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Parameter Estimation Assignment

Q1 Let $(x_1, x_2, \ldots)$ be a random sample of size n taken from a normal population with parameters: mean = θ_1 and variance = θ_2. Find the maximum likelihood estimates of these two parameters.

A1
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
(PDF of normal distribution)

$\mu = \theta_1$, $\sigma^2 = \theta_2$

$$f(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$f(x_i) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x-\theta_1)^2}{2\theta_2}}$$

Likelihood function

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i)$$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} (\theta_2)^{-1/2} \prod_{i=1}^{n} (2\pi)^{-1/2} \prod_{i=1}^{n} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = (\theta_2)^{-n/2} (2\pi)^{-\frac{n}{2}} e^{-\sum_{i=1}^{n}\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = (\theta_2)^{-\frac{n}{2}} (2\pi)^{-\frac{n}{2}} e^{\left(-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2\right)} \quad \text{— (1)}$$

Taking log both sides in eq^n (1)

$$\ln(L(\theta_1, \theta_2)) = \ln\left[(\theta_2)^{-\frac{n}{2}} (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}\right]$$

$$Z = \ln L(\theta_1, \theta_2) = -\frac{n}{2}\ln\theta_2 - \frac{n}{2}\ln(2\pi) - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i - \theta_1)^2 \quad -(2)$$

Differentiating eq^n (2) w.r.t θ_1

$$\frac{\partial Z}{\partial \theta_1} = \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i - \theta_1)$$

Now, $\frac{\partial Z}{\partial \theta_1} = 0$

$$\Rightarrow \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i - \theta_1) = 0$$

$$\Rightarrow \frac{1}{\theta_2}\left(\sum_{i=1}^{n} x_i - n\theta_1\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - n\theta_1 = 0$$

$$\Rightarrow n\theta_1 = \sum_{i=1}^{n} x_i$$

$$\Rightarrow \theta_1 = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Rightarrow \theta_1 = \bar{x}_n \Rightarrow \boxed{\theta_{1_{MLE}} = \bar{x}_n} \quad -(3)$$

Differentiating eq^n (2) w.r.t θ_2

$$\frac{\partial Z}{\partial \theta_2} = \frac{-n}{2\theta_2} + \frac{1}{2(\theta_2)^2}\sum_{i=1}^{n}(x_i - \theta_1)^2$$

Now, $\frac{\partial Z}{\partial \theta_2} = 0$

$$\Rightarrow \frac{-n}{2\theta_2} + \frac{1}{2(\theta_2)^2}\sum_{i=1}^{n}(x_i-\theta_1)^2 = 0$$

$$\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

From eq^n ③ $\theta_1 = \bar{x}_n$

$$\boxed{\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2}$$

Q2 Let $X_1, X_2, \ldots, X_n$ be a random sample from $B(m,\theta)$ distribution where $\theta \in \Theta = (0,1)$ is unknown and 'm' is a known positive integer. Compute value of θ using M.L.E.

A2 $f(x) = {}^nC_x\, p^x (1-p)^{n-x}$ (PDF of binomial distribution)

Here, $n = m$, $p = \theta$.

$$f(x) = {}^mC_x\, \theta^x (1-\theta)^{m-x} \Rightarrow f(x_i) = {}^mC_{x_i}\, \theta^{x_i}(1-\theta)^{m-x_i}$$

Likelihood function

$$L(m,\theta) = \prod_{i=1}^{n} f(x_i)$$

$$L(m,\theta) = \prod_{i=1}^{n} {}^mC_{x_i}\, \theta^{x_i}(1-\theta)^{m-x_i}$$

$$L(m,\theta) = \prod_{i=1}^{n} {}^mC_{x_i} \prod_{i=1}^{n}\theta^{x_i} \prod_{i=1}^{n}(1-\theta)^{m-x_i}$$

$$L(m,\theta) = \prod_{i=1}^{n} {}^mC_{x_i}\, \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{mn-\sum_{i=1}^{n} x_i} = L$$

Taking log both sides

$$\ln(L(m,\theta)) = \ln\left[\prod_{i=1}^{n} {}^mC_{x_i}\, \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{mn-\sum_{i=1}^{n} x_i}\right]$$

$$\ln(L(m,\theta)) = \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\right) + \ln\left(\theta^{\sum_{i=1}^{n} x_i}\right) + \ln\left[(1-\theta)^{mn - \sum_{i=1}^{n} x_i}\right]$$

$$Z = \ln(L(m,\theta)) = \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\right) + \sum_{i=1}^{n} x_i \ln\theta + \left(mn - \sum_{i=1}^{n} x_i\right)\left(\ln(1-\theta)\right) \quad -①$$

Differentiating eq^n ① w.r.t θ

$$\frac{\partial Z}{\partial \theta} = \frac{1}{\theta}\sum_{i=1}^{n} x_i + \left(\frac{\sum_{i=1}^{n} x_i - mn}{1-\theta}\right)$$

Now, $\frac{\partial Z}{\partial \theta} = 0$

$$\Rightarrow \frac{1}{\theta}\sum_{i=1}^{n} x_i + \left(\frac{\sum_{i=1}^{n} x_i - mn}{1-\theta}\right) = 0$$

$$\frac{\sum_{i=1}^{n} x_i - mn}{\theta - 1} = \left(\frac{\sum_{i=1}^{n} x_i}{\theta}\right)$$

$$1 - \frac{mn}{\sum_{i=1}^{n} x_i} = \frac{\theta - 1}{\theta}$$

$$+\frac{mn}{\sum_{i=1}^{n} x_i} = \frac{+1}{\theta}$$

$$\theta = \frac{\bar{x}_n}{m}$$

$$\boxed{\theta_{MLE} \in (0,1) = \frac{\bar{x}_n}{m}}$$