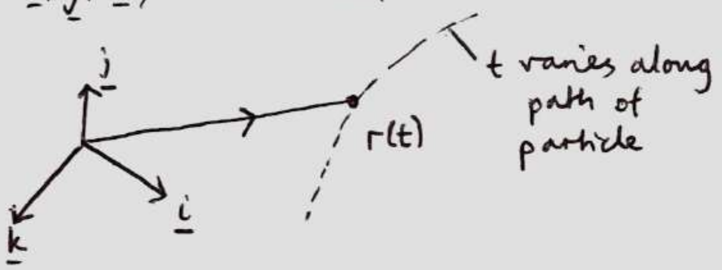
Dynamics and Relativity - Lecture 1

21/1/21

1. Newtonian Dynamics

1.1 Particles

A particle is an object of negligible size - may have mass $m > 0$.
May have charge q.
Position is described by a position vector $\underline{r}(t)$ or $\underline{x}(t)$, wrt an origin.
The Cartesian components of $\underline{r}(t)$ are given by (x, y, z): $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$
$(\underline{i}, \underline{j}, \underline{k})$ orthonormal



Choice of axes defines frame of reference S.

Velocity $\underline{u} = \frac{d}{dt}\underline{r}(t) = \dot{\underline{r}}(t)$, tangent to trajectory

Momentum $\underline{p} = m\underline{u} = m\dot{\underline{r}}$

Acceleration $\dot{\underline{u}} = \ddot{\underline{r}} = \frac{d^2\underline{r}}{dt^2}$

Product rules : scalar $f(t)$, vector $\underline{g}(t)$, $\underline{h}(t)$

$$\frac{d}{dt}(f\underline{g}) = \frac{df}{dt}\underline{g} + f\frac{d\underline{g}}{dt} \qquad \frac{d}{dt}(\underline{g}\times\underline{h}) = \frac{d\underline{g}}{dt}\times\underline{h} + \underline{g}\times\frac{d\underline{h}}{dt}$$

$$\frac{d}{dt}(\underline{g}\cdot\underline{h}) = \frac{d\underline{g}}{dt}\cdot\underline{h} + \underline{g}\cdot\frac{d\underline{h}}{dt} \qquad \text{(check by components)}$$

1.2 Newton's Laws

1. There exist inertial frames in which a particle remains at rest or moves at constant velocity unless acted on by a force. (Galileo's law of inertia)

2. In an inertial frame $\frac{d\underline{p}}{dt} = \underline{F}$

3. To every action, there is an equal and opposite reaction.

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1.3 Inertial frames and Galilean transformations

In inertial frame acceleration is zero if force is zero

$\ddot{\underline{r}} = \underline{0} \iff \underline{F} = \underline{0}$

Inertial frames are not unique. If S is an inertial frame, then any other frame S' moving with constant velocity relative to S is also an inertial frame.

$S\ (x, y, z, t)$ $\quad S'\ (x', y', z', t')$ may take

$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$ for relative motion in x-direction. (speed v)

in general: $\underbrace{\underline{r}'}_{S'} = \underbrace{\underline{r}}_{S} - \underbrace{\underline{v}}_{\text{vel. of } S' \text{ relative to } S}t$

This transformation is called a boost.

For a particle with position of $\underline{r}(t)$ in S, it has position $\underline{r}'(t')$ in S'. Then $\underline{u} = \dot{\underline{r}}(t)$, $\underline{a} = \ddot{\underline{r}}(t)$ in S.

$\underline{u}' = \underline{u} - \underline{v}$, $\underline{a}' = \underline{a}$ are relationships for velocity / acc^n.

inertial frame

A general Galilean transformation preserves inertial frames, and combines boosts with some combination of:

- translation of space $\underline{r}' = \underline{r} - \underline{r}_0$ ← constant
- translation of time $t' = t - t_0$
- rotations & reflections in space: $\underline{r}' = R\underline{r}$

orthogonal matrix

This set generates the Galilean group

For any Galilean transformation: $\ddot{\underline{r}} = \underline{0} \iff \ddot{(\underline{r}')} = \underline{0}$

S inertial $\iff$ S' inertial

Galilean relativity

Principle of Galilean relativity is that laws of Newtonian physics are the <u>same</u> in all inertial frames.

Laws of physics look the same at any point in space or time, or facing whatever direction, or moving with whatever constant velocity.

Any set of equations describing Newtonian physics has Galilean invariance.

Measure of <u>acc^n</u> <u>is</u> absolute, but measure of velocity is <u>not</u>.

1.4 Newton's Second Law

For particle subject to force $\underline{F}$, momentum $\underline{p}$ satisfies $\underline{F} = \frac{d\underline{p}}{dt}$

$(\underline{p} = m\underline{u} = m\dot{\underline{r}})$ Here assume m is constant.

$$m\frac{d\underline{u}}{dt} = m\ddot{\underline{r}} = \underline{F}$$ Mass is a measure of inertia.

If $\underline{F}$ is specified as a function of $\underline{r}, \dot{\underline{r}}, t$ i.e. $\underline{F}(\underline{r}, \dot{\underline{r}}, t)$ then have <u>2nd order DE</u> for $\underline{r}(t)$.

$$m\ddot{\underline{r}} = m\frac{d^2\underline{r}}{dt^2} = \underline{F}\left(\underline{r}, \frac{d\underline{r}}{dt}, t\right)$$

Need initial position $\underline{r}(0)$ and initial velocity $\frac{d}{dt}\underline{r}(\underline{0})$.

Then have unique solution. Path of particle is <u>determined</u> (at <u>all</u> times, $\geqslant 0$ or < 0).

1.5 Examples

1) Gravitational force

2 particles at $\underline{r}_1$ and $\underline{r}_2$ respectively

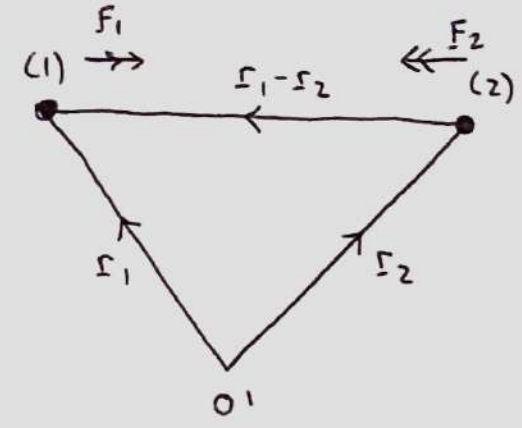
Force on particle 1 is $$\underline{F}_1 = -\frac{Gm_1 m_2(\underline{r}_1 - \underline{r}_2)}{|\underline{r}_1 - \underline{r}_2|^3} = -\underline{F}_2$$

m_1, m_2 masses, G = gravitational constant

$\underline{F}_1 = -\underline{F}_2$ proportional to $|\underline{r}_1 - \underline{r}_2|^{-2}$ (inverse square law)

note $(\underline{r}_1 - \underline{r}_2)$ specifies direction; attractive force from $-$ sign

Newton's 3rd holds here.



G is a dimensional constant

dimensions $L^3 M^{-1} T^{-2}$

$\underline{F}_1$, $\underline{F}_2$ don't depend on velocities of particles

but EM forces do.

2) Electromagnetic forces

Consider a particle with electric charge q, in electric field $\underline{E}(\underline{r}, t)$ and magnetic field $\underline{B}(\underline{r}, t)$

$$\underline{F}(\underline{r}, \dot{\underline{r}}, t) = q\left(\underline{E}(\underline{r}, t) + \dot{\underline{r}} \times \underline{B}(\underline{r}, t)\right) \quad \text{Lorentz force law}$$

Consider $\underline{E}$, $\underline{B}$ as given.

<u>Example</u> $\underline{E} = \underline{0}$, $\underline{B}$ = constant vector

Equation of motion: $m\ddot{\underline{r}} = q\dot{\underline{r}} \times \underline{B}$, a 2nd order ODE to solve

Choose axes with $\underline{B} = B\hat{\underline{z}}$, hence $m\ddot{z} = 0 \Rightarrow z = \underbrace{z_0 + ut}_{\text{constants}}$

$\underline{m\ddot{x} = qB\dot{y}}$, $\underline{m\ddot{y} = -qB\dot{x}}$

convenient to define $\omega = \dfrac{qB}{m}$ then get
$$\boxed{\begin{aligned} x &= x_0 - \alpha\cos(\omega(t-t_0)) \\ y &= y_0 + \alpha\sin(\omega(t-t_0)) \end{aligned}}$$

x_0, y_0, α, t_0 constants from ICs

Circular motion in x, y; constant velocity in z $\Rightarrow$ <u>helical</u> motion.

Clockwise when viewed from direction of $\underline{B}$.

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2. Dimensional Analysis

3 main basic quantities : L length, M mass, T time

Dimensions of some quantity $[x]$ can be expressed in terms of L, M, T

e.g. density $[\text{density}] = ML^{-3}$

$[\text{force}] = MLT^{-2}$

Only powers of these quantities allowed

Units e.g. SI units: L m (metres), M kg (kilograms), T s (seconds)

Force $F = \dfrac{Gm_1 m_2}{r^2}$ $\Rightarrow$ dimensions of G :

$$G = \frac{F \cdot r^2}{m_1 m_2} = \frac{M \cdot L}{T^2} \cdot L^2 \cdot \frac{1}{M^2} = \frac{L^3}{MT^2} \quad m^3 kg^{-1} s^{-2}$$

Scaling Dimensional quantity Y depends on other D.Qs $X_1, X_2, \dots, X_n$

Let dimensions be $[Y] = L^a M^b T^c$

$[X_i] = L^{a_i} M^{b_i} T^{c_i} \quad (i = 1, \dots, n)$

If $n \leq 3$ then $Y = C X_1^{p_1} X_2^{p_2} X_3^{p_3}$ and p_1, p_2, p_3 can be determined by balancing dimensions

$L^a M^b T^c = (L^{a_1} M^{b_1} T^{c_1})^{p_1} \times \dots$

Hence
$$a = a_1 p_1 + a_2 p_2 + a_3 p_3$$
$$b = b_1 p_1 + b_2 p_2 + b_3 p_3$$
$$c = c_1 p_1 + c_2 p_2 + c_3 p_3$$

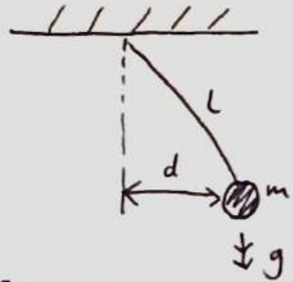
unique solution for p_1, p_2, p_3 if dimensions of X_1, X_2, X_3 are "(linearly) independent".

If $n > 3$ then X_1, X_2, X_3 are not dimensionally independent (only 3 - L, M, T). Choose X_1, X_2, X_3 (dim. indep) and $n-3$ dimensionless quantities $\lambda_1 = \dfrac{X_4}{X_1^{q_{11}} X_2^{q_{12}} X_3^{q_{13}}}$, $\lambda_2 = \dfrac{X_5}{X_1^{q_{21}} X_2^{q_{22}} X_3^{q_{23}}}$

q_{mn} chosen to balance dimensions. Then $Y = X_1^{p_1} X_2^{p_2} X_3^{p_3} C(\lambda_1, \lambda_2, \dots, \lambda_{n-3})$

(Bridgman's theorem)

Example simple pendulum $n = 4$



How does period of oscillation p depend on m, l, d, g?

$[p] = T \quad [m] = M \quad [g] = LT^{-2} \quad [l] = [d] = L$

$n = 4$: form 1 dimensionless group $\quad p = f\left(\frac{d}{l}\right) m^{p_1} l^{p_2} g^{p_3}$

$\therefore \quad T = M^{p_1} L^{p_2} \left(\frac{L}{T^2}\right)^{p_3}. \quad M: p_1 = 0 \quad L: p_2 + p_3 = 0$

$T: 1 = -2p_3 \Rightarrow p_1 = 0, \; p_2 = \frac{1}{2}, \; p_3 = -\frac{1}{2}$

so $\underline{p = f\left(\frac{d}{l}\right) l^{1/2} g^{-1/2}}$ useful: $d \to 2d, \; l \to 2l \Rightarrow p \to \sqrt{2}\,p$

$d \to 2d, \; l \to l$ can't say - don't know form of $f\left(\frac{d}{l}\right)$

Example 2 Taylor's estimate of energy of first atomic explosion

$R(t)$ size as function of time : $\underline{L}$ time t $\underline{T}$

air density $\rho \quad \frac{M}{L^3}$, energy $E \quad \frac{ML^2}{T^2}$

$R = C t^\alpha \rho^\beta E^\gamma \; : \; L = T^\alpha \frac{M^\beta}{L^{3\beta}} \frac{M^\gamma L^{2\gamma}}{T^{2\gamma}}$:

$M: \beta + \gamma = 0, \quad L: -3\beta + 2\gamma = 1, \quad T: \alpha - 2\gamma = 0$

$\Rightarrow \gamma = \frac{1}{5}, \quad \beta = -\frac{1}{5}, \quad \alpha = \frac{2}{5}$

hence $\underline{R(t) = C t^{2/5} \rho^{-1/5} E^{1/5}}$

Taylor verified $\frac{2}{5}$ power law and estimated value of E

$$E = \frac{\rho R^5}{C^5 t^2}$$: if $C \sim 1$ then $E \sim 10^{14}$ J

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3 Forces

3.1 Force and potential energy in 1 spatial dimension

Consider mass m moving in straight line with position $x(t)$.

Force depends only on position x, not on velocity or on time.

Force $F(x)$: define potential energy $V(x)$ by

$$F(x) = -\frac{dV}{dx} \quad \text{or} \quad V(x) = -\int^x F(x)\, dx$$

lower limit omitted – arbitrary constant in V

Eqn of motion – Newton's 2nd : $m\ddot{x} = -\dfrac{dV}{dx}$

Define kinetic energy $T = \frac{1}{2}m\dot{x}^2$

Total energy $E = T + V = \frac{1}{2}m\dot{x}^2 + V(x)$ (conserved) $\dfrac{dE}{dt} = 0$

as $$\frac{dE}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}m\dot{x}^2 + V(x)\right) = m\dot{x}\ddot{x}\ \frac{dV}{dx}\dot{x} = \dot{x}\left(m\ddot{x} + \frac{dV}{dx}\right) = 0.$$

For conservation of $\frac{1}{2}m\dot{x}^2 + \Phi$, require $\dot{x}F = -\dfrac{d\Phi}{dt}$

(Φ may depend on $x, \dot{x}, t$) but usually there is no such Φ if F depends on $\dot{x}$ or t.

Example harmonic oscillator $F(x) = -kx$ (e.g. Hooke's law)

so $V(x) = -\int^x (-kx')\, dx' = \frac{1}{2}kx^2$ (with appropriate $+c$)

Seek explicit expression for $x(t)$

$$m\ddot{x}(t) = -kx \quad \Rightarrow \quad x(t) = A\cos\omega t + B\sin\omega t \qquad \text{for } \omega = \sqrt{\tfrac{k}{m}}$$

$$\dot{x}(t) = -\omega A\sin\omega t + \omega B\cos\omega t \qquad A, B \text{ from ICs}$$

Can check $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ is conserved.

In 1D, conservation of energy gives useful information about motion.

Conservation of energy is a 1st integral of Newton's 2nd :

$$E = \tfrac{1}{2}m\dot{x}^2 + V(x) \Rightarrow \dot{x} = \pm\sqrt{\tfrac{2}{m}(E - V(x))} \qquad \text{if we know } E \text{ from ICs}$$

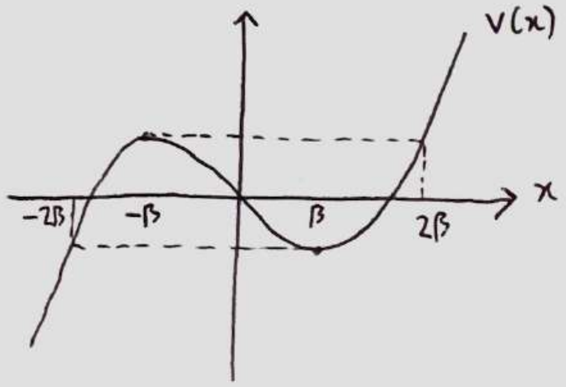
This implies $\pm \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}\left(E - V(x)\right)}} = t - t_0$ with $x = x_0$ when $t = t_0$.

This gives an <u>implicit</u> solution for $x(t)$.

In principle we can evaluate integral for $x(t)$ - not as easy in practice

Qualitatively:

<u>Example</u> $V(x) = \lambda\left(x^3 - 3\beta^2 x\right)$ λ, β constants, > 0



What happens if we release particle <u>from rest</u> at $x = x_0$ for different choices of x_0?

From rest $\Rightarrow$ $\underline{E = V(x_0)}$

so $\underline{V(x) \leq V(x_0)}$ $\forall x$.

<u>Case 1</u> $x_0 < -\beta$ particle moves to left with $x(t) \to -\infty$ as $t \to \infty$.

<u>Case 2</u> $-\beta < x_0 < 2\beta$ particle confined to $-\beta \leq x(t) < 2\beta$

<u>Case 3</u> $2\beta < x_0$ particle moves to left, reaches $x = -\beta$ and continues with $x(t) \to -\infty$ as $t \to \infty$.

<u>Special case 1</u>: $x_0 = -\beta$ $\frac{dV}{dx} = 0$ so no force $\Rightarrow$ no acc^n

particle remains at $x = -\beta$

<u>Special case 2</u>: $x_0 = \beta$: particle remains at $x = \beta$

<u>Special case 3</u>: $x_0 = 2\beta$: particle moves to left and comes to rest at $x = -\beta$ (E conserved).

How long does it take to come to rest?

Write down integral expression relating x and t:

$$\int_{x(t)}^{2\beta} \frac{d\tilde{x}}{\sqrt{\frac{2\lambda}{m}\left(2\beta^3-\tilde{x}^3+3\beta^2\tilde{x}\right)}} = t \qquad (x(0)=2\beta)$$

$$= \int_{x(t)}^{2\beta} \frac{1}{\sqrt{\frac{2\lambda}{m}}}\,\frac{1}{\tilde{x}+\beta}\cdot\frac{1}{(2\beta-\tilde{x})^{1/2}}\,d\tilde{x}$$

<u>diverges</u> as $\tilde{x} \to -\beta$

so particle takes "infinite time" to come to rest at $x=-\beta$.

(logarithmic behaviour)

3.2 Equilibrium Points

A point at which a particle can stay at rest for all time
($x = \pm\beta$ in Lecture 4)

Here $\underline{V'(x) = 0}$. Analyse motion close to eq. point at $\underline{x = x_0}$

$V'(x_0) = 0$. Assume $x - x_0$ is small - expand $V(x)$ as Taylor series.

$$V(x) \approx V(x_0) + (x - x_0)\cancel{V'(x_0)} + \frac{1}{2}(x-x_0)^2 V''(x_0) + \dots$$

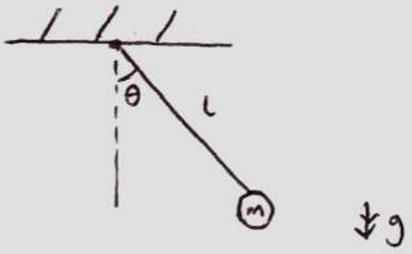
so $$m\ddot{x} = -V'(x) \approx \underline{-(x-x_0)V''(x_0)}$$

If $V''(x_0) > 0$: local minimum, harmonic oscillator eqn, frequency is $\sqrt{V''(x_0)/m}$ (stable equilibrium - start close $\Rightarrow$ stay close)

If $V''(x_0) < 0$: local maximum, exponentially increasing / decaying solutions - almost always the ICs excite exponential growth with rate $\gamma = \sqrt{-V''(x_0)/m}$

If $V''(x_0) = 0$ must take higher terms

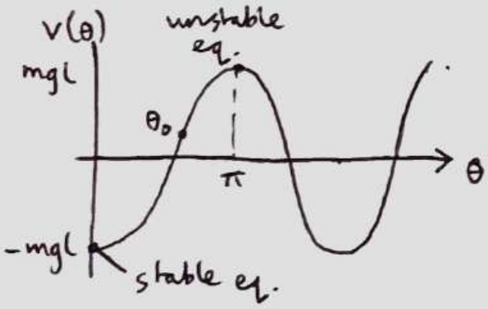
Example Pendulum:



Newton's 2nd: $ml\ddot{\theta} = -mg\sin\theta$ (see later; acc^n $\perp$ to string)

$$ml\ddot{\theta} = -mg\sin\theta = -\frac{d}{d\theta}(-mg\cos\theta)$$

$$\Rightarrow E = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \qquad \text{check } \frac{dE}{dt} = 0$$



If $-mgl < E < mgl$: oscillates about position of stable equilibrium

$E > mgl$: either $\dot{\theta} > 0$ or $\dot{\theta} < 0$ for all time

What is the period of oscillations? $\dot{\theta} = 0$ at $\theta = \theta_0$ - dependence of period on θ_0?

oscillates
$$\theta_0 \to 0 \to -\theta_0 \to 0 \to \theta_0 \Rightarrow P = 4\int_0^{\theta_0} \frac{d\theta}{\left(\frac{2gl(\cos\theta - \cos\theta_0)}{l^2}\right)^{1/2}}$$
$\dot{\theta}$ from E eqn

(see lecture 4)

$$= 4\left(\frac{l}{g}\right)^{1/2} \int_0^{\theta_0} \frac{d\theta}{(2\cos\theta - 2\cos\theta_0)^{1/2}} = \underline{4\left(\frac{l}{g}\right)^{1/2} F(\theta_0)}$$

(Recall $P = \left(\frac{l}{g}\right)^{1/2} H\left(\frac{d}{l}\right)$ from dim. analysis)

If θ_0 is small: $F(\theta_0) \simeq \int_0^{\theta_0} \frac{d\theta}{(\theta_0^2 - \theta^2)^{1/2}} = \frac{\pi}{2}$

Then $\underline{P \simeq 2\pi \sqrt{\frac{l}{g}}}$ for small θ_0.

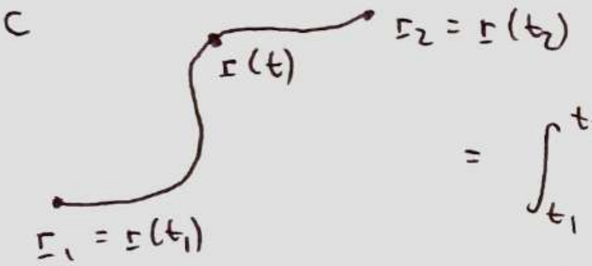
3.3 Force / potential in 3D

Consider particle in 3D under force $\underline{F}$. $m\ddot{\underline{r}} = \underline{F}$

$T = \frac{1}{2} m |\dot{\underline{r}}|^2 = \frac{1}{2} m |\underline{u}|^2$ then $\frac{dT}{dt} = m\dot{\underline{r}} \cdot \ddot{\underline{r}} = \underline{F} \cdot \dot{\underline{r}} = \underline{\underline{F} \cdot \underline{u}}$

rate of working of force on particle

Consider total work done by force on particle along a finite path:



Total work $\int_{t_1}^{t_2} \underline{F} \cdot \underline{u} \, dt$

$$= \int_{t_1}^{t_2} \underline{F} \cdot \dot{\underline{r}} \, dt = \int_{\underline{r}(t_1)}^{\underline{r}(t_2)} \underline{F} \cdot d\underline{r} \leftarrow \text{along } C$$

line integral

Total work $= \int_{t_1}^{t_2} \underline{F} \cdot d\underline{r} = \int_{t_1}^{t_2} F_x \, dx + F_y \, dy + F_z \, dz$

$(\underline{F} = (F_x, F_y, F_z))$

Now suppose $\underline{F}$ is a function of $\underline{r}$ only i.e. $\underline{F}(\underline{r})$ defines a <u>force field</u>

A <u>conservative</u> force field is such that $\underline{\underline{F}(\underline{r}) = -\nabla V(\underline{r})}$

for some function $V(\underline{r})$ (components: $F_i = \frac{\partial V}{\partial x_i}$)

If $\underline{F}$ is conservative then E is conserved.

Proof $\frac{dE}{dt} = \frac{dT}{dt} + \frac{d}{dt}V(r) = m\dot{\underline{r}}\cdot\ddot{\underline{r}} + \nabla V\cdot\dot{\underline{r}}$

$$= \dot{\underline{r}}\cdot(m\ddot{\underline{r}} + \nabla V) = \dot{\underline{r}}\cdot(m\ddot{\underline{r}} - \underline{F}) = 0.$$

To find work done by <u>conservative</u> force:

$$W = \int_C \underline{F}\cdot d\underline{r} = -\int_C \nabla V\cdot d\underline{v} = \underline{V(\underline{r}_1) - V(\underline{r}_2)}$$

follows from properties of ∇

This is independent of path taken: depends only on endpts $\underline{r}_1, \underline{r}_2$

<u>Corollary:</u> if C is closed then no net work is done.

In general a given $\underline{F}(\underline{r})$ is not conservative i.e. there is no function $V(\underline{r})$ with $\underline{F} = -\nabla V$.

Condition: $\underline{F}(\underline{r})$ is conservative when $\nabla\times\underline{F}(\underline{r}) = \underline{0}$ see VC
(curl operator)

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3.4 Gravity

$$\underline{F}(\underline{r}) = -\frac{GMm}{|\underline{r}|^3}\cdot\underline{r} = -\frac{GMm}{r^2}\hat{\underline{r}} \qquad \text{(radial direction)}$$

$\underline{r}$ is position vector of mass m relative to M.

$\underline{F}(\underline{r}) = -\nabla V$ with $V(\underline{r}) = -\frac{GMm}{r}$ so gravitational force is <u>conservative</u>.

Often define a "gravitational potential" $\Phi_g(\underline{r}) = -\frac{GM}{r}$

"gravitational field" $\underline{g} = -\nabla\Phi_g(\underline{r}) = -\frac{GM}{r^2}\hat{\underline{r}}$ (easier sometimes to work without m)

Effect on mass m: $V(\underline{r}) = m\Phi_g(\underline{r})$, $\underline{F}(\underline{r}) = m\underline{g}$

Can generalise to grav. potential associated with many masses M_i at $\underline{r}_i$

$$\Phi_g(\underline{r}) = -\sum_{i=1}^{N}\frac{GM_i}{|\underline{r}-\underline{r}_i|} \qquad \underline{g} = -\sum_{i=1}^{N}\frac{GM_i(\underline{r}-\underline{r}_i)}{|\underline{r}-\underline{r}_i|^3}$$

Could extend to a continuous distribution: sum $\to$ integral

In particular for uniform spherical mass distribution centred at 0: outside distribution, have $\Phi_g(\underline{r}) = -\frac{GM}{r}$ $\quad M$ = total mass

behaves as point mass (see VC)

Gravitational Mass vs Inertial Mass

Inertial: $m\ddot{\underline{r}} = \underline{F}$

Gravitational: $\underline{F} = -\frac{GMm}{r^2}\hat{\underline{r}}$

regard as same for this course ($\sim$ within $1/10^{12}$) see GR

(1) 1D approximation

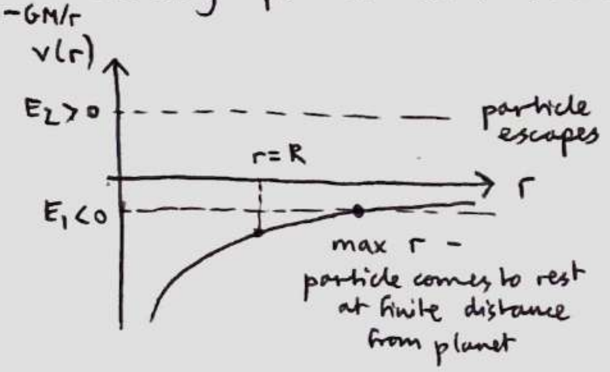
Consider mass m at height z above surface of planet mass M, radius R $(z \ll R)$

potential $V(R+z) = -\frac{GMm}{R+z} = -\frac{GMm}{R} + \frac{GMmz}{R^2} + \ldots$ (binomial exp)

Potential energy in uniform gravitational field with $g = \frac{GM}{R^2} \approx 9.8\,ms^{-2}$ for Earth

(2) Escape velocity

Leaving planet with velocity v.



E conserved:

$$E = T + V = \frac{1}{2}mv^2 - \frac{GM}{R}$$

Escape if $v \geq \sqrt{\frac{2GM}{R}}$

= escape velocity

(to escape: need $E \geq 0$ so KE is ≥ 0 at ∞, when $V \to 0$)

3.5 Electromagnetic Forces

Force $\underline{F}$ on charge q particle is $\underline{\underline{F} = q\underline{E} + q\underbrace{\underline{u}}_{\dot{\underline{r}}} \times \underline{B}}$ (Lorentz force law)

$\underline{E}$ electric field, $\underline{B}$ magnetic field

If $\underline{E}$, $\underline{B}$ time independent then $\underline{E} = -\nabla \underbrace{\Phi_e(\underline{r})}_{\text{electrostatic potential}}$

then qE is conservative.

<u>Claim</u> for time indep. $\underline{E}(r)$, $\underline{B}(r)$, energy of particle moving under Lorentz force law is constant.

<u>Proof</u> $E = \frac{1}{2}m|\dot{\underline{r}}|^2 + q\Phi_e(\underline{r}) = T + V$

$$\frac{dE}{dt} = m\dot{\underline{r}} \cdot \ddot{\underline{r}} + q\dot{\underline{r}}\,\nabla\Phi_e(\underline{r}) = \dot{\underline{r}} \cdot (m\ddot{\underline{r}} + q\nabla\Phi_e)$$

$$= \dot{\underline{r}}\,(\cancel{q\underline{E}} + q\dot{\underline{r}} \times B + \cancel{q\nabla\Phi_e}) = q\dot{\underline{r}} \cdot (\dot{\underline{r}} \times \underline{B}) = 0.$$

i.e. rate of working of magnetic field is 0.

<u>Point charges</u> Charge Q particle at origin generates potential $\Phi_e(\underline{r}) = \frac{Q}{4\pi\varepsilon_0 r}$

$$\underline{E}(\underline{r}) = -\nabla\Phi_e = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\underline{r}}$$

Force on particle with charge q located at $\underline{r}$:

$$\underline{F} = -q\nabla\Phi_e = \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{\underline{r}} \quad \text{"Coulomb force"}$$

may be attractive or repulsive depending on sign of Qq

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3.6 Friction

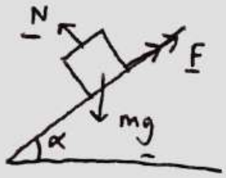
A contact force between two bodies or solid / surrounding fluid

Dry friction

Normal force : perpendicular to contact surface

Tangential force : resists tangential motion (sliding / slipping)

Static friction : if no sliding occurs



$\underline{N}$ normal, $\underline{F}$ friction

$$\underline{|\underline{F}| \leq \mu_s |\underline{N}|}$$

μ_s = coeff. of static friction

Block can remain static provided $\alpha \leq \tan^{-1}(\mu_s)$ (check)

Kinetic friction : if block starts to slide, there is a kinetic frictional force

$|\underline{F}| = \underset{\substack{\uparrow \\ \text{coeff of kinetic friction}}}{\mu_K} |\underline{N}|$. expect $\mu_s > \mu_K > 0$

Fluid drag solid moving through fluid

Linear drag : $\underline{\underline{F} = -k_1 \underline{u}}$ $\underline{u}$ rel. velocity of solid to fluid, k_1 constant

Relevant to "small" objects moving through viscous fluid.

Stokes drag law : $k_1 = 6\pi\eta R$ η viscosity, R radius of sphere

Quadratic drag : $\underline{\underline{F} = -k_2 |\underline{u}| \underline{u}}$ for large bodies moving through less viscous fluid

typically $k_2 = \rho_{fluid} C_D R^2$ C_D drag coefficient

A body loses KE as a result of a drag force

rate of working $\underline{F} \cdot \underline{u} = k_1 |\underline{u}|^2$ (linear) or $\underline{F} \cdot \underline{u} = -k_2 |\underline{u}|^3$ (quadratic)

Example 1 damped oscillator : $m\ddot{x} = -kx - \underline{\lambda \dot{x}}$

Example 2 Projectile under uniform gravity, experiencing linear drag

$$m \frac{d^2 \underline{x}}{dt^2} = m\underline{g} - k \frac{d\underline{x}}{dt} \qquad \underline{x} = \underline{0}, \ \dot{\underline{x}} = \underline{u} = \underline{U} \text{ at } t = 0$$

Solve for $\underline{u}$: $m\dot{\underline{u}} = m\underline{g} - k\underline{u}$ I.F. $\frac{d}{dt}(\underline{u} e^{kt/m}) = m\underline{g}\, e^{kt/m}$

$$\Rightarrow \underline{u} = \frac{m\underline{g}}{k} + \underset{\substack{\uparrow \\ \text{constant}}}{\underline{C}} e^{-kt/m} = \frac{m\underline{g}}{k} + \left(\underline{U} - \frac{m\underline{g}}{k}\right) e^{-kt/m}$$

Integrate for $\underline{x}(t)$: $\underline{x}(t) = \frac{m\underline{g}t}{k} - \frac{m}{k}\left(\underline{U} - \frac{m\underline{g}}{k}\right)e^{-kt/m} + \underset{\uparrow \text{ by ICs}}{D}$

$$\Rightarrow \underline{x}(t) = \frac{m\underline{g}t}{k} + \frac{m}{k}\left(\underline{U} - \frac{m\underline{g}}{k}\right)\left(1 - e^{-kt/m}\right)$$

Consider components: choose $\underline{U} = (U\cos\theta, 0, U\sin\theta)$, $\underline{g} = (0, 0, -g)$

$u_1 = U\cos\theta\, e^{-kt/m}$ $\quad u_2 = 0 \quad$ $u_3 = \left(U\sin\theta + \frac{mg}{k}\right)e^{-kt/m} - \frac{mg}{k}$

Note terminal velocity is $(0, 0, -mg/k)$ as $t \to \infty$

$x = \frac{mU\cos\theta}{k}\left(1 - e^{-kt/m}\right) \quad y = 0 \quad z = -\frac{mgt}{k} + \frac{m}{k}\left(U\sin\theta + \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right)$

Range: value of x when z returns to 0

$R(U, \theta, m, k, g)$

Dimensionless group $\frac{kU}{mg} = \frac{U/g}{m/k}$

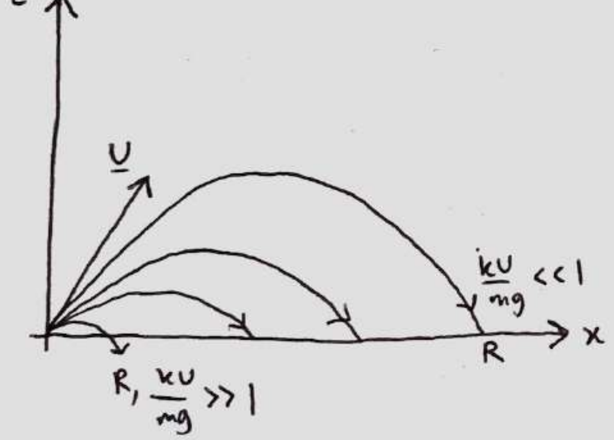
$\frac{U}{g}$ time to reduce initial velocity under gravity

$\frac{m}{k}$ time to achieve terminal velocity

From dimensional analysis: $R = \frac{U^2}{g} F\left(\theta, \frac{kU}{mg}\right)$ find F:

If $\frac{kU}{mg} << 1$: $R = \frac{U^2}{g} \cdot 2\sin\theta\cos\theta$ strong friction

If $\frac{kU}{mg} >> 1$: $R = \frac{U^2}{g}\left(\frac{mg}{kU}\cos\theta\right)$ weak friction



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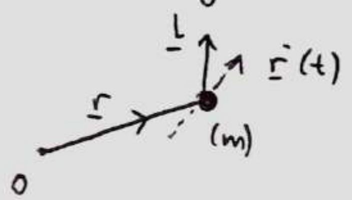
3.7 Angular Momentum

Define angular momentum for particle mass m moving under force $\underline{F}$, position $\underline{r}(t)$, velocity $\dot{\underline{r}}(t)$

$$\underline{L} = \underline{r} \times \underline{p} = \underline{r} \times m\dot{\underline{r}}$$



Then $$\frac{d\underline{L}}{dt} = m\dot{\underline{r}} \times \dot{\underline{r}} + m\underline{r} \times \ddot{\underline{r}} = m\underline{r} \times \ddot{\underline{r}} = \underline{r} \times \underline{F} = \underline{G}$$

called torque or moment of force F

Values of $\underline{L}$ and $\underline{G}$ depend on choice of origin so refer to as AM/M about origin.

If $\underline{r} \times \underline{F} = \underline{0}$ then angular momentum is constant.

$\underline{L}$ about some suitably chosen point may be constant.

4 Orbits

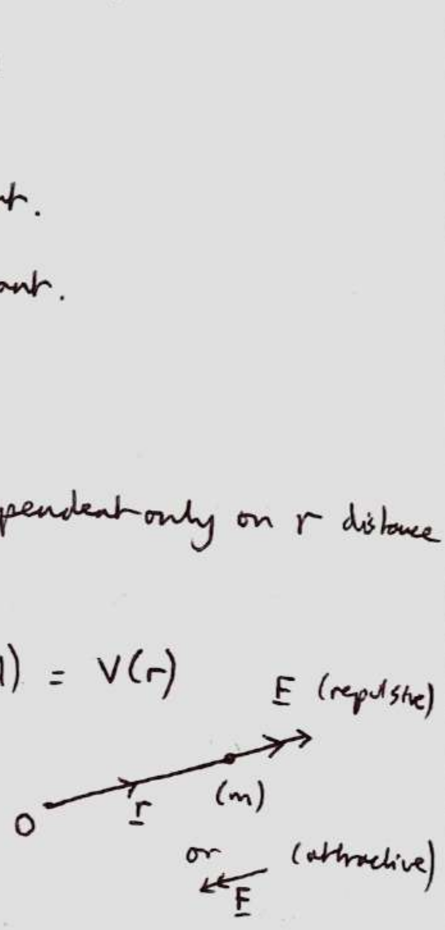
The basic problem: $m\ddot{\underline{r}} = -\nabla V(r)$

particle moves in force resulting from potential dependent only on r distance

4.1 Central forces

These are conservative forces with $V(\underline{r}) = V(|\underline{r}|) = V(r)$

$$\underline{F}(\underline{r}) = -\nabla V(|\underline{r}|) = -\frac{dV}{dr}\hat{\underline{r}}$$



Check: $r^2 = x^2 + y^2 + z^2$

$$2r\nabla r = (2x, 2y, 2z) = 2\underline{x}$$
$$\Rightarrow \nabla r = \frac{\underline{x}}{r} = \hat{\underline{r}}$$

Consider $\underline{L}$, about O: $$\frac{d\underline{L}}{dt} = \underline{r} \times \underline{F} = \underline{r} \times \left(-\frac{dV}{dr}\hat{\underline{r}}\right) = 0$$

so $\underline{L}$ is conserved for a central force.

So $\underline{L}$ is constant, also $\underline{L} \cdot \underline{r} = 0$ (from defn of $\underline{L}$)

Hence motion is in a plane through the origin and orientation of plane is set by ICs for $\underline{L}$.

4.2 Polar coordinates

Choose z-axis s.t. orbit lies in $z=0$: use (x,y) plane $x = r\cos\theta$, $y = r\sin\theta$

Define unit vectors in direction of increase of r, θ : $\underline{e}_r$, $\underline{e}_\theta$

$$\underline{e}_r = \hat{\underline{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \underline{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Note: $\dfrac{d}{d\theta}\underline{e}_r = \dfrac{d}{d\theta}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \underline{\underline{e}_\theta}$, $\quad \dfrac{d}{d\theta}\underline{e}_\theta = \underline{-\underline{e}_r}$

For moving particle θ is a function of position, and hence of time $r(t), \theta(t)$

$$\frac{d\underline{e}_r}{dt} = \frac{d\theta}{dt}\frac{d\underline{e}_r}{d\theta} = \underline{\underline{e}_\theta \frac{d\theta}{dt}}, \qquad \frac{d\underline{e}_\theta}{dt} = \frac{d\theta}{dt}\frac{d\underline{e}_\theta}{d\theta} = \underline{-\underline{e}_r \frac{d\theta}{dt}}$$

<u>Now:</u> $\underline{r} = r\underline{e}_r \Rightarrow \dot{\underline{r}} = \dot{r}\underline{e}_r + r\dfrac{d\underline{e}_r}{dt} = \underline{\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta}$

$\dot{r}$ radial comp. of velocity, $r\dot{\theta}$ angular comp. of velocity

and have $\underline{\dot{\theta} = \text{angular velocity}}$

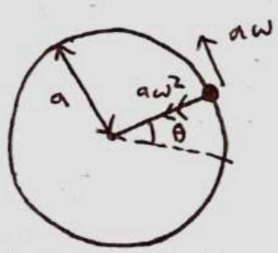
Then $\ddot{\underline{r}} = \dfrac{d}{dt}\left(\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta\right) = \ddot{r}\underline{e}_r + \dot{r}\underbrace{\dot{\underline{e}}_r}_{\dot{\theta}\underline{e}_\theta} + \dot{r}\dot{\theta}\underline{e}_\theta + r\ddot{\theta}\underline{e}_\theta + r\dot{\theta}\underbrace{\dot{\underline{e}}_\theta}_{-\dot{\theta}\underline{e}_r}$

$$= \underline{\left(\ddot{\underline{r}} - r\dot{\theta}^2\right)\underline{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\underline{e}_\theta}$$ acceleration in r and θ components

<u>Example</u> constant $\dot{\theta}$ circular motion

$$r = a, \quad \dot{\theta} = \omega \qquad \dot{r} = 0, \quad \ddot{\theta} = 0, \quad \ddot{r} = 0$$

Plug in: $\underline{\dot{\underline{r}} = a\omega\underline{e}_\theta}$, $\quad \underline{\ddot{\underline{r}} = -a\omega^2\underline{e}_\theta}$



centripetal acceleration

Newton's 2nd implies force is required to maintain circular motion (centripetal force).

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4.3 Motion in a central force field

Newton's 2nd: $m\ddot{\underline{r}} = \underline{F} = -\nabla V = -\frac{dV}{dr}\underline{e}_r$

Lecture 8: ~~mat~~ $m(\ddot{r} - r\dot{\theta}^2)\underline{e}_r + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{e}_\theta = -\frac{dV}{dr}\underline{e}_r$

radial   angular   radial

Angular component $m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 = \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$

equivalent to $mr^2\dot{\theta} = \text{constant}$

Recall $\underline{L} = m\underline{r} \times \dot{\underline{r}} = mr\underline{e}_r \times (\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta) = mr^2\dot{\theta}\underline{e}_z$

i.e. $|\underline{L}|$ is constant: $\underline{r^2\dot{\theta} = h}$ (constant).

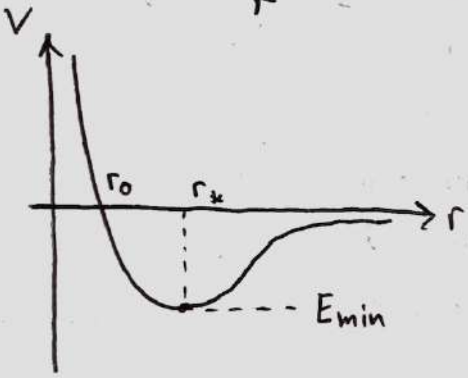
Radial component: $m\ddot{r} - mr\dot{\theta}^2 = -\frac{dV}{dr} \Rightarrow m\ddot{r} = -\frac{dV}{dr} + \frac{mh^2}{r^3} = -\frac{dV_{eff}}{dr}$

with $V_{eff}(r) = V(r) + \frac{1}{2}\frac{mh^2}{r^2}$ <u>effective potential</u>

Consider energy: $T + V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{1}{2}\frac{mh^2}{r^2} + V(r)}_{V_{eff}(r)}$

<u>Example</u> inverse square law force

$V(r) = -\frac{GMm}{r}$ $V_{eff}(r) = -\frac{GMm}{r} + \frac{1}{2}\frac{mh^2}{r^2}$ (for given h)



$V_{eff}(r_0) = 0, \quad V_{eff}'(r_*) = 0$

$r_0 = \frac{h^2}{2GM} \qquad r_* = \frac{h^2}{GM}$

$E_{min} = -\frac{m(GM)^2}{2h^2}$

What is possible motion of m?

$E = E_{min}$: $r(t) = r_*$ $\dot{\theta} = \frac{h}{r_*^2}$ circular motion, constant $\dot{\theta}$

$E_{min} < E < 0$: $r(t)$ oscillates between min, max values

$\dot{\theta}$ varies

$0 \leq E$, $r(t) \to \infty$ as $t \to \infty$ - escape to infinity

r_{min} is called the periapsis / perihelion (sun), perigee (Earth)

r_{max} is called the apoapsis, aphelion, apogee

4.5 Stability of orbits (circular)

Consider general $V(r)$: is there a circular orbit, and if so is it stable?

$\underline{L} \neq \underline{0}$, given

For circular orbit $r(t) = r_* =$ constant requires $\ddot{r} = 0 \Rightarrow \boxed{V_{eff}'(r_*) = 0}$

Stable if V_{eff} is a minimum at r_*, else unstable for max

Rewrite in terms of $V(r)$: $\underline{V'(r_*) - \frac{mh^2}{r_*^3} = 0}$ for circular

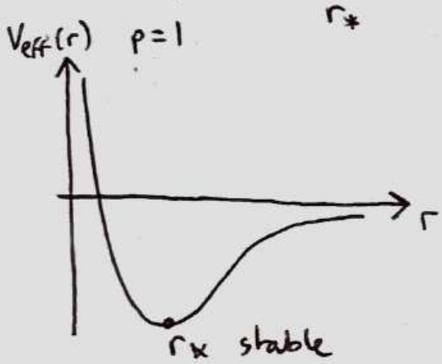
Stable if $V''(r_*) + \frac{3mh^2}{r_*^4} > 0$, i.e. $V''(r_*) + \frac{3V'(r_*)}{r_*} > 0$

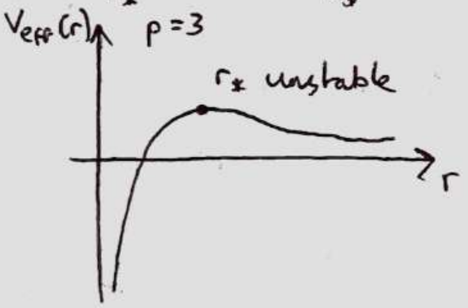
Example $V(r) = -\frac{km}{r^p}$ $p > 0$, $k > 0$

For circular motion: $\frac{pkm}{r_*^{p+1}} - \frac{mh^2}{r_*^3} = 0 \Rightarrow r_*^{p-2} = \frac{pk}{h^2}$

$\Rightarrow r_* = \left(\frac{pk}{h^2}\right)^{1/p-2}$ so circular orbit $\forall p \neq 2$

$V''(r_*) + \frac{3V'(r_*)}{r_*} = \frac{-kmp(p+1)}{r_*^{p+2}} + \frac{3kmp}{r_*^{p+2}} = \frac{-p(p-2)k}{r_*^{p+2}}$ >0 if $0<p<2$ (stable), <0 if $p>2$ (unstable)





4.6 The orbit equation

Could determine $r(t)$ via $E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r) =$ constant (scalar r as V_{eff} applies to radial r)

and use $r(t)^2\dot{\theta} = h$ to deduce $\theta(t)$ but often not analytically possible

Better: θ indep. variable: write $\frac{d}{dt} = \frac{d\theta}{dt}\cdot\frac{d}{d\theta} = \frac{h}{r^2}\frac{d}{d\theta}$

Newton's 2nd: $m\frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{d}{d\theta}r\right) - \frac{mh^2}{r^3} = F(r)$

suggests using $u = \frac{1}{r}$ as variable: gives

$$mhu^2\frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right) - mh^2u^3 = F(u^{-1})$$

$$\Rightarrow \underline{\frac{d^2u}{d\theta^2} + u = -\frac{1}{mh^2u^2}F(u^{-1})}$$ — $\underline{\text{orbit equation}}$ (Binet eqn)

Solve for $u(\theta)$ then use $\dot{\theta} = hu^2$ etc.

4.7 The Kepler problem

Orbit problem for special case of gravitational central force

$$F(r) = -\frac{mk}{r^2} \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{k}{h^2} \quad (u = r^{-1}) \quad \text{linear in } u$$

Solving: $u = \frac{k}{h^2} + A\cos(\theta - \theta_0)$ WLOG take $A \geq 0$.

$A = 0$: $u = \frac{k}{h^2}$ circular orbit, $A > 0$: u max when $\theta = \theta_0$ (r min) periapsis

WLOG let θ_0 and write in r: $r = \frac{1}{u} = \frac{l}{1 + e\cos\theta}$ $\quad l = h^2/k$, $e = Ah^2/k$

a conic section : e = eccentricity

Cartesian : $r(1 + e\cos\theta) = l \Rightarrow r = l - ex \Rightarrow x^2 + y^2 = (l - ex)^2$

$\Rightarrow (1 - e^2)x^2 + y^2 + 2elx = l^2 \quad (\dagger)$ $\quad 0 < e < 1$ ellipse : $\frac{l}{1+e} \leq r \leq \frac{l}{1-e}$

Rewrite: $\frac{(x + ea)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a = \frac{l}{1 - e^2}, \quad b = \frac{l}{\sqrt{1 - e^2}}$ $\quad e = 0$ circle $a = b =$ radius

$e > 0$: origin lies at one focus - e and l determine a and b.

$e > 1$: hyperbola with $r \to \infty$ as $\theta \to \pm\alpha = \pm\cos^{-1}\left(-\frac{1}{e}\right) \in \left(\frac{\pi}{2}, \pi\right)$

Rewrite $(\dagger)$: $\frac{(x - ea)^2}{a^2} - \frac{y^2}{b^2} = 1 \quad a = \frac{l}{e^2 - 1}, \quad b = \frac{l}{\sqrt{e^2 - 1}}$

Represents incoming body with large velocity - asymptotes $y = \pm\frac{b}{a}(x - ea)$

i.e. $bx \mp ay = eba$, normal vectors $\frac{(b, \mp a)}{\sqrt{a^2 + b^2}}$

$\perp$ distance between incoming trajectory and mass is $\underline{r} \cdot \underline{n} = (x, y) \cdot \underline{n}$

$$= \frac{bx \mp ay}{\sqrt{a^2 + b^2}} = \frac{eba}{\sqrt{a^2 + b^2}} = b \quad \text{(impact parameter)}$$

$e = 1$ parabola : $r = \frac{l}{1 + \cos\theta}$ $\quad r \to \infty$ as $\theta \to \pm\pi$

$$y^2 = 2l(l - x)$$

Recall

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{mk}{r}$$

$$= \frac{1}{2}mh^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) - mku \quad \left(\dot{r} = -\frac{h\,du}{d\theta}\right)$$

$$= \frac{1}{2}mh^2\left(e^2\sin^2\theta + (1 + e\cos\theta)^2\right)\frac{1}{l^2} - \frac{mk}{l}(1 + \cos\theta)$$

$$= \frac{mk}{2l}(e^2 - 1) \quad (l = h^2/k)$$

$E > 0$ for $e > 1$ - unbounded

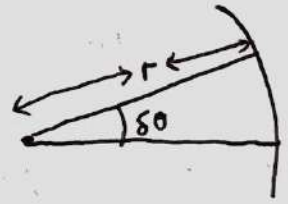
$E < 0$ for $e < 1$ - bounded

Kepler's Laws of Planetary Motion

1) Orbit of planet is ellipse with sun at focus
2) Line between planet and sun sweeps out equal area in equal times
3) Square of period $P \propto$ cube of semi-major axis $\therefore$ $P^2 \propto a^3$

1 - ~~consitent~~ consistent with the solution for bound orbits

2 -



$\delta\theta$ small change in θ in time δt

area $\approx \frac{1}{2} r^2 \delta\theta$! rate of change is $\frac{1}{2} r^2 \dot{\theta}$

$= \frac{1}{2} h$ (h = angular momentum) constant

Follows from conservation of angular momentum.

3 - note area of ellipse is $\pi a b = \frac{h}{2} P$ (see above)

recall $b^2 = a^2(1-e^2)$ $\quad h^2 = kl = ka(1-e^2)$ - check

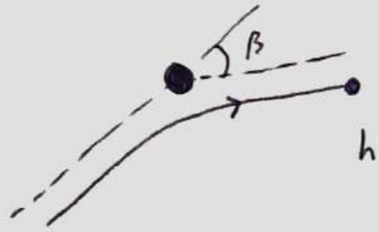
now consider $P^2 = \left(\frac{2\pi a b}{h}\right)^2 = \frac{4\pi^2 a^2 \cdot a^2(1-e^2)}{ka(1-e^2)}$

$= \frac{4\pi^2 a^3}{k} \propto a^3$. (constant of proportionality indep. of mass)

Dynamics and Relativity - Lecture 11

4.8 Rutherford Scattering

+ve charge fired towards another fixed +ve charge - what is scattering angle β?



$h = |\underline{r} \times \dot{\underline{r}}|$

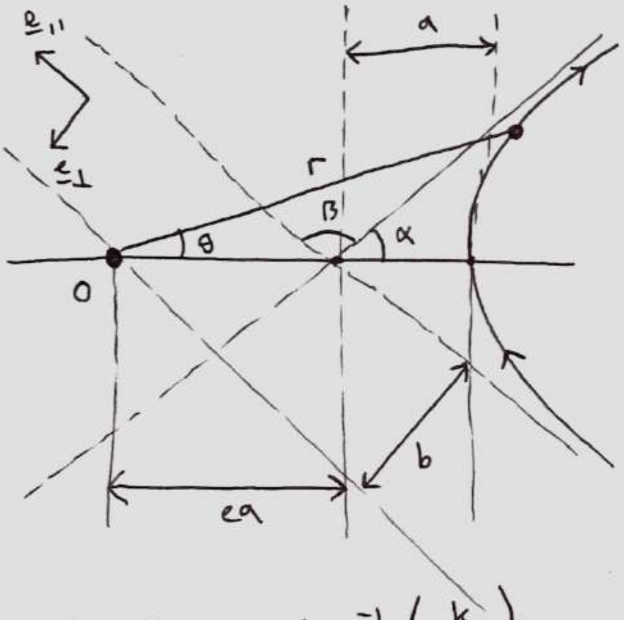
Consider motion in repulsive square law force $V(r) = \frac{mk}{r}$, $F(r) = \frac{mk}{r^2}$

Solution of orbit eqn $u = \frac{-k}{h^2} + A\cos(\theta - \theta_0)$ wlog $\theta_0 = 0$, $A \geq 0$

Rewrite as $r = \frac{l}{e\cos\theta - 1}$ $\quad l = \frac{h^2}{k}$ $\quad e = \frac{Ah^2}{k}$ requires $e > 1$ for $r > 0$ for some θ

Then $r \to \infty$ as $\theta \to \pm\alpha$ with $\alpha = \cos^{-1}\left(\frac{1}{e}\right) \in (0, \frac{\pi}{2})$ - hyperbolic

Cartesian: $\frac{(x - ea)^2}{a^2} - \frac{y^2}{b^2} = 1$, $\quad a = \frac{l}{e^2 - 1}$, $\quad b = \frac{l}{\sqrt{e^2 - 1}}$



on incoming asymptote

$\dot{\underline{r}} \approx v\underline{e}_{\parallel}$

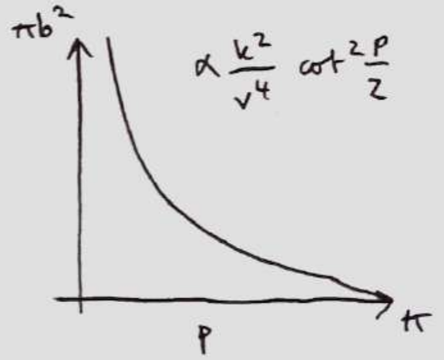
$\underline{r} \approx ()\,\underline{e}_{\parallel} + b\,\underline{e}_{\perp}$

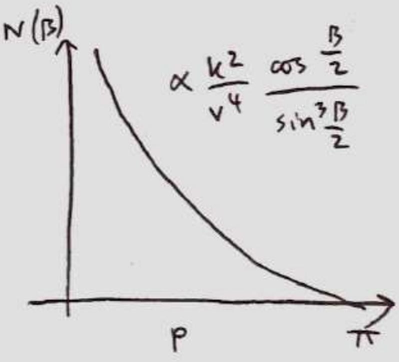
$\Rightarrow \quad h = |\underline{r} \times \dot{\underline{r}}| = bv$

$$b = \frac{l}{\sqrt{e^2 - 1}} = \frac{l}{\tan\alpha}$$

$$= \frac{l}{\tan\left(\frac{\pi}{2} - \beta\right)} = \frac{h^2}{k}\tan\frac{\beta}{2} = \frac{v^2 b^2}{k}\tan\frac{\beta}{2}$$

$$\Rightarrow \quad \beta = 2\tan^{-1}\left(\frac{k}{bv^2}\right)$$





prob. density of scattering angle

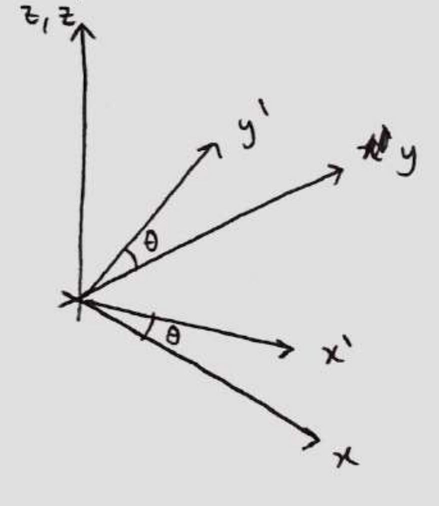
Cumulative distribution for β

5 Rotating frames of reference

Newton's laws valid only in inertial frames

A rotating frame is non-inertial - modified eqns of motion

Let S be an inertial frame, and S' be a non-inertial rotating frame rotating about z in S - with angular velocity $\omega = \dot{\theta}$



(x, y)-plane and (x', y') plane in same plane

Basis for S: $\underline{e}_i = \{\underline{\hat{x}}, \underline{\hat{y}}, \underline{\hat{z}}\}$
$\underline{e}_1, \underline{e}_2, \underline{e}_3$

Basis for S': $\underline{e}_i' = \{\underline{\hat{x}}', \underline{\hat{y}}', \underline{\hat{z}}'\}$
$\underline{e}_1', \underline{e}_2', \underline{e}_3'$

Consider particle at rest in S'. Viewed in S we have

$$\left(\frac{d\underline{r}}{dt}\right)_S = \underline{\omega} \times \underline{r} \quad \text{where } \underline{\omega} = \omega \underline{\hat{z}}$$

(Angular velocity vector aligned with axis of rotation, $|\underline{\omega}| = \omega$, viewed from direction of vector rotation AC if $\omega > 0$)

Same formula applies to any vector fixed in S', in particular to basis vectors $\underline{e}_i'$

i.e. $\left(\frac{d}{dt} \underline{e}_i'\right)_S = \underline{\omega} \times \underline{e}_i'$ (note $\frac{d}{dt} \underline{e}_3' = 0$ here)

Consider general time dependent vector $\underline{a}$ $\quad \underline{a}(t) = \sum_{i=1}^{3} a_i'(t) \underline{e}_i'(t)$

expression of $\underline{a}$ in terms of components defined in S'

Consider rate of change:

$$\left(\frac{d}{dt} \underline{a}(t)\right)_{S'} = \sum_{i=1}^{3} \left(\frac{d}{dt} a_i(t)\right) \underline{e}_i'(t) \quad \text{as observed in } S'$$

How about observed in S?

$$\left(\frac{d}{dt} \underline{a}(t)\right)_S = \sum_{i=1}^{3} \left(\frac{d}{dt} a_i'(t)\right) \underline{e}_i'(t) + \sum_{i=1}^{3} a_i'(t) \overbrace{\frac{d}{dt} \underline{e}_i'(t)}^{\omega \times \underline{e}_i'}$$

from rate of change of $\underline{e}_i'$

$$= \left(\frac{d}{dt}\underline{a}\right)_{S'} + \underline{\omega} \times \underline{a}$$

key identity - relates rate of change of vectors seen in S' to rate of change seen in S.

$$\left(\frac{d}{dt}\underline{a}(t)\right)_S = \left(\frac{d}{dt}\underline{a}(t)\right)_{S'} + \underline{\omega} \times \underline{a}$$

Apply to position vector $\underline{r}$:

$$\left(\frac{d\underline{r}}{dt}\right)_S = \left(\frac{d\underline{r}}{dt}\right)_{S'} + \underline{\omega} \times \underline{r} \quad \text{(velocity)}$$

difference depends on position

Now apply to velocity - allow $\underline{\omega}(t)$

$$\left(\frac{d^2r}{dt^2}\right)_S = \left\{\left(\frac{d}{dt}\right)_{S'} + \underline{\omega}\times\right\}\left\{\left(\frac{d}{dt}\right)_{S'} + \underline{\omega}\times\right\}\underline{r}$$

$$= \left(\frac{d^2}{dt^2}\underline{r}\right)_{S'} + 2\underline{\omega} \times \left(\frac{d}{dt}\underline{r}\right)_{S'} + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad (\text{acc}^n)$$

Equation of motion in rotating frame:

$$m\left(\frac{d^2r}{dt^2}\right)_S = \underline{F} = m\left(\left(\frac{d^2\underline{r}}{dt^2}\right)_{S'} + 2\underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_{S'} + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})\right)$$

additional forces are "fictitious forces"

Must take fictitious forces into account in rotating / non-inertial frame

Coriolis force: $-2m\underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_{S'}$

Euler force: $-m\dot{\underline{\omega}} \times \underline{r}$ - usually 0 e.g. when $\dot{\underline{\omega}} = 0$

Centrifugal force: $-m\underline{\omega} \times (\underline{\omega} \times \underline{r})$

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Rotating frames cont.

Coriolis force $-2m\underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_S$, Euler force $-m\dot{\underline{\omega}} \times \underline{r}$ (often 0)

Centrifugal force $-m\underline{\omega} \times (\underline{\omega} \times \underline{r})$

Centrifugal force

$$-m\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -m((\underline{\omega}\cdot\underline{r})\underline{\omega} - \omega^2 \underline{r}) = m\omega^2(\underline{r} - \hat{\underline{\omega}}(\hat{\underline{\omega}}\cdot\underline{r}))$$

$\hat{\underline{\omega}}$ unit vector in direction of $\underline{\omega}$ $= m\omega^2 \underline{r}_\perp$ $\underline{r}_\perp$ part of $\underline{r}$ $\perp$ to $\underline{\omega}$

Note that $|\underline{r}_\perp|$ is perp. distance from point to axis

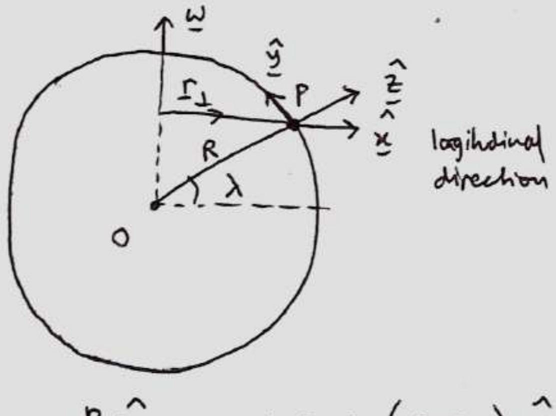
Centrifugal force is $\perp$ to rotation axis and directed <u>away</u> from it.
Magnitude $\propto$ distance from axis

Note: $\underline{r}_\perp^2 = \underline{r}^2 - (\underline{r}\cdot\hat{\underline{\omega}})^2 = |\hat{\underline{\omega}} \times \underline{r}|^2$

$\nabla \underline{r}_\perp^2 = 2\underline{r} - 2\hat{\underline{\omega}}(\hat{\underline{\omega}}\cdot\underline{r}) = 2\underline{r}_\perp$ hence $m\omega^2\underline{r}_\perp = \nabla\left(\frac{1}{2}m\omega^2\underline{r}_\perp^2\right)$

centrifugal force conservative

On a rotating planet, convenient to combine CF force and grav. force into "effective gravity" : $\underline{g}_{eff} = \underline{g} + \omega^2 \underline{r}_\perp$



point P at latitude λ

$\hat{\underline{x}}$ horizontal, eastward axis

$\hat{\underline{y}}$ horizontal, axis northward

$\hat{\underline{z}}$ vertical

$\underline{r} = R\hat{\underline{z}}$ $\underline{\omega} = \omega(\cos\lambda\,\hat{\underline{y}} + \sin\lambda\,\hat{\underline{z}})$

$\underline{g}_{eff} = -g\hat{\underline{z}} + \omega^2\underline{r}_\perp$

$\omega^2 R\cos\lambda\,(\hat{\underline{z}}\cos\lambda - \hat{\underline{y}}\sin\lambda)$

$= \hat{\underline{z}}(\omega^2 R\cos^2\lambda - g) - \hat{\underline{y}}\,\omega^2 R\cos\lambda\sin\lambda$

Angle between $\underline{g}_{eff}$ and $\hat{\underline{z}}$ $= \alpha = \tan^{-1}\left(\frac{\omega^2 R\cos\lambda\sin\lambda}{g - \omega^2 R\cos^2\lambda}\right)$

Coriolis force

$$-2m\underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_{S'} = -2m\underline{\omega} \times \underline{v} \qquad \underline{v} \text{ velocity observed in rotating frame}$$

proportional to $\underline{v}$ and perpendicular to velocity – hence this force does no work on the particle.

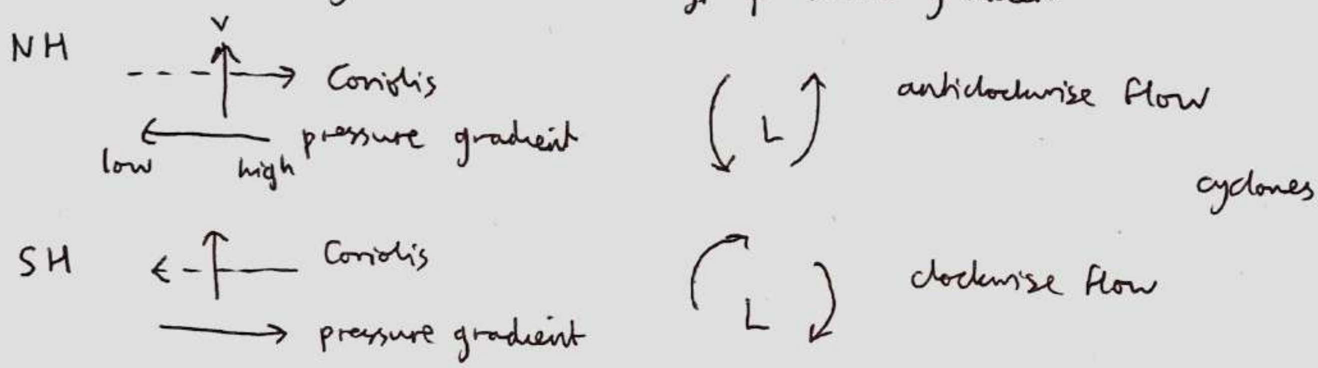
On rotating planet, $\underline{v} = v_x \hat{\underline{x}} + v_y \hat{\underline{y}}$ tangential to surface

angular velocity $\underline{\omega} = \omega(\cos\lambda\, \hat{\underline{y}} + \sin\lambda\, \hat{\underline{z}})$

hence $-2m\underline{\omega} \times \underline{v} = \underbrace{2m\omega\sin\lambda\,(v_y \hat{\underline{x}} - v_x \hat{\underline{y}})}_{\text{horizontal}} + \underbrace{2m\omega\cos\lambda\, v_x \hat{\underline{z}}}_{\text{vertical}}$

Horizontal Coriolis force gives acc^n to the right of the horizontal velocity in Northern Hemisphere, left in Southern Hemisphere

Can be balanced by another force e.g. pressure gradient



Anticyclones

<u>Example problem</u> (not horizontal) Ball dropped from top of tower – where does it land? taking account of rotation, $\underline{m = 1}$

$$\ddot{\underline{r}} = \underline{g} - 2\underline{\omega} \times \dot{\underline{r}} - \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

rotation slow: $\omega^2 R / g$ is small

Work to <u>first order</u> in ω: $\ddot{\underline{r}} = \underline{g} - 2\underline{\omega} \times \dot{\underline{r}} + O(\omega^2)$

$$\dot{\underline{r}} = \underline{g}t - 2\underline{\omega} \times \underline{r} + O(\omega^2) + \underbrace{2\underline{\omega} \times \underline{r}(0)}_{\text{constant from ICs}}$$

Hence neglecting $O(\omega^2)$:

$$\ddot{\underline{r}} = \underline{g} - 2\underline{\omega} \times \underline{g}t + O(\omega^2)$$

$$\underline{r} = \underline{g}\frac{t^2}{2} - \frac{1}{3}\underline{\omega} \times \underline{g}t^3 + \underline{r}(0) + O(\omega^2)$$

Now consider $\underline{g} = (0, 0, -g)$ $\quad \underline{w} = (0, w, 0)$ hence $\underline{r}(0) = (0, 0, R+h)$

equator

Hence $\underline{r}(t) = \left(0, 0, -\frac{gt^2}{2}\right) + (0, 0, R+h) + \frac{1}{3} wg (t^3, 0, 0)$

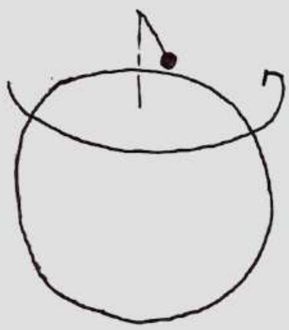
Particle hits ground when $h = \frac{gt^2}{2}$ $\quad t = (2h/g)^{1/2}$

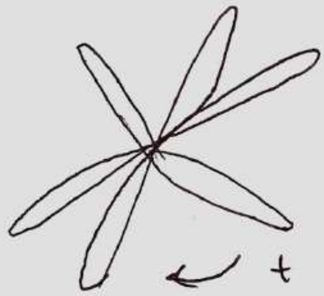
horizontal displacement $\underline{\frac{1}{3} wg \left(\frac{2h}{g}\right)^{3/2}}$

hits ground to east of base of tower

(consistent with conservation of angular momentum)

Foucault Pendulum





consider pendulum at North pole - swings in plane fixed in inertial frame, Earth rotates relative to this - observer on Earth sees plane of rotation move to west

At North Pole plane of rotation observed to rotate once per day - explained by fictitious force

At general latitude: plane completes circuit in $\frac{1}{\sin \lambda}$ days

(consider dynamics of pendulum - include Coriolis force - hence latitudinal dependence)

6 Systems of particles

We have considered dynamics of particles - now move onto systems of particles.

Consider system of N particles, mass m_i, position $\underline{r}_i(t)$, momentum $\underline{p}_i(t) = m_i \dot{\underline{r}}_i$

Newton's 2nd law applies to ith particle individually: $m_i \ddot{\underline{r}}_i = \dot{\underline{p}}_i = \underline{F}_i$

Distinguish between external and internal force: $\underline{F}_i = \underline{F}_i^{ext} + \sum_{j=1}^{N} \underline{F}_{ij}$

$\underline{F}_{ij}$: force exerted on ith by jth particle, $\underline{F}_{ii} = 0$

$\underline{F}_i^{ext}$ external force exerted on ith particle $\quad \boxed{F_{ij} = -F_{ji}}$

e.g. gravitation $\underline{F}_{ij} = \dfrac{-GM_iM_j(\underline{r}_i - \underline{r}_j)}{|\underline{r}_i - \underline{r}_j|^3} \quad \underline{F}_{ji} = \dfrac{-GM_iM_j(\underline{r}_j - \underline{r}_i)}{|\underline{r}_i - \underline{r}_j|^3}$

6.1 Centre of mass

Total mass $M = \sum_{i=1}^{n} m_i$

Centre of mass located at $\underline{R} = \frac{1}{M} \sum_{i=1}^{n} m_i \underline{r}_i$ "weighted"

Total linear momentum

$$\underline{P} = \sum_{i=1}^{n} m_i \dot{\underline{r}}_i = \sum_{i=1}^{n} \underline{p}_i = M\dot{\underline{R}}$$

Then

$$\dot{\underline{P}} = M\ddot{\underline{R}} = \sum_{i=1}^{n} \dot{\underline{p}}_i = \sum_{i=1}^{n} \underline{F}_i^{ext} + \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} \underline{F}_{ij}}_{=\underline{0} \text{ (consider pairwise sum)}}$$

$$= \sum_{i=1}^{n} \underline{F}_i^{ext}$$

Centre of mass moves as if it's position of mass M under influence of $\underline{F}^{ext} = \sum_{i=1}^{n} \underline{F}_i^{ext}$, extending N's 2nd to system of particles.

If $\underline{F}^{ext} = \underline{0}$ then $\dot{\underline{P}} = 0$: total momentum conserved.

There will be a "centre-of-mass" frame with origin at centre of mass - inertial frame.

In this frame $\dot{\underline{R}} = 0$, so can take $\underline{R} = \underline{0}$

Now consider motion relative to centre of mass.

6.2 Motion relative to centre of mass

Let $\underline{r}_i = \underline{R} + \underline{s}_i$ $\quad\quad$ $\underline{s}_i$ position rel. to CoM

$$\sum_{i=1}^{n} m_i \underline{s}_i = \sum_{i=1}^{n} m_i (\underline{r}_i - \underline{R}) = \underbrace{\sum_{i=1}^{n} m_i \underline{r}_i}_{M\underline{R}} - \underbrace{\sum_{i=1}^{n} m_i \underline{R}}_{M\underline{R}} = \underline{0}$$

$$\frac{d}{dt}\left(\sum_{i=1}^{n} m_i \underline{s}_i\right) = 0$$

Total linear momentum

$$\underline{P} = \sum_{i=1}^{n} m_i (\dot{\underline{R}} + \dot{\underline{s}}_i) = \sum_{i=1}^{n} m_i \dot{\underline{R}}_i = M\dot{\underline{R}}$$

6.3 Angular momentum

Total angular momentum $\underline{L} = \sum_{i=1}^{n} \underline{r}_i \times \underline{p}_i$ (about 0)

$$\dot{\underline{L}} = \underbrace{\cancel{\sum_{i=1}^{n} \dot{\underline{r}}_i \times \underline{p}_i}}_{0} + \underline{\sum_{i=1}^{n} \underline{r}_i \times \dot{\underline{p}}_i} = \sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i^{ext} + \sum_{i=1}^{n} \underline{r}_i \times \sum_{j=1}^{n} \underline{F}_{ij}$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{r}_i \times \underline{F}_{ij} + \underline{r}_j \times \underline{F}_{ji}) (\underline{r}_i - \underline{r}_j) \underline{F}_{ij}$$

$= 0$ e.g. if $\underline{F}_{ij} \parallel \underline{r}_i - \underline{r}_j$. (*)

If (*) holds: $\underline{\dot{\underline{L}} = \sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i^{ext} = \underline{G}^{ext}}$ total external torque

sum of moments of ext. forces

Now return to motion / position relative to centre of mass.

Total angular momentum

$$\underline{L} = \sum_{i=1}^{n} m_i (\underline{R} + \underline{s}_i) \times (\dot{\underline{R}} + \dot{\underline{s}}_i)$$

$$= \sum_{i=1}^{n} m_i (\underline{R} \times \dot{\underline{R}}) + \underbrace{\cancel{\sum_{i=1}^{n} m_i \underline{R} \times \dot{\underline{s}}_i}}_{0} + \underbrace{\cancel{\sum_{i=1}^{n} m_i \underline{s}_i \times \dot{\underline{R}}}}_{0} + \sum_{i=1}^{n} m_i \underline{s}_i \times \dot{\underline{s}}_i$$

by $\underline{s}_i$ property earlier

so $\underline{L}$ of mass M at $\underline{R}$ and moving with $\dot{\underline{R}}$ $\underline{+}$ angular momentum associated with motion of particles rel. to CoM.

6.4 Energy

Total KE $\quad T = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\underline{r}}_i^{\,2} = \sum_{i=1}^{n} \frac{1}{2} m_i (\dot{\underline{R}} + \dot{\underline{s}}_i)^2$

$$= \underbrace{\frac{1}{2} \dot{\underline{R}}^2 \sum_{i=1}^{n} m_i}_{\text{KE of mass } M \text{ moving with vel } \dot{R}} + \underbrace{\cancel{\sum_{i=1}^{n} m_i \dot{\underline{s}}_i \cdot \dot{\underline{R}}}}_{=0} + \underbrace{\sum_{i=1}^{n} \frac{1}{2} m \dot{\underline{s}}_i^{\,2}}_{\text{KE assoc. with particle motion rel. to CoM}}$$

Is energy conserved? $\quad \frac{dT}{dt} = \frac{d}{dt} \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\underline{r}}_i^{\,2} = \sum_{i=1}^{n} m_i \dot{\underline{r}}_i \cdot \ddot{\underline{r}}_i$

$$= \sum_{i=1}^{n} \dot{\underline{r}}_i \cdot \underline{F}_i^{\,ext} + \sum_{i=1}^{n} \dot{\underline{r}}_i \cdot \sum_{j=1}^{n} \underline{F}_{ij}$$

$$\uparrow$$

$$\sum_{i=1}^{n} \sum_{j>i}^{n} (\dot{\underline{r}}_i - \dot{\underline{r}}_j) \cdot \underline{F}_{ij}$$

If ext. forces defined by potential $\underline{F}_i^{\,ext} = -\nabla_{\underline{r}_i} V_i$

and int. forces defined by potential $\underline{F}_{ij} = -\nabla_{\underline{r}_i} V(\underline{r}_i - \underline{r}_j)$

then $\frac{dT}{dt} = -\frac{d}{dt} \sum_{i=1}^{n} V^{ext}(\underline{r}_i) - \frac{d}{dt} \sum_{i=1}^{n} \sum_{j>i} V(\underline{r}_i - \underline{r}_j)$

so if this holds, then energy is conserved.

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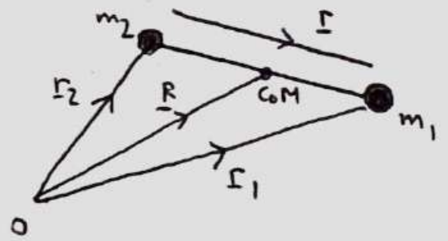
6.3 Two body problem



No external forces

Centre of mass at $\underline{R} = \frac{1}{M}(m_1 \underline{r}_1 + m_2 \underline{r}_2)$

$M = m_1 + m_2$

$\underline{r} = \underline{r}_1 - \underline{r}_2$

$\underline{r}_1 = \underline{R} + \frac{m_2}{M}\underline{r}, \quad \underline{r}_2 = \underline{R} - \frac{m_1}{M}\underline{r}$

$\underline{F}^{ext} = \underline{0}$ hence $\ddot{\underline{R}} = \underline{0}$ centre of mass moves with constant v

Consider $\ddot{\underline{r}} = \ddot{\underline{r}}_1 - \ddot{\underline{r}}_2 = \frac{\underline{F}_{12}}{m_1} - \frac{\underline{F}_{21}}{m_2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\underline{F}_{12}$

Equivalent: $\mu \ddot{\underline{r}} = \underline{F}_{12}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$ "reduced mass" $\mu < m_1, m_2$

Standard Newton's 2nd, particle mass μ

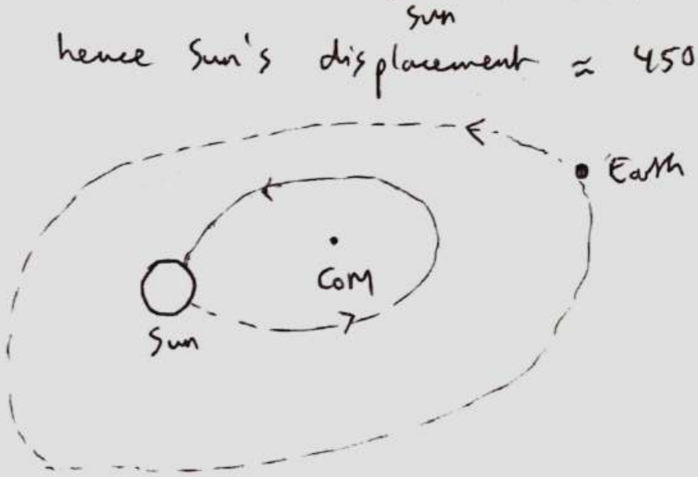
Gravitational force: $\mu \ddot{\underline{r}} = -\frac{G m_1 m_2}{|\underline{r}|^3}\underline{r}$ hence $\ddot{\underline{r}} = -\frac{G(m_1+m_2)\underline{r}}{|\underline{r}|^3}$

This is motion of particle under gravitational force due to mass $M = m_1 + m_2$ fixed at origin.

Consider Earth – Sun orbit, both move about CoM, same shape of orbit with different size

Ratio of masses $\frac{\text{Earth}}{\text{Sun}} \approx 3 \times 10^{-4}$, Earth-Sun distance $\approx 1.5 \times 10^7$ km

hence Sun's displacement ≈ 450 km.



Reduced mass

$\mu = \frac{m_1 m_2}{m_1 + m_2}$

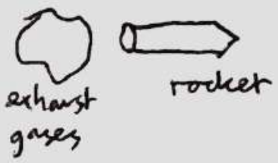
KE: $T = \frac{1}{2} M \dot{\underline{R}}^2 + \frac{1}{2}\mu \dot{\underline{r}}^2$

$L = M\underline{R} \times \dot{\underline{R}} + \mu \underline{r} \times \dot{\underline{r}}$

Special forms of L13 eqns for 2 bodies

6.4 Variable mass problems

Rocket problem

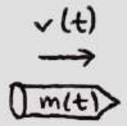


Rocket-only system is not closed wrt mass

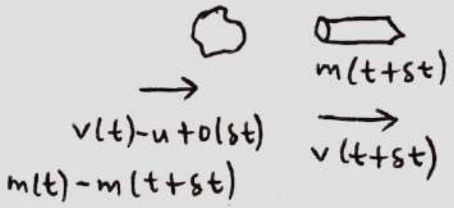
Cannot apply Newton's 2nd to just rocket

speed $v(t)$, mass $m(t)$, expels mass at velocity u relative to rocket

Time t



Time $t+\delta t$



Total momentum conserved:

Total p at $t+\delta t$ is $m(t+\delta t)\,v(t+\delta t)$

$$+\left(m(t)-m(t+\delta t)\right)\left(v(t)-u+o(\delta t)\right)$$

Change in p from t to $t+\delta t$ is

$$m(t+\delta t)\,v(t+\delta t)+\left(m(t)-m(t+\delta t)\right)\left(v(t)-u+o(\delta t)\right)-m(t)v(t)$$

$$\approx \left(\frac{dm}{dt}u+m\frac{dv}{dt}\right)\delta t+o(\delta t^2) = 0$$

Let $\delta t\to 0$: $\quad \underline{\dfrac{dm}{dt}u+m\dfrac{dv}{dt}=0}$ $\quad$ rocket equation

(Generalise to $\dfrac{dm}{dt}u+m\dfrac{dv}{dt}=F_{ext}$)

If $F_{ext}=0$ then $m\dfrac{dv}{dt}=-\dfrac{dm}{dt}u$

hence $v(t)=v(0)+u\log\left(\dfrac{m(0)}{m(t)}\right)$ $\quad$ solving (integrating)

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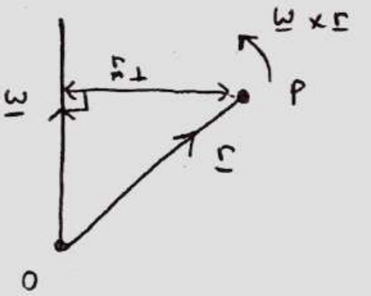
7 Rigid bodies

Multiparticle system s.t. $|\underline{r}_i - \underline{r}_j| = \text{constant} \ \forall i, j$

Possible motion is translation / rotation (isometries of Euclidean space)

7.1 Angular velocity

$\dot{\underline{r}} = \underline{\omega} \times \underline{r}$



$|\dot{\underline{r}}| = \omega r_\perp$, $r_\perp$ being perp. distance to axis of rotation

If particle has ~~too~~ mass m then

$$T = \frac{1}{2} m \dot{\underline{r}}^2 = \frac{1}{2} m (\underline{\omega} \times \underline{r}) \cdot (\underline{\omega} \times \underline{r})$$
$$= \underline{\frac{1}{2} m \omega^2 r_\perp^2}$$

Note that if $\underline{\omega} = \omega \underline{n}$ then $r_\perp = |\underline{n} \times \underline{r}|$.

$T = \frac{1}{2} m r_\perp^2 \omega^2 = \underline{\frac{1}{2} I \omega^2}$ where I is moment of inertia about axis of rotation : $I = m r_\perp^2$

7.2 Moment of inertia for a rigid body

N particle rigid body rotating about axis through origin with angular velocity $\underline{\omega}$

For each particle: $\underline{\dot{\underline{r}}_i = \underline{\omega} \times \underline{r}_i}$

note $\frac{d}{dt} |\underline{r}_i - \underline{r}_j|^2 = 2(\underline{r}_i - \underline{r}_j) \cdot (\dot{\underline{r}}_i - \dot{\underline{r}}_j)$

$= 2(\underline{r}_i - \underline{r}_j) \cdot (\underline{\omega} \times (\underline{r}_i - \underline{r}_j)) = 0$ - consistent with rigid body property

Consider KE:

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\underline{r}}_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i |\underline{\omega} \times \underline{r}_i|^2 = \sum_{i=1}^{n} \frac{1}{2} m_i |\underline{n} \times \underline{r}_i|^2 \cdot \omega^2$$

Then $T = \frac{1}{2} \sum_{i=1}^{N} m_i |\underline{\omega} \times \underline{r}_i|^2 = \frac{1}{2} \omega^2 \sum_{i=1}^{N} m_i |\underline{n} \times \underline{r}_i|^2$

$= \underline{\frac{1}{2} I \omega^2}$

where I = moment of inertia of body for rotation about axis $\underline{n}$ through origin.

Now consider angular momentum $\underline{L} = \sum_{i=1}^{N} m_i \underline{r}_i \times \underbrace{(\underline{\omega} \times \underline{r}_i)}^{\dot{\underline{r}}_i}$

Consider $\underline{\omega} = \omega \underline{n}$, then $\underline{L} = \omega \sum_{i=1}^{N} m_i \underline{r}_i \times (\underline{n} \times \underline{r}_i)$

Now consider part of $\underline{L}$ parallel to rotation axis $\underline{L} \cdot \underline{n}$

$\underline{L} \cdot \underline{n} = \omega \sum_{i=1}^{N} m_i \underline{n} \cdot (\underline{r}_i \times (\underline{n} \times \underline{r}_i)) = \omega \sum_{i=1}^{N} m_i |\underline{n} \times \underline{r}_i|^2$

$= \omega \sum_{i=1}^{N} m_i (r_i)_\perp^2 = \underline{I \omega}$

Component of angular momentum in direction of rotation axis is $\underline{I\omega}$.

In general $\underline{L}$ is not parallel to rotation axis

But have $\underline{L} = \sum_{i=1}^{N} m_i \left(|\underline{r}_i|^2 \underline{\omega} - (\underline{r}_i \cdot \underline{\omega}) \underline{r}_i \right)$ linear function of $\underline{\omega}$

i.e. $\underline{L} = \underline{\underline{I}}\, \underline{\omega}$

$\underline{\underline{I}}$: matrix like object

e.g. $L_\alpha = I_{\alpha\beta} \omega_\beta$ (Σ convention)

symmetric tensor $\quad I_{\alpha\beta} = \sum_{i=1}^{N} m_i \left\{ |\underline{r}_i|^2 \delta_{\alpha\beta} - (\underline{r}_i)_\alpha (\underline{r}_i)_\beta \right\}$

In general there are 3 directions s.t. $\underline{\underline{I}} \cdot \underline{\omega} \parallel \underline{\omega}$ (principle axes of tensor)

If body rotates about a principle axis then $\underline{L} \parallel \underline{\omega}$.

Recap : Rotate in direction s.t. $\underline{L} \parallel \underline{\omega}$: $\underline{L} = I(\underline{n})\,\underline{\omega}$

$$T = \frac{1}{2} I \omega^2, \quad \underline{L} = I \underline{\omega}$$

7.3 Calculating moment of inertia

Consider body occupying volume V with mass density $\rho(\underline{r})$

Total mass $M = \int_V \rho(\underline{r})\, dV$, centre of mass positive vector is $\underline{R} = \frac{1}{M} \int \underline{r}\, \rho(\underline{r})\, dV$

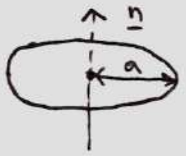
Moment of inertia about axis $\underline{n}$:

$$I = \int_V \rho(\underline{r}) |\underline{r}_\perp|^2 \, dV = \int_V \rho(\underline{r}) |\underline{n} \times \underline{r}|^2 \, dV$$

Can be made specific to surfaces / line integrals.

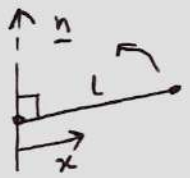
Now calculate I :

1) Uniform thin ring mass M, radius a, about axis through centre perpendicular to plane



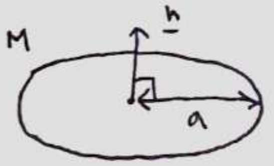
$$\rho = \frac{M}{2\pi a}, \quad I = \int_0^{2\pi} \underbrace{\left(\frac{M}{2\pi a}\right)}_{\rho} \cdot \underbrace{a^2}_{r_\perp^2} \cdot \underbrace{a\, d\theta}_{dl} = \underline{Ma^2}$$

2) Uniform thin rod, mass M, length l, rotation about axis through one end



$$\rho = \frac{M}{l}, \quad I = \int_0^l \left(\frac{M}{l}\right) x^2 \, dx = \underline{\frac{1}{3} M l^2}$$

3) uniform thin disc



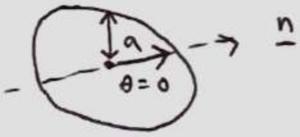
Use area integral $dV \to dA$

$\rho = \frac{M}{\pi a^2}$ mass / unit area

$$I = \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \left(\frac{M}{\pi a^2}\right) r^2 \underbrace{r\,dr\,d\theta}_{dA}$$

$$I = \frac{M}{\pi a^2} \int_0^a r^2 \cdot r\,dr \int_0^{2\pi} d\theta = \frac{M}{\pi a^2} \frac{a^4}{4} 2\pi = \underline{\frac{1}{2} Ma^2}$$

4) Uniform thin disc, mass M, $r = a$, axis of rotation through centre but in plane of disc

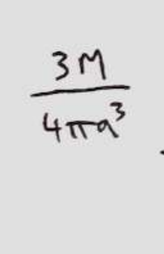


$\perp$ distance to axis for (r, θ) is $r \sin\theta$

$$I = \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \left(\frac{M}{\pi a^2}\right) r^2 \sin^2\theta \; r\,dr\,d\theta$$

$$= \frac{M}{\pi a^2} \int_0^a r^3\,dr \int_{\theta=0}^{2\pi} \sin^2\theta\,d\theta = \frac{M}{\pi a^2} \cdot \frac{a^4 \pi}{4} = \frac{1}{4} Ma^2$$

5) Uniform sphere



Density $\frac{M}{4\pi a^3/3}$

$$I = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{M}{\frac{4\pi a^3}{3}} r^2 \sin^3\theta \; r^2\,dr\,d\theta\,d\phi$$

$$= \frac{3M}{4\pi a^3} \int_0^a r^4\,dr \int_0^{\pi} \sin^3\theta\,d\theta \int_0^{2\pi} d\phi = \frac{3M}{4\pi a^3} \times \frac{a^5}{5} \times \frac{4}{3} \times 2\pi$$

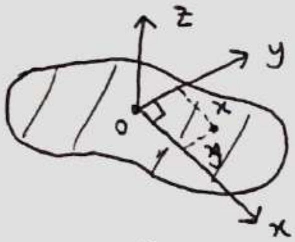
$$= \frac{2Ma^2}{5}$$

General results

Perpendicular Axes Theorem

2D body (lamina) : $I_z = I_x + I_y$ ← I about z-axis, etc ⊥ plane

I_x, I_y are I about x, y axes lying in the plane



Proof Let A be lamina as shown

Then $I_x = \int_A \rho y^2 \, dA, \quad I_y = \int_A \rho x^2 \, dA$

$$I_z = \int_A \rho r^2 \, dA = \int_A \rho (x^2 + y^2) \, dA = I_x + I_y \qquad \square$$

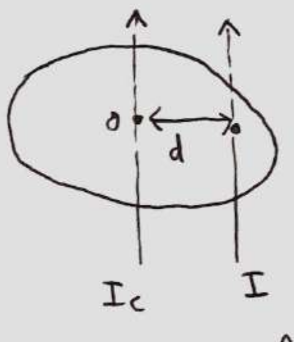
Example of symmetric case

$$I_z = \frac{1}{2} M a^2 = 2 \left(\frac{1}{4} M a^2 \right) = 2 I_x = 2 I_y$$

Parallel Axes Theorem

Rigid body of mass M : I_c about axis through centre of mass

Then I about a parallel axis a distance d from the axis through the centre of mass is $\underline{I = I_c + M d^2}$



Proof Choose Cartesian axes, origin 0 at CoM and z direction ∥ axis of rotation

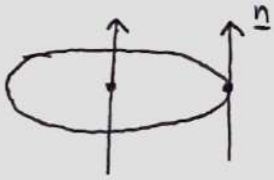
Let 2nd point $(d, 0, 0)$ be point the 2nd axis passes through. Denote volume by V

Then $I_c = \int_V \rho (x^2 + y^2) \, dV$

$$I = \int \rho \left((x-d)^2 + y^2 \right) dV = \int \rho (x^2 + y^2) \, dV - 2 \underbrace{\int \rho x d \, dV}_{= 0 \text{ : origin is CoM}} + d^2 \int \rho \, dV$$

$= I_c + M d^2$ as required.

Example uniform thin disc, mass M, radius a



$\underline{n} \leftarrow$ I about this axis?

$$I = \underbrace{I_c}_{\frac{1}{2}Ma^2} + Ma^2$$

(a^2 is d^2 with d distance between axes)

so $\underline{I = \frac{3}{2}Ma^2}$.

7.3 Motion of rigid body

$$\underline{r}_i = \underline{R} + \underline{s}_i \qquad \dot{\underline{r}}_i = \dot{\underline{R}} + \dot{\underline{s}}_i \qquad \left(\sum_{i=1}^{N} m_i \underline{r}_i = M\underline{R}\right)$$

If body rotates about CoM with $\underline{\omega}$ A.V:

$$\dot{\underline{s}}_i = \underline{\omega} \times \underline{s}_i, \qquad \dot{\underline{r}}_i = \dot{\underline{R}} + \underline{\omega} \times \underline{s}_i$$

Then kinetic energy is $T = \frac{1}{2}M\dot{\underline{R}}^2 + \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\underline{s}}_i^2$

$= \frac{1}{2}M\dot{\underline{R}}^2 + \frac{1}{2}I_c\omega^2$ (I_c is I about axis $\parallel \underline{\omega}$ through CoM)

$T =$ translation KE + rotational KE

Recall from 6.1:

$$\dot{\underline{P}} = \underbrace{\underline{F}}_{\sum \text{ext. force}}, \qquad \dot{\underline{L}} = \underbrace{\underline{G}}_{\sum \text{ext. torque}}$$

determine translational and rotational motion

(Can sometimes use E conservation)

$\underline{L}$ and $\underline{G}$ depend on choice of origin

Take origin to be any point fixed in inertial frame,

or define $\underline{L}$ and $\underline{G}$ wrt centre of mass – eqn still holds

$$\frac{d}{dt}\left[\left(M\underline{R} \times \dot{\underline{R}}\right) + \sum_{i=1}^{N} m_i \underline{s}_i \times \dot{\underline{s}}_i\right] = \underline{G} \quad \text{ext. torque about } \underline{O}$$

$$= \cancelto{0}{M\dot{\underline{R}} \times \dot{\underline{R}}} + M\underline{R} \times \ddot{\underline{R}} + \frac{d}{dt}\left(\sum_{i=1}^{n} m_i \underline{s}_i \times \dot{\underline{s}}_i\right)$$

$$= \underline{R} \times \underline{F}^{ext} + \frac{d}{dt}\left(\sum_{i=1}^{n} m_i \underline{s}_i \times \dot{\underline{s}}_i \right)$$

Hence

$$\underbrace{\frac{d}{dt}\left(\sum_{i=1}^{N} m_i \underline{s}_i \times \dot{\underline{s}}_i \right)}_{\substack{\text{angular momentum} \\ \text{about centre of mass} \\ (\text{not including } \frac{d}{dt})}} = \underline{G} - \underline{R} \times \underline{F}^{ext}$$

$$= \sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i^{ext} - \underline{R} \times \underline{F}^{ext}$$

$$= \sum_{i=1}^{N} (\underline{r}_i - \underline{R}) \times \underline{F}_i^{ext} = \text{ext. torque about CoM}$$

Uniform gravitational field

Total grav. force + torque acting on rigid body are the same as those that would act on particle of mass M at CoM.

$$\underline{F} = \sum_{i=1}^{N} \underline{F}_i^{ext} = \sum_{i=1}^{N} m_i \underline{g} = M\underline{g}$$

$$\underline{G} = \sum_{i=1}^{N} \underline{G}_i^{ext} = \sum_{i=1}^{N} \underline{r}_i \times m_i \underline{g} = \left(\sum_{i=1}^{N} m_i \underline{r}_i \right) \times \underline{g}$$

$$= M\underline{R} \times \underline{g}$$ as expected. Grav. torque about CoM is 0:

$$\sum_{i=1}^{N} \underline{s}_i \times m_i \underline{g} = \left(\sum_{i=1}^{N} m_i \underline{s}_i \right) \times \underline{g} = \underline{0}$$

Grav. potential $V^{ext} = \sum_{i=1}^{N} V_i^{ext} = -\sum_{i=1}^{N} m_i \underline{r}_i \cdot \underline{g} = -M\underline{R} \cdot \underline{g}$

Example stick thrown into air

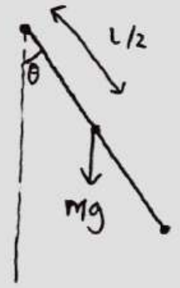
CoM follows parabola

ω of stick about CoM is constant as grav. torque about CoM is 0.

Dynamics and Relativity - Lecture 17

Example Rod of length l, mass M, fixed at pivot point O.

Density of rod is uniform.



Consider angular momentum about O. $\omega = \dot\theta$

$L = I\dot\theta$

$$L = I\dot\theta = \frac{1}{3}Ml^2\dot\theta \quad \text{(see earlier example for } I\text{)}$$

Gravitational torque about O:

$$G = -Mg\cdot\frac{l}{2}\sin\theta \qquad \text{(torque associated with force at pivot is 0)}$$

$$\dot L = G \Rightarrow I\ddot\theta = -Mg\frac{l}{2}\sin\theta \Rightarrow \ddot\theta = -\frac{3g}{2l}\sin\theta$$

equivalent to a simple pendulum of length $\frac{2l}{3}$, $\omega = \sqrt{\frac{3g}{2l}}$, $T \approx 2\pi\sqrt{\frac{2l}{3g}}$

Recap: A Newton's 2nd for CoM + $\dot L$ about CoM

or B $\dot L$ about fixed point

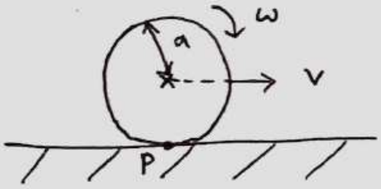
or C conservation of energy

Alternatively using energy: $T + V = \frac{1}{2}I\dot\theta^2 - \frac{Mgl}{2}\cos\theta = E$

E constant: $$\frac{dE}{dt} = I\dot\theta\ddot\theta + \frac{Mgl\dot\theta}{2}\sin\theta = \dot\theta\underbrace{\left(I\ddot\theta + \frac{Mgl}{2}\sin\theta\right)}_{\text{EoM}} = 0$$

7.4 Sliding vs Rolling

Consider cylinder or sphere, radius a, moving along stationary horizontal surface



General motion: combination of rotation of CoM (ω) and translation of CoM (v)

P = point of contact

The horizontal velocity of point of contact on sphere: $v_{slip} = v - a\omega$

Point of contact P slips, may be a kinetic frictional force

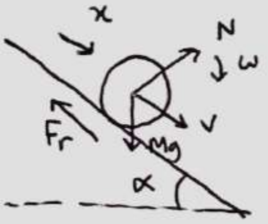
A pure sliding motion is $\omega = 0$, $v_{slip} = v \neq 0$

A pure rolling motion is $v \neq 0$, $\omega \neq 0$, $\underbrace{v_{slip} = v - a\omega = 0}_{\text{no slip condition}}$

The point of contact P is stationary.

A rolling body can be described instantaneously as rotating about the point of contact with angular velocity ω.

<u>Example</u> rolling downhill



Cylinder / sphere of mass M, radius a, rolling down ~~a~~ inclined plane at angle α to the horizontal

x distance down slope travelled by CoM : $v = \dot{x}$

<u>Rolling ~~a~~ or no slip condition: $v - a\omega = 0$</u>

Analyse using energy:

$$T = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2 = \frac{1}{2}\left(M + \frac{I}{a^2}\right) v^2$$

Normal / Frictional forces do no work since they act at point P and $v_{slip} = 0$. Hence energy is conserved, with potential energy

$V = -Mgx \sin\alpha$ so $T + V = \frac{1}{2}\left(M + \frac{I}{a^2}\right) v^2 - Mgx\sin\alpha$

$$\frac{d}{dt}(T+V) = \left(M + \frac{I}{a^2}\right)\dot{x}\ddot{x} - Mg\dot{x}\sin\alpha = 0$$

Hence $\left(M + \frac{I}{a^2}\right)\ddot{x} = Mg\sin\alpha$

Rotational contribution implies acc^n smaller than it would be for a frictionless particle.

e.g. for cylinder $I = \frac{1}{2} M a^2 \Rightarrow \underline{\ddot{x} = \frac{2}{3} g \sin\alpha}$

<u>Alternatively</u>: analyse using forces and torques

$M\dot{v} = Mg\sin\alpha - F_r$

$\dot{L}$ about CoM: $I\dot{\omega} = aF_r$ torque due to F_r

Rolling condition : $\dot{v} = a\dot{\omega} \Rightarrow \frac{I\dot{v}}{a} = aF_r$

Hence $M\dot{v} = Mg\sin\alpha - \frac{I\dot{v}}{a^2} \Rightarrow \left(M + \frac{I}{a^2}\right)\dot{v} = \underline{Mg\sin\alpha}$

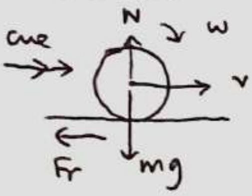
<u>Another alternative</u>: torque about P

$I_P = I + Ma^2$ (by parallel axis theorem)

$I_P\dot{\omega} = Mga\sin\alpha$, $\quad v = a\omega$

so $(Ma^2 + I)\frac{\dot{v}}{a} = Mga\sin\alpha \Rightarrow \left(M + \frac{I}{a^2}\right)\dot{v} = \underline{Mg\sin\alpha}$

<u>Example</u> sliding to rolling transition



Ball hit <u>centrally</u> by cue: no torque initially

$v = v_0, \ \omega = 0$ at $t = 0$

Kinetic frictional force: $F_r = \mu_k N = \mu_k Mg$

Linear: $M\dot{v} = -F_r$ $\qquad$ Angular: $I\dot{\omega} = aF_r$

Recall $I = \frac{2}{3}Ma^2$ for sphere. Hence $v = v_0 - \mu_k g t$

$$\omega = \left(\frac{5}{2a}\right)\mu_k g t$$

combining eqns and solving, using ICs.

Hence $v_{slip} = v - a\omega = v_0 - \frac{7}{2}\mu_k g t$ so there is slipping when $v_{slip} > 0$, for $0 \leq t < \frac{2v_0}{7\mu_k g}$

Rolling begins when $t = \frac{2v_0}{7\mu_k g} = t_{roll}$: in <u>this</u> phase E is conserved, F_r does no work

Note when $t = t_{roll}$ $\quad T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$$= \frac{1}{2}M\left(1 + \frac{2}{5}\right)v_{roll}^2 = \underline{\frac{5}{7}\left(\frac{1}{2}Mv_0^2\right)} \qquad \frac{2}{7} \text{ energy "lost"}$$

Check this:

$$\int_0^{t_{roll}} F_r v_{slip}\, dt = \int_0^{t_{roll}} F\left(v_0 - \frac{7}{2}\mu g t\right) dt$$

$$= \frac{1}{7}Mv_0^2 = \underline{\frac{2}{7}\left(\frac{1}{2}Mv_0^2\right)} \text{ as expected.}$$

Dynamics and Relativity - Lecture 18

2/3/21

Postulates of Special Relativity:

1) The laws of physics are the same in every inertial frame
2) The speed of light in a vacuum is the same in every inertial frame

8.2 The Lorentz Transformation

Consider S, S' inertial, coincidental origin $x = t = 0$, $x' = t' = 0$
$t = t' = 0$

S' moving with speed v in x direction relative to S

Constant velocity paths are straight lines in (x, t) and (x', t') planes so transformation $(x,t) \to (x', t')$ is linear.

O' moves with speed v in S, hence $x' = \gamma(x - vt)$
for $\gamma = \gamma(|v|)$ (no preferred direction)

O moves with speed $-v$ in S' hence $x = \gamma(|v|)(x' + vt')$

Consider light ray passing through origin at $t = t' = 0$

Light ray eqn: $x = ct$ in S, $x' = ct'$ in S'

Hence $x = ct = \gamma(x' + vt') = \gamma(c + v)t'$ (1)

$x' = ct' = \gamma(x - vt) = \gamma(c - v)t$ (2)

For consistency, $\gamma^2 \left(1 - \frac{v}{c}\right)\left(1 + \frac{v}{c}\right) = 1$ (check by (1)x(2))

$$\Rightarrow \gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},$$ the Lorentz factor

So we have (1) $x = \gamma(x' - vt')$ (2) $x' = \gamma(x - vt)$

(1) $\Rightarrow vt' = \frac{x}{\gamma} - x' = \frac{x}{\gamma} - \gamma(x - vt) \Rightarrow t' = \frac{t - vx/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$

so have $t' = \gamma\left(t - \frac{vx}{c^2}\right)$ $\qquad$ $t = \gamma\left(t' + \frac{vx'}{c^2}\right)$

These define the Lorentz transformation.

$\gamma(v) \approx 1$ for low velocities, $\gamma \to \infty$ as $|v| \to c$

Check on speed of light

(A) light ray travelling in x direction

In S: $x = ct$, $y = 0$, $z = 0$

In S': $x' = \gamma(x - vt) = \gamma(c - v)t$

$t' = \gamma\left(t - \frac{vx}{c^2}\right), = \gamma\left(1 - \frac{v}{c}\right)t \qquad y' = 0, z' = 0$

$$\frac{x'}{t'} = \frac{\gamma(c-v)}{\gamma\left(1 - \frac{v}{c}\right)} = c \text{ as required.}$$

(B) light ray travelling in y direction in S

In S: $y = ct$, $x = 0$, $z = 0$

In S': $x' = \gamma(x - vt)$, $t' = \gamma\left(t - \frac{vx}{c^2}\right)$, $y' = y$, $z' = z$

hence $x' = -\gamma vt$, $t' = \gamma t$, $y' = ct$, $z' = 0$

Consider speed of light ray: $(\text{speed})^2 = (x\text{ comp})^2 + (y\text{comp})^2$

$$= v^2 + \frac{c^2}{\gamma^2} = v^2 + c^2\left(1 - \frac{v^2}{c^2}\right) = \underline{c^2}$$

speed still c but direction changes.

General property of Lorentz transformations:

$$c^2 t'^2 - r'^2 = c^2 t'^2 - (x'^2 + y'^2 + z'^2)$$

$$= \cancel{z'} \; c^2\gamma^2\left(t - \frac{vx}{c^2}\right)^2 - (x - vt)^2\gamma^2 - y^2 - z^2$$

$$= c^2\gamma^2\left(t^2 - \frac{2vxt}{c^2} + \frac{v^2x^2}{c^2}\right) - \gamma^2(x^2 - 2vxt + v^2t^2) - y^2 - z^2$$

$$= c^2t^2 - x^2 - y^2 - z^2 = \underline{c^2 t - r^2}$$

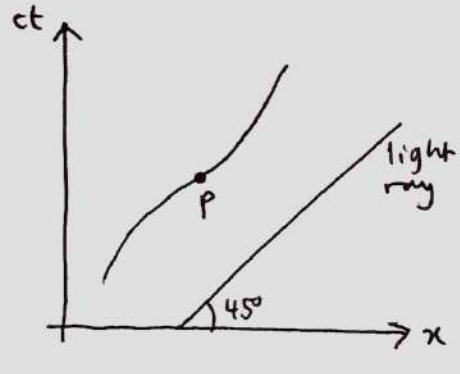
so $r' = ct' \Leftrightarrow r = ct$ invariant quantity

Dynamics and Relativity - Lecture 19

8.3 Spacetime diagrams

1 spatial dimension x, time t in inertial frame S

Plot x on horizontal axis and ct on vertical axis



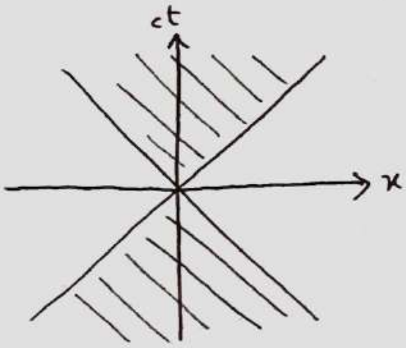
Convention: Minkowski spacetime

Each point P in spacetime represents an event labelled (x, ct)

Moving particle traces out a world line straight line if uniform velocity

Light ray in x-direction is straight line $x = ct$ (45° from axes)

Particles cannot travel at $v > c$ so world lines are restricted to certain regions

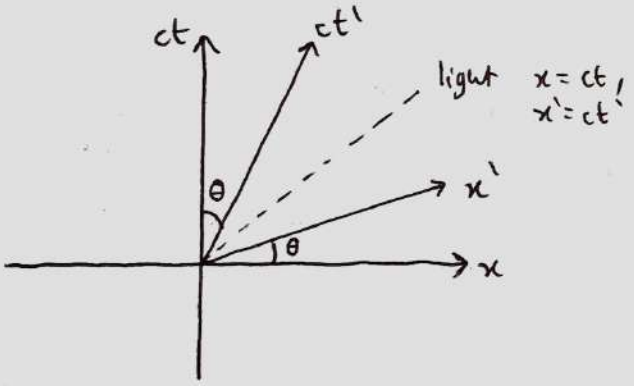


/// region allowed for particle that is at origin at $t=0$, $x=0$

We can also draw axes of frame S', moving at speed v relative to S

t' axis corresponds to $x' = 0$ $(x = vt)$ or $x = \frac{v}{c} ct$

x' axis corresponds to $t' = 0$ hence $ct = \frac{vx}{c}$



x' and ct' axis both symmetric about $x = ct$

so $x = ct \iff x' = ct'$

Comparison of velocities

Consider particle moving with constant velocity u' in S', S' travels at velocity v wrt S. What is velocity of "u" measured in S?

Consider world line of particle in S': $x' = u't'$ (constant vel. in 1 frame corresponds to constant vel. in other - inertial)

in S: $x = ut$

Lorentz transformation: $x = \gamma(x' + vt') = \gamma(u' + v)t'$

$t = \gamma(t' + vx'/c^2) = \gamma(1 + vu'/c^2)t'$. Eliminate t':

Hence $u = \dfrac{u' + v}{1 + \frac{u'v}{c^2}}$. Invert by swapping sign of v: $u' = \dfrac{u - v}{1 - \frac{uv}{c^2}}$

$u', v \ll c$: recover Galilean result

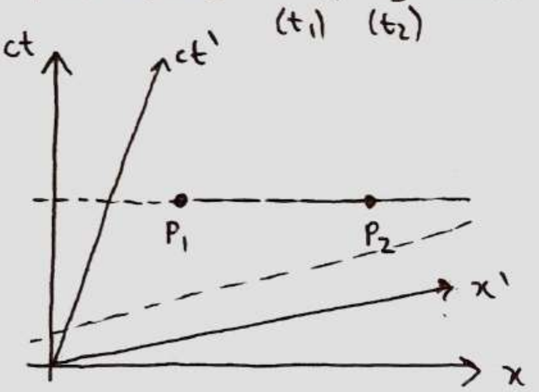
Note: $\cancel{c - u = c - \frac{u' + v}{}}$ $c - u = c - \dfrac{u' + v}{1 - \frac{u'v}{c^2}}$

$= \dfrac{(c - u')(c - v)}{1 + \frac{u'v}{c^2}} > 0$ if $u' < c$, $v < c$ so $\underline{c > u}$

i.e. cannot exceed speed of light through successive boosts.

8.3 Simultaneity and causality

Two events P_1, P_2 are simultaneous in S if $t_1 = t_2$
(t_1) (t_2)



surface of simultaneity in a frame is parallel to the x axis or the x' axis

P_1, P_2 simultaneous in S but not S'

P_2 occurs before P_1 in S':

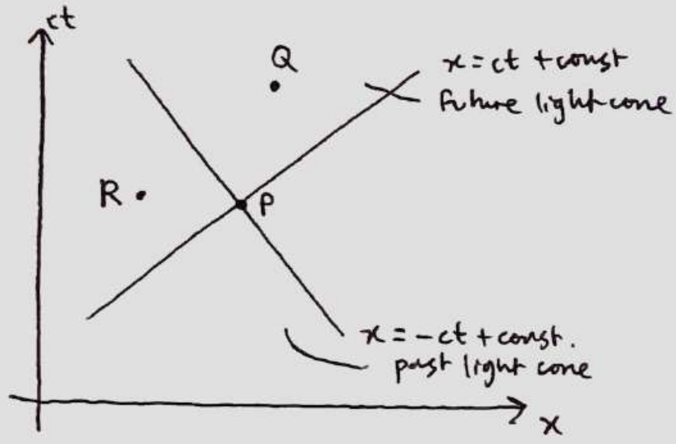
P_1 (x_1, ct), P_2 (x_2, ct) $\Rightarrow$ $t_1' = \gamma\left(t - \frac{vx_1}{c^2}\right)$, $t_2' = \gamma\left(t - \frac{vx_2}{c^2}\right)$

$x_2 > x_1 \Rightarrow t_2' < t_1'$ - simultaneity is frame dependent

Causality Different observers may disagree on time ordering of events but we can construct a viewpoint where cause and effect are consistent.

Lines of simultaneity cannot be ~~inclinde~~ inclined at more than 45° to x.

Lines / surfaces emerging from an event P at angle 45° to x axis form a light cone (past and future)



All observers agree that Q occurs after P, but ordering of P and R in time depends on frame.

(since line QP has slope $> 45°$ but all "x" axes must have slope $< 45°$)

If no information travels faster than c, then P and R <u>cannot</u> influence each other.

P can <u>only</u> influence events inside its future light cone

P can <u>only</u> be influenced by events inside its past light cone

<u>Time dilation</u>

Consider a clock stationary in S' ticking at constant intervals $\Delta t'$.
What is time interval between ticks viewed in S?

Lorentz transf: $\quad t = \gamma \left(t' + \frac{vx'}{c^2} \right) \quad x'$ fixed - stationary in S'

hence $\Delta t = \gamma \Delta t'$

"moving clocks run slower"

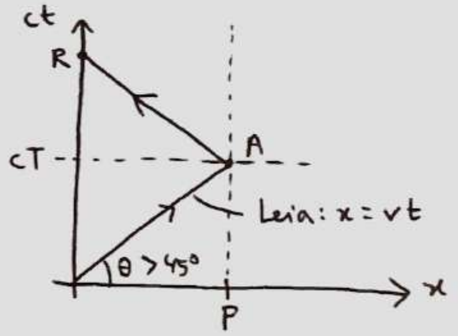
Proper time is time measured in the rest frame of a particular moving object.

Dynamics and Relativity - Lecture 20

Time Dilation - Twin Paradox

2 twins Luke and Leia: Luke stays at home, Leia travels at constant speed to distant planet P then returns home.

In Luke's frame



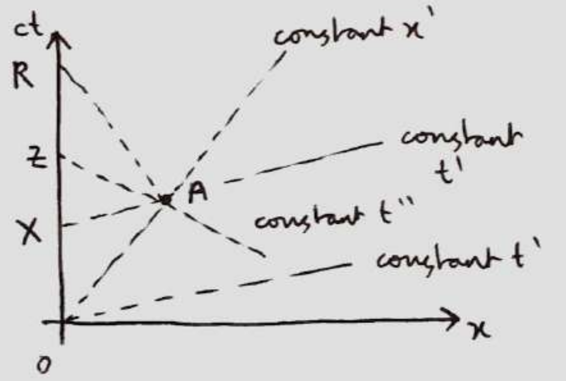
Leia's arrival at P - event A has $(x, ct) = (vT, cT)$

Time experienced by Leia on outward journey:

$$T' = \gamma\left(T - \frac{v}{c}\cdot vT\right) = \frac{T}{\gamma}$$

On Leia's return (R) Luke has aged by 2T but Leia has aged by $\frac{2T}{\gamma} < 2T$, so Leia is younger than Luke when she returns by time dilation.

From Leia's perspective, Luke has travelled away at speed v and returned at speed v, so he should be younger. Why is it not symmetric?



In frame of Leia's outward journey:

$$A: \quad x' = 0, \quad t' = \frac{T}{\gamma}$$

$$X: \quad x = 0, \quad t' = \frac{T}{\gamma}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \Rightarrow t' = t\gamma \quad (x = 0)$$

$$\Rightarrow t = \frac{t'}{\gamma} = \frac{T}{\gamma^2}$$

Each thinks the other has aged <u>less</u> by factor $\frac{1}{\gamma}$.

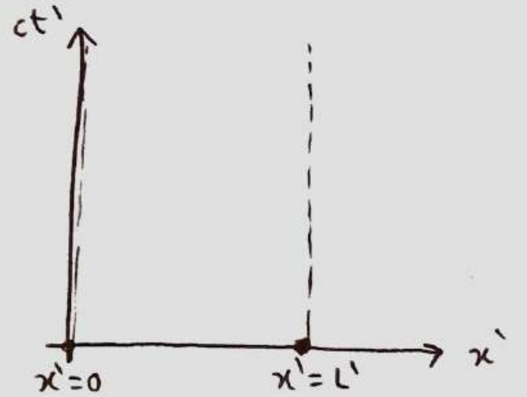
Return journey:

Luke sees Leia aging from $A \to R$, Leia sees Luke aging from $Z \to R$. (Z is simultaneous with A in frame moving with Leia on her return journey, i.e. moving speed $-v$ relative to Luke's frame)

Leia's change of speed makes her see Luke age instantaneously from X to Z - the frame is <u>not inertial</u>.

Length Contraction

A rod of length L' is stationary in S'. What is its length in S?



Length is distance between the two ends at the same instant in time.

$$x' = 0 \Rightarrow \gamma(x - vt) = 0$$

$$x' = L' \Rightarrow \gamma(x - vt) = L'$$

Hence L = length in S $= \underline{\dfrac{L'}{\gamma} < L'}$

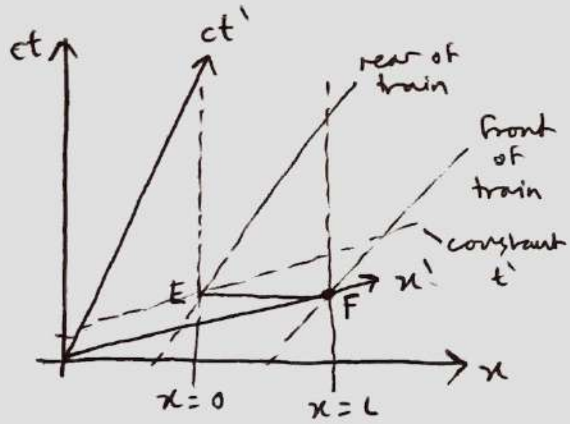
Moving objects appear contracted in length along direction of travel.

<u>Proper length</u> is length measured in rest frame of object.

<u>Example</u>: Does a train of length $2L$ fit alongside a platform of length L if it is travelling at a speed such that $\gamma = 2$?

For observers on platform, train contracts to length L - it fits

For observers on train, the platform contracts from L to $L/2$ - doesn't fit



S: platform $\quad S'$: train

Platform : $x = 0$ (rear), $x = L$ (front)

Train : $x' = 0$ (rear), $x' = 2L$ (front)

E is event when rear of train is coincident with rear of platform : assume this occurs at $t = t' = 0$

Front of train is $x' = 2L$, front of platform is $x = L$

E and F are simultaneous in S : $x' = \gamma(x - vt)$, $2L = \gamma(x - vt)$,

$$x = L \text{ so } 2L = \gamma(L - vt)$$

$$\Rightarrow t = 0 \quad (\gamma = 2)$$

and $x = \gamma(x' + vt')$, $L = \gamma(2L + vt')$, $\Rightarrow t' = \dfrac{L - 2\gamma L}{v} = \dfrac{-3L}{2v} < 0$

F occurs before E in S'

i.e. rear of train coincident with rear of platform <u>after</u> front of train is coincident with front of platform, i.e. <u>doesn't fit</u>.

Problem: simultaneity is frame dependent.

9/13/21

Dynamics and Relativity - Lecture 21

Geometry of spacetime

Invariant interval - 2 events P, Q $(ct_1, x_1), (ct_2, x_2)$ respectively

Time separation $\Delta t = t_1 - t_2$, space separation $\Delta x = x_1 - x_2$

Invariant interval between P, Q defined by $\underline{(\Delta s)^2 = c^2 \Delta t^2 - \Delta x^2}$

All inertial observers measure the same Δs

In 3D space: $\underline{\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2}$

If separation is very small - have expression for infinitesimal:

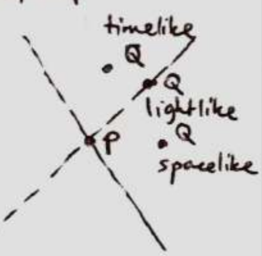
$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$

Spacetime topologically equivalent to $\mathbb{R}^4$ with distance measure ds^2

(Minkowski spacetime - $1+3$)

Invariant interval $\Delta s^2 = c^2 \Delta t^2 - \Delta \underline{x}^2$

If $\underline{\Delta s^2 > 0}$, then events are <u>time-like</u> separated - there exists a frame where they occur at the same <u>space</u> position (inside light cone of P - time ordering unambiguous)



If $\underline{\Delta s^2 < 0}$, then events are <u>space-like</u> separated - there exists a frame where they occur at the same time (outside light cone of P - time ordering observer dependent)

If $\underline{\Delta s^2 = 0}$, then events are <u>light-like</u> separated - each lies on the light cone of the other

The Lorentz group

The coordinates of an event P in frame S can be written as a 4-vector

$$X^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \begin{array}{l} -\mu=0 \\ -\mu=1 \\ -\mu=2 \\ -\mu=3 \end{array}$$

Invariant interval between P and O can be written as $X \cdot X = X^T \underset{\substack{\uparrow \\ \text{matrix}}}{\eta} X = X^\mu \eta_{\mu\nu} X^\nu$

where $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ - Minkowski metric

so $X \cdot X = c^2 t^2 - x^2 - y^2 - z^2$

$X \cdot X > 0$ timelike, $X \cdot X = 0$ lightlike, $X \cdot X < 0$ spacelike

Lorentz transformation is a linear transformation taking comp. of X in S to comp. X' in S'.

Represented by 4×4 matrix Λ : $X' = \Lambda X$

Lorentz transformations can be defined as the set of Λ that leaves $X\cdot X = X'\cdot X'$ $\forall\, X$ $\Rightarrow$ $\Lambda^T \eta \Lambda = \eta$.

Two classes of possible Λ are

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}, \quad R^T R = I \ \text{(rotation/reflection)}, \qquad \text{or} \qquad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \beta = \frac{v}{c}, \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

(Lorentz boost)

The set of all possible Λ satisfying the condition is the Lorentz group $O(1,3)$.

Subgroup with det 1 is proper Lorentz group $SO(1,3)$

Subgroup preserving time direction and spatial orientation is restricted Lorentz group $SO^+(1,3)$ - generated by rotations, boosts

<u>Rapidity</u> - way of representing Lorentz transformations

Focus on 2×2 submatrix operating on (ct, x)

$$\Lambda(\beta) = \begin{pmatrix} \gamma & -\gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$$ Consider combining 2 boosts:

$$\Lambda(\beta_1)\Lambda(\beta_2) = \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & -\gamma_1\gamma_2(\beta_1+\beta_2) \\ -\gamma_1\gamma_2(\beta_1+\beta_2) & \gamma_1\gamma_2(1+\beta_1\beta_2) \end{pmatrix} = \Lambda\left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)$$

Recall for spatial rotations $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\quad R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2)$

For Lorentz boosts define ~~ϕ~~ ϕ s.t. $\underline{\beta = \tanh\phi}$

$\Rightarrow \gamma = \cosh\phi, \quad \gamma\beta = \sinh\phi$

Then composition law is $\Lambda(\phi_1)\Lambda(\phi_2) = \Lambda(\phi_1+\phi_2)$

Suggests Lorentz boosts can be regarded as "hyperbolic rotations" in spacetime.

Dynamics and Relativity – Lecture 22 11/3/21

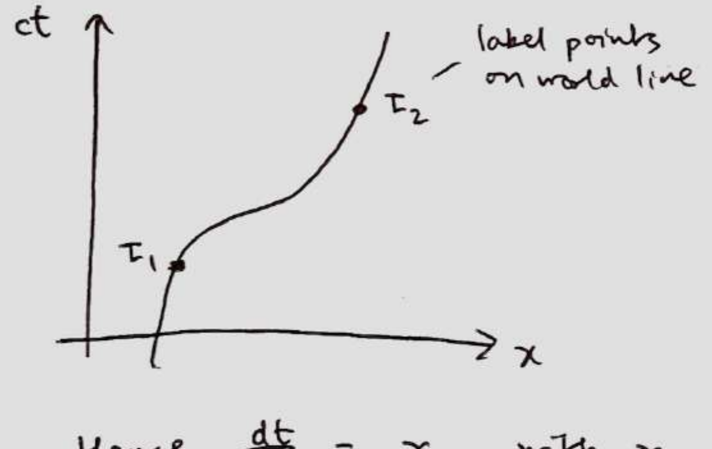
8.6 Relativistic Kinematics

A particle moves along a trajectory $\underline{x}(t)$. Its velocity is $\dot{\underline{x}}(t) = \underline{u}(t)$
Path in spacetime parametrised by t. But both $\underline{x}$ and t vary under Lorentz transformation.

Proper time First consider a particle at rest in S' with $\underline{x}' = \underline{0}$.
The invariant interval between points on its world line is $\Delta s^2 = c^2 \Delta t'^2$ (no $\Delta x'^2$)

Define proper time by $\Delta \tau = \frac{1}{c} \Delta s$. This is the change in time experienced in the rest frame of the particle. But $\Delta \tau = \frac{1}{c} \Delta s$ holds in all frames (Δs invariant).
Note Δs is real in all frames – the interval is timelike.

The world line of the particle can now be parametrised in terms of τ.



Infinitesimal changes:

$$d\tau = \frac{ds}{c} = \frac{1}{c}\sqrt{c^2 dt^2 - |d\underline{x}|^2}$$
$$= \frac{1}{c}\sqrt{c^2 dt^2 - |\underline{u}|^2 dt^2}$$
$$= \left(1 - \frac{u^2}{c^2}\right)^{1/2} dt$$

Hence $\frac{dt}{d\tau} = \gamma_{\underline{u}}$ with $\gamma_{\underline{u}} = \frac{1}{\sqrt{1 - u^2/c^2}}$

Total time observed by particle moving along world line is

$$T = \int d\tau = \int \frac{dt}{\gamma_u}$$

4-velocity

Position 4-vector is $X(\tau) = \begin{pmatrix} ct(\tau) \\ \underline{x}(\tau) \end{pmatrix}$

4-velocity is $\frac{d}{d\tau} X(\tau) = U = \begin{pmatrix} c\, dt/d\tau \\ d\underline{x}/d\tau \end{pmatrix} = \frac{dt}{d\tau}\begin{pmatrix} c \\ \underline{u} \end{pmatrix} = \gamma_{\underline{u}} \begin{pmatrix} c \\ \underline{u} \end{pmatrix}$

Another notation: $(ct, \underline{x})$, $(\gamma_{\underline{u}} c, \gamma_{\underline{u}} \underline{u})$

$\uparrow$ position 4-vector $\qquad \uparrow$ 4-velocity

If frames S and S' are such that X and X' are components of position vector, then $\underline{X' = \Lambda X}$, $\underline{U' = \Lambda U}$ relates components.

(Any quantity that transforms this way is a 4-vector).

U is a 4-vector since X is a 4-vector and τ is invariant,

i.e. $\dfrac{dX}{d\tau}$ is a 4-vector, but not $\dfrac{dX}{dt}$ – t not invariant

The scalar product $U \cdot U$ is invariant under Lorentz transformations

$U \cdot U = U' \cdot U'$

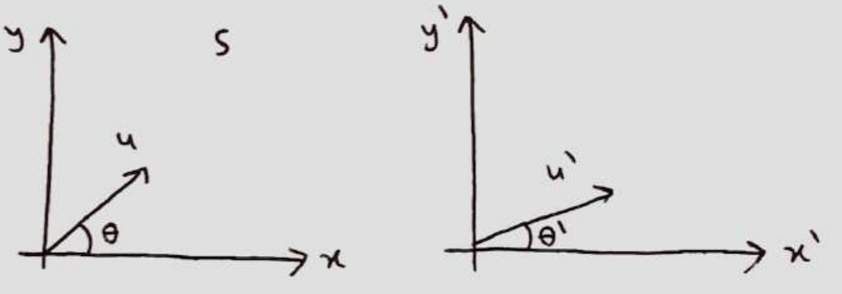
In the rest frame of a particle moving with 4-velocity U, $U = \begin{pmatrix} c \\ \underline{0} \end{pmatrix}$,

hence $U \cdot U = c^2 \Rightarrow U \cdot U = c^2$ in all frames.

$\left(c^2 = \gamma^2 (c^2 - \underline{u}^2) = c^2 \right)$

<u>Transformations of velocities as 4-vectors</u>

Consider example: transformation $S \to S'$, S' moving at speed v in x-direction relative to S.



$$U' = \begin{pmatrix} \gamma_{\underline{u}'} c \\ \gamma_{\underline{u}'} u' \cos\theta' \\ \gamma_{\underline{u}'} u' \sin\theta' \\ 0 \end{pmatrix}$$

with $\underline{U' = \Lambda U}$

where
$$\Lambda = \begin{pmatrix} \gamma_v & -\gamma_v v/c & 0 & 0 \\ -\gamma_v \frac{v}{c} & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In frame S, 4-velocity U is $\begin{pmatrix} \gamma_{\underline{u}} c \\ \gamma_{\underline{u}} u \cos\theta \\ \gamma_{\underline{u}} u \sin\theta \\ 0 \end{pmatrix}$

From $\underline{U' = \Lambda U}$ we get

$$u' \cos\theta' = \frac{u\cos\theta - v}{1 - \frac{uv\cos\theta}{c^2}}, \qquad \tan\theta' = \frac{u\sin\theta}{\gamma_v (u\cos\theta - v)}$$

(check - compare 2nd and 3rd components)

This corresponds to rules for velocity addition - see earlier

There is change in perceived angle of velocity - <u>aberration</u>

Applies when $u = c$, i.e. aberration applies to light rays -
apparent direction of stars changes as result of Earth's orbital motion.

Dynamics and Relativity - Lecture 23 13/3/21

8.7 Relativistic physics

4-momentum of particle of mass m, 4-velocity U is $P = mU = m\gamma_{\underline{u}}\begin{pmatrix} c \\ \underline{u} \end{pmatrix}$

4 components P^0, P^1, P^2, P^3

For P to be a 4-vector, m must be an invariant - m is rest mass (mass measured in rest frame)

4-momentum of system $= \sum$ 4-momentum of particles

Spatial components of P ($\mu = 1, 2, 3$) are relativistic 3-momentum $\underline{p} = \gamma_{\underline{u}} m\underline{u}$

Apparent / relativistic mass $\gamma_{\underline{u}} m$ ($\to \infty$ as $|\underline{u}| \to c$)

$$P^0 = \gamma_{\underline{u}} mc = \frac{mc}{\sqrt{1 - \frac{\underline{u}^2}{c^2}}} = \frac{1}{c}\Big(mc^2 + \underbrace{\tfrac{1}{2}m\underline{u}^2}_{KE} + \dots\Big)$$

Natural interpretation of P^0 is as an energy : $P = \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix}$

with $E = \gamma_{\underline{u}} mc^2 = mc^2 + \frac{1}{2}m\underline{u}^2 + \dots$ ($E \to \infty$ as $|\underline{u}| \to c$ for $m \neq 0$)

P sometimes called energy-momentum 4-vector

For a stationary particle with rest mass m, have $E = mc^2$

Implication: mass is a form of energy

$$E = mc^2 + \underbrace{(\gamma_{\underline{u}} - 1)mc^2}_{\text{Relativistic KE}}$$

recall: 4-vectors are invariant under LT (defined)

Since $P \cdot P = \frac{E^2}{c^2} - |\underline{p}|^2$ is Lorentz invariant $\Rightarrow P \cdot P = m^2c^2$

$$\Rightarrow \frac{E^2}{c^2} = p^2 + m^2c^2$$

- useful relation between 3-momentum and energy.

Summary: 4-momentum $P = \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix}$

E relativistic energy, $P \cdot P$ invariant $\Rightarrow P \cdot P = \frac{E^2}{c^2} - |\underline{p}|^2 = m^2c^2$

m = rest mass, mass is a form of energy

Massless particles

Zero rest mass particles, e.g. photons

These can have nonzero momentum / nonzero energy, as they travel at the speed of light - " $\gamma_{\underline{u}}$ infinite "

In this case $\underline{P \cdot P = 0}$ $(m = 0)$

These particles travel along light-like trajectory and hence they have no proper time (can't transform to rest frame)

$$\frac{E^2}{c^2} = |p|^2 \Rightarrow \underline{E = |\underline{p}|c}$$

Then $P = \frac{E}{c}\begin{pmatrix} 1 \\ \underline{n} \end{pmatrix}$ with $\underline{n}$ in direction of travel.

Newton's 2nd in SR

$$\frac{dP}{d\tau} = F \Rightarrow F \text{ is also a 4-vector}$$

(check)

Relation between 4-force F and 3-vector force $\underline{F}$ is $F = \gamma_{\underline{u}}\begin{pmatrix} \underline{F} \cdot \underline{u} / c \\ \underline{F} \end{pmatrix}$

Then $\frac{dE}{d\tau} = \gamma_{\underline{u}} \frac{\underline{F} \cdot \underline{u}}{}$, $\frac{d\underline{p}}{d\tau} = \gamma_{\underline{u}} \underline{F}$ ← (gamma factor added to classical eqns)

so $\frac{dE}{dt} = \underline{F} \cdot \underline{u}$, $\frac{d\underline{p}}{dt} = \underline{F}$

Equivalently for rest mass m, $F = mA$ with $A = \frac{dU}{d\tau}$ 4-acceleration

8.8 Applications to particle physics

Consider conservation of $P = \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix}$ for a system of particles

Often convenient to write down conservation of 4-momentum in the centre of momentum frame (CM) i.e. the frame in which $\underline{P^1 = P^2 = P^3 = 0}$. (May be problematic for massless particles)

8.8 Application to particle physics

- Consider conservation of 4-momentum – $P = \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix}$ for system
- Convenient to write down in centre of momentum frame $P^1 = P^2 = P^3 = 0$

Example 1 Particle Decay

Particle of mass m_1 decays into 2 particles of mass m_2 and m_3

$P_1 = P_2 + P_3$ conservation (P_i = 4-momentum of i^{th} particle subscript)

0 component: $E_1 = E_2 + E_3$

1, 2, 3 – components: $\underline{p}_1 = \underline{p}_2 + \underline{p}_3$

Centre of momentum frame: $\underline{p}_1 = \underline{0}$ (frame in which m_1 is initially at rest)

Hence $\underline{p}_2 = -\underline{p}_3$.

Then in this frame $\frac{E_1}{c} \overset{E=mc^2,\ \text{rest}}{=} m_1 c = \frac{E_2}{c} + \frac{E_3}{c}$ $(*)$

We have $\frac{E_2}{c} = \sqrt{\underline{p}_2^2 + m_2^2 c^2}$, $\frac{E_3}{c} = \sqrt{\underline{p}_3^2 + m_3^2 c^2}$

then $(*) = \sqrt{p_2^2 + m_2^2 c^2} + \sqrt{p_3^2 + m_3^2 c^2} \geq m_2 c + m_3 c$ (as $p_2^2, p_3^2 \geq 0$)

i.e. $\underline{m_1 \geq m_2 + m_3}$ (recall mass may not be conserved)

Example 2 Higgs to photon decay

$h \to \gamma\gamma$ (2 photons)

Conservation of 4-momentum: $P_h = P_{\gamma_1} + P_{\gamma_2}$

In the rest frame of h: $P_h = \begin{pmatrix} m_h c \\ \underline{0} \end{pmatrix} = \begin{pmatrix} E_{\gamma_1}/c \\ \underline{p}_{\gamma_1} \end{pmatrix} + \begin{pmatrix} E_{\gamma_2}/c \\ \underline{p}_{\gamma_2} \end{pmatrix}$

(1,2,3 comp): $\underline{0} = \underline{p}_{\gamma_1} + \underline{p}_{\gamma_2}$ Recall $\frac{E^2}{c^2} = \underline{p}^2 + m^2c^2$

(0 comp) $\frac{E_{\gamma_1}}{c} = |\underline{p}_{\gamma_1}|$, $\frac{E_{\gamma_2}}{c} = |\underline{p}_{\gamma_2}|$ by $m = 0$ for γ
(rest)

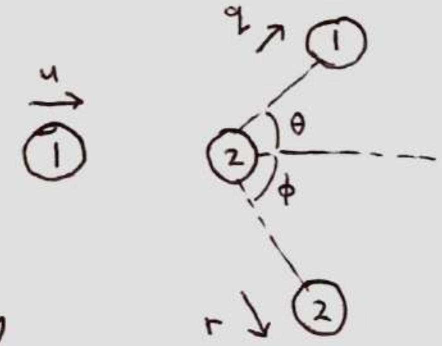
$\Rightarrow E_{\gamma_1} = E_{\gamma_2} \Rightarrow E_{\gamma_1} = E_{\gamma_2} = \frac{1}{2} m_h c^2$ (0 comp).

Note: mass not conserved

<u>Example 3</u> Particle scattering

2 identical particles collide (no decay)

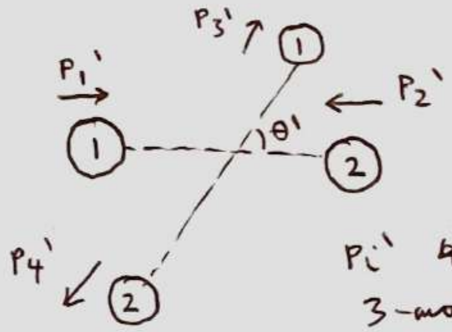
S laboratory frame



u, q, r
3-velocity

2 initially at rest

S` centre of momentum frame



P_i\` ~~ea.~~ are 3-momenta

Conservation of 4-momentum: $P_1 + P_2 = P_3 + P_4$

1 after collision   2 after collision

What is relation between θ and ϕ?

Because 1 and 2 have the same mass, their speeds are equal both before and after the collision.
CoM frame

Let speeds before and after be v and w respectively.

Using S\` frame: $P_1\text{`} + P_2\text{`} = P_3\text{`} + P_4\text{`}$

$$P_1\text{`} = \begin{pmatrix} m\gamma_v c \\ m\gamma_v v \\ 0 \\ 0 \end{pmatrix} \quad P_2\text{`} = \begin{pmatrix} m\gamma_v c \\ -m\gamma_v v \\ 0 \\ 0 \end{pmatrix} \quad \text{before}$$

$$P_3\text{`} = \begin{pmatrix} m\gamma_w c \\ m\gamma_w w\cos\theta\text{'} \\ m\gamma_w w\sin\theta\text{'} \\ 0 \end{pmatrix} \quad P_4\text{`} = \begin{pmatrix} m\gamma_w c \\ -m\gamma_w w\cos\theta\text{'} \\ -m\gamma_w w\sin\theta\text{'} \\ 0 \end{pmatrix}$$

Consider 0 component: $2m\gamma_v c = 2m\gamma_w c \Rightarrow \underline{v = w}$.

Apply Lorentz transformation to go $S' \to S$:

$$\Lambda = \begin{pmatrix} \gamma_v & \gamma_v v/c & 0 & 0 \\ \gamma_v v/c & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

determined by requirement that particle 2 before collision in lab frame, so must go to a frame moving at v relative to S'

$$P_1 = \Lambda P_1' = \begin{pmatrix} m\gamma_v^2 (c + v^2/c) \\ m\gamma_v^2 (v+v) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m\gamma_u c \\ m\gamma_u u \\ 0 \\ 0 \end{pmatrix}$$

Particle 1 after collision as seen in S:

$$P_3 = \begin{pmatrix} m\gamma_q c \\ m\gamma_q q\cos\theta \\ m\gamma_q q \sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} m\gamma_v^2 \left(c + \frac{v^2}{c}\cos\theta'\right) \\ m\gamma_v^2 (v + v\cos\theta') \\ m\gamma_v v \sin\theta' \\ 0 \end{pmatrix}$$

from specification of problem in S Lorentz on P_3'

Hence from 1 and 2 components:

$$\tan\theta = \frac{m\gamma_v v \sin\theta'}{m\gamma_v^2 v(1+\cos\theta')} = \underline{\frac{1}{\gamma_v}\tan\left(\frac{\theta'}{2}\right)}$$

(note $\frac{1}{2}$ angle identity)

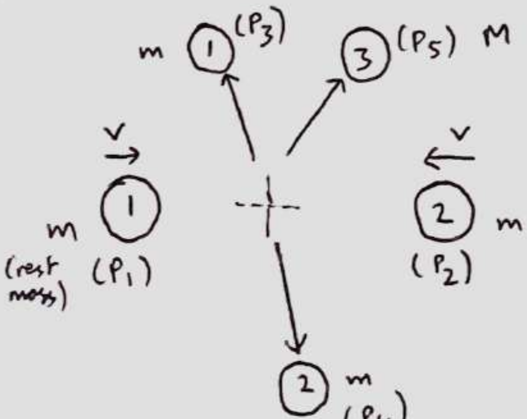
Similarly $\tan\phi = \dfrac{m\gamma_v v\sin\theta'}{m\gamma_v^2 v(1-\cos\theta')} = \underline{\dfrac{1}{\gamma_v}\cot\left(\dfrac{\theta'}{2}\right)}$

by going through same process with P_2 and P_4 rather than P_1 and P_3.

Hence $\tan\theta\tan\phi = \underline{\dfrac{1}{\gamma_v^2} = \dfrac{2}{1+\gamma_u} \le 1}$ see (†) in Ex 4

(Newtonian limit: $\gamma_v = 1 \Rightarrow \tan\theta\tan\phi = 1 \Rightarrow$ angle between trajectories is $\frac{\pi}{2}$) - relativistic: <u>less</u> than $\frac{\pi}{2}$

<u>Example 4</u> Particle creation



Use centre of momentum frame

Conservation of 4-momentum:

$$P_1 + P_2 = \underset{(1)}{P_3} + \underset{(2)}{P_4} + \underset{(3)}{P_5}$$

Let $v, -v$ be velocities of 1, 2 before collision

Hence $P_1 + P_2 = \begin{pmatrix} 2m\gamma_v c \\ \underline{0} \end{pmatrix} = \begin{pmatrix} E_3/c + E_4/c + E_5/c \\ \underline{0} \end{pmatrix}$

$$\underbrace{E_3 + E_4 + E_5}_{2m\gamma_v c} \geqslant (m + m + M)c^2 = (2m + M)c^2$$

$\Rightarrow \gamma_v \geqslant \left(1 + \frac{M}{2m}\right)$ for particle creation to be possible

i.e. initial KE in CM frame must satisfy $2m(\gamma_v - 1)c^2 \geqslant Mc^2$

Now transform to a frame where one particle moves with velocity u and the other is at rest.

Then $u = \dfrac{2v}{1 + \frac{v^2}{c^2}}$ (velocity composition rule) $(\dagger)$

$\Rightarrow \gamma_u = 2\gamma_v{}^2 - 1$ (check algebraically)

so $\gamma_u = 2\gamma_v{}^2 - 1 \geqslant 2\left(1 + \frac{M}{2m}\right)^2 - 1 = 1 + \frac{2M}{m} + \frac{M^2}{2m^2}$

Hence in this frame, KE $mc^2(\gamma_u - 1) \geqslant mc^2\left(\frac{2M}{m} + \frac{M^2}{2m^2}\right) \geqslant 2Mc^2 + \frac{M^2c^2}{2m} \geqslant Mc^2$ may be significantly greater than mc^2

Hence it is much easier to create new particles by colliding particles in beams than colliding one beam with a fixed target.