Lecture Content Overview - Michaelmas Term 2021

Contents

1 Analysis and Topology (2020)

- **Lecture 1.** Definition of pointwise and uniform convergence of functions with examples. Proof that the uniform limit of continuous functions is continuous.
- **Lecture 2.** Proof that uniform limit preserves boundedness and integrability. Proof of integrability of uniform limit of sum of integrable functions. Uniform limit of sum of continuously differentiable functions. Uniformly Cauchy sequence definition and general principle of uniform convergence. Weierstrass M-test.
- **Lecture 3.** Weierstrass M-test examples. Proof of uniform convergence inside the radius of convergence. Uniform continuity definition. Proof that continuous implies uniformly continuous on a closed bounded interval.
- **Lecture 4.** Definition of a metric space and examples. Subspaces and product spaces.
- **Lecture 5.** Convergence and uniqueness of limits in general metric spaces, with examples. Continuity in general metric spaces; equivalence with the sequence statement of continuity. Compositions of continuous functions being continuous.
- **Lecture 6.** Continuous, isometric, Lipschitz and uniformly continuous functions between metric spaces. Topology of metric spaces - neighbourhoods and open sets. Continuity in terms of topology. Union and intersection properties of open sets.
- **Lecture 7.** Proof that complement of closed sets are open. Homeomorphisms and equivalent/Lipschitz equivalent metrics. Cauchy sequences in general metric spaces; proof that convergent $\implies$ Cauchy $\implies$ bounded. Completeness of metric spaces.
- **Lecture 8.** Proof that l_∞ is complete in the uniform metric. Completeness and closure of subspaces. Proof that several spaces are complete in the uniform metric. The contraction mapping theorem.
- **Lecture 9.** Application of the contraction mapping theorem. The Lindelöf-Picard theorem and special cases.
- **Lecture 10.** Definition of topological spaces and examples; metrisable spaces. Indiscrete, discrete and cofinite topologies. Hausdorff spaces. Neighbourhoods and properties of open and closed sets. Convergence in topological spaces and uniqueness of limits in Hausdorff spaces.
- **Lecture 11.** Interior and closure of a subset of a topological space. Dense and separable subsets. Subspace topology. Bases for a topology. Second countable topological spaces.

- **Lecture 12.** Continuous functions between topological spaces. Product topologies.
- **Lecture 13.** Quotient topologies and examples; continuity and openness. Proof that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1. Hausdorff spaces and closure.
- **Lecture 14.** Connectedness of topological spaces; relation to intervals. Conditions for connectedness of spaces and subspaces. Proof that connectedness is a topological property; examples.
- **Lecture 15.** Connectedness of union of subsets and connectedness of product space. Connected components. Path connectedness. Proof that open subsets of $\mathbb{R}^n$ are connected iff they are path-connected. Proof that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic for $n \geq 2$.
- **Lecture 16.** Compactness of topological spaces. "Bounded and attains bounds" property. Compactness of subspaces. Proof that $[0, 1]$ is compact; more examples. Proof that compactness is a topological property.
- **Lecture 17.** The topological inverse function theorem and applications. Tychonov's theorem. The Heine-Borel theorem. Local uniform convergence.
- **Lecture 18.** Sequential compactness. Totally bounded spaces and properties. Equivalence of compactness, sequential compactness and completeness/total boundedness for metric spaces.
- **Lecture 19.** The space of linear maps as a metric space. Differentiability in $\mathbb{R}^n$ in terms of linear maps, with examples.
- **Lecture 20.** More linear map examples. General definition of differentiability. Proof that differentiable implies continuous. Proof of chain rule. Proof that components of a differentiable function are differentiable. Proof of sum and product properties.
- **Lecture 21.** Partial derivatives, including expression in terms of basis. The Jacobian matrix.
- **Lecture 22.** Mean value inequality proof. Proof of inverse function theorem.
- **Lecture 23.** Second derivatives, including expression as a bilinear map. Examples. Proof of symmetry of mixed partial derivatives.
- **Lecture 24.** Local maxima, minima and stationary points. Expression of the second derivative in terms of an expansion. Positive and negative definiteness and relation to nature of stationary points.

2 Linear Algebra

- **Lecture 1.** Definitions of vector spaces and subspaces, and properties. Sum of subspaces; quotient spaces.
- **Lecture 2.** Span, linear independence and basis. Steinitz exchange lemma.
- **Lecture 3.** Dimension (well-defined). Basic properties of dimensions of subspaces. Direct sums and propositions relating basis and subspaces.
- **Lecture 4.** Definition of linear map; extension by linearity. Isomorphisms and proof that n-dimensional vector spaces over a field are isomorphic. Image, kernel and isomorphism theorem. Rank-nullity theorem proof.
- **Lecture 5.** Matrices and linear maps; using matrices to represent linear maps. Bases and dimension.
- **Lecture 6.** Change of basis/equivalent matrices and related propositions. Proof that column and row rank are equal.

- **Lecture 7.** Similarity of matrices. Elementary column and row operations, with corresponding elementary matrices. Constructive proof of equivalence with I_r block matrix. Gauss's pivot algorithm and finding matrix inverse.
- **Lecture 8.** Dual spaces and linear forms. Dual basis. Annihilator definition and properties. Dual map and representation of linear map in dual basis.
- **Lecture 9.** Properties of the dual map and double dual.
- **Lecture 10.** Annihilator intersection properties. Bilinear forms and properties; matrix representation. Degeneracy of bilinear forms. Orthogonal forms and change of basis for bilinear forms.
- **Lecture 11.** Similar matrices and trace. Definition of determinant and examples. Properties of volume form; proof that determinant is a volume form.
- **Lecture 12.** Proof that there is only one volume form up to a constant. Product property of determinants; other basic properties. Determinant of block triangular matrices.
- **Lecture 13.** More volume form properties. Adjugate matrix definition.
- **Lecture 14.** Cramer's rule. Eigenvectors and eigenvalues. Condition for triangulable matrices.
- **Lecture 15.** Diagonalisation and projection operators. Simultaneous diagonalisation.
- **Lecture 16.** Minimal polynomial definition. Cayley-Hamilton theorem and multiplicities of eigenvalues.
- **Lecture 17.** Inequalities for algebraic and geometric multiplicities, and multiplicity of root of minimal polynomial. Jordan normal form definition and generalised eigenspace decomposition. More on projection operators.
- **Lecture 18.** Bilinear forms and quadratic forms. Polarisation identity and diagonalisation of symmetric bilinear forms.
- **Lecture 19.** Sylvester's law and sesquilinear forms.
- **Lecture 20.** Hermitian forms and polarisation identity; Hermitian formulation of Sylvester's law. Sylvester form of skew-symmetric matrices. Inner product spaces and examples.
- **Lecture 21.** Gram-Schmidt algorithm. Orthogonal complement and projection.
- **Lecture 22.** Orthogonal complement and projection map. Adjoint maps and isometries.
- **Lecture 23.** Spectral theory: properties of self-adjoint and unitary operators and proof of spectral theorem in each case.
- **Lecture 24.** Application of spectral theory to bilinear forms. Simultaneous diagonalisation corollary.

3 Quantum Mechanics (2020)

- **Lecture 1.** Historical introduction to quantum mechanics.
- **Lecture 2.** More historical introduction. Foundations of quantum mechanics: three postulates including TDSE.
- **Lecture 3.** Proof that TDSE preserves normalisation of wavefunction. Principle of superposition. Expectation and operators.

- **Lecture 4.** Hamiltonian and TISE. Stationary states.
- **Lecture 5.** Solving TISE in one dimension; infinite and finite potential well. Proof that for even potential with non-degenerate eigenvalues, eigenfunctions are even or odd.
- **Lecture 6.** TISE for a free particle and probability density. Scattering states; reflection and transmission coefficients. Scattering on a potential step.
- **Lecture 7.** Scattering off a potential barrier.
- **Lecture 8.** Solving the TISE for the harmonic oscillator. Power series solution and parity. Proof that non-terminating series is not normalisable. Hermite polynomials.
- **Lecture 9.** Axioms for quantum mechanics. Hermitian operators and properties.
- **Lecture 10.** Postulates of quantum mechanics from experiment. Expectation values and commutators.
- **Lecture 11.** Uncertainty principle (generalised) and Heisenberg uncertainty principle.
- **Lecture 12.** Ehrenfest theorem.
- **Lecture 13.** Schrödinger equation in 3D for spherically symmetric potentials. Examples for potential step and radial wavefunction of the hydrogen atom.
- **Lecture 14.** More on radial wavefunction of the hydrogen atom. Proof that for a normalisable solution a power series must terminate.
- **Lecture 15.** Angular momentum in quantum mechanics; operator and properties. Associated Legendre equation.
- **Lecture 16.** Full wavefunction of the hydrogen atom.

4 Methods

- **Lecture 1.** Function inner product and orthogonality properties for sin and cos functions. Fourier series definition; inner product to obtain coefficients.
- **Lecture 2.** Dirichlet conditions for well-behaved function Fourier series. Examples and exercises for several functions. Conditions for integrability and differentiability term-by-term. Parseval's theorem.
- **Lecture 3.** Half-range Fourier series. Self-adjoint (Hermitian) matrix properties. Solving inhomogeneous ODEs with Fourier series.
- **Lecture 4.** Sturm-Liouville form; transforming ODEs. Self-adjoint operator definition and required boundary conditions. Properties of self-adjoint operators.
- **Lecture 5.** Eigenfunction expansions and finding coefficients. Completeness and Parseval's identity. Bessel's inequality and error. Finding Legendre polynomials as solutions of ODE.
- **Lecture 6.** Properties of Legendre polynomials. Eigenfunction expansion in terms of Legendre polynomials. General method for solving inhomogeneous ODEs using S-L theory. Integral solution and Green's function. PDEs on bounded domains; derivation of wave equation.
- **Lecture 7.** Solving wave equation by separation of variables. Initial conditions and temporal solutions.
- **Lecture 8.** Oscillation energy of a vibrating string. Wave reflection and transmission; density discontinuity conditions. Solving wave equation in plane polar coordinates.

- **Lecture 9.** Asymptotic behaviour of Bessel functions. 2D wave equation and vibrating drum.
- **Lecture 10.** Derivation of the diffusion equation. Similarity solutions and separation of variables.
- **Lecture 11.** Laplace's equation in Cartesian and plane polar coordinates; steady heat conduction.
- **Lecture 12.** Laplace's equation in cylindrical and spherical coordinates. Generating function for Legendre polynomials.
- **Lecture 13.** Dirac delta function and standard properties.
- **Lecture 14.** Complex Fourier series and general eigenfunctions. Green's function and static forces on a string. Defining properties of Green's function.
- **Lecture 15.** Constructing Green's function (boundary value problem) using properties. Inhomogeneous boundary conditions and higher-order ODEs.
- **Lecture 16.** Eigenfunction expansion for Green's function. Constructing Green's function (initial value problem). Introduction to Fourier transforms.
- **Lecture 17.** Relation between Fourier transforms and series. Properties of Fourier transform. Fourier inversion theorem. Convolution and Parseval's theorem.
- **Lecture 18.** Fourier transforms of Dirac delta and Heaviside functions. Signal processing and general transfer functions for ODEs.
- **Lecture 19.** Damped oscillator example. Discrete Fourier transform and Nyquist frequency. The sampling theorem.
- **Lecture 20.** More on discrete Fourier transforms. PDEs on unbounded domains; well-posed Cauchy problems and method of characteristics.
- **Lecture 21.** Method of characteristics continued. Inhomogeneous 1st order PDE example. Hyperbolic, parabolic and elliptic cases.
- **Lecture 22.** Transforming to characteristic coordinates. General solution for wave equation. Diffusion equation and Fourier transforms.
- **Lecture 23.** Gaussian pulse example. Forced diffusion equation. Forced wave equation. Poisson's equation.
- **Lecture 24.** Green's identities. Dirichlet's Green's function. Method of images and Laplace's equation on half-space.

5 Markov Chains

- **Lecture 1.** Markov chain definition; stochastic and transition matrices, and basic results. Independence of sequences. Statement of simple Markov property.
- **Lecture 2.** Proof of simple Markov property. Powers of transition matrix; proof of $\mathbb{P}(X_n = x) = (\lambda P^n)_x$ and solving recurrences. General procedure for finding P^n.
- **Lecture 3.** Example of finding P^n. Communicating classes and equivalence relation. Hitting times; hitting time vector as minimal non-negative solution to system of equations.
- **Lecture 4.** The rest of the proof of the previous theorem, and example applications. Birth and death chain. Expected hitting time as minimal non-negative solution to system of equations.

- **Lecture 5.** Continuation of proof from L4 for expectation of hitting time. Stopping times and strong Markov property. Application to simple random walk. Transience and recurrence.
- **Lecture 6.** Transience and recurrence properties. Proof that communicating states are either recurrent or transient. Recurrence of finite closed classes.
- **Lecture 7.** Proof of Polya's theorem for recurrence and transience of simple random walks.
- **Lecture 8.** Probability distributions and invariants. Conditions for unique invariant distributions. Invariance of $\nu_k(k)$.
- **Lecture 9.** Perron-Frobenius theorem. Results about irreducible chains and invariant measures. Positive and null recurrence.
- **Lecture 10.** Transience of simple random walk on $\mathbb{Z}^3$. Time reversibility and detailed balance equations. Example in $\mathbb{Z}_n$ and random walk on a graph.
- **Lecture 11.** Aperiodic chains and properties.
- **Lecture 12.** Proof that if P is irreducible, aperiodic and null-recurrent, then $P^n(x, y) \to 0$ as $n \to \infty$. The ergodic theorem.