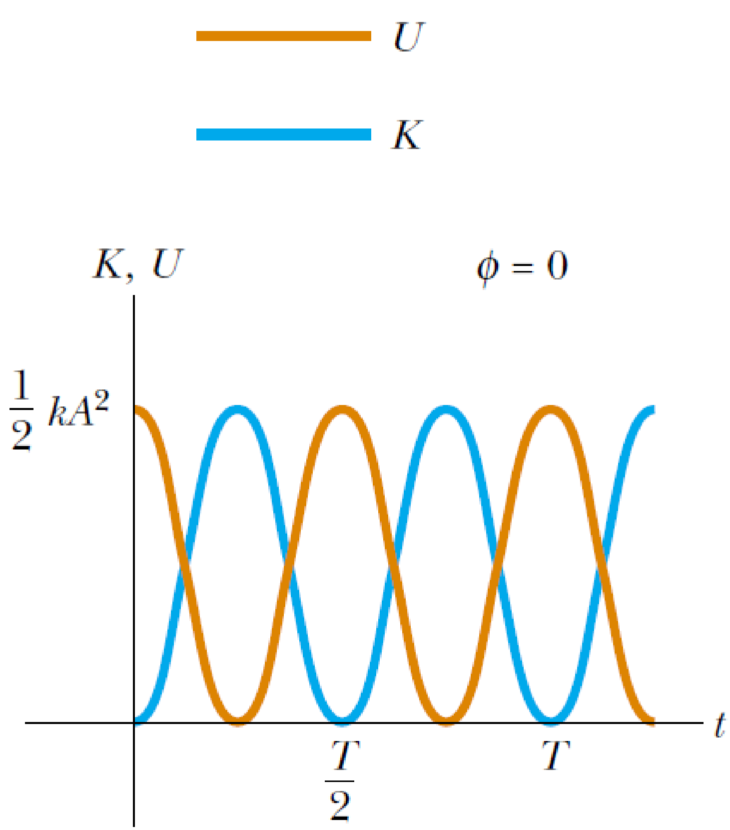
$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos(\omega t + \phi)$$

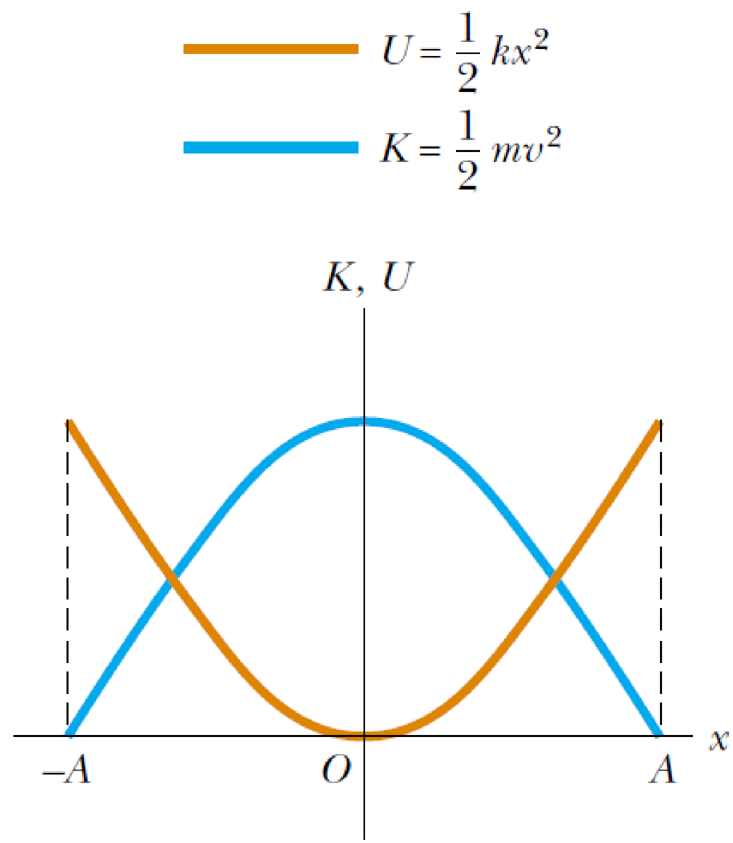
$$E = K + U$$

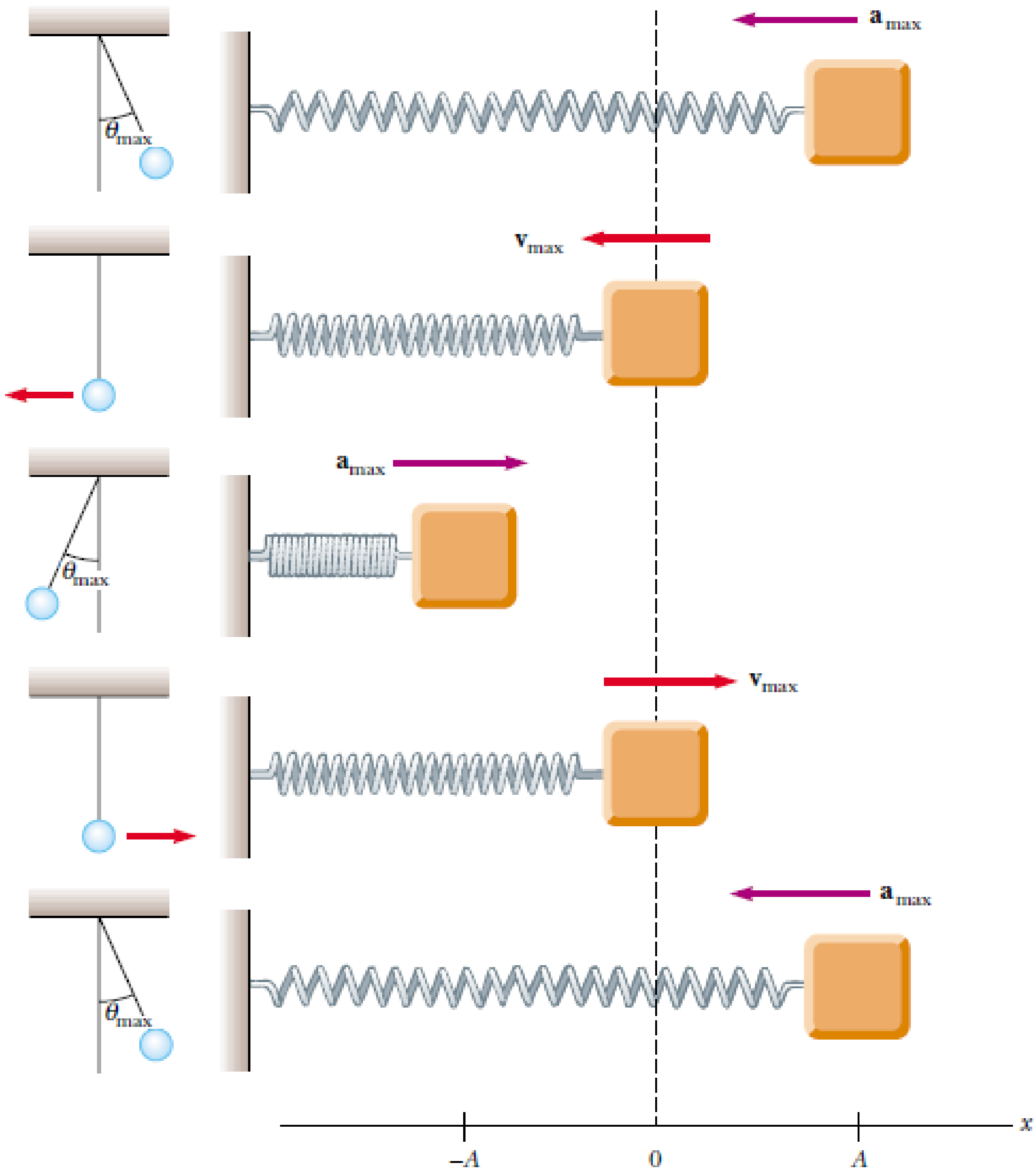
$$E = \frac{1}{2}kA^2$$

The total mechanical energy of a SHO is a constant of the motion and is proportional ot the square of the amplitude.

Energy of the SHO






$\mathbf{a}_{max}$
θ_{max}
$\mathbf{v}_{max}$
$\mathbf{a}_{max}$
θ_{max}
$\mathbf{v}_{max}$
$\mathbf{a}_{max}$
θ_{max}
x
$-A$
0
A

Example: Horizontal Oscillations

A 0.5 kg cart connected to a light spring for which the force constant is 20 N/m oscillates on a horizontal, frictionless air track.

a) Calculate the total energy of the system and the maximum speed of the cart if the amplitude of the motion is 3.00 cm.
b) What is the velocity of the cart when the position is 2.00 cm?
c) Calculate the kinetic ad potential energies of the system when the position is 2.00 cm.

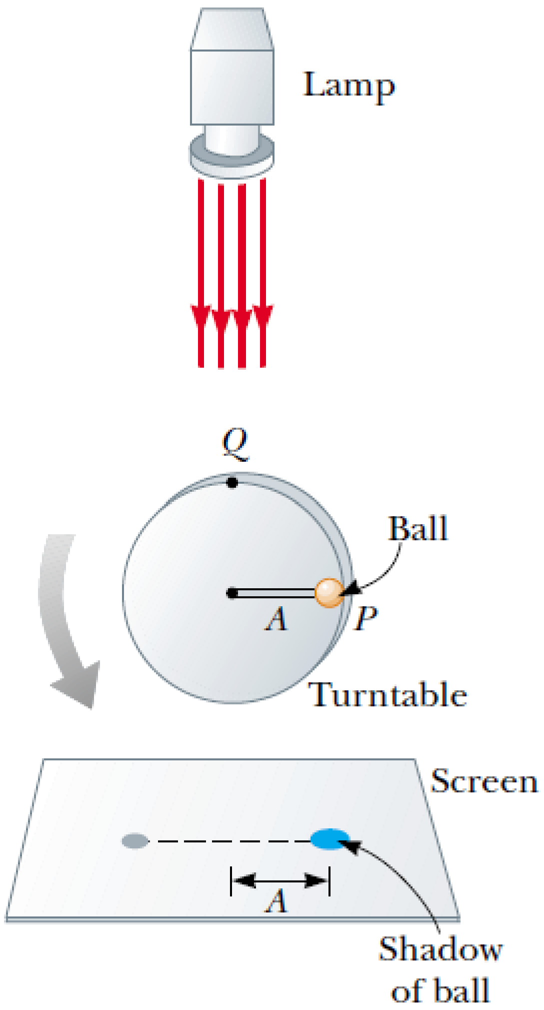
SHM vs UCM

How does rotational motion relate to simple harmonic motion?

As the turntable rotates with constant angular speed, the shadow of the ball moves back and forth in simple harmonic motion.

Simple harmonic motion along a straight line can be represented by the projection of uniform circular motion along a diameter of a reference circle.

Uniform circular motion can be considered a combination of two simple harmonic motions, one along the x axis and one along the y axis, with the two differing in phase by $\pi/2$.



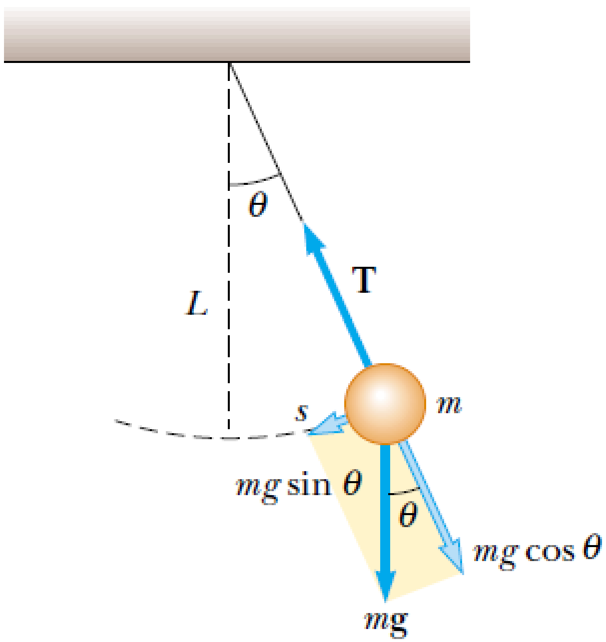
Example: Circular Motion

An object rotates CCW in a circle of radius 3.00 m with a constant angular speed of 8.00 rad/s. Initially, the object has an x coordinate of 2.00 m and is moving to the right.

a) What is the x coordinate as a function of time?
b) What are the x components of the object's velocity and acceleration at any time t?

The Pendulum

Simple pendulum – particle-like bob of mass m suspended by a light string of length L that is fixed at the upper end.



We assume the angular displacements are small ($< 10°$)

$$F_t = -mg\sin\theta = m\frac{d^2s}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

If θ is our position variable instead of x, we should expect that the left side of the equation should be proportional to θ in order to comply with SHM. This is not the case here.

But in the **small angle approximation, $\boldsymbol{\sin\theta \approx \theta}$:**

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

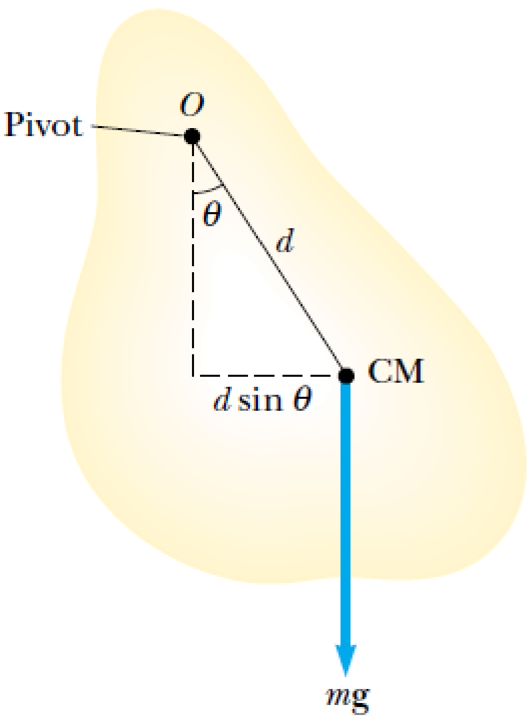
$$\boxed{\begin{gathered}\omega = \sqrt{\frac{g}{L}} \\ T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}\end{gathered}}$$

Physical Pendulum

Physical pendulum – when a hanging object oscillates about a fixed axis that doesn't pass through its centre of mass and the object can't be approximated as a point mass.

The gravitational force provides a torque about an axis O defined by the moment of inertia I:

$$\Sigma\tau = I\alpha$$



$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

Or in the small angle approximation,

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta$$

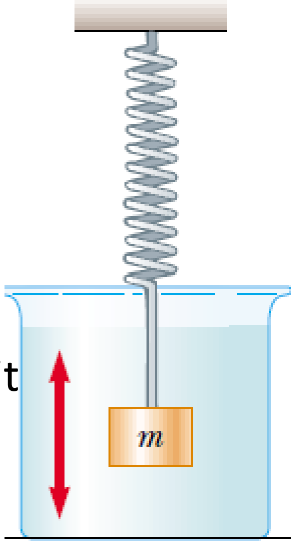
Where, $\omega = \sqrt{\frac{mgd}{I}}$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

Example: Swinging Rod

A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane. Find an expression for the period of oscillation if the amplitude of the motion is small and give a value for a rod of length 7 m and mass 22 kg.

Damped Oscillations



In real systems (non-ideal ones), nonconservative forces like friction will inhibit the motion → the mechanical energy of the system decreases over time.

We add a generic retarding force that is velocity dependent and is parametrized by:

$$R = -bv$$

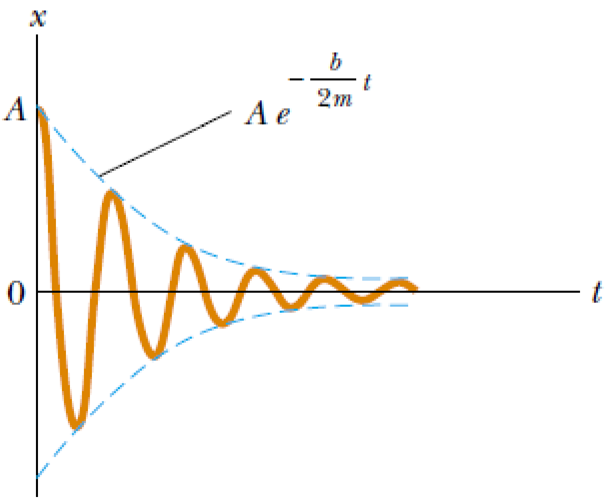
Then,

$$\Sigma F_x = -kx - bv_x = ma_x$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$



Consider the limiting conditions on b…

Damped Oscillations

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

ω_0 is referred to as the natural frequency.

Underdamped – when the magnitude of the maximum retarding force $bv_{max} < kA$

Critically damped – when b reaches a critical value such that $\frac{b_c}{2m} = \omega_0$, the system does not oscillate

Overdamped – when the maximum retarding force is greater than the restoring force $bv_{max} > kA$ and $\frac{b}{2m} > \omega_0$

Damped Oscillations

a – underdamped; b – critically damped; c - overdamped