Analysis and Topology - Lecture 1 13/7/21

1 Uniform convergence and uniform continuity

Recall from IA: $x_n \to x$ as $n \to \infty$ if

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geqslant N$, $|x_n - x| < \varepsilon$

We want to define $\underline{f_n \to f}$ for $\underline{\text{functions}}$.

Definition Given a set S and functions $f_n : S \to \mathbb{R}$, $n \in \mathbb{N}$, $f : S \to \mathbb{R}$:

We say that $f_n \to f$ pointwise on S as $n \to \infty$ if

$\forall x \in S$, $f_n(x) \to f(x)$.

Remarks

1. f_n may converge at different speeds for different x
2. Can replace $\mathbb{R}$ with $\mathbb{C}$

Examples

1. $f_n(x) = x^n$ for $x \in [0,1]$, $n \in \mathbb{N}$

   For $0 \leqslant x < 1$, $f_n(x) = x^n \to 0$, $f_n(1) = 1$ $\forall n$

   So with $f(x) = \begin{cases} 0 & \text{if } 0 \leqslant x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

   have $f_n \to f$ (note f_n cts, f not cts).

2. $f_n(x) = x^2 e^{-nx}$ $x \in [0, \infty)$ $n \in \mathbb{N}$

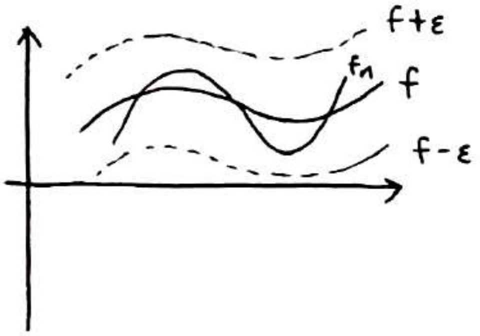
   For $x > 0$, $0 \leqslant f_n(x) = \dfrac{x^2}{e^{nx}} = \dfrac{x^2}{1 + nx + \frac{n^2x^2}{2} + \dots} \leqslant \dfrac{x}{n}$

   $\dfrac{x}{n} \to 0$ as $n \to \infty$

   so by squeeze theorem $f_n \to 0$ pointwise on $[0, \infty)$.

<u>Definition</u> (Uniform convergence). Given a set S and functions $f_n : S \to \mathbb{R}$, $n \in \mathbb{N}$, $f : S \to \mathbb{R}$, we say that $f_n \to f$ <u>uniformly</u> as $n \to \infty$ if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N,\ \forall x \in S :$$
$$|f_n(x) - f(x)| < \varepsilon .$$



Note here f_n lies between $f+\varepsilon$ and $f-\varepsilon$ for all $n \geq N$ corresponding to ε.

<u>Remarks</u>

1. N depends <u>only</u> on ε, not on x.
2. Uniform convergence $\Rightarrow$ pointwise convergence
3. Can replace $\mathbb{R}$ with $\mathbb{C}$ or replace with a subset of the domain
4. An equivalent definition of $f_n \to f$ uniformly on S:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N,\ \sup_{x \in S} |f_n(x) - f(x)| < \varepsilon$$

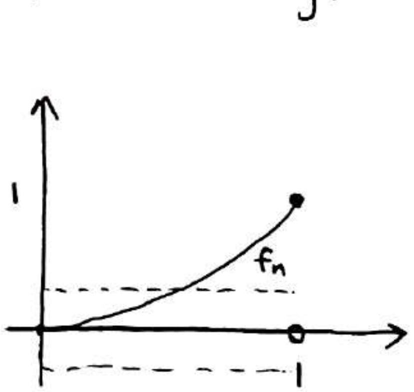
or shorter: $$\sup_{x \in S} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty .$$

<u>Examples</u>

1. $f_n(x) = x^n$ for $x \in [0,1]$, $n \in \mathbb{N}$

We know $f_n \to f$ pointwise with $$f(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x = 1 \end{cases}$$

Limits are unique, so <u>if</u> f_n converges uniformly, then $f_n \to f$ uniformly.



Let $\varepsilon = \frac{1}{2}$. For any $n \in \mathbb{N}$, setting $x = \varepsilon^{1/n}$ we have $f_n(x) = \varepsilon$, and so

$|f_n(x) - f(x)| \geqslant \varepsilon$. ($f(x) = 0$ for $x \neq 1$)

So $f_n \not\to f$ uniformly on $[0,1]$.

Alternatively:

Since $f_n(1) = 1$ and f_n is continuous, there exists $\delta > 0$ s.t. $|f_n(x) - 1| < \frac{1}{2}$ for $x \in (1-\delta, 1+\delta)$.

So for any $x \in [0,1]$ with $1-\delta < x < 1$, we have $|f_n(x) - f(x)| \geqslant \varepsilon$.

2. $f_n(x) = x^2 e^{-nx}$ for $x \in [0, \infty)$ $n \in \mathbb{N}$

We saw that $f_n \to f = 0$ pointwise on $[0, \infty)$.

$$0 \leqslant f_n(x) = \frac{x^2}{e^{nx}} = \frac{x^2}{1 + nx + \frac{n^2x^2}{2} + \dots} \leqslant \frac{x^2}{nx} = \frac{x}{n}$$

But we still have an x, so take the $\frac{n^2x^2}{2}$ instead:

$$0 \leqslant f_n(x) = \frac{x^2}{e^{nx}} = \frac{x^2}{1 + nx + \frac{n^2x^2}{2} + \dots} \leqslant \frac{x^2}{\frac{n^2x^2}{2}} = \underbrace{\frac{2}{n^2}}_{\text{indep. of } x}$$

Thus $\sup_{x \in [0,\infty)} |f_n(x) - f(x)| = \sup_{x \in [0,\infty)} f_n(x) \leqslant \frac{2}{n^2} \to 0$ as $n \to \infty$

So $f_n \to 0$ <u>uniformly</u> on $[0, \infty)$.

Generally, a good method is to find an upper bound for the f_n that is independent of x.

<u>A general question:</u> Does (f_n) converge uniformly on S?

First check if f_n converges pointwise. If not, then f_n does not converge uniformly.

If it does, find pointwise limit f and then check

$$\sup_{x \in S} |f_n(x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

<u>Remark</u> What does it mean if $f_n \not\to f$ uniformly on S?

The negation:

$\exists \, \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists \, n \geq N$, $\exists \, x \in S$ s.t.

$|f_n(x) - f(x)| \geq \varepsilon$.

<u>Theorem 1</u> Let $S \subseteq \mathbb{R}$ or $\mathbb{C}$. Suppose $f_n \to f$ uniformly on S. If f_n is continuous for all $n \in \mathbb{N}$, then f is continuous.

(Idea: given $a \in S$, want $f(x) \approx f(a)$ if $x \approx a$. First choose large n so $f_n(x) \approx f(x)$ $\forall x \in S$. f_n is continuous so $f_n(x) \approx f_n(a)$ if $x \approx a$. So if $x \approx a$ then $f(x) \approx f_n(x) \approx f_n(a) \approx f(a)$).

<u>Proof</u> Fix $a \in S$ and $\varepsilon > 0$. We want f cts at a, i.e. $\forall x \in S$ $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Since $f_n \to f$ uniformly on S, may <u>fix</u> $n \in \mathbb{N}$ with

$$\forall x \in S, \quad |f_n(x) - f(x)| < \varepsilon.$$

Since f_n is cts, there exists $\delta > 0$ with

$$\forall x \in S, \quad |x-a| < \delta \Rightarrow |f_n(x) - f_n(a)| < \varepsilon.$$

Thus for any $x \in S$, if $|x-a| < \delta$ then

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|$$
$$< 3\varepsilon.$$

$\square$

(Note this comes from triangle inequality).

<u>Remarks</u>

1. The uniform limit may not preserve differentiability.

2. It follows from thm 1 that x^n does not converge uniformly on $[0,1]$ (pointwise limit not cts).

3. The proof is sometimes called a 3ε-proof.

4. We have

$$\lim_{x \to a} \Big(\underset{\substack{\uparrow \\ \text{u.c.}}}{\lim_{n \to \infty}} f_n(x) \Big) = \lim_{x \to a} f(x) \underset{\substack{\uparrow \\ \text{continuity}}}{=} f(a) = \lim_{n \to \infty} f_n(a)$$

$$= \lim_{n \to \infty} \Big(\lim_{x \to a} f_n(x) \Big)$$

Lemma 2 Let S be a set and let f_n be a bounded function on S for every $n \in \mathbb{N}$. If $f_n \to f$ uniformly on S, then f is also bounded on S.

Proof Fix $n \in \mathbb{N}$ with $|f_n(x) - f(x)| \leq 1 \quad \forall x \in S$. We can do this as $f_n \to f$ uniformly on S. Since f_n is bounded, there is an $M \in \mathbb{R}$ with $|f_n(x)| \leq M$ $\forall x \in S$. So $\forall x \in S$, we have

$$|f(x)| \underset{\Delta \text{ineq}}{\leq} |f(x) - f_n(x)| + |f_n(x)| \leq 1 + M. \qquad \square$$

Recall from IA: Take $f : [a,b] \to \mathbb{R}$ bounded.

Given a dissection D of $[a,b]$

Recall Riemann's criterion: f integrable iff $\forall \varepsilon > 0$, $\exists D$ s.t. $S(f,D) - s(f,D) < \varepsilon$.

$$S(f,D) - s(f,D) = \sum_{k=1}^{n} (x_k - x_{k-1}) \underbrace{\left(\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f \right)}_{\text{"oscillation" of function}} < \varepsilon$$

Also for $I \subseteq [a,b]$ we have

$$\sup_I f - \inf_I f = \sup_{x,y \in I} (f(x) - f(y)) = \sup_{x,y \in I} |f(x) - f(y)|$$

This is called the oscillation of f on I.

Theorem 3 Suppose $f_n : [a,b] \to \mathbb{R}$ is Riemann integrable $\forall n \in \mathbb{N}$. If $f_n \to f$ uniformly on $[a,b]$, then f is also Riemann integrable on $[a,b]$ with

$$\int_a^b f_n \to \int_a^b f \quad \text{as } n \to \infty.$$

Proof Each f_n is bounded, so by Lemma 2 f is bounded. Now fix $\varepsilon > 0$. Since $f_n \to f$ uniformly on $[a,b]$, we can fix $n \in \mathbb{N}$ with $|f_n(x) - f(x)| < \varepsilon \ \forall x \in [a,b]$.

f_n integrable, so $\exists D$ with $S(f_n, D) - s(f_n, D) < \varepsilon$.

If I is a subinterval of D, then for any $x, y \in I$ we have

$$\begin{aligned} |f(x) - f(y)| &\overset{\Delta\text{ineq}}{\leq} |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< |f_n(x) - f_n(y)| + 2\varepsilon \end{aligned}$$

Hence

$$\sup_{x,y \in I} |f(x) - f(y)| \leq \sup_{x,y \in I} |f_n(x) - f_n(y)| + 2\varepsilon$$

Multiplying both sides by length of I and summing over all subintervals I of D, we get

$$\begin{aligned} S(f, D) - s(f, D) &\leq S(f_n, D) - s(f_n, D) + 2\varepsilon(b-a) \\ &< (2(b-a) + 1)\varepsilon. \end{aligned}$$

So f satisfies Riemann's criterion: is integrable.

Finally, we estimate

$$\left| \int_a^b f_n - \int_a^b f \right| \leq \int_a^b |f_n - f| \leq (b-a) \sup_{[a,b]} |f_n - f| \to 0$$

as $n \to \infty$ by <u>uniform</u> convergence. $\square$

This says $\displaystyle\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$

in this case.

<u>Corollary 4</u> Let $f_n : [a,b] \to \mathbb{R}$ be integrable $\forall n \in \mathbb{N}$.
If $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly* on $[a,b]$ then

$\displaystyle\sum_{n=1}^{\infty} f_n(x)$ defines an integrable function on $[a,b]$ and

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx.$$

<u>Proof</u> Define $\displaystyle F_n(x) = \sum_{k=1}^{n} f_k(x)$ for $x \in [a,b]$, $n \in \mathbb{N}$

*So $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a,b]$ means that

(F_n) converges uniformly on $[a,b]$. So for each $x \in [a,b]$
the series $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ is convergent and the function

$\displaystyle x \mapsto \sum_{n=1}^{\infty} f_n(x)$ is the uniform limit of (F_n) on $[a,b]$.

But we can write this out in a <u>shorter way</u>:

Each F_n is integrable (finite sum of f_i) and $\int_a^b F_n = \sum_{k=1}^{n} \int_a^b f_k$

So by Theorem 3, $\sum_{n=1}^{\infty} f_n(x)$ is integrable, so

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \lim_{n\to\infty} \int_a^b F_n(x)\,dx$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b f_k(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx \qquad \square$$

<u>Theorem 5</u> Let (f_n) be a sequence of <u>ctsly differentiable</u> functions on $[a,b]$. Assume further that

(i) $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[a,b]$, and

(ii) there exists $c \in [a,b]$ s.t. $\sum_{n=1}^{\infty} f_n(c)$ converges

(note: don't need pointwise convergence - just one point).

Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a continuously differentiable function f on $[a,b]$ and moreover

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \qquad \forall x \in [a,b].$$

<u>Informally</u>: $\frac{d}{dx}\left(\sum_{n=1}^{\infty} f_n(x)\right) = \sum_{n=1}^{\infty} \frac{df_n}{dx}$. (under these <u>conditions</u>).

<u>Proof</u> Let $g(x) = \sum_{n=1}^{\infty} f_n'(x)$ for $x \in [a,b]$.

Idea: solve $f' = g$ with I.C. $f(c) = \sum_{n=1}^{\infty} f_n(c)$ by (ii).

Since $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to $g(x)$, and f_n'

is cts for all $n \in \mathbb{N}$, using Thm 1 g is cts $\Rightarrow$ integrable.

Let $\lambda = \sum_{n=1}^{\infty} f_n(c)$ and define

$$f(x) = \lambda + \int_c^x g(t)\, dt \quad \text{for } x \in [a,b].$$

As g is cts, by FTC have $f' = g$, $f(c) = \lambda$.

This solves $f' = g$.

By the FTC, $f_k(x) = f_k(c) + \int_c^x f_k'(t)\, dt$

for $x \in [a,b]$, $k \in \mathbb{N}$.

Given $\varepsilon > 0$, by assumptions there is $N \in \mathbb{N}$ s.t.

$$\left| \lambda - \sum_{k=1}^{n} f_k(c) \right| < \varepsilon \quad \forall n \geq N$$

$$\left| g(t) - \sum_{k=1}^{n} f_k'(t) \right| < \varepsilon \quad \forall n \geq N,\ \forall t \in [a,b].$$

So $\forall n \geq N$, $x \in [a,b]$:

$$\left| f(x) - \sum_{k=1}^{n} f_k(x) \right| = \left| \lambda + \int_c^x g(t)\, dt - \sum_{k=1}^{n} \left(f_k(c) + \int_c^x f_k'(t) \right) \right|$$

$$= \left| \lambda - \sum_{k=1}^{n} f_k(c) + \int_c^x \left(g(t) - \sum_{k=1}^{n} f_k'(t) \right) dt \right|$$

$$\leq \left| \lambda - \sum_{k=1}^{n} f_k(c) \right| + \left| \int_c^x \left(g(t) - \sum_{k=1}^{n} f_k'(t) \right) dt \right|$$

$$< \varepsilon + (b-a)\varepsilon$$

So $\sum_{k=1}^{n} f_k(x) \to f(x)$ uniformly on $[a,b]$.

Have seen already that f is diff'ble, $f' = g$ is cts. $\square$

Recall from Analysis I: (x_n) is Cauchy if

$\forall \varepsilon > 0, \ \exists N \in \mathbb{N}$ s.t. $\forall m, n \geqslant N, \ |x_m - x_n| < \varepsilon$.

Every Cauchy sequence is convergent.

<u>Definition</u> Let (f_n) be a sequence of scalar functions on a set S. We say (f_n) is <u>uniformly Cauchy</u> on S if

$\forall \varepsilon > 0, \ \exists N \in \mathbb{N}$ s.t. $\forall m, n \geqslant N, \ \forall x \in S$

we have $|f_m(x) - f_n(x)| < \varepsilon$.

<u>Theorem 6</u> (General Principle of Uniform Convergence)

If (f_n) is a uniformly Cauchy sequence of scalar functions on a set S, then (f_n) converges uniformly to some function f on S.

Proof First show (f_n) converges pointwise on S. Fix $x \in S$.

Given $\varepsilon > 0$, since (f_n) is uniformly Cauchy, there is $N \in \mathbb{N}$ s.t. $\forall m, n \geq N$, $\forall t \in S$

$|f_m(t) - f_n(t)| < \varepsilon$.

In particular, for all $m, n \geq N$, we have $|f_m(x) - f_n(x)| < \varepsilon$. So $(f_n(x))$ is a Cauchy sequence, so is convergent.

Setting $f(x) = \lim_{n \to \infty} f_n(x)$ proves pointwise convergence (do $\forall x$). ✓

Claim $f_n \to f$ uniformly on S.

Given $\varepsilon > 0$, since (f_n) is uniformly Cauchy, there is $N \in \mathbb{N}$ s.t.

$\forall m, n \geq N$, $\forall x \in S$, $|f_m(x) - f_n(x)| < \varepsilon$.

Fix $n \geq N$, $x \in S$. Since $|f_m(x) - f_n(x)| < \varepsilon$ $\forall m \geq N$, letting $m \to \infty$, we obtain $|f(x) - f_n(x)| \leq \varepsilon$.

Since this holds $\forall n \geq N$, $\forall x \in S$, we are done. $\square$

Corollary 7 (Weierstrass M-test)

Let (f_n) be a sequence of scalar functions on a set S.

Let $\sum_n M_n$ be a convergent series of non-negative reals.

Suppose that $|f_n(x)| \leq M_n$ $\forall x \in S$, $n \in \mathbb{N}$.

Then $\sum_n f_n(x)$ converges uniformly on S.

(Note: when we say an infinite series converges uniformly, what we mean is that the sequence of partial sums converges uniformly).

<u>Proof</u> Set $F_n(x) = \sum_{k=1}^{n} f_k(x)$ for $x \in S$, $n \in \mathbb{N}$.

Need F_n to be uniformly convergent.

For $n \geq m$, $x \in S$ we have

$$|F_m(x) - F_n(x)| \underset{(\Delta \text{ineq})}{\leq} \left| \sum_{k=m+1}^{n} f_k(x) \right| \underset{(\Delta \text{ineq})}{\leq} \sum_{k=m+1}^{n} |f_k(x)|$$

$$\leq \underbrace{\sum_{k=m+1}^{n} M_k}_{\text{indep. of } x,}$$

and $\sum_n M_n$ is convergent

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ s.t. $\sum_{k=N}^{\infty} M_k < \varepsilon$. (can do: convergent)

Then by above, for every $x \in S$ and $n \geq m \geq N$,

have $|F_m(x) - F_n(x)| \leq \sum_{k=m+1}^{n} M_k < \varepsilon$.

So (F_n) is uniformly Cauchy, so by Thm 6

it is <u>uniformly</u> convergent. $\square$

Analysis and Topology - Lecture 3

14/7/21

Consider a power series $\sum_{n=0}^{\infty} c_n (z-a)^n$

R radius of convergence : if $|z-a| < R$ it converges
if $|z-a| > R$ it diverges.

We write $D(a,R) = \{ z \in \mathbb{C} : |z-a| < R \}$
open disk centred on a, radius R

The power series does not generally converge uniformly in D:

Examples

1. $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ $\quad |z| < 1 \quad R = 1$

Consider $f_n : D(0,1) \to \mathbb{C}$ with $f_n(z) = \frac{z^n}{n^2}$

Note: $|f_n(z)| \leq \frac{1}{n^2} \quad \forall |z| < 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

By Weierstrass M-test, power series converges uniformly on $D(0,1)$.

2. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$

Note: $\left| \sum_{n=0}^{N} z^n \right| \leq N+1$ for $|z| < 1$, $N \in \mathbb{N}$
(by Δ ineq)

So partial sum functions are bounded on $D(0,1)$ but $\frac{1}{1-z}$ is not bounded on $D(0,1)$ so by Lemma 2 the series does not converge uniformly.

Alternatively we can use the sup method in L1:

$$\sup_{|z|<1} \left| \sum_{n=0}^{N} z^n - \frac{1}{1-z} \right| = \sup_{|z|<1} \left| \frac{z^{N+1}}{1-z} \right|$$

unbounded, close to $|z|=1$

<u>Theorem 8</u> Let power series $\sum_{n=0}^{\infty} c_n (z-a)^n$ have RoC R.

Then for any r with $0 < r < R$, the power series converges uniformly on $D(a,r)$.

<u>Proof</u> Fix $w \in D(a,R)$ with $r < |w-a| < R$.

Set $\rho = \frac{r}{|w-a|}$. Since $\sum_{n=0}^{\infty} c_n (w-a)^n$ converges,

have $c_n (w-a)^n \to 0$ as $n \to \infty$.

So the sequence $\left(c_n (w-a)^n \right)_{n=0}^{\infty}$ is <u>bounded</u>.

So $\exists\, M \geq 0$ s.t. $|c_n (w-a)^n| \leq M \quad \forall n$.

Now for any $z \in D(a,r)$ we have

$$|c_n (z-a)^n| = |c_n (w-a)^n| \left| \frac{(z-a)^n}{(w-a)^n} \right| \leq M \frac{r^n}{|(w-a)|^n} = M\rho^n$$

Since $\sum_{n=0}^{\infty} M\rho^n$ converges, by Weierstrass M-test,

the power series $\sum_{n=0}^{\infty} c_n (z-a)^n$ converges <u>uniformly</u>

on $D(a,r)$. $\square$

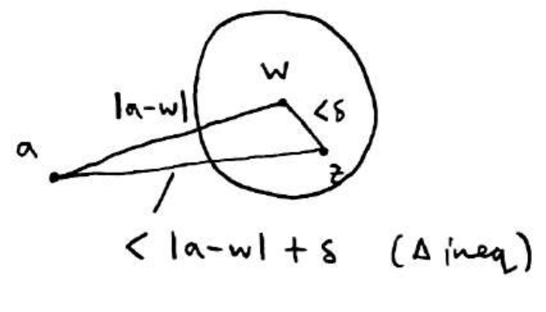
Remarks

1. The function $f : D(a,R) \to \mathbb{C}$, $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ is continuous on $D(a,r)$ for any $0<r<R$ using Thm 1. Since $D(a,R) = \bigcup_{0<r<R} D(a,r)$, f is continuous on $D(a,R)$.

2. The power series $\sum_{n=1}^{\infty} c_n\, n\, (z-a)^{n-1}$ also has radius of convergence R (differentiate - from 1A). So this converges uniformly on $D(a,r)$ for any $0<r<R$. By an analogue of Thm 5, f is complex differentiable on $D(a,R)$.

$$f'(z) = \frac{d}{dz}\left(\sum_{n=0}^{\infty} c_n (z-a)^n \right) = \sum_{n=1}^{\infty} c_n\, n (z-a)^{n-1}$$

3. Given $w \in D(a,R)$, fix r with $|w-a| < r < R$ and $\delta > 0$ with $|w-a| + \delta < r$. Then $D(w,\delta) \subseteq D(a,r)$.



$$|z-a| \leq |z-w| + |w-a| < \delta + |w-a| < r$$

so $z \in D(a,r)$.

So $\sum_{n=0}^{\infty} c_n (z-a)^n$ converges uniformly on $D(w,\delta)$.

__Definition__ A subset U of $\mathbb{C}$ is __open__ if for every $w \in U$, there is a $\delta > 0$ such that $D(w, \delta) \subseteq U$.

__Definition__ Let U be an open subset of $\mathbb{C}$. A sequence (f_n) of scalar functions on U converges __locally uniformly__ on U if for every $w \in U$ there is a $\delta > 0$ s.t. $D(w, \delta) \subseteq U$ __and__ $f_n \to f$ __uniformly__ on $D(w, \delta)$.

We proved on previous page that a power series converges __locally uniformly__ inside the radius of convergence.

__Uniform Continuity__

__Definition__ Let $U \subseteq \mathbb{R}$ or $\mathbb{C}$, f a scalar function on U. We say f is __uniformly continuous__ on U if

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x, y \in U$,

$$|y - x| < \delta \implies |f(y) - f(x)| < \varepsilon$$

Note δ depends __only__ on ε, not on x.

__Examples__

1. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = -3x + 7$. Given $\varepsilon > 0$, set some δ with that $\forall x, y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| = 3|x - y| < 3\delta$. Setting $\underline{\delta = \frac{\varepsilon}{3}}$ works. f __is__ uniformly continuous.

2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

Given $\varepsilon = 1$, we try and find $\delta > 0$ that works.

Take some $x > 0$ and $y = x + \frac{\delta}{2}$

Then $|x-y| = \frac{\delta}{2} < \delta$ and

$$|f(x) - f(y)| = \left(x + \frac{\delta}{2}\right)^2 - x^2 = \underbrace{x\delta}_{\text{depends on } x} + \frac{\delta^2}{4}$$

So for any $\delta > 0$, if $x = \frac{1}{\delta}$, $y = x + \frac{\delta}{2}$ then

$|x-y| < \delta$ but $f(x) - f(y) \underline{>} \varepsilon = 1$.

f is continuous but <u>not</u> uniformly continuous.

<u>Theorem 9</u> Let f be a scalar function on a <u>closed</u> and <u>bounded</u> interval $[a,b]$. Then if f is continuous on $[a,b]$ <u>then</u> f is <u>uniformly</u> continuous on $[a,b]$.

<u>Proof</u> Argue by contradiction.

Suppose there <u>is</u> $\varepsilon > 0$ s.t.

$\forall \delta > 0$, $\exists x, y \in [a,b]$ s.t. $|x-y| < \delta$, $|f(x) - f(y)| \geq \varepsilon$.

In particular, $\forall n \in \mathbb{N}$, $\exists x_n, y_n \in [a,b]$ with

$$\underbrace{|x_n - y_n| < \frac{1}{n}}_{(*)} \text{ and } |f(x_n) - f(y_n)| \geq \varepsilon.$$

By Bolzano-Weierstrass there are subsequences of (x_n) and (y_n) that converge to $x, y \in [a,b]$

closed, bounded so $x \in [a,b]$
by $(*)$ can choose <u>$y = x$</u>

Then

$$|\underbrace{y_{k_n}}_{\text{subseq. term}} - x| \leq |y_{k_n} - x_{k_n}| + |\underbrace{x_{k_n}}_{\text{subseq. term}} - x| \leq \frac{1}{n} + |x_{k_n} - x| \to 0$$

But f is cts at x, so $f(x_{k_n}) \to f(x)$, $f(y_{k_n}) \to f(x)$

Hence $\varepsilon \leq |f(x_{k_n}) - f(y_{k_n})| \to |f(x) - f(x)| = 0$ ※

$\square$

Corollary 10 If $f : [a,b] \to \mathbb{R}$ is continuous, then it's integrable.

Proof f is bounded and attains its bounds.

Given $\varepsilon > 0$: f is uniformly continuous by Thm 9 so we have that $\exists\, \delta > 0$ with

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon .$$

Choose dissection D of $[a,b]$ so all the subintervals of D have length strictly less than δ. (e.g. $\frac{b-a}{n} < \delta$)

If $I \subseteq D$ (I an interval) then $\forall x, y \in I$, have

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon .$$

So $$\sup_I f - \inf_I f = \sup_{x,y \in I} |f(x) - f(y)| \leq \varepsilon$$

Multiplying both sides by length of I and summing over all I, we get

$$S(f, D) - s(f, D) \leq (b-a)\,\varepsilon .$$

$\square$

Analysis and Topology - Lecture 4 14/7/21

2 Metric Spaces

In $\mathbb{R}$ or $\mathbb{C}$, $|x-y|$ measures distance between x and y. We can generalise this concept.

Definition Let M be a set. A metric on M is a function $d : M \times M \to \mathbb{R}$ satisfying

(i) $\forall x, y \in M$, $d(x,y) \geqslant 0$ and $d(x,y) = 0$ iff $x = y$ (positivity)

(ii) $\forall x, y \in M$, $d(x,y) = d(y,x)$ (symmetry)

(iii) $\forall x, y, z \in M$, $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

A metric space is a pair $(\underset{\text{set}}{M}, \underset{\text{metric on } M}{d})$

Examples

1. $M = \mathbb{R}$ or $\mathbb{C}$: $d(x,y) = |x-y|$ "usual/standard metric"

2. $M = \mathbb{R}^n$ or $\mathbb{C}^n$: Euclidean norm/length

$$\|x\| = \|x\|_2 = \left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2}$$

$$d(x,y) = d_2(x,y) = \|x-y\| = \left(\sum_{k=1}^{n} |x_k - y_k|^2 \right)^{1/2}$$

Also called l_2-norm

3. $M = \mathbb{R}^n$ or $\mathbb{C}^n$ $\quad$ ~~ℓ_2~~ ℓ_1 – norm

$$\|x\|_1 = \sum_{k=1}^{n} |x_k|$$

We can generalise ℓ_p – metric:

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \qquad d_p(x,y) = \|x-y\|_p$$

4. Letting $p \to \infty$ we get ℓ_∞ – metric:

$$\|x\|_\infty = \max_{1 \le k \le n} |x_k| \qquad d_\infty(x,y) = \max_{1 \le k \le n} |x_k - y_k|$$

5. Let S be a set. Denote by $\underline{\ell_\infty(S)}$ the set of all bounded scalar functions on S.

The $\underline{\ell_\infty\text{-norm}}$ (or uniform norm / sup norm) of $f \in \ell_\infty(S)$ is

$$\|f\| = \|f\|_\infty = \underline{\sup\{|f(x)| : x \in S\}}$$

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le \|f\| + \|g\|$$

so $\|f+g\| \le \|f\| + \|g\|$ so $\underline{d(f,g) = \|f-g\|_\infty}$ is a metric on $\ell_\infty(S)$.

Note $\ell_\infty^n = \ell_\infty(\{1,2,\dots,n\})$.

Often ℓ_∞ just means $\underline{\ell_\infty(\mathbb{N})}$, the space of bounded scalar <u>sequences</u>.

6. $C[a,b]$ is space of cts scalar functions on $[a,b]$

Let $p = 1$ or 2 : define L_p-norm by

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt \right)^{1/p}$$

7. M a set, $x, y \in M$:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$ discrete metric

8. Group $G = \langle S \rangle$

$$d(x,y) = \min \{ n : \exists s_1, \dots, s_n \in S \text{ s.t. } y = x s_1 \dots s_n \}$$

9. p prime : for $x, y \in \mathbb{Z}$ write $\underline{x - y = p^n m}$

with $m \in \mathbb{Z}$, $n \in \mathbb{Z}$, $n \geqslant 0$, $p \nmid m$

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ p^{-n} & \text{if } x \neq y \end{cases}$$ p-adic metric

This is an <u>ultrametric</u>: $d(x,z) \leqslant \max \{ d(x,y), d(y,z) \}$
($\Rightarrow \Delta$ ineq).

<u>Subspaces</u>

(M, d) metric space, $N \subseteq M$:

Then $\underline{d \restriction_{N \times N}}$ (restriction of d to $N \times N$) is a metric on N.

1. $\mathbb{Q}$ with $d(x,y) = |x - y|$ is a subspace of $\mathbb{R}$

2. $C[a,b]$ with uniform metric is a subspace of $\ell_\infty([a,b])$.

cts functions on $[a,b]$ $\Rightarrow$ bounded

bounded functions on $[a,b]$

Products

no "standard" metric

(M, d), (M', d') : can define metrics on $M \times M'$ e.g.

$$d_1((x, x'), (y, y')) = d(x, y) + d'(x', y')$$

or $$d_2((x, x'), (y, y')) = (d(x, y)^2 + d'(x', y')^2)^{1/2}$$

or $$d_\infty((x, x'), (y, y')) = \max\{d(x, y), d'(x', y')\}$$

Denote $(M \times M', d_p)$ by $\underline{M \oplus_p M'}$

Can be generalised to any finite product.

Note $d_\infty \leq d_2 \leq d_1 \leq 2 d_\infty$.

Examples

$\mathbb{R} \oplus_1 \mathbb{R} = l_1^2$, $\mathbb{R} \oplus_2 \mathbb{R} \oplus_2 \mathbb{R} = l_2^3$

Generally $\mathbb{R} \oplus_\infty \dots \oplus_\infty \mathbb{R} = l_\infty^n$

But $\mathbb{R} \oplus_1 \mathbb{R} \oplus_2 \mathbb{R}$ makes no sense as

$(\mathbb{R} \oplus_1 \mathbb{R}) \oplus_2 \mathbb{R}$ and $\mathbb{R} \oplus_1 (\mathbb{R} \oplus_2 \mathbb{R})$ are different.

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Convergence

Let M be a metric space and let (x_n) be a sequence in M.
We say $\underline{x_n \to x}$ as $n \to \infty$ if

$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, \quad d(x_n, x) < \varepsilon$.

If such an x exists in M, then (x_n) is convergent.

Note: $\quad x_n \to x \iff d(x_n, x) \to 0$

Lemma 1 (Uniqueness of limit)
Suppose $x_n \to x$ and $x_n \to y$. Then $x = y$ (in M).

Proof Suppose not. Set $\varepsilon = d(x,y)/3$. We can choose N_1, N_2 with $\forall n \geq N_1 \; d(x_n, x) < \varepsilon$ and $\forall n \geq N_2 \; d(x_n, y) < \varepsilon$.
Fix $n \geq \max\{N_1, N_2\}$. Then

$d(x,y) \leq d(x, x_n) + d(x_n, y) < 2\varepsilon < d(x,y)$. $\square$

Examples

1. Usual meaning in $\mathbb{R}$ and $\mathbb{C}$
2. Constant sequences converge.
3. Suppose $x_n \to x$ in a discrete metric space. Then $\exists N \in \mathbb{N}$ s.t. $d(x_n, x) < 1$ for $n \geq N$ and hence $d(x_n, x) = 0 \quad \forall n \geq N$ i.e. eventually constant.

4. 3-adic metric on $\mathbb{Z}$: $3^n \to 0$ as $n \to \infty$ since $d(3^n, 0) = 3^{-n} \to 0$ as $n \to \infty$.

5. Let S be a set. In $\ell_\infty(S)$ (bounded scalar functions with uniform metric), $f_n \to f$ iff we have

$$d(f_n, f) = \|f_n - f\|_\infty = \sup_S |f_n - f| \to 0 \text{ as } n \to \infty$$

which is true iff $f_n \to f$ uniformly on S.

Note: if $f_n(x) = x + \frac{1}{n}$, $f(x) = x$ $x \in \mathbb{R}$, $n \in \mathbb{N}$: have $f_n \to f$ uniformly on $\mathbb{R}$, but $f_n, f \notin \ell_\infty(\mathbb{R})$.

6. Consider $\mathbb{R}^{\mathbb{N}}$ (all real sequences). Check that for sequences $x = (x_k)$, $y = (y_k)$:

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \min\{1, |x_k - y_k|\}$$

defines a metric on $\mathbb{R}^{\mathbb{N}}$.

Then a sequence $(x^{(n)})$ in $\mathbb{R}^{\mathbb{N}}$ converges to $x \in \mathbb{R}^{\mathbb{N}}$ iff $x_k^{(n)} \to x_k$ as $n \to \infty$ for each $k \in \mathbb{N}$.

(Note we write $x^{(n)} = (x_k^{n})_{k=1}^{\infty}$)

7. $f_n(x) = x^n$ $x \in [0, 1]$ $n \in \mathbb{N}$

(f_n) is a sequence in $C[0,1]$. Recall: (f_n) converges pointwise but not uniformly on $[0,1]$ so (f_n) does not converge in the uniform metric.

However, $d_1(f_n, 0) = \|f_n\|_1 = \int_0^1 |f_n| = \frac{1}{n+1} \to 0$

so $f_n \to 0$ in $C[0,1]$ in the L_1 - metric.

8. Let N be a subspace of the metric space M and (x_n) a sequence in N. If $(x_n) \to x$ in N then $(x_n) \to x$ in M. Converse is <u>false</u>.

9. M, M' metric spaces - consider $N = M \oplus_p M'$ $(p = 1, 2 \text{ or } \infty)$. Take (a_n) a sequence in N: write $a_n = (x_n, x_n')$ with $x_n \in M$, $x_n' \in M'$

Then $a_n \to a$ in $N \iff x_n \to x$ in M <u>and</u> $x_n' \to x'$ in M'

This holds because

$$\max\{d(x_n, x), d'(x_n', x')\} = d_\infty(a_n, a) \leqslant d_p(a_n, a)$$
$$\leqslant d_1(a_n, a) = d(x_n, x) + d'(x_n', x')$$

<u>Continuity</u>

Consider $f: M \to M'$. For $a \in M$, f is <u>continuous</u> at a if

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in M$,

$$d(x, a) < \delta \implies d'(f(x), f(a)) < \varepsilon.$$

Note δ depends on ε <u>and</u> on a.

For subset $N \subset M$, f is cts on N if f is cts at every $a \in N$.

If f is cts on N, then $f\restriction_N : N \to M'$ is cts, but the converse is not true.

f cts on N means

$\forall a \in N$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t $\underline{\forall x \in M}$ if $d(x, a) < \delta$ then $d'(f(x), f(a)) < \varepsilon$

But $f\restriction_N : N \to M'$ cts means

$\forall a \in N$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\underline{\forall x \in N}$ if $d(x, a) < \delta$ then $d'(f(x), f(a)) < \varepsilon$.

Example: $M = \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

Take $N = [0, \infty)$ then $f\restriction_N : [0, \infty) \to \mathbb{R}$, $x \mapsto 1$ is constant so continuous. But f is not continuous on N, e.g. at 0.

Proposition 2 Let $f : M \to M'$ be a function, fix $a \in M$. The following are equivalent:

(i) f is continuous at a

(ii) if $x_n \to a$ in M, then $f(x_n) \to f(a)$ in M'.

Proof (i) $\Rightarrow$ (ii): given $\varepsilon > 0$, as f is cts at a, $\exists \delta > 0$ s.t. $d'(f(x), f(a)) < \varepsilon$ for any $x \in M$ with $d(x, a) < \delta$. If $x_n \to a$ then there is $N \in \mathbb{N}$ with $d(x_n, a) < \delta$ for $n \geq N$.

So $d'(f(x_n), f(a)) < \varepsilon$ $\forall n \geq N$, so $f(x_n) \to f(a)$ as $n \to \infty$. ✓

(ii) $\Rightarrow$ (i) : Suppose f is <u>not</u> cts at a. This means

$\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x \in M$ s.t. $d(x,a) < \delta$, $d'(f(x), f(a)) \geq \varepsilon$.

Fix such a "bad" ε. Then for every $n \in \mathbb{N}$, $\exists x_n \in M$ s.t. $d(x_n, a) < \underset{\leq\delta}{\frac{1}{n}}$ and $d'(f(x_n), f(a)) \geq \varepsilon$.

So $x_n \to a$ but $f(x_n) \not\to f(a)$. $\square$

<u>Corollary 3</u> f, g scalar functions on metric space M
Let $a \in M$, assume f and g continuous at a.
Then $f+g$ and fg are also continuous at a.
Also, if $N = \{x \in M : g(x) \neq 0\}$, $a \in N$, $\frac{f}{g} : N \to M'$ is also continuous at a.

<u>Proof</u> Assume $x_n \to a$ in M. Then $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$. Hence

$$(f+g)(x_n) = f(x_n) + g(x_n) \to f(a) + g(a) = (f+g)(a).$$

Similar argument for fg and f/g. $\square$

<u>Proposition 4</u> $f : M \to M'$, $g : M' \to M''$
If f is cts at a and g is cts at $f(a)$ then the <u>composition</u> $g \circ f : M \to M''$ is cts at a.

<u>Proof</u> Given $\varepsilon > 0$, since g is cts at $f(a)$, $\exists \delta > 0$ s.t.

$\forall y \in M'$ $d'(y, f(a)) < \delta \Rightarrow d''(g(y), g(f(a))) < \varepsilon$

Since f is cts at a, there is $\eta > 0$ s.t.

$\forall x \in M$ $d(x,a) < \eta \Rightarrow d'(f(x), f(a)) < \delta$

$\Rightarrow d''(g(f(x)), g(f(a))) < \varepsilon$. $\square$

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Examples

1. Constant functions are continuous: $f: M \to M'$, $f(x) = b$ for all $x \in M$. Indeed, $d'(f(x), f(a)) = d'(b, b) = 0$, for any $x, a \in M$.

2. Identity functions are continuous: $f: M \to M$, $f(x) = x$ $\forall x \in M$ since $d'(f(x), f(a)) = d(x, a)$ $\forall x, a \in M$

3. Composing functions: polynomials, rational functions, and uniform limits of such functions e.g. exp are continuous.

4. (M, d) metric space: the metric d is a function
$d: M \oplus_p M \to \mathbb{R}$ $(p = 1, 2 \text{ or } \infty)$
For $\underline{x} = (x, x')$, $\underline{y} = (y, y')$ in $M \oplus_p M$:
$$|d(\underline{x}) - d(\underline{y})| = |d(x, x') - d(y, y')| \leqslant d(x, y) + d(x', y')$$
from Δ ineq
$$= d_1(\underline{x}, \underline{y}) \leqslant 2 d_p(\underline{x}, \underline{y})$$
Then d is <u>cts</u> (e.g. $\delta = \frac{\varepsilon}{2}$ works).

<u>Definitions</u> A map $f: M \to M'$ is

(i) <u>isometric</u> if $d'(f(x), f(y)) = d(x, y)$ $\forall x, y \in M$

(ii) <u>Lipschitz</u> if $\exists\, C \geqslant 0$ s.t. $d'(f(x), f(y)) \leqslant C d(x, y)$
e.g. example 4 above, $C = 2$

(iii) <u>uniformly continuous</u> if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.
$\forall x, y \in M$, $d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$

isometric $\Rightarrow$ Lipschitz $\underset{\delta = \frac{\varepsilon}{C}}{\Rightarrow}$ uniformly continuous $\Rightarrow$ continuous

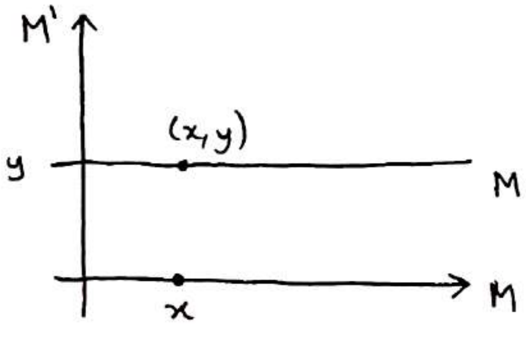
Isometric maps are injective, not necessarily surjective

A bijective isometric map is called an <u>isometry</u>

If there is an isometry $M \to M'$ then M and M' are said to be <u>isometric</u>.

<u>Examples</u> (cont.)

5. Let M, M' be metric spaces. Fix $y \in M'$ and consider $f : M \to M \oplus_p M'$, $\quad f(x) = (x, y)$



For $x, z \in M$ we have

$$d_p(f(x), f(z)) = d_p((x,y),(z,y)) = \underline{d(x,z)}$$

So f is <u>isometric</u> and $M \times \{y\}$ is an isometric copy of M. This generalises to any finite number of metric spaces.

e.g. take $\underline{a} \in \mathbb{R}^n$, $\quad x \mapsto (a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n)$ from $\mathbb{R} \to \mathbb{R}^n$ is <u>isometric</u>.

6. M, M' metric spaces, consider $q : M \oplus_p M' \to M$, $(x, x') \mapsto x$ and $q' : M \oplus_p M' \to M'$, $(x, x') \mapsto x'$

For $\underline{x} = (x, x')$, $\underline{y} = (y, y')$ in $M \oplus_p M'$ we have

$$d(q(\underline{x}), q(\underline{y})) = d(x,y) \leq \max\{d(x,y), d'(x',y')\}$$
$$= d_\infty(\underline{x}, \underline{y}) \leq d_p(\underline{x}, \underline{y})$$

So q is 1-Lipschitz.

The same holds for q'.

For instance $\mathbb{C}^n \to \mathbb{C}$, $(z_1, \dots, z_n) \mapsto z_k$ is continuous so by Corollary 3, polynomials in any number of variables are also continuous.

The topology of Metric Spaces

In a metric space, continuity at a point / convergence of a sequence depends on the set of points close to some x.

Definition Let (M, d) be a metric space, $x \in M$.
Define the open ball in M with centre x and radius r by
$$D_r(x) = \{ y \in M : d(y, x) < r \} \qquad (\text{sometimes } D_r^M(x))$$

Note $x_n \to x$ in $M \iff \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $x_n \in D_\varepsilon(x)$

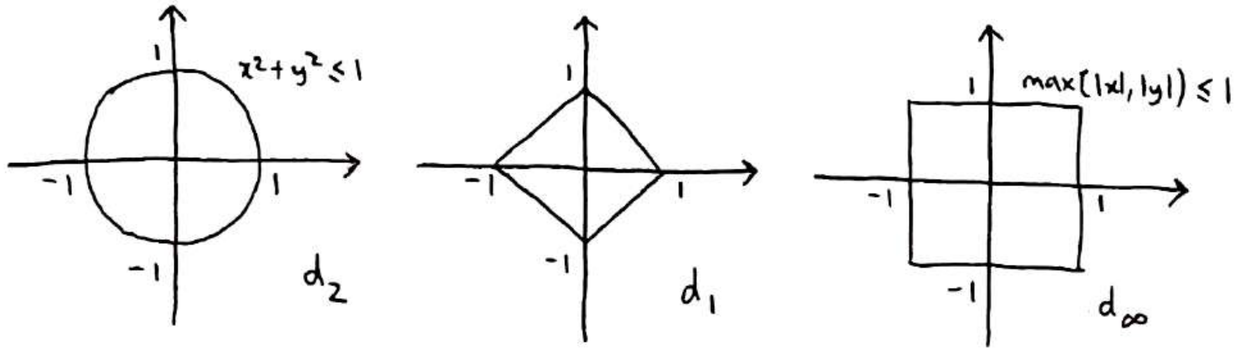
and $f : M \to M'$ is continuous at x iff $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $f(D_\delta(x)) \subseteq D_\varepsilon(f(x))$

Definition The closed ball in M with centre x, radius r is $B_r(x) = \{ y \in M : d(y, x) \leq r \}$.

Examples

1. In $\mathbb{R}$: $D_r(x) = (x-r, x+r)$, $B_r(x) = [x-r, x+r]$
2. In $\mathbb{C}$, $D_r(x)$, $B_r(x)$ are the open and closed discs with centre x, radius r

3. In $\mathbb{R}^2$ these are pictures of $\underline{B_1(0)}$ (closed unit ball):



4. If M is a discrete metric space then $D_1(x) = \{x\}$ and $B_1(x) = M$.

Note: $D_r(x) \subseteq B_r(x) \subseteq D_s(x)$ whenever $r < s$

and $d(x,y) = \min\{r \geq 0 : y \in B_r(x)\}$.

$d(x,y) = \inf\{r > 0 : y \in D_r(x)\}$

<u>Definitions</u> Let M be a metric space, let $U \subset M$.

Given $x \in M$, we say that U is a <u>neighbourhood</u> of x in M if $\exists r > 0$ s.t. $\underline{D_r(x) \subset U}$

or equivalently $\exists r > 0$ s.t. $B_r(x) \subset U$.

We say U is <u>open</u> in M if

$\forall x \in U,\ \exists r > 0$ s.t. $D_r(x) \subseteq U$

(i.e. U is a <u>neighbourhood of all of its points</u>).

<u>Note</u>: for $x \in M$, $r > 0$, $D_r(x)$ and $B_r(x)$ are both neighbourhoods of x.

__Example__ Consider closed upper half-plane

$H = \{ z \in \mathbb{C} : \operatorname{Im}(z) \geq 0 \} \subseteq \mathbb{C}$

If $\operatorname{Im}(w) > 0$, then $D_\delta(w) \subseteq H$ for $\delta = \operatorname{Im}(w)$

so H is a neighbourhood of w.

If $\operatorname{Im}(w) = 0$ then $w - \frac{\delta}{2} i \in D_\delta(w) \setminus H$ for any $\delta > 0$, so H is not a neighbourhood of w.

__Lemma 5__ Open balls are open.

__Proof__ M metric space, $x \in M$, $r > 0$. Fixing $y \in D_r(x)$:

Take $\delta = r - d(y, x) > 0$ (as $y \in D_r(x)$).

For any $\underline{z \in D_\delta(y)}$, we have

$d(z, x) \leq d(z, y) + d(y, x) < \delta + d(y, x) = r$

and hence $\underline{z \in D_r(x)}$. So $\underline{D_\delta(y) \subset D_r(x)}$

so open balls are open. $\square$

__Corollary 6__ Let U be a subset of M, let $x \in M$. Then U is a neighbourhood of x if and only if there is an open subset V of M such that $x \in V \subseteq U$.

__Proof__ If U is a neighbourhood of x, then by definition $D_r(x) \subseteq U$ for some $r > 0$. So taking $V = D_r(x)$, we have $x \in V \subset U$ and by Lemma 5, V is open.

Conversely, if $x \in V \subset U$ for some open subset V of M, then by definition of open set, there is $r > 0$ with $D_r(x) \subset V$.

So $D_r(x) \subset U$ so U is a neighbourhood of x. $\square$

Proposition 7 In a metric space M the following are equivalent:

(i) $x_n \to x$

(ii) For all neighbourhoods U of x in M, there exists $N \in \mathbb{N}$ s.t. $\forall n \geq N$, $x_n \in U$.

(iii) For all open sets U in M with $x \in U$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$.

Proof First recall (i) can be written as

(i) $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $x_n \in D_\varepsilon(x)$

(i) $\Rightarrow$ (ii) Take U a neighbourhood of x. There is some $\varepsilon > 0$ with $D_\varepsilon(x) \subset U$. Since $x_n \to x$, there is $N \in \mathbb{N}$ s.t. $\forall n \geq N$, we have $x_n \in D_\varepsilon(x) \subset U$. $\square$

(ii) $\Rightarrow$ (iii) Any open set U with $x \in U$ is by definition a neighbourhood of x. $\square$

(iii) $\Rightarrow$ (i) For any $\varepsilon > 0$, the ball $D_\varepsilon(x)$ is an open set containing x. $\square$

Proposition 8 $f : M \to M'$ function between metric spaces

(a) For $x \in M$ the following are equivalent:

(i) f is continuous at x

(ii) $\forall$ neighbourhoods V of $f(x)$ in M', $\exists$ a neighbourhood U of x in M with $f(U) \subset V$

(to be continued)

<u>and</u> (iii) $\forall$ neighbourhoods V of $f(x)$ in M',

$f^{-1}(V)$ is a neighbourhood of x in M.

(b) The following are equivalent:

(i) f is continuous

(ii) $\forall$ open sets V in M', the set $f^{-1}(V)$ is open in M.

(The inverse image of an open set is open).

<u>Proof</u> Part (a): first recall (i) is

(i) $\forall \varepsilon > 0, \ \exists \delta > 0$ s.t. $\underline{f(D_\delta(x)) \subset D_\varepsilon(f(x))}$

<u>(i) $\Rightarrow$ (ii)</u> Given a neighbourhood V of $f(x)$ in M', there is an $\varepsilon > 0$ with $D_\varepsilon(f(x)) \subset V$. Since f is continuous at x, there is $\delta > 0$ with $f(D_\delta(x)) \subset D_\varepsilon(f(x))$. Set $\underline{U = D_\delta(x)}$. Then U is a neighbourhood of x in M and $f(U) \subset D_\varepsilon(f(x)) \subset V$.

<u>(ii) $\Rightarrow$ (iii)</u> Given a neighbourhood V of $f(x)$ in M', by (ii) there is a neighbourhood U of x in M with $f(U) \subset V$. By defn of neighbourhood, there is $r > 0$ with $D_r(x) \subset U$. It follows that $\underline{D_r(x) \subset U \subset f^{-1}(V)}$ so $f^{-1}(V)$ is a neighbourhood of x in M.

<u>(iii) $\Rightarrow$ (i)</u> Given $\varepsilon > 0$, the set $\underline{V = D_\varepsilon(f(x))}$ is a neighbourhood of $f(x)$ in M' so $f^{-1}(V)$ is a neighbourhood of x in M by (iii).

PTO

By definition there is a $\delta > 0$ s.t. $D_\delta(x) \subset f^{-1}(V)$.

Hence $f(D_\delta(x)) \subset V = D_\varepsilon(f(x))$. So f is cts. $\square$

(b) $\underline{(i) \Rightarrow (ii)}$: Let V be an open set in M'.

Fix $x \in f^{-1}(V)$. Then $f(x) \in V$ and so V is a neighbourhood of $f(x)$ in M'. Since f is continuous at x, it follows from (a) that $f^{-1}(V)$ is a neighbourhood of x in M. This holds for <u>any</u> $x \in f^{-1}(V)$ so $f^{-1}(V)$ is open in M.

$(ii) \Rightarrow (i)$: Let $x \in M$, let $\varepsilon > 0$. Since $V = D_\varepsilon(f(x))$ is open in M', we have that $f^{-1}(V)$ is open in M by assumption. Since $f(x) \in V$, we have $x \in f^{-1}(V)$. Thus there is a $\delta > 0$ s.t. $D_\delta(x) \subset f^{-1}(V)$. Hence $f(D_\delta(x)) \subset V = D_\varepsilon(f(x))$.

So f is cts at every $x \in M$. $\square$

<u>Definition</u> The <u>topology</u> of a metric space M is the family of all open subsets of M.

<u>Proposition 9</u> The topology of M satisfies the following:

(i) $\emptyset$ and M are open

(ii) If U_i is open for all $i \in I$ (index set) then $\underline{\bigcup_{i \in I} U_i}$ is open

(iii) If U and V are open, then $\underline{U \cap V}$ is open.

<u>Proof</u> Part (i) is trivial

To see (ii), fix $x \in \bigcup_{i \in I} U_i$. Then for some $j \in I$,

we have $x \in U_j$. Since U_j is open, there exists $r > 0$

with $D_r(x) \subset U_j \subset \bigcup_{i \in I} U_i$ as required.

For (iii) fix $x \in U \cap V$. Since U and V are open,

there are $r, s > 0$ with $D_r(x) \subset U$, $D_s(x) \subset V$.

Then $t = \min\{r, s\}$ is <u>strictly</u> positive and

$$D_t(x) = D_r(x) \cap D_s(x) \subset U \cap V$$

so $U \cap V$ is open. $\square$

Definition M metric space, $A \subset M$. We say that A is closed in M (or A is a closed subset of M) if for every sequence (x_n) in A that converges in M, we have $\lim_{n \to \infty} x_n \in A$.

Lemma 10 Closed balls are closed.

Proof Let M be a metric space, $z \in M$, $r > 0$, $x_n \in B_r(z)$ $\forall n$. Suppose $x_n \to x$ in M.

Then $d(x, z) \leq d(x, x_n) + d(x_n, z) \leq d(x, x_n) + r \to r$ as $n \to \infty$. So $d(x, z) \leq r \Rightarrow x \in B_r(z)$. $\square$

Examples

$[0,1] = B_{1/2}(1/2)$ closed, $(0,1) = D_{1/2}(1/2)$ open

$(0,1]$ neither open nor closed

Lemma 11 M metric space, $A \subset M$. Then A is closed in M if and only if $M \setminus A$ is open in M.

Proof Suppose A is closed, $x \in M \setminus A$. Seek $r > 0$ with $D_r(x) \subset M \setminus A$ (to show $M \setminus A$ open). If no such r exists, then $D_{1/n}(x) \cap A \neq \emptyset$ $\forall n \in \mathbb{N}$. So there exists $x_n \in A$ with $d(x_n, x) < \frac{1}{n}$ $\forall n$. So have a sequence $(x_n) \in A$ with $x_n \to x \notin A$ ※ A being closed.

Conversely, assume $M \setminus A$ is open, and assume A is not closed for contradiction. So there is $(x_n) \in A$ with $x_n \to x \notin A$.
Since $M \setminus A$ is open, $\exists r > 0$ with $D_r(x) \subset M \setminus A$.
Since $x_n \to x$, there is $N \in \mathbb{N}$ with $x_n \in D_r(x) \subset M \setminus A$ $\forall n \geq N$. This contradicts assumption that $(x_n) \in A$. ※
So A is closed.

$\square$

<u>Example</u> M a <u>discrete</u> metric space, let $A \subset M$. For any $x \in A$, $D_1(x) = \{x\} \subset A$ so A is open.
Hence <u>every</u> subset of M is <u>open</u>. By Lemma 11, also <u>every</u> subset of M is <u>closed</u>.

<u>Definition</u> A function $f: M \to M'$ between metric spaces is called a <u>homeomorphism</u> if f is a <u>bijection</u> and both f and f^{-1} are <u>continuous</u>.

<u>Example</u> $(0, \infty)$ and $(0,1)$ homeomorphic:

$$x \mapsto \frac{1}{x+1} \quad \text{and} \quad x \mapsto \frac{1}{x} - 1$$

<u>Remarks</u>

1. Isometry $\Rightarrow$ homeomorphism (converse false)
2. Identity map is a homeomorphism
3. Inverse of a HM is a HM, composite of HM is HM
4. $f: M \to M'$ a HM: if U is an <u>open</u> subset of M, then $f(U) = (f^{-1})^{-1}(U)$ is open in M'.

Conversely, if V is an open subset of M' then $f^{-1}(V)$ is open in M, so there is a bijection between the topologies of M and M'.

5. $f: M \to M'$ a HM: then (x_n) converges in M iff $(f(x_n))$ converges in M'.

Also $g: N \to M$ cts iff $f \circ g$ cts

$h: N \to M'$ cts iff $f^{-1} \circ h$ cts

6. A continuous bijection is not necessarily a homeomorphism

e.g. identity $(\mathbb{R}, \text{discrete}) \to (\mathbb{R}, \text{euclidean})$

Definition Let d, d' be metrics on a set M. We say d and d' are equivalent $(d \sim d')$ if (M, d) and (M, d') have the same open sets (i.e. same topology).

Remark All the following are equivalent to $d \sim d'$:

1. Identity map $\mathrm{Id}: (M, d) \to (M, d')$ is a homeomorphism
2. (M, d) and (M, d') have the same convergent sequences
3. For every metric space N and every $f: N \to M$, f is cts wrt d iff it is cts wrt d'.
4. Same applies for functions away from M.

Definitions

Let d, d' be metrics on a set M. We say d and d' are uniformly equivalent ($d \sim_u d'$) if

$Id : (M, d) \to (M, d')$ and $Id : (M, d') \to (M, d)$

are uniformly continuous.

Say d, d' are Lipschitz equivalent ($d \sim_{Lip} d'$) if both identity maps are Lipschitz. (That is: $\exists\, a, b > 0$ with $a\, d(x,y) \leq d'(x,y) \leq b\, d(x,y)$).

for all $x, y \in M$.

Examples

1. Given (M, d): $d'(x,y) = \min\{1, d(x,y)\}$ is uniformly equivalent.

2. Given M, M': metrics d_1, d_2, d_∞ on $M \times M'$ are pairwise Lipschitz equivalent.

3. Uniform metric, L_1-metric on $C[0,1]$ not equivalent

4. Usual metric and discrete metric on $\mathbb{R}$ not equivalent.

3 Completeness and the Contraction Mapping Theorem

Definition (x_n) sequence in metric space M.

(x_n) is Cauchy if:

$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N, \; d(x_m, x_n) < \varepsilon$

(x_n) is bounded if:

$\exists z \in M, \; r > 0$ s.t. $\forall n \in \mathbb{N}, \; x_n \in B_r(z)$

<u>Note</u>: if $w \in M$, $R = r + d(z,w)$ then $B_r(z) \subset B_R(w)$.

So if $M = \mathbb{R}^n$, $\mathbb{C}^n$ or $C[a,b]$ with l_p-metric or L_p-metric, and $\|\cdot\|$ is the corresponding norm, then (x_n) is bounded iff $\exists\, r > 0$ s.t. $\underline{\|x_n\| \leq r}$ $\forall n$.

<u>Lemma 12</u> Convergent $\Rightarrow$ Cauchy $\Rightarrow$ bounded

<u>Proof</u> (x_n) sequence in metric space M

For convergent $\Rightarrow$ Cauchy: suppose $x_n \to x \in M$.

Given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $d(x_n, x) < \frac{\varepsilon}{2}$ $\forall n \geq N$.

Then for $m, n \geq N$:

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \varepsilon \quad \checkmark$$

For Cauchy $\Rightarrow$ bounded: assume (x_n) is Cauchy. Then $\exists\, N \in \mathbb{N}$ s.t. $d(x_m, x_n) \leq 1$ $\forall m, n \geq N$.

We must have $\underline{x_n \in B_1(x_N)}$ $\forall n \geq N$ (just a specific case).

Choosing $r = \max\{d(x_1, x_N), \ldots, d(x_{N-1}, x_N), 1\}$

gives $x_n \in B_r(x_N)$ $\forall n \in \mathbb{N}$. $\square$

<u>Remarks</u>

1. Bounded $\not\Rightarrow$ Cauchy e.g. $0, 1, 0, 1, \ldots$
2. Cauchy $\not\Rightarrow$ convergent e.g. sequence $\left(\frac{1}{n}\right)$ in $(0, \infty)$ because $\left(\frac{1}{n}\right) \xrightarrow{\text{in } \mathbb{R}} 0$ but $0 \notin (0, \infty)$.

<u>Definition</u> A metric space is <u>complete</u> if <u>every</u> Cauchy sequence converges. (e.g. $\mathbb{R}$ and $\mathbb{C}$).

<u>Proposition 13</u> If M, M' are complete, so is $M \oplus_p M'$. ($p = 1, 2, \infty$)

<u>Proof</u> Take Cauchy sequence $a_n = (x_n, x_n')$

Given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $d_p(a_m, a_n) < \varepsilon \quad \forall m, n \geqslant N$

So $d(x_m, x_n) \leqslant d_p(a_m, a_n) < \varepsilon \quad \forall m, n \geqslant N$.

So (x_n) and similarly (x_n') are Cauchy. Since M, M' are complete, $\exists\, x, x'$ s.t. $x_n \to x$, $x_n' \to x'$.

Then $d_p(a_n, a) \leqslant d(x_n, x) + d'(x_n', x') \to 0$ as $n \to \infty$

where $a = (x, x')$.

So $a_n \to a$ in $M \oplus_p M'$. $\square$

<u>Note</u> (a_n) Cauchy <u>iff</u> (x_n) and (x_n') Cauchy.

<u>Corollary 14</u> $\mathbb{R}^n$, $\mathbb{C}^n$ complete in l_p-metric for $p = 1, 2, \infty$. In particular, an n-dimensional real or complex Euclidean space is complete.

Theorem 15 Let S be a set. Then the space $\ell_\infty(S)$ is complete in the uniform metric D.

Proof Let (f_n) be Cauchy in $\ell_\infty(S)$. Then given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N$, we have

$$D(f_m, f_n) = \sup_{x \in S} |f_m(x) - f_n(x)| < \varepsilon$$

and hence $\forall x \in S, \ |f_m(x) - f_n(x)| < \varepsilon$

So (f_n) is uniformly Cauchy. By Thm 1.6, (f_n) converges uniformly to a scalar function f on S. By Lemma 1.2, f is bounded i.e. $f \in \ell_\infty(S)$.

Since $f_n \to f$ uniformly on S, given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, have $\forall x \in S$, $|f_n(x) - f(x)| < \varepsilon$ and hence $D(f_n, f) = \sup_{x \in S} |f_n(x) - f(x)| \leq \varepsilon$

So $f_n \to f$ in the uniform metric. $\square$

Proposition 16 Let N be a subspace of a metric space M.

(i) If N is complete, then N is closed in M

(ii) If M is complete and N is closed in M, then N is complete.

In particular, in a complete metric space, a subspace is closed if and only if it is complete.

Proof (i) Given (x_n) in N, assume $x_n \to x$ in M. Then (x_n) is Cauchy in M by Lemma 12, so Cauchy in N. N is complete so $x_n \to y \in N$, so $x_n \to y \in M$. So $x = y$ so $\underline{x \in N}$ as required.

(ii) Let (x_n) be Cauchy in N. Then (x_n) is Cauchy in M, so $x_n \to x \in M$. Since N is closed, $x \in N$. So $x_n \to x \in N$ so x_n is convergent. $\square$

Definition (M, d) metric space. Define

$$C_b(M) = \{ f \in l_\infty(M) : f \text{ continuous} \}$$

In real case:

$$C_b(M) = \{ f : M \to \mathbb{R} : f \text{ bounded, continuous} \}$$

Note $C_b(M)$ is a subspace of $l_\infty(M)$ in uniform metric D.

Theorem 17 $C_b(M)$ is complete in the uniform metric for any metric space M.

Proof By Thm 15, Prop 16 (ii) it is enough to show that $C_b(M)$ is closed in $l_\infty(M)$.

Let (f_n) be a sequence in $C_b(M)$, let $f \in l_\infty(M)$, and assume $f_n \to f$. Need to show $f \in C_b(M)$ i.e. that f is cts.

PTO

Fix $a \in M$, $\varepsilon > 0$. Since $f_n \to f$, we can choose a large $n \in \mathbb{N}$ s.t.

$$D(f_n, f) = \sup_{x \in M} |f_n(x) - f(x)| < \varepsilon$$

Since f_n is cts at a, $\exists\, \delta > 0$ s.t. $\forall x \in M$,

$d(x, a) < \delta \Rightarrow |f_n(x) - f_n(a)| < \varepsilon$

So for any $x \in M$, if $d(x, a) < \delta$ then

$$\begin{aligned} |f(x) - f(a)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &\leq 3\varepsilon \end{aligned}$$

So f is continuous. (Compare with Ch 1 proof). $\square$

<u>Corollary 18</u> For any closed, bounded interval $[a, b]$ in $\mathbb{R}$, the space $C[a, b]$ is complete in the uniform metric.

<u>Definitions</u> Let S be a set, (N, e) a metric space.

Define $\ell_\infty(S, N) = \{ f : S \to N \mid f \text{ bounded} \}$

Note f is bounded if $\exists\, y \in N$, $r > 0$ s.t. $\forall x \in S$, $f(x) \in B_r(y)$.

Now assume $g : S \to N$ is bounded: $\exists\, z \in N$, $s > 0$ s.t. $g(x) \in B_s(z)$ $\forall x \in S$. Then $\forall x \in S$,

$$\begin{aligned} e(f(x), g(x)) &\leq e(f(x), y) + e(y, z) + e(z, g(x)) \\ &\leq r + e(y, z) + s \end{aligned}$$

So $\sup_{x \in S} e(f(x), g(x)) = \underline{D(f, g)}$ exists, which is a <u>metric</u> on <u>$\ell_\infty(S, N)$</u>.

With (M, d) also a metric space, define

$$C_b(M, N) = \{f : M \to N \mid f \text{ bounded, cts}\}$$

which is a subspace of $l_\infty(M, N)$.

<u>Theorem 19</u> S a set, (M, d), (N, e) metric spaces with N complete. Then

(i) $l_\infty(S, N)$ is complete in the uniform metric

(ii) $C_b(M, N)$ is complete in the uniform metric.

<u>Proof</u> (i) Take Cauchy sequence (f_k) in $l_\infty(S, N)$. First show (f_k) pointwise convergent. Fix $x \in S$, $\varepsilon > 0$. Since (f_k) is Cauchy, $\exists K \in \mathbb{N}$ s.t. $D(f_i, f_j) < \varepsilon$ $\forall i, j \geq K$. So

$$e(f_i(x), f_j(x)) \leq D(f_i, f_j) < \varepsilon \qquad \forall i, j \geq K$$

so $(f_k(x))$ is Cauchy in $\underset{\text{complete}}{N}$, so convergent in N.

Then define $f : S \to N$ by $f(x) = \lim_{k \to \infty} f_k(x)$

Then show f is bounded.

Lemma 12: (f_k) is bounded in $l_\infty(S, N)$, so $\exists g \in l_\infty(S, N)$, $r > 0$ with $f_k \in B_r(g)$ $\forall k \in \mathbb{N}$.

And g is bounded so $\exists y \in N$, $s > 0$ with $g(x) \in B_s(y)$ for all $x \in S$. For any $x \in S$, $k \in \mathbb{N}$:

$$e(f_k(x), y) \leq e(f_k(x), g(x)) + e(g(x), y) \leq D(f_k, g) + s$$
$$\leq r + s$$

Hence $f_k(x) \in B_{r+s}(y)$. By Lemma 2.10, the set $B_{r+s}(y)$ is closed in N. So $f(x) = \lim\limits_{k\to\infty} f_k(x) \in B_{r+s}(y)$ for all $x \in S$, so f is bounded : have $f \in L_\infty(S, N)$.

It remains to show that $f_k \to f$ in <u>uniform</u> metric in $L_\infty(S, N)$. Let $\varepsilon > 0$.

Since (f_k) is Cauchy, $\exists K \in \mathbb{N}$ s.t. $D(f_i, f_j) < \varepsilon$ $\forall i, j \geq K$.

Fix $x \in S$ and $i \geq K$. Then

$e(f_i(x), f_j(x)) \leq D(f_i, f_j) < \varepsilon \qquad \forall j \geq K$.

Since $f_j(x) \to f(x)$ and e is cts, we have $e(f_i(x), f(x)) \leq \varepsilon$ which holds for any $x \in S$.

So $D(f_i, f) \leq \varepsilon$ for any $i \geq K$.

So $f_i \to f$ in the <u>uniform</u> metric D. ✓

(ii) Since $L_\infty(M, N)$ is complete, by Prop 16 it's sufficient to prove $C_b(M, N)$ is closed in $L_\infty(M, N)$.

Let (f_k) be a sequence in $C_b(M, N)$ with $f \in L_\infty(M, N)$ and assume $f_k \to f$ in the uniform metric.

Need to show that $f \in C_b(M, N)$ i.e. f is cts.

Fix $a \in M$, $\varepsilon > 0$. Since $f_k \to f$, can choose a large $k \in \mathbb{N}$ s.t.

PTO

$$D(f_k, f) = \sup_{x \in M} e(f_k(x), f(x)) < \varepsilon$$

f cts at a : $\exists\, \delta > 0$ s.t.

$\forall x \in M$, $d(x, a) < \delta \Rightarrow e(f_k(x), f_k(a)) < \varepsilon$

So for any $x \in M$, if $d(x, a) < \delta$ then

$$e(f(x), f(a)) \leqslant e(f(x), f_k(x)) + e(f_k(x), f_k(a)) + e(f_k(a), f(a)) \leqslant \underline{3\varepsilon}$$

So f is continuous. □

<u>Definition</u> A map $f : M \to M'$ between metric spaces is a <u>contraction mapping</u> if

$\exists\, \lambda < 1$ s.t. $\forall x, y \in M$, $\underline{d'(f(x), f(y)) \leqslant \lambda d(x, y)}$

i.e. a Lipschitz map with constant $\lambda < 1$.

So contraction $\Rightarrow$ <u>cts</u>.

<u>Theorem 20</u> (Contraction Mapping Theorem)

Let M be a non-empty complete metric space and let $f : M \to M$ be a contraction mapping. Then f has a <u>unique</u> fixed point $z \in M$ with $f(z) = z$.

<u>Proof</u> Fix $\lambda < 1$ s.t. $d(f(x), f(y)) \leqslant \lambda d(x, y)$ $\forall x, y \in M$.

<u>Uniqueness:</u> if $f(z) = z$, $f(w) = w$ then

$$d(z, w) = d(f(z), f(w)) \leqslant \lambda d(z, w)$$

Then since $\lambda < 1$, have $z = w$.

Existence: Fix arbitrary $x_0 \in M$ and recursively define $(x_n)_{n=1}^{\infty}$ by $x_n = f(x_{n-1})$.

For any $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq \lambda\, d(x_{n-1}, x_n)$$

Recursively $d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)$

Use Δ ineq: for any $m, n \in \mathbb{N}$, $n \geq m$ we have

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \leq \sum_{k=m}^{n-1} \lambda^k d(x_0, x_1)$$

$$= \frac{\lambda^m - \lambda^n}{1-\lambda} d(x_0, x_1) \leq \underline{\frac{\lambda^m}{1-\lambda} d(x_0, x_1)}$$

As $m \to \infty$, $\frac{\lambda^m}{1-\lambda} \to 0$ so given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $\frac{\lambda^m}{1-\lambda} d(x_0, x_1) < \varepsilon$ $\forall m \geq N$, then by above

$\underline{d(x_m, x_n) < \varepsilon}$ whenever $n \geq m \geq N$.

So (x_n) is Cauchy. As M is complete, $\underline{x_n \to z}$ for some $z \in M$.

Since f is continuous, $f(x_n) \to f(z)$ as $n \to \infty$.

But $f(x_n) = x_{n+1} \to z$ also, so $\underline{f(z) = z}$. $\square$

Remarks

1. Shows existence of fixed point <u>and</u> way to approximate. Letting $n \to \infty$ we get from the $d(x_m, x_n)$ ineq:

$$d(x_m, z) \leq \frac{\lambda^m}{1-\lambda} d(x_0, x_1) \qquad \forall m \in \mathbb{N}$$

which converges exponentially <u>fast</u>.

2. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$, $x \mapsto \frac{x}{2}$ is a contraction $(\lambda = \frac{1}{2})$ but has no fixed point so <u>completeness</u> is necessary.

3. $f : \mathbb{R} \to \mathbb{R}$ $\quad x \mapsto x+1$ is isometric $(\lambda = 1)$ with no fixed point : need $\lambda < 1$.

4. $f : [1, \infty) \to [1, \infty)$ $\quad x \mapsto x + \frac{1}{x}$ satisfies $|f(x) - f(y)| < |x - y| \quad \forall x, y \in [1, \infty)$ but has no fixed point.

Application $y_0 \in \mathbb{R}$ initial value problem $f'(t) = f(t^2)$ with $f(0) = y_0$ has a unique solution on $[0, \frac{1}{2}]$.

Step 1 f solves the problem iff

$$f \in C[0, \tfrac{1}{2}] \quad \text{and} \quad f(t) = y_0 + \int_0^t f(s^2)\, ds \quad \forall t \in [0, \tfrac{1}{2}]$$

Step 2 $M = C[0, \frac{1}{2}]$ is a nonempty, complete metric space in the uniform metric D.

Define $T : M \to M$ by

$$T(g)(t) = y_0 + \int_0^t g(s^2)\, ds$$

This is well-defined since $s \mapsto g(s^2)$ is continuous. $T(g)$ is continuous and differentiable with $(Tg)'(t) = g(t^2)$. Hence $Tg \in M$. By Step 1, f solves the problem iff $f \in M$ and $Tf = f$

Step 3 Use contraction mapping theorem. Check T is a contraction mapping: for $g, h \in M$ we estimate

$$|(Tg)(t) - (Th)(t)| = \left| \int_0^t g(s^2) - h(s^2)\, ds \right|$$

$$\leq t \sup_{s \in [0, \frac{1}{2}]} |g(s^2) - h(s^2)| \leq \frac{1}{2} D(g, h)$$

Taking sup over all $t \in [0, \frac{1}{2}]$: $D(Tg, Th) \leq \frac{1}{2} D(g, h)$ so T is a contraction mapping, with a unique fixed point satisfying $Tf = f$. So have a unique solution.

<u>Remark</u> Same proof works on $[0, \delta]$ with $\delta < 1$.
For $\delta < \mu < 1$, we have $f_\mu \restriction_{[0,\delta]} = f_\delta$ by uniqueness.
So there is a unique solution on $[0, 1)$.

<u>Theorem 21</u> (Lindelöf - Picard) Given $n \in \mathbb{N}$, $a, b, R \in \mathbb{R}$ with $a < b$, $R > 0$, $y_0 \in \mathbb{R}^n$ and a continuous function ϕ

$$\phi : [a, b] \times B_R(y_0) \to \mathbb{R}^n$$

Assume $\exists K > 0$ s.t. $\underline{\| \phi(t, x) - \phi(t, y) \| \leq K \| x - y \|}$ for all $t \in [a, b]$, $x, y \in B_R(y_0)$.

Then $\exists \varepsilon > 0$ s.t. for any $t_0 \in [a, b]$, the IVP

$$f'(t) = \phi(t, f(t)), \quad f(t_0) = y_0$$

has a <u>unique</u> solution on $[c, d] = [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a, b]$.

<u>Notes</u>

1. $f : [c, d] \to \mathbb{R}^n$ solving IVP requires condition that f takes values in $B_R(y_0)$

2. The assumption on ϕ is called a Lipschitz condition in the second variable.

3. Given a function $f : [c, d] \to \mathbb{R}^n$ we let
$f_k = q_k \circ f : [c, d] \to \mathbb{R}$ (q_k maps $(y_1, \dots, y_n) \mapsto y_k$)
f is differentiable if it is componentwise differentiable.

(Define $v = \int_c^d f(t)\,dt$).

Also if f is cts, then each f_k is cts $\Rightarrow$ integrable.

Define integral coordinate-wise: then observe that

$$\|v\|^2 = \sum_{k=1}^{n} v_k^2 = \sum_{k=1}^{n} v_k \int_c^d f_k(t)\,dt$$

$$= \int_c^d \sum_{k=1}^{n} v_k f_k(t)\,dt = \int_c^d v \cdot f(t)\,dt$$

$$\underset{\text{C-S}}{\leq} \int_c^d \|v\|\,\|f(t)\|\,dt = \|v\| \int_c^d \|f(t)\|\,dt$$

So $\|v\| = \left\| \int_c^d f(t)\,dt \right\| \leq \int_c^d \|f(t)\|\,dt$.

<u>Proof</u> of Thm 21

By Lemma 2.10, $B_R(y_0)$ is a closed subset of $\mathbb{R}^n$.

So ϕ is continuous on the closed, bounded subset $[a,b] \times B_R(y_0)$ of $\mathbb{R}^{n+1}$ so ϕ is <u>bounded</u>.

So can take $C = \sup\{\|\phi(t,x)\| : t \in [a,b],\ x \in B_R(y_0)\}$

Then set $\varepsilon = \min\left\{\frac{R}{C}, \frac{1}{2K}\right\}$. Will show this works.

Given $t_0 \in [a,b]$, set $[c,d] = [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a,b]$.

Need to prove $\exists!$ diffble function $f : [c,d] \to \mathbb{R}^n$ s.t. $f(t_0) = y_0$, $f'(t) = \phi(t, f(t))$ $\forall\, t \in [c,d]$.

U.M uniform metric

Since $B_R(y_0)$ is closed in $\mathbb{R}^n$ and $\mathbb{R}^n$ is complete, also $B_R(y_0)$ is <u>complete</u> (Prop. 16). By Thm 19, the space $M = C([c,d], B_R(y_0))$ is <u>complete</u> in U.M. <u>D</u>.

Also $M \neq \emptyset$.

Next note f solves IVP on $[c,d]$ <u>iff</u> (by FTC) coordinate wise

$f \in M$ <u>and</u> $\quad f(t) = y_0 + \int_{t_0}^{t} \phi(s, f(s))\, ds \quad \forall t \in [c,d]$.

Next define $T: M \to M$ by

$$T(g)(t) = y_0 + \int_{t_0}^{t} \phi(s, g(s))\, ds \qquad \text{for } t \in [c,d]$$

Note $T(g)(t)$ makes sense as $s \mapsto \phi(s, g(s))$ is cts, $T(g)$ is cts $[c,d] \to \mathbb{R}^n$ with $(Tg)'(t) = \phi(t, g(t))$ FTC

And $T(g)$ takes values in $B_R(y_0)$ since

$$\| T(g)(t) - y_0 \| = \left\| \int_{t_0}^{t} \phi(s, g(s))\, ds \right\| \leq |t - t_0| \sup_{s \in [c,d]} \|\phi(s, g(s))\|$$

$\leq \varepsilon C \leq \underline{R}$ as required, for all $t \in [c,d]$.

So $T(g) \in M$.

Now f solves the IVP iff $f \in M$ and $\underline{Tf = f}$.

So we <u>just</u> need T to be a contraction mapping.

Let $g, h \in M$. Note that, for all $s \in [c,d]$:

$$\| \phi(s, g(s)) - \phi(s, h(s)) \| \leqslant K \| g(s) - h(s) \| \leqslant K D(g,h)$$

So $\forall t \in [c,d]$, have

$$\| (Tg)(t) - (Th)(t) \| = \left\| \int_{t_0}^{t} \phi(s, g(s)) - \phi(s, h(s)) \, ds \right\|$$

$$\leqslant |t - t_0| K D(g,h) \leqslant \varepsilon K D(g,h)$$

Taking sup over $t \in [c,d]$, get

$$D(Tg, Th) \leqslant \varepsilon K D(h,g) \leqslant \underline{\tfrac{1}{2} D(g,h)} \quad \text{(by min. defn of } \varepsilon).$$

So T is a contraction mapping, so we are done. $\square$

__Remarks__

1. In general - not possible to get global soln on $[a,b]$
2. Thm 21 handles n^{th} order ODEs as well.

__Special case__

Given $n \in \mathbb{N}$, $a, b, R \in \mathbb{R}$, $a < b$, $R > 0$:

$z = (z_0, z_1, \ldots, z_{n-1}) \in \mathbb{R}^n$ and cts function

$\psi : [a,b] \times B_R(z) \to \mathbb{R}$, assume for some $K > 0$

that $|\psi(t,x) - \psi(t,y)| \leqslant K \|x - y\| \quad \forall t \in [a,b], \; x, y \in B_R(z)$.

Then $\exists\, \varepsilon > 0$ s.t. $\forall\, t_0 \in [a,b]$, the n^{th} order IVP $(*)$

$$g^{(n)}(t) = \psi(t, g(t), g^{(1)}(t), g^{(2)}(t), \ldots, g^{(n-1)}(t))$$

and $g^{(j)}(t_0) = z_j$ for $0 \leq j \leq n-1$

has a <u>unique</u> soln on $[c,d] = [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a,b]$.

(Implicitly, assuming $(g(t), g^{(1)}(t), \ldots, g^{(n-1)}(t)) \in B_R(z)$).

<u>Proof</u> Define $\phi : [a,b] \times B_R(z) \to \mathbb{R}^n$ by

$$\phi(t, x_0, x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, \psi(t, x_0, x_1, \ldots, x_{n-1}))$$

for $t \in [a,b]$ and $x = (x_0, x_1, \ldots, x_{n-1}) \in B_R(z)$

Then ϕ is cts with $\|\phi(t,x) - \phi(t,y)\| \leq (K+1)\|x-y\|$

$\forall\, t \in [a,b]$, $x, y \in B_R(z)$.

By Lindelöf - Picard, $\exists\, \varepsilon > 0$ s.t.

$$f'(t) = \phi(t, f(t)), \quad f(t_0) = z \qquad (\overset{*}{*})$$

has a <u>unique</u> soln f on $[c,d]$. Let $f_0, f_1, \ldots, f_{n-1}$ be components of f. Since f solves $(\overset{*}{*})$, each f_j is diff^{ble} and

$$(f_0'(t), \ldots, f_{n-1}'(t)) = f'(t) = \phi(t, f(t))$$
$$= (f_1(t), \ldots, f_{n-1}(t), \psi(t, f_0(t), \ldots, f_{n-1}(t))) \qquad (\dagger)$$

Compare coordinates in $(\dagger)$:

g is an n-times differentiable function $[c,d] \to \mathbb{R}$ with $\underline{g^{(j)} = f_j}$ for $0 \leq j < n$.

(This must hold since (†) specifies a solution to (*)).

We have also

$$g^{(n)}(t) = f_{n-1}'(t) = \Psi(t, f_0(t), f_1(t), \dots, f_{n-1}(t))$$
$$= \Psi(t, g(t), g^{(1)}(t), \dots, g^{(n-1)}(t)) \qquad \forall t \in [c,d].$$

Finally since $f(t_0) = z$, have $g^{(j)}(t_0) = f_j(t_0) = z_j$ for $0 \le j \le n-1$ so g solves (*) showing existence.

<u>Uniqueness</u>: if $\tilde{g}$ also solves (*) then we can check $\tilde{f} : [c,d] \to B_R(z)$, $\tilde{f}(t) = (\tilde{g}(t), \tilde{g}^{(1)}(t), \dots, \tilde{g}^{(n-1)}(t))$ solves $\binom{*}{*}$ whose soln is unique.

So $\tilde{f} = f \Rightarrow \tilde{g} = g$. $\square$

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4 Topological Spaces

Definition Let X be a set. A topology on X is a family τ of subsets of X such that

(i) $\emptyset, X \in \tau$

(ii) if $U_i \in \tau$ $\forall i \in I$ index set, then $\bigcup_{i \in I} U_i \in \tau$

(iii) If $U, V \in \tau$ then $U \cap V \in \tau$ (note: only finite intersection)

Members of τ are called open sets. (open in X) or τ-open

Definition A topological space is a pair (X, τ) where X is a set, and τ is a topology on X.

Examples

1. Metric topologies: (M, d) metric space. Recall $U \subset M$ is open in M if $\forall x \in U$, $\exists r > 0$ s.t. $B_r(x) \subset U$.
   Sometimes say d-open to emphasise metric openness
   Prop 2.9: Family of d-open sets is a topology on M (metric topology).

Definition A topological space is metrisable if there is a metric d on X s.t. τ is the metric topology induced on d.

Hence for $U \subset X$, have U τ-open $\Leftrightarrow$ U d-open.
Equivalent metrics induce the same topology.

<u>Examples</u> (cont.)

2. <u>Indiscrete topology</u> on X $\{\emptyset, X\}$: if $|X| \geq 2$ then this is <u>not</u> metrisable. To see this, take any metric d on X, fix $x \neq y$ in X, set $r = d(x,y)$ and note $U = D_r(x)$ is d-open (Lemma 2.5), non-empty (contains x) and $U \neq X$ (as $y \notin U$).

<u>Definition</u> τ_1, τ_2 topologies on X
Say τ_1 is <u>coarser</u> than τ_2 (or τ_2 is <u>finer</u> than τ_1) if $\tau_1 \subset \tau_2$ i.e. every τ_1-open set is τ_2-open.
Indiscrete topology is the <u>coarsest</u> topology.

3. Discrete topology is power set $P(X)$ of X
Metrisable by <u>discrete</u> metric. (Every set is open)

4. The <u>cofinite</u> topology on a set X is

$$\tau = \{\emptyset\} \cup \{U \subset X : U \text{ is cofinite in } X\}$$

Say $U \subset X$ is <u>cofinite</u> in X if <u>$X \setminus U$</u> is <u>finite</u>.
If X is finite, this is the discrete topology.
If X is infinite, cofinite topology is <u>not</u> metrisable: if $x \neq y$ in X and U, V are open sets with $x \in U$ and $y \in V$, then $U \cap V \neq \emptyset$.

Definition A topological space X is said to be Hausdorff if for all $x \neq y$ in X, there exist disjoint open sets U and V in X s.t. $x \in U$ and $y \in V$.

Note infinite set with cofinite topology is not Hausdorff.

Proposition 1 Every metric space is Hausdorff.

Proof Given $x \neq y$ in M, fix $r > 0$ with $2r < d(x,y)$. Then $U = D_r(x)$, $V = D_r(y)$ are open in M with $x \in U$, $y \in V$ and $U \cap V = \emptyset$ (else if $z \in U \cap V$ then
$d(x,y) \leqslant d(x,z) + d(z,y) < 2r < d(x,y)$) $\divideontimes$ $\square$

Definition A subset A of a topological space (X, τ) is closed in X if $X \setminus A$ is open in X.

Proposition 2 The collection of closed sets in a T.S. X satisfies

(i) $\emptyset$, X closed

(ii) If A_i $(i \in I)$ are closed then $\bigcap_{i \in I} A_i$ is closed

(iii) If A, B are closed then $A \cup B$ is closed.

Examples

1. In a discrete T.S., every subset is closed
2. In cofinite topology on X, A is closed iff $A = X$ or A is finite.

Definition X a topological space, $U \subset X$, $x \in X$

Say U is a **neighbourhood** of x in X if $\exists$ open subset V of X with $x \in V \subset U$.

Proposition 3 Let $U \subset X$. Then U is open in X iff U is a neighbourhood of x for all $x \in U$.

Proof If U is open and $x \in U$: then $V = U$ is open with $x \in V \subset U$. So U is a neighbourhood of x. ✓

Now if $\forall x \in U$, $\exists V_x$ in X with $x \in V_x \subset U$: then $U = \bigcup_{x \in U} V_x$ so U is open in X, proving the converse. ✓

$\square$

Definition (x_n) sequence in top. space X, let $x \in X$

We say $x_n \to x$ if for all neighbourhoods U of X, $\exists N \in \mathbb{N}$ s.t. $\forall n \geqslant N$, $x_n \in U$.

Or equivalently:

$\forall$ open sets U with $x \in U$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geqslant N$, $x_n \in U$.

Examples

1. Eventually constant sequences are convergent
2. In an indiscrete space, every sequence converges to every point

3. X set with cofinite topology. Assume $x_n \to x$ in X. For any $y \neq x$, the set $U = X \setminus \{y\}$ is a neighbourhood of x. So $N_y = \{n \in \mathbb{N} : x_n = y\}$ is finite. Conversely, if for some $x \in X$, the set N_y is finite for all $y \neq x$, then $x_n \to x$. So if N_y is finite for all $y \in X$, then $x_n \to x$ $\underline{\forall}\, x \in X$.

<u>Proposition 4</u> Limits are <u>unique</u> in a Hausdorff space.

<u>Proof</u> Assume $x, y \in X$, $x \neq y$. Then $\exists$ disjoint open sets U, V with $x \in U$, $y \in V$. Since $x_n \to x$, $\exists N_1$ s.t. $x_n \in U$ $\forall n \geqslant N_1$. Similarly since $x_n \to y$, $\exists N_2$ s.t. $x_n \in V$ $\forall n \geqslant N_2$. So for $n = \max\{N_1, N_2\}$, have $x_n \in U \cap V$ $\#$.

$\square$

<u>Remark</u> Recall that a subset A of a <u>metric</u> space X is closed in X <u>iff</u> whenever $(x_n) \to x$ for $(x_n) \in A$, $x \in X$, the limit x belongs to A.

The <u>forward</u> implication is also true in an <u>arbitrary</u> topological space.

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Definitions Let A be a subset of a topological space X. The interior of A in X $\text{int}(A)$ or A° is the set

$$\text{int}(A) = \bigcup \{ U \subset X : U \text{ is open in } X \text{ and } U \subset A \}$$

The closure of A in X, denoted $\text{Cl}(A)$ or $\bar{A}$ is

$$\text{Cl}(A) = \bar{A} = \bigcap \{ F \subset X : F \text{ closed in } X \text{ and } A \subset F \}$$

(Note X is closed in X and $A \subset X$).

Remarks

1. A° is open, $A^\circ \subset A$ and if U is open with $U \subset A$ then $U \subset A^\circ$ so A° is the largest open set in A. So A is open $\iff A = A^\circ$.

2. $\bar{A}$ is closed, $A \subset \bar{A}$ and if F is closed with $A \subset F$ then $\bar{A} \subset F$ so $\bar{A}$ is the smallest closed set containing A.
   So A is closed $\iff A = \bar{A}$.

Proposition 5 X topological space, $A \subset X$. Then

(a) $A^\circ = \{ x \in X : A \text{ is a neighbourhood of } x \}$

(b) $\bar{A} = \{ x \in X : \forall \text{ neighbourhoods } U \text{ of } x,\ U \cap A \neq \emptyset \}$

PTO

<u>Proof</u> (a) For $x \in X$ we have

$$x \in A^\circ \iff \exists \text{ open } U \text{ with } U \subset A \text{ and } x \in U$$
$$\iff A \text{ is a neighbourhood of } x \quad \checkmark$$

(b) If $x \notin \bar{A}$, then $U = X \setminus \bar{A}$ is open with $x \in U$ so U is a neighbourhood of x, and $U \cap A = \emptyset$.
Conversely, if there is a neighbourhood U of x with $U \cap A = \emptyset$, then there is an open set V with $x \in V \subset U$. Then $V \cap A = \emptyset$ so $A \subset X \setminus V$.
Since $X \setminus V$ is closed, $\bar{A} \subset X \setminus V$ so $x \notin \bar{A}$. $\square$

<u>Examples</u>

1. In $\mathbb{R}$, if $A = [0,1) \cup \{2\}$ then $A^\circ = (0,1)$ and $\bar{A} = [0,1] \cup \{2\}$.

2. $\mathbb{Q}^\circ = \emptyset$ and $\bar{\mathbb{Q}} = \mathbb{R}$

3. $\mathbb{Z}^\circ = \emptyset$ and $\bar{\mathbb{Z}} = \mathbb{Z}$

<u>Note</u> In a metric space, $x \in \bar{A} \iff \exists\, (x_n)$ in A s.t. $x_n \to x$
In a topological space, $\Leftarrow$ is true, but $\Rightarrow$ not true in general.

In <u>metric</u> spaces, convergent sequences determine the topology.

<u>Definitions</u> Let X be a topological space. A subset A of X is <u>dense</u> in X if $\underline{\bar{A} = X}$.

X is <u>separable</u> if there is a countable subset A of X that is dense in X.

<u>Examples</u> $\mathbb{R}$ is separable since $\mathbb{Q}$ is dense in $\mathbb{R}$. Similarly $\mathbb{R}^n$ is separable for any $n \in \mathbb{N}$. But an uncountable discrete topological space is <u>not</u> separable.

<u>Subspaces</u>

Let (X, τ) be a topological space, let $Y \subset X$. The <u>subspace topology</u> on Y is the topology $\{U \cap Y : U \in \tau\}$.

So for $U \subset Y$, have U open in Y iff $\exists$ open $V \subset X$ with $U = V \cap Y$.

<u>Example</u>

$X = \mathbb{R}$, $Y = [0,2]$, $U = (1,2]$. Then $U \subset Y \subset X$ and U is open in Y since $U = (1,3) \cap Y$. But U is <u>not</u> open in X.

<u>Remarks</u>

A subspace of a subspace is a subspace: X a T.S. with $Z \subset Y \subset X$. Then topology of Z induced by topology of X is the <u>same</u> as the topology of Z induced by topology of Y.

Let (M, d) be a metric space and $N \subset M$.
The metric d induces the metric topology of M. This induces the relative topology on N.
The restriction of d to N is a metric on N which induces metric topology on N. These two topologies are the <u>same</u>: as we have $D_r^N(x) = D_r^M(x) \cap N$ for $x \in N$, $r > 0$.

<u>Proposition 6</u> Let X be a topological space, $A \subset Y \subset X$.
(i) A closed in $Y \Leftrightarrow \exists$ closed set B in X with $A = B \cap Y$
(ii) Closure of A in Y is intersection with Y of the closure of A in X.

<u>Proof</u> (i) A closed in Y: $Y \setminus A$ open in Y, so $Y \setminus A$ is $U \cap Y$ for some U open in X. Then $A = (X \setminus U) \cap Y$ and $X \setminus U$ is closed in X $\Rightarrow$ ✓

<u>Conversely</u>, assume $A = B \cap Y$, B is closed in X
Then $Y \setminus A = (X \setminus B) \cap Y$ and $X \setminus B$ is open in X.
So $Y \setminus A$ is open in Y, so A is closed in Y.
(ii) Note: $\bar{A}^X \cap Y \supset A$, $\bar{A}^X \cap Y$ closed in Y by (i)
So $\bar{A}^X \cap Y \supset \bar{A}^Y$.
Conversely, $\bar{A}^Y$ closed in Y: by (i) $\exists$ closed $V \subset X$ s.t. $\bar{A}^Y = V \cap Y$. Then $V \supset A$ so $A \supset \bar{A}^X$ since V is closed in X.

So $\bar{A}^Y = V \cap Y \supset \bar{A}^X \cap Y$. $\square$

<u>Definitions</u>

Let (X, τ) be a topological space. Then a <u>base</u> for τ is a family $\tilde{B} \subset \tau$ s.t. $\forall\, U \in \tau$, $\exists\, C \subset \tilde{B}$ s.t. $U = \bigcup_{B \in C} B$.

<u>Note</u>: if $\tilde{B}$ is a base for τ then $\tau = \{ \bigcup_{B \in C} B : C \subset \tilde{B} \}$.

Also if $\tilde{B}$ is a base for τ then for every $U \subset X$:

$U \in \tau \iff \forall\, x \in U,\ \exists\, B \in \tilde{B}$ s.t. $x \in B \subset U$.

<u>Examples</u>

1. The family of <u>open balls</u> in a metric space is a base for the metric topology
2. In $\mathbb{R}$: family $\{ (a, b) : a < b \}$ is a base for the usual topology
3. In a discrete space X, any base contains $\{ \{x\} : x \in X \}$ which is a base for X.

<u>Lemma 7</u> Let X be a set with $\tilde{B} \subset \mathcal{P}(X)$.

If $X \in \tilde{B}$ <u>and</u> $\forall\, B_1, B_2 \in \tilde{B}$ we have $B_1 \cap B_2 \in \tilde{B}$, then there is a <u>unique</u> topology τ on X such that $\tilde{B}$ is a base for τ.

PTO

Proof If such topology τ exists, then by defn of base:

$\tau = \{ U \subset X : \forall x \in U, \ \exists B \in \tilde{\mathcal{B}} \text{ s.t. } x \in B \subset U \}$

If $U \in \tilde{\mathcal{B}}$ then for each $x \in U$, can take $B = U$ and get $x \in B \subset U$, so $\underline{\tilde{\mathcal{B}} \subset \tau}$. So enough to check that τ as defined above is a topology on X.

Now $\emptyset \in \tau$, $X \in \tau$ as $X \in \tilde{\mathcal{B}}$ by assumption

Let $U_i \in \tau$ for $i \in I$, let $x \in \bigcup_{i \in I} U_i$.

Then $x \in U_j$ for some $j \in I$ so $\exists B \in \tilde{\mathcal{B}}$ with $x \in B \subset U_j \subset \bigcup_{i \in I} U_i$. So $\bigcup_{i \in I} U_i \in \tau$.

Finally, let $U_1, U_2 \in \tau$, let $x \in U_1 \cap U_2$. Then for each $i = 1, 2$, since $x \in U_i \in \tau$, there exists $B_i \in \tilde{\mathcal{B}}$ with $x \in B_i \subset U_i$. So $x \in B_1 \cap B_2 \subset U_1 \cap U_2$. By assumption $B_1 \cap B_2 \in \tilde{\mathcal{B}}$ so $\underline{U_1 \cap U_2 \in \tau}$. $\square$

Remarks

The conditions in Lemma 7 are sufficient but not necessary

The following conditions are necessary and sufficient:

(i) $\bigcup_{B \in \tilde{\mathcal{B}}} B = X$

(ii) $\forall U, V \in \tilde{\mathcal{B}}, \ \forall x \in U \cap V, \ \exists W \in \tilde{\mathcal{B}}$ s.t. $x \in W \subset U \cap V$.

<u>Definition</u> A topological space (X, τ) is <u>second countable</u> if there is a countable base for τ.

<u>Example</u>

$\mathbb{R}$ is second countable: family of open intervals (a, b) with $a, b \in \mathbb{Q}$ is a countable base.

Similarly $\mathbb{R}^n$ is second countable.

Definition A function $f: X \to Y$ between topological spaces is *continuous* if $f^{-1}(V)$ is open in X for every open subset V of Y.

Note By Prop 2.8, this agrees with metric defn.

Examples

1. Constant functions are cts: if $f: X \to Y$ is constant then for any $V \subset Y$, have $f^{-1}(V)$ either X or $\emptyset$ depending if the constant value is in V or not.

2. Identity function $\mathrm{Id}: X \to X$ is cts

3. If Y is a subspace of X, then $\iota: Y \to X$ inclusion is cts. If V is open in X, then $\iota^{-1}(V) = V \cap Y$ which is open in Y by defn of subspace topology.
   So for any cts $f: X \to Z$, the restriction $f\restriction_Y : Y \to Z$ is also cts since $f\restriction_Y = f \circ \iota$. (See next proposition).

Proposition 8 $f: X \to Y$ between topological spaces

(i) f is continuous $\iff$ $f^{-1}(V)$ is closed in X for all closed sets V in Y

(ii) If $\tilde{\mathcal{B}}$ is a base for Y, then f is continuous $\iff$ $f^{-1}(B)$ is open in X for every $B \in \tilde{\mathcal{B}}$

(iii) If $g : Y \to Z$ is another continuous function between topological spaces, then $g \circ f : X \to Z$ is also continuous.

<u>Proof</u> (i) If f is cts, V is a closed subset of Y:
Then $Y \setminus V$ is open in Y so $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is open in X. So $f^{-1}(V)$ is closed in X. Reverse implication is similar.

(ii) Members of $\widetilde{\mathcal{B}}$ are open in Y by defn, so given condition is necessary. Conversely, assuming the condition, if V is open in Y then there is a subfamily $\mathcal{C} \subset \widetilde{\mathcal{B}}$ with $V = \bigcup_{B \in \mathcal{C}} B$. So $f^{-1}(V) = \bigcup_{B \in \mathcal{C}} f^{-1}(B)$ which is a union of open sets in X, so open in X.

(iii) Given an open set W in Z, have
$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. This is open in X, since $V = g^{-1}(W)$ is open in Y by continuity of g, so $f^{-1}(V)$ is open in X by continuity of f. $\square$

<u>Remarks</u>

1. There is a notion of continuity at a point. Let $f : X \to Y$ be a function between topological space, and let $x \in X$. We say f is continuous at x if for every neighbourhood V of $f(x)$ in Y, there is a neighbourhood U of x in X such that $f(U) \subset V$.

Equivalently, $f^{-1}(V)$ is a neighbourhood of x in X for every neighbourhood V of $f(x)$ in Y.

Then f is cts iff f is cts at every $x \in X$.

If f is cts at x and $x_n \to x$ in X, then $f(x_n) \to f(x)$.
The converse is <u>not</u> generally true.

<u>Definition</u> A function $f: X \to Y$ between topological spaces is a <u>homeomorphism</u> if f is a bijection and f, f^{-1} are continuous.

<u>Examples</u>

1. Being <u>metrisable</u> is a topological invariant.

Assume $f: (X, \underset{\substack{\uparrow \\ \text{metric}}}{d}) \to (Y, \underset{\substack{\uparrow \\ \text{topology}}}{\tau})$ is a HM.

Define $d'(y, z) = d(f^{-1}(y), f^{-1}(z))$ for $y, z \in Y$.
A routine check shows d' is a metric.

Now let $\underset{\substack{\uparrow \\ \text{(same in terms of sets)}}}{g} = f$ with $g: (X, d) \to (Y, d')$.

Note g is defined to be <u>isometric</u> so is a HM.

And $\mathrm{Id} = g \circ f^{-1}: (Y, \tau) \to (Y, d')$ is a HM.

So if $U \subset Y$, then U is <u>τ-open</u> <u>iff</u> U is <u>d'-open</u>.

$\square$

2. "Being a complete metric space" is not a topological invariant.

<u>Note</u> Let $f : X \to Y$ be a cts bijection between T.S. f need not be a homeomorphism.

f is a HM iff f^{-1} is cts i.e. $\forall$ open U in X, the set $(f^{-1})^{-1}(U) = f(U)$ is <u>open</u> in Y.

<u>Definition</u> A function $f : X \to Y$ between T.S. is <u>open</u> if $f(U)$ is open in Y for <u>every</u> open subset U of X.

<u>Remark</u> A homeomorphism is a <u>continuous</u> and <u>open</u> bijection.

<u>Product topology</u>

X, Y topological spaces. Consider the family

$$\tilde{B} = \{ U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y \}$$

of subsets of $X \times Y$.

Note that $X \times Y \in \tilde{B}$. Also for <u>open sets</u> U, U' in X and V, V' in Y, we have

$$U \times V \cap U' \times V' = (U \cap U') \times (V \cap V')$$ so $\tilde{B}$ is closed under taking intersections.

By Lemma 7, $\exists !$ topology on $X \times Y$ for which $\tilde{B}$ is a base, called the <u>product topology</u> on $X \times Y$.

<u>Note</u> For a subset $W \subset X \times Y$, we have

W is open $\iff$ $\forall z \in W$, $\exists$ open U in X, V in Y s.t.
$$z \in U \times V \subset W.$$

<u>Examples</u>

Let (M, d), (M', d') be metric spaces. Consider metric topology on M, M' which induces a product topology on $M \times M'$. Let $W \subset M \times M'$.

Then W is open in the product topology iff
$\forall (x, x') \in W$, have $(x, x') \in U \times U' \subset W$ for some open sets U in M, U' in M'
which is <u>equivalent</u> to
$\forall (x, x') \in W$, $(x, x') \in D_r(x) \times D_r(x') \subset W$ for some $r > 0$.

Observe that $D_r(x) \times D_r(x') = D_r(x, x')$ in $M \oplus_\infty M'$.
So W is open in the product topology iff W is d_∞-open.
So product topology on <u>product</u> of <u>metrisable spaces</u> is <u>metrisable</u>.

e.g. taking product of two copies of $\mathbb{R}$ with its usual topology, we deduce that the product topology on $\mathbb{R}^2$ is the standard Euclidean topology.

Proposition 9 X, Y topological spaces. The coordinate projections

$q_X : X \times Y \to X, \ (x, y) \mapsto x$ and

$q_Y : X \times Y \to Y, \ (x, y) \mapsto y$ satisfy

(i) q_X, q_Y are continuous

(ii) Given a topological space Z and a function $f : Z \to X \times Y$, we have

f is cts $\iff$ $q_X \circ f : Z \to X$ and $q_Y \circ f : Z \to Y$ are cts.

Proof (i) If U is an open subset of X, then $q_X^{-1}(U) = U \times Y$, open in $X \times Y$. Thus q_X is cts. (Similar argument for q_Y).

(ii) By (i), if f is cts then so are $q_X \circ f$, $q_Y \circ f$.

For converse, set $g = q_X \circ f$, $h = q_Y \circ f$ and assume g and h are cts.

Note that $f(z) = (g(z), h(z))$ for $z \in Z$.

Let W be a member of the defining base of the product topology on $X \times Y$, i.e. $W = U \times V$ where U is open in X, V is open in Y. Then for $z \in Z$, have $f(z) \in W$ iff $g(z) \in U$, $h(z) \in V$. So $f^{-1}(W) = g^{-1}(U) \cap h^{-1}(V)$, which is open in Z since g and h are cts.

Finally by Prop 8 (ii), f is continuous. $\square$

Remarks

1. All of above extends to any finite number of topological spaces. Given $n \in \mathbb{N}$, spaces $X_1, \dots, X_n$:

$$\tilde{B} = \{ U_1 \times \dots \times U_n : U_i \text{ is open in } X_i \text{ for } 1 \leq i \leq n \}$$

of subsets $X = X_1 \times \dots \times X_n$ is a base for a unique topology on X (product topology).

A subset $W \subset X$ is open in X iff

$\forall (x_1, \dots, x_n) \in W$, $\exists$ open $U_i \subset X_i$ s.t.

$(x_1, \dots, x_n) \in U_1 \times \dots \times U_n \subset W$.

As before, product of metrisable spaces is metrisable using any of the metrics d_1, d_2, d_∞ on the product.

The analogue of Prop. 9 holds.

Quotient topology

Let X be a topological space, R equivalence relation on X

For $x \in X$ write $q(x)$ for equivalence class:

$q(x) = \{ y \in X : y \sim x \}$

The set X/R of all E.Cs is the quotient set of X by R.

The map $q : X \to X/R$, $x \mapsto q(x)$ is the quotient map

Define quotient topology on X/R by

$$\{ V \subset X/R : q^{-1}(V) \text{ is open in } X \}$$

Check this is a topology:

(i) $q^{-1}(\emptyset) = \emptyset$ and $q^{-1}(X/R) = X$ so $\emptyset$, X/R are open

(ii) $q^{-1}\left(\bigcup_{i \in I} V_i \right) = \bigcup_{i \in I} q^{-1}(V_i)$ so union of open sets is open

(iii) $q^{-1}(U \cap V) = q^{-1}(U) \cap q^{-1}(V)$ so U, V open $\Rightarrow U \cap V$ open.

Remarks

1. Quotient map $q : X \to X/R$ is continuous: if V is open in X/R then by defn $q^{-1}(V)$ is open in X.

2. For $x \in X$, $t \in X/R$, have $x \in t \iff t = q(x)$ because ECs partition X. Hence for $V \subset X/R$:

$$q^{-1}(V) = \{ x \in X : \underbrace{q(x)}_{\text{this is just } q(x)} \in V \} = \{ x \in X : \exists t \in V \text{ s.t. } x \in t \}$$
$$= \bigcup \{ t : t \in V \}$$

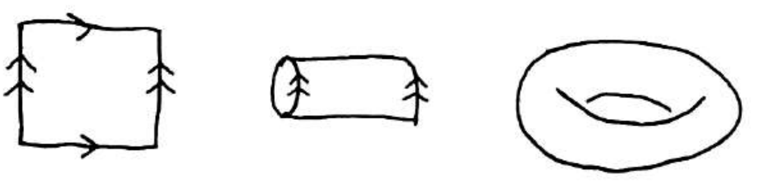
Examples

1. $\mathbb{R}$ in usual topology is an abelian group under addition. Then $\mathbb{Z} \leqslant \mathbb{R}$, can form quotient group $\mathbb{R}/\mathbb{Z}$ a quotient space with relation $x \sim y \iff x - y \in \mathbb{Z}$. Elements of $[0,1]$ represent different cosets except for $0 + \mathbb{Z} = 1 + \mathbb{Z}$. There are no other cosets so $\mathbb{R}/\mathbb{Z}$ is a circle i.e. $\mathbb{R}/\mathbb{Z}$ in quotient topology is homeomorphic to a circle (S^1).

2. Consider group $\mathbb{R}/\mathbb{Q}$ - note $q: \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ is a ~~homo~~ homomorphism. Assume that $V \subset \mathbb{R}/\mathbb{Q}$ is <u>open</u> and <u>nonempty</u>. Then $q^{-1}(V)$ is a nonempty open subset of $\mathbb{R}$ so $(a,b) \subset q^{-1}(V)$ for some $a<b$ in $\mathbb{R}$. Given $x \in \mathbb{R}$, $\exists\, r \in \mathbb{Q} \cap (a-x, b-x)$. So $x + r \in (a,b)$ so $q(x) = q(x+r) \in V$ and $x \in q^{-1}(V)$. We have shown $q^{-1}(V) = \mathbb{R}$ and thus $V = q(q^{-1}(V)) = \underline{\mathbb{R}/\mathbb{Q}}$. So the quotient topology is the <u>indiscrete</u> topology.
Hence a quotient of a <u>metrisable</u> space need <u>not</u> be <u>metrisable</u>.

3. Unit square $[0,1] \times [0,1]$ in $\mathbb{R}^2$ with this E.R:

$$(x,y) \sim (x',y') \iff \begin{cases} (x,y) = (x',y') & \text{or} \\ x = x',\ \{y,y'\} = \{0,1\} & \text{or} \\ \{x,x'\} = \{0,1\},\ y = y' \end{cases}$$



Recall from IA N&S:

Let X be a set, R an eq. rel. on X. Quotient map $q : X \to X/R$

Suppose Y is another set and $f : X \to Y$ respects R:

$\forall x, y \in X$, $\quad x \sim y \Rightarrow f(x) = f(y)$

Then there is a unique function $\tilde{f} : X/R \to Y$ s.t. $f = \tilde{f} \circ q$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\
& X/R \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

This diagram commutes

To show it commutes: given $z \in X/R$, write $z = q(x)$ for some $x \in X$, then set $\tilde{f}(z) = f(x)$.

Note: $\quad \text{im}(f) = \text{im}(\tilde{f})$ so f surjective $\Rightarrow \tilde{f}$ surjective

And if f fully respects R i.e. $\forall x, y \in X$,

$x \sim y \Leftrightarrow f(x) = f(y)$ then $\tilde{f}$ is injective.

Proposition 10 $\quad X, Y$ topological spaces, R an E.R on X, $q : X \to X/R$ quotient map. Let $f : X \to Y$ respect R. Then with $\tilde{f} : X/R \to Y$, $\tilde{f} \circ q = f$:

(i) If f is continuous then $\tilde{f}$ is continuous

(ii) If f is an open map then $\tilde{f}$ is an open map.

Proof $\quad$ Let V be an open subset of Y. Is $\tilde{f}^{-1}(V)$ open in X/R? We have $q^{-1}(\tilde{f}^{-1}(V)) = (\tilde{f} \circ q)^{-1}(V) = f^{-1}(V)$ which is open in X since f is continuous.

PTO

Hence $\tilde{f}^{-1}(V)$ is open in X/R so $\tilde{f}$ is continuous.

(ii) Let V be an open subset of X/R. Then $U = q^{-1}(V)$ is open in X and $q(U) = q(q^{-1}(V)) = V$. Hence $\tilde{f}(V) = \tilde{f}(q(U)) = f(U)$ which is open in Y. So $\tilde{f}$ is open. $\square$

<u>Example</u> $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1

Define $f : \mathbb{R} \to S^1$ by $f(t) = (\cos(2\pi t), \sin(2\pi t))$

Then f is continuous, surjective and for $s, t \in \mathbb{R}$, have $s - t \in \mathbb{Z} \iff f(s) = f(t)$ so by Prop 10, there is a unique map $\tilde{f} : \mathbb{R} \to \mathbb{Z} \to S^1$ with $f = \tilde{f} \circ q$ and $\tilde{f}$ is a <u>continuous bijection</u>.

Then check f is <u>open</u>. Let U be an open subset of $\mathbb{R}$. Suppose for * that $f(U)$ is not open in S^1. Then its complement is not closed, so it contains a sequence (z_n) that converges to some $z \in f(U)$. For each $n \in \mathbb{N}$, choose $x_n \in [0,1]$ with $z_n = f(x_n)$. Note $x_n \notin U$ since $z_n \notin f(U)$. Passing to a subseq, by B-W can assume $x_n \to x$ for some $x \in [0,1]$. By continuity $f(x_n) \to f(x)$ so $f(x) = z$. Since $z \in f(U)$ we have $z = f(y)$ for some <u>$y \in U$</u>. Then $k = y - x \in \mathbb{Z}$. Now $f(x_n + k) = f(x_n) = z_n \notin f(U)$ so $x_n + k \notin U$.

PTO

But $x_n + k \to x + k = y$. Since $\mathbb{R} \setminus U$ is closed, we have $\underline{y \notin U}$ ※

<u>Proposition 11</u> X topological space, R an E.R on X

(a) If X/R is Hausdorff then R is closed in $X \times X$

(b) If R is closed in $X \times X$ and $q : X \to X/R$ is open then X/R is Hausdorff.

<u>Proof</u> (a) Given $(x, y) \in X \times X \setminus R$, have $x \not\sim y$ so $q(x) \neq q(y)$. So there exist <u>disjoint</u> <u>open</u> sets S, T in X/R with $q(x) \in S$, $q(y) \in T$. Then $U = q^{-1}(S)$, $V = q^{-1}(T)$ are disjoint open sets in X with $x \in U$, $y \in V$. So for all $a \in U$, $b \in V$, have $q(a) \in S$, $q(b) \in T$ so $a \not\sim b$ i.e. $(a, b) \notin R$. So $(x, y) \in U \times V \subset X \times X \setminus R$. So R has open complement, so R is <u>closed</u>. ✓

(b) Given $z \neq w$ in X/R, choose $x, y \in X$ with $z = q(x)$, $w = q(y)$. Then $(x, y) \notin R$ i.e. the open set $X \times X \setminus R$ contains (x, y). So there exist open sets U, V in X s.t. $(x, y) \in U \times V \subset X \times X \setminus R$. Since q is an <u>open</u> map, $q(U), q(V)$ are open sets in X/R with $z = q(x) \in q(U)$, $w = q(y) \in q(V)$. Finally $q(U) \cap q(V) = \emptyset$ as otherwise, $\exists\, a \in U, b \in V$ with $q(a) = q(b)$ and thus $(a, b) \in R \cap U \times V$ contradicting choice of U, V. ✓ $\square$

5 Connectedness

Recall intermediate value theorem (IVT) from IA.

Note A subset I of $\mathbb{R}$ is an interval if the following holds: for all $x, y, z \in \mathbb{R}$, if $x < y < z$ and $x, z \in I$ then $y \in I$. So IVT says the continuous image of an interval is an interval. For what topological spaces does the IVT hold?

Example The function

$$f : X = [0,1) \cup (1,2] \to \mathbb{R} \qquad x \mapsto \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x \in (1,2] \end{cases}$$

is continuous but its image is not an interval.

Definition A topological space X is disconnected if there exist $U, V \subset X$ with

- $U \neq \emptyset$ and $V \neq \emptyset$
- U and V open in X
- $U \cap V = \emptyset$
- $U \cup V = X$

In this case, U and V disconnect X.

If X is not disconnected then X is connected.

"X is disconnected if it can be partitioned into two disjoint, nonempty, open sets"

Theorem 1 For a topological space X these are equivalent:

(i) X is connected

(ii) $f: X \to \mathbb{R}$ continuous $\Rightarrow$ $f(X)$ is an interval

(iii) $f: X \to \mathbb{Z}$ continuous $\Rightarrow$ f is constant

Proof (i) $\Rightarrow$ (ii): Suppose $f: X \to \mathbb{R}$ is cts but $f(X)$ is not an interval. So $\exists\, a < b < c$ in $\mathbb{R}$ with $a, c \in f(X)$, $b \notin f(X)$. Choose $x, z \in X$ with $f(x) = a$, $f(z) = c$. Take $U = f^{-1}(-\infty, b)$ and $V = f^{-1}(b, \infty)$ which are nonempty ($x \in U$, $z \in V$), open (as f is cts) and disjoint with $U \cup V = X$ since $b \notin f(X)$. So U and V disconnect X contradicting (i). ✓

(ii) $\Rightarrow$ (iii) Inclusion map $\iota: \mathbb{Z} \to \mathbb{R}$ is continuous. So if $f: X \to \mathbb{Z}$ is cts, then so is $g = \iota \circ f: X \to \mathbb{R}$. Then $g(X)$ is an interval by (ii) but $g(X) = f(X) \subset \mathbb{Z}$. So f must have been constant. ✓

(iii) $\Rightarrow$ (i) Assume U and V disconnect X. Define

$$f: X \to \mathbb{Z} \qquad x \mapsto \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in V \end{cases}$$

Then for any $A \subset \mathbb{Z}$, have $f^{-1}(A)$ one of $\emptyset, U, V, X$ depending if $A \cap \{0,1\}$ is $\emptyset$, $\{0\}$, $\{1\}$ or $\{0,1\}$. So f is continuous but not constant as U, V are nonempty. ✓ ※

□

<u>Corollary 2</u> Let $X \subset \mathbb{R}$. Then X is connected iff X is an interval.

<u>Proof</u> $\Rightarrow$: the inclusion map $\iota : X \to \mathbb{R}$ is cts so by Thm 1 (ii) its image X is an interval.

Alternatively if X is not an interval then there exist $a < b < c$ in $\mathbb{R}$ with $a, c \in X$, $b \notin X$. It follows that $U = (-\infty, b) \cap X$ and $V = (b, \infty) \cap X$ disconnect X.

$\Leftarrow$: by IVT, (ii) in Thm 1 holds, so X is connected. (See lecture slides for a direct proof). $\square$

<u>Examples</u>

1. Any indiscrete space is connected
2. The cofinite topology on any infinite set is connected
3. The discrete topology on a set X of size at least 2 is disconnected (by $\{x\}$ and $X \setminus \{x\}$ for any $x \in X$).

<u>Lemma 3</u> Let Y be a subspace of a topological space X. Then Y is disconnected iff there exist open subsets U and V of X with $U \cap Y \neq \emptyset$, $V \cap Y \neq \emptyset$, $U \cap V \cap Y = \emptyset$ and $Y \subset U \cup V$.

<u>Proof</u> $\Rightarrow$ Let $U', V' \subset Y$ disconnect Y. By defn of subspace topology, there are open sets U, V in X with $U' = U \cap Y$, $V' = V \cap Y$.

PTO

Then $U \cap Y = U' \neq \emptyset$, $V \cap Y = V' \neq \emptyset$,

$U \cap V \cap Y = U' \cap V' = \emptyset$ and $Y = U' \cup V' \subset U \cup V$. ✓

$\Leftarrow$ Now assume that U and V are open subsets of X with $U \cap Y \neq \emptyset$, $V \cap Y \neq \emptyset$, $U \cap V \cap Y = \emptyset$ and $Y \subset U \cup V$. Then $U' = U \cap Y$, $V' = V \cap Y$ <u>disconnect</u> Y. ✓

□

<u>Proposition 4</u> Let Y be a subspace of a topological space X. If Y is connected then its <u>closure</u> $\overline{Y}$ in X is connected.

<u>Proof</u> Assume $\overline{Y}$ is disconnected. Then by Lemma 3, there are open subsets U, V in X with $U \cap \overline{Y} \neq \emptyset$, $V \cap \overline{Y} \neq \emptyset$, $U \cap V \cap \overline{Y} = \emptyset$ and $\overline{Y} \subset U \cup V$.

Then $U \cap Y \cap V = \emptyset$ and $Y \subset U \cup V$. If we also have $U \cap Y \neq \emptyset$ and $V \cap Y \neq \emptyset$ then Y is disconnected by Lemma 3. <u>So</u> one of $U \cap Y$, $V \cap Y$ is empty.

WLOG take $U \cap Y = \emptyset$, so $Y \subset \underset{\uparrow \text{closed}}{X \setminus U}$.

So $\overline{Y} \subset X \setminus U$ contradicting $U \cap \overline{Y} \neq \emptyset$. * □

<u>Remarks</u>

1. Generally if Y is connected and $Y \subset Z \subset \overline{Y}$ then Z is connected. The closure of Y in Z is $\overline{Y} \cap Z$ (Prop 4.6).

2. Thm 1 (iii) can also be used to prove Prop 4.

__Theorem 5__ Let $f : X \to Y$ be __continuous__. Then if X is connected, then $f(X)$ is also connected.

__Proof__ Assume U, V are open subsets of Y that disconnect $f(X)$. Then $f^{-1}(U)$, $f^{-1}(V)$ are open in X since f is cts. Also $f^{-1}(U)$, $f^{-1}(V)$ are __nonempty__ as $U \cap f(X) \neq \emptyset$ and $V \cap f(X) \neq \emptyset$. But if $x \in f^{-1}(U) \cap f^{-1}(V)$ then $f(x) \in U \cap V \cap f(X) = \emptyset$

This contradiction shows that $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

Finally $f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = X$

since $U \cup V \supset f(X)$.

So $f^{-1}(U)$, $f^{-1}(V)$ __disconnect__ X ⋇ □

__Remarks__

1. Connectedness is a topological property (invariant)
2. If $f : X \to Y$ is cts and $A \subset X$ is a connected subset of X, then $f(A)$ is connected. Apply Thm 5 to $f\restriction_A : A \to Y$.

__Corollary 6__ The quotient space of a connected space is connected.

__Proof__ Exercise. □

<u>Example</u> Consider the subset $Y = \{(x, \sin\frac{1}{x}) : x > 0\}$ of $\mathbb{R}^2$. The function $f : (0, \infty) \to \mathbb{R}^2$ given by $x \mapsto (x, \sin\frac{1}{x})$ is continuous and hence its image Y is connected. By Prop 4, $\overline{Y}$ is also connected.

Note that
$$\overline{Y} = Y \cup \{(0, y) : -1 \leq y \leq 1\}$$

Indeed let $-1 \leq y \leq 1$. For $n \in \mathbb{N}$, the function $x \mapsto \frac{1}{x}$ maps $(0, \frac{1}{n})$ onto (n, ∞) and hence $\sin\left(\frac{1}{x_n}\right) = y$ for some $x_n \in (0, \frac{1}{n})$. Then $(x_n, \sin\frac{1}{x_n}) \to (0, y)$ so $(0, y) \in \overline{Y}$. This shows that $\widetilde{Y} := Y \cup \{(0, y) : -1 \leq y \leq 1\}$ is contained in $\overline{Y}$.

Conversely, $\overline{Y} \subset \widetilde{Y}$ follows if we show $\widetilde{Y}$ is closed. So assume $(x_n, y_n) \to (x, y)$ in $\mathbb{R}^2$ with $(x_n, y_n) \in \widetilde{Y}$ for all $n \in \mathbb{N}$. Since $x_n \geq 0$ and $-1 \leq y_n \leq 1$ for all n, we have $x \geq 0$, $-1 \leq y \leq 1$. So if $x = 0$, then $(x, y) \in \widetilde{Y}$. Otherwise $x > 0$, so $x_n > 0$ for all large n. Now if $x_n > 0$ then $y_n = \sin\frac{1}{x_n}$.

Thus $y_n \to \sin\frac{1}{x}$, $(x, y) = (x, \sin\frac{1}{x}) \in \widetilde{Y}$.

Lemma 7 Let X be a topological space and let $\tilde{A}$ be a family of connected subsets of X. Assume that $A \cap B \neq \emptyset$ for all $A, B \in \tilde{A}$. Then $\bigcup_{A \in \tilde{A}} A$ is connected.

Proof Set $Y = \bigcup_{A \in \tilde{A}} A$ and assume that $f: Y \to \mathbb{Z}$ is continuous. We show f must be constant. Then Thm 1 will show that Y is connected.

For each $A \in \tilde{A}$, the restriction $f\restriction_A : A \to \mathbb{Z}$ is cts, so constant by Thm 1. For any $A, B \in \tilde{A}$, $f\restriction_A$ and $f\restriction_B$ take the same constant value since $A \cap B \neq \emptyset$. So f is constant, so by Thm 1 Y is connected. $\square$

Theorem 8 Let X and Y be connected topological spaces. Then $X \times Y$ is connected in the product topology.

Proof WLOG, take $X, Y \neq \emptyset$. Fix $x_0 \in X$ and consider the map $f: Y \to X \times Y$, $y \mapsto (x_0, y)$. We show that f is continuous. Given open sets U in X and V in Y, if $x_0 \in U$ then $f^{-1}(U \times V) = V$ and if $x_0 \notin U$ then $f^{-1}(U \times V) = \emptyset$. By Prop 4.8 (ii), f is continuous.

Since Y is connected, f is cts, by Thm 5 the image of f i.e. $\{x_0\} \times Y$ is connected. Similarly for any $y_0 \in Y$, the subspace $X \times \{y_0\}$ of $X \times Y$ is connected.

Now assume for contradiction that U and V <u>disconnect</u> $X \times Y$. Fix $x_0 \in X$. Since $\{x_0\} \times Y$ is connected, U, V cannot disconnect $\{x_0\} \times Y$. Hence either U or V has empty intersection with $\{x_0\} \times Y$. WLOG suppose $\{x_0\} \times Y \subset U$. Similarly for any $y \in Y$, the set $X \times \{y\}$ is contained either in U or V and since $(x_0, y) \in U$, must have $X \times \{y\} \subset U$. So $X \times Y \subset U$ and $V = \emptyset$ ※ $\square$

(See lecture slides for another proof).

<u>Example</u> $\mathbb{R}^n$ is connected for any $n \in \mathbb{N}$

<u>Remarks</u> In the proof we showed that $y \mapsto (x, y)$ is a continuous map $Y \to X \times Y$ with image $\{x\} \times Y$. In fact this map is injective with inverse $\{x\} \times Y \to Y$ being the second coordinate projection, which is continuous. So $\{x\} \times Y$ is <u>homeomorphic</u> to Y for any $x \in X$.

2. Converse of Thm 8 holds if X and Y are nonempty: if $X \times Y$ is connected then so are X and Y (apply Thm 5 to coordinate projections).

<u>Components</u>

Define relation $\sim$ on a topological space:

$$\forall x, y \in X, \quad x \sim y \iff \exists \text{ connected subset } A \text{ of } X \text{ s.t. } x, y \in A$$

Check this is an equivalence relation: reflexivity and symmetry are immediate. Transitivity: if $x \sim y$ and $y \sim z$ then $\exists$ connected subsets A, B of X with $x, y \in A$, $y, z \in B$. Since $A \cap B \neq \emptyset$, $A \cup B$ is connected (Lemma 7) so $x \sim z$.

$\square$

The equivalence classes of form $C_x = \{y \in X : x \sim y\}$ are called the <u>connected components</u> of X.

<u>Proposition 9</u> Connected components of a topological space X are non-empty, maximal (wrt inclusion) connected subsets of X. Connected components are closed and partition X.

<u>Proof</u> Let C be a connected component of X. Then $C = C_x$ for some $x \in X$, <u>$C \neq \emptyset$</u> as $x \in C$.

If A is a connected subset of X with $x \in A$ then $y \sim x$ for all $y \in A$ and hence $A \subset C$. So if A is a connected subset of X with $A \supset C$ then <u>$A = C$</u>.

For each $y \in C$ there is a connected subset A_y of X with $x, y \in A_y$. Then $A_y \cap A_{y'} \neq \emptyset$ for all $y, y' \in C$ so $A = \bigcup_{y \in C} A_y$ is connected by Lemma 7 and $A \supset C$.

By above, $A = C$ so C is connected.

By Prop 4, the closure $\overline{C}$ of C in X is also connected.

By maximality of C, have $C = \overline{C}$ and C is closed.

Finally, X is the disjoint union of components since equivalence classes partition. $\square$

Definition Let X be a topological space. For $x, y \in X$, a path from x to y in X is a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.
We say X is path-connected if for all $x, y \in X$ there is a path from x to y in X.

Theorem 10 Every path-connected topological space is connected.

Proof Assume X is path-connected but not connected. Let U, V disconnect X. Fix $x \in U$, $y \in V$ and a cts function $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then can check $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ disconnect $[0,1]$ contradicting Corollary 2. $\square$

The converse is false: connected $\not\Rightarrow$ path-connected
See Slide 144 for example.

The set $\{(x, \sin\frac{1}{x}) : x > 0\} \cup \{(0, y) : -1 \leq y \leq 1\}$ is connected but not path-connected.

Lemma 11 Let X be a topological space, let A, B be closed subsets with $X = A \cup B$. If $f : X \to Y$ satisfies $f\restriction_A : A \to Y$ and $f\restriction_B : B \to Y$ continuous, then f is continuous. "Gluing lemma"

<u>Proof</u> Let V be a closed subset of Y. Since $f\restriction_A$ is continuous, it follows that $(f\restriction_A)^{-1}(V) = A \cap f^{-1}(V)$ is closed in A, so also closed in X (Prop 4.6).
Similarly $B \cap f^{-1}(V)$ is closed in X. So

$$f^{-1}(V) = (A \cap f^{-1}(V)) \cup (B \cap f^{-1}(V))$$

is also closed in X. Then by Prop 4.8 (i) f is <u>cts</u>. $\square$

<u>Path-connected components</u>

Let X be a topological space. Define relation on X:

$$x \sim y \iff \exists \text{ path in } X \text{ from } x \text{ to } y$$

This is <u>reflexive</u>: for $x \in X$ constant function with value x is continuous. ✓

<u>Symmetric</u>: if γ is a path from x to y then $t \mapsto \gamma(1-t)$ is a path from y to x ✓

<u>Transitive</u>: If γ is a path $x \to y$, δ is a path $y \to z$ then

$$\eta(t) = \begin{cases} \gamma(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \delta(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

is a path from x to z. Note it's well-defined at $\frac{1}{2}$ since $\gamma(1) = \delta(0) = y$. Also $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$ are closed in $[0,1]$ and $[0,1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$ and $\eta\restriction_{[0,\frac{1}{2}]}$, $\eta\restriction_{[\frac{1}{2},1]}$ are continuous. So by Lemma 11, η is continuous. ✓

So this is an equivalence relation.

Equivalence classes of this relation are path-connected components of X. From fact that $\sim$ is an E.R, have that path-connected components are path-connected.

Theorem 12 An open subset U of $\mathbb{R}^n$ is connected if and only if it is path-connected.

Proof $\Leftarrow$ true in general by Thm 10.

$\Rightarrow$ Fix $x_0 \in U$ and set $P = \{x \in U : \exists \text{ path } x_0 \text{ to } x\}$.
In other words P is the path-connected component of x_0 in U. We show that P is both open and closed in U.
Since P, $U \setminus P$ cannot disconnect U, one must be empty.
Since $x_0 \in P$, have $P = U$ so U is path-connected.

Fix $x \in P$. Since U is open, there is $r > 0$ with $D_r(x) \subset U$. Now for any $y \in D_r(x)$, there is a path from x to y inside $D_r(x)$ (and hence inside U).
So $D_r(x) \subset P$ so P is open in U.

Now fix $x \in U \setminus P$. Choose $r > 0$ with $D_r(x) \subset U$.
As before, for any $y \in D_r(x)$, have $y \sim x$. Hence if there exists $y \in D_r(x) \cap P$, then $y \sim x$ and $y \sim x_0$ which gives the contradiction that $x \in P$.
It follows that $D_r(x) \subset U \setminus P$ so $U \setminus P$ is open in U.
So P is closed in U.

$\square$

__Application__ For $n \geq 2$, $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.

__Proof__ Assume that $f : \mathbb{R} \to \mathbb{R}^n$ is a homeomorphism with inverse $g : \mathbb{R}^n \to \mathbb{R}$. Then $f\restriction_{\mathbb{R}\setminus\{0\}}$ is a continuous bijection from $\mathbb{R}\setminus\{0\}$ onto $\mathbb{R}^n\setminus\{f(0)\}$ whose inverse is $g\restriction_{\mathbb{R}^n\setminus\{f(0)\}}$ which is also continuous. Thus, $\mathbb{R}\setminus\{0\}$ and $\mathbb{R}^n\setminus\{f(0)\}$ are __homeomorphic__. Now the non-interval $\mathbb{R}\setminus\{0\}$ is __not__ connected. But $\mathbb{R}^n\setminus\{f(0)\}$ is path-connected, and hence __connected__. Connectedness is a topological property ⋇ $\square$

Analysis and Topology - Lecture 16 21/7/21

6 Compactness

__Question__ For which topological spaces X is it true that every continuous function $X \to \mathbb{R}$ is __bounded__?

__Some answers:__

1. For finite spaces

2. For spaces X with the following property:

For every cts function $f: X \to \mathbb{R}$, $\exists\, n \in \mathbb{N}$ and subsets $A_1, \dots, A_n$ of X with $X = \bigcup_{k=1}^{n} A_k$, f bounded on each A_k

__Note__ Let X be a topological space, $f: X \to \mathbb{R}$ cts

For $x \in X$, let $U_x = f^{-1}(f(x)-1, f(x)+1)$.

Then U_x is open, $x \in U_x$ and for every $y \in U_x$, we have

$$|f(y)| \leq |f(y)-f(x)| + |f(x)| \leq |f(x)|+1$$

So $X = \bigcup_{x \in X} U_x$ and f is bounded on each U_x.

If there exists a finite set $F \subset X$ with $X = \bigcup_{x \in F} U_x$ then f is bounded.

PTO

Definitions

Let X be a topological space. An open cover for X is a family $\tilde{U}$ of open subsets of X with $\bigcup_{U \in \tilde{U}} U = X$.

A subcover of an open cover $\tilde{U}$ for X is a subfamily $\tilde{V} \subset \tilde{U}$ with $\bigcup_{U \in \tilde{V}} U = X$.

A topological space is compact if every open cover for X has a finite subcover.

Theorem 1 Let X be a compact topological space and $f : X \to \mathbb{R}$ a continuous function. Then f is bounded, and moreover, if $X \neq \emptyset$, then f attains its bounds.

Proof Boundedness proof was essentially given in the Note on the previous page (see slide 150 for another proof).

Now let $m = \inf_X f$, $M = \sup_X f$. Suppose there is no $y \in X$ with $f(y) = m$. Then $f(y) > m$ $\forall y \in X$. Let $y \in X$. Choose $\alpha_y \in \mathbb{R}$ with $m < \alpha_y < f(y)$ and set $U_y = f^{-1}(\alpha_y, \infty)$. Then U_y is open, contains y and f is bounded below by α_y on U_y. In particular, $\{U_y : y \in X\}$ is an open cover for X. Hence there is a finite $F \subset X$ with $X = \bigcup_{y \in F} U_y$.

PTO

Let $\alpha = \min\{\alpha_y : y \in F\}$. Then $\alpha > m$ and f is bounded below by α on X. This contradiction also shows that $\underline{m = f(y)}$ for some $y \in X$.

Similarly, M is also attained. $\square$

"Compactness is the next best thing after finiteness"

<u>Lemma 2</u> Let Y be a subspace of a top. space X. Then Y is compact iff for any family $\tilde{U}$ of open subsets of X satisfying $Y \subset \bigcup_{U \in \tilde{U}} U$, there is a finite subfamily $\tilde{V} \subset \tilde{U}$ with $Y \subset \bigcup_{U \in \tilde{V}} U$.

<u>Proof</u> $\Rightarrow$ Let $\tilde{U}$ be a family of open subsets of X satisfying $Y \subset \bigcup_{U \in \tilde{U}} U$. Then $\{U \cap Y : U \in \tilde{U}\}$ is an open cover for Y. Since Y is compact, there is a finite subcover: there is a finite $\tilde{V} \subset \tilde{U}$ with $Y = \bigcup_{U \in \tilde{V}} U \cap Y$. So $Y \subset \bigcup_{U \in \tilde{V}} U$. ✓

$\Leftarrow$ Let $\tilde{W}$ be an open cover for Y. For each $W \in \tilde{W}$, fix open set $\tilde{\tilde{W}}$ in X s.t. $W = \tilde{\tilde{W}} \cap Y$. Then $\tilde{U} = \{\tilde{\tilde{W}} : W \in \tilde{W}\}$ is a family of open subsets of X s.t. $Y \subset \bigcup_{U \in \tilde{U}} U$. By assumption there is a <u>finite</u> subfamily of $\tilde{U}$ whose union still contains Y.

PTO

So there is a finite $\widetilde{V} \subset \widetilde{W}$ such that $\bigcup_{W \in \widetilde{V}} \widetilde{\widetilde{W}} \supset Y$.

So $\bigcup_{W \in \widetilde{V}} W = Y$ i.e. $\widetilde{V}$ is a finite subcover of $\widetilde{W}$. $\square$

<u>Theorem 3</u> The unit interval $[0,1]$ is compact.

<u>Proof</u> Let $\widetilde{U}$ be a family of open subsets of $\mathbb{R}$ such that $[0,1] \subset \bigcup_{U \in \widetilde{U}} U$.

For $I \subset [0,1]$, we say $\widetilde{U}$ finitely covers I if there is a finite subfamily $\widetilde{V} \subset \widetilde{U}$ satisfying $I \subset \bigcup_{U \in \widetilde{V}} U$.

We now make an observation. Suppose $I = J \cup K$ for subsets I, J, K of $[0,1]$. If $\widetilde{U}$ finitely covers J and K then $\widetilde{U}$ finitely covers I.

Now assume that $\widetilde{U}$ does not finitely cover $[0,1]$. Then at least one of the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ cannot be finitely covered by $\widetilde{V}$. Call that interval $[a_1, b_1]$. Then putting $c = \frac{1}{2}(a_1 + b_1)$, at least one of $[a_1, c]$ and $[c, b_1]$ cannot be finitely covered by $\widetilde{U}$. Call that interval $[a_2, b_2]$. Inductively, we obtain

$$[0,1] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots$$

s.t. for each $n \in \mathbb{N}$, $[a_n, b_n]$ cannot be covered finitely by $\widetilde{U}$, and $b_n - a_n = 2^{-n}$.

Since (a_n) is increasing, bounded above: it converges to some $x \in [0,1]$. Then $b_n = a_n + 2^{-n} \to x$ as well.

Since $[0,1] \subset \bigcup_{U \in \tilde{\mathcal{U}}} U$, there exists $U \in \tilde{\mathcal{U}}$ with $x \in U$. Since U is open, there exists $\varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset U$. Since $a_n, b_n \to x$, there exists $n \in \mathbb{N}$ with $a_n, b_n \in (x-\varepsilon, x+\varepsilon)$. It follows that $[a_n, b_n] \subset (x-\varepsilon, x+\varepsilon) \subset U$ contradicting that $[a_n, b_n]$ cannot be finitely covered by $\tilde{\mathcal{U}}$.

So $[0,1]$ is <u>compact</u> by Lemma 2. $\square$

<u>Examples</u>

1. Finite spaces are compact.
2. Any set X with cofinite topology is compact. Indeed, given an open cover $\tilde{\mathcal{U}}$ for X, fix a nonempty $U \in \tilde{\mathcal{U}}$. Then $F = X \setminus U$ is finite. For each $x \in F$, fix $U_x \in \tilde{\mathcal{U}}$ with $x \in U_x$. Then $\{U\} \cup \{U_x : x \in F\}$ is a finite subcover of $\tilde{\mathcal{U}}$.
3. If $x_n \to x$ in a T.S. X, then $Y = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ is compact. To see this, assume $\tilde{\mathcal{U}}$ is a family of open subsets of X covering Y. Then $x \in U$ for some $U \in \tilde{\mathcal{U}}$. Since $x_n \to x$, there exists $N \in \mathbb{N}$ with $x_n \in U$ for $n > N$. Now choose $U_k \in \tilde{\mathcal{U}}$ with $x_k \in U_k$ for $k = 1, \dots, N$. Then $\{U\} \cup \{U_k : 1 \leq k \leq N\}$ is a finite subcover.

4. An infinite set with the discrete topology is not compact. Indeed, open cover $\{ \{x\} : x \in X \}$ has no finite subcover.

5. $\mathbb{R}$ is not compact. Indeed, $\{ (-n, n) : n \in \mathbb{N} \}$ is an open cover without a finite subcover.

Proposition 4 Let Y be a subspace of a topological space X.

(a) If X is compact and Y is closed in X, then Y is compact.

(b) If X is Hausdorff and Y is compact, then Y is closed in X.

Proof (a) Let $\tilde{U}$ be a family of open subsets of X covering Y (i.e. $Y \subset \bigcup_{U \in \tilde{U}} U$). Then $\tilde{U} \cup \{ X \setminus Y \}$ is an open cover for X. Since X is compact, this has a finite subcover. So there is a finite subfamily $\tilde{V}$ of $\tilde{U}$ with $\tilde{V} \cup \{ X \setminus Y \}$ still an open cover for X. It follows that $\tilde{V}$ covers Y. By Lemma 2, it follows that Y is compact.

(b) Fix $x \in X \setminus Y$. For each $y \in Y$, since $x \neq y$, there exist some disjoint open sets U_y, V_y in X such that $x \in U_y$ and $y \in V_y$. Then $\{ V_y : y \in Y \}$ is a family of open sets in X covering Y. Since Y is compact, there is a finite $F \subset Y$ s.t. $Y \subset \bigcup_{y \in F} V_y$.

PTO

It follows that $U = \bigcap_{y \in F} U_y$ is an open set containing x and disjoint from Y. So $X \setminus Y$ is a neighbourhood of x, and thus $X \setminus Y$ is open $\Rightarrow$ Y is closed. $\square$

Proposition 5 Let $f: X \to Y$ be a continuous function between topological spaces. If X is compact, then $f(X)$ is compact.

Proof Let $\tilde{U}$ be a family of open subsets of Y covering $f(X)$. Now since f is continuous, $f^{-1}(U)$ is open in X for every $U \in \tilde{U}$ and moreover $X = f^{-1}\left(\bigcup_{U \in \tilde{U}} U\right) = \bigcup_{U \in \tilde{U}} f^{-1}(U)$. Thus we have that $\{f^{-1}(U) : U \in \tilde{U}\}$ is an open cover for X. Since X is compact, this has a finite subcover. So there is a finite subfamily $\tilde{V} \subset \tilde{U}$ such that $X = \bigcup_{U \in \tilde{V}} f^{-1}(U)$. It follows that $f(X) \subset \bigcup_{U \in \tilde{V}} U$. Now from Lemma 2 again, we deduce that $f(X)$ is compact. $\square$

Remarks

1. Compactness is a topological property.
2. If $f: X \to Y$ is continuous and A is a compact subset of X, then $f(A)$ is compact – simply apply Prop. 5 to the restriction $f\restriction_A : A \to Y$.

Example

For $a < b$ in $\mathbb{R}$, the unit interval $[0,1]$ is homeomorphic to the interval $[a,b]$ via the map $t \mapsto (1-t)a + tb$. Hence $[a,b]$ is compact.

Corollary 6 Let R be an equivalence relation on a compact topological space X. Then X/R is compact in the quotient topology. $\square$

Analysis and Topology - Lecture 17

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Theorem 7 (Topological inverse function theorem)

Let $f: X \to Y$ be a continuous bijection between topological spaces. If X is compact and Y is Hausdorff, then f is an open map, and hence a homeomorphism.

Proof Let U be an open subset of X. Set $K = X \setminus U$.

Since f is a bijection, it follows that $f(U) = Y \setminus f(K)$.

So it is enough to show that $f(K)$ is closed in Y.

Since K is a closed subset of the compact space X, it is compact (Prop 4(a)).

Since f is continuous, $f(K)$ is the continuous image of the compact set K, and hence compact (Prop 5).

Since $f(K)$ is a compact subspace of the Hausdorff space Y, it is closed (Prop 4(b)).

$\square$

Application The quotient space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the unit circle S^1.

Proof Recall that $f: \mathbb{R} \to S^1$, $f(t) = (\cos(2\pi t), \sin(2\pi t))$ fully respects the coset relation ($s \sim t$ i.e. $s - t \in \mathbb{Z}$ iff $f(s) = f(t)$), is cts and surjective. So there is a unique map $\tilde{f}: \mathbb{R}/\mathbb{Z} \to S^1$ s.t. $f = \tilde{f} \circ q$ where q is quotient map, and $\tilde{f}$ is a cts bijection (see Prop 4.10).

Note: for any $x \in \mathbb{R}$, we have $q(x) = q(x - \lfloor x \rfloor)$. Thus $\mathbb{R}/\mathbb{Z} = q(\mathbb{R}) = q([0,1])$. Since $[0,1]$ is compact (Theorem 3) and since q is continuous, it follows that $\mathbb{R}/\mathbb{Z}$ is compact (Prop 5). Also S^1 is a metric space and hence Hausdorff. So $\tilde{f}$ is a homeomorphism by Theorem 7. $\square$

Theorem 8 (Tychonov's theorem) The product of compact topological spaces is compact in the product topology.

Proof Let X, Y be compact topological spaces. We will show that $X \times Y$ is compact in the product topology. (Then by induction any finite product of compact spaces is compact).

Let $\tilde{W}$ be an open cover for $X \times Y$. Need to show that $\tilde{W}$ has a finite subcover. Define

$$\tilde{U} = \{ U \times V : U \text{ open in } X,\ V \text{ open in } Y,\ \exists W \in \tilde{W} \text{ s.t. } U \times V \subset W \}$$

Then note $\tilde{U}$ is an open cover for $X \times Y$. Indeed given $z \in X \times Y$, there exists $W \in \tilde{W}$ with $z \in W$, and by defn of product topology, there exist open sets U, V in X, Y respectively s.t. $z \in U \times V \subset W$.

Claim It is enough to show that $\tilde{U}$ has a finite subcover.

<u>Proof of claim</u> Assume that for some $n \in \mathbb{N}$, $\exists$ open sets $U_1, \ldots, U_n$ in X and open sets $V_1, \ldots, V_n$ in Y s.t. $U_i \times V_i \in \tilde{U}$ for $1 \leq i \leq n$ and $X \times Y = \bigcup_{i=1}^{n} U_i \times V_i$.

For each $i = 1, \ldots, n$ we can choose $W_i \in \tilde{W}$ with $U_i \times V_i \subset W_i$. Then $X \times Y = \bigcup_{i=1}^{n} W_i$ and thus $\{W_1, \ldots, W_n\}$ is a <u>finite subcover of $\tilde{W}$.</u> ✓

Now fix $x \in X$. From proof of Thm 5.8, we know $\{x\} \times Y$ is a <u>cts</u> image of Y so <u>compact</u> by Prop. 5. Since $\tilde{U}$ covers $\{x\} \times Y$, there is a finite subfamily of $\tilde{U}$ covering $\{x\} \times Y$ (Lemma 2). So there exist $n_x \in \mathbb{N}$ and open sets $U_{x,1}, U_{x,2}, \ldots, U_{x,n_x}$ in X and open sets $V_{x,1}, V_{x,2}, \ldots, V_{x,n_x}$ in Y s.t. $U_{x,i} \times V_{x,i} \in \tilde{U}$ for $1 \leq i \leq n_x$ and $\{x\} \times Y$ is a subset of $\bigcup_{i=1}^{n_x} U_{x,i} \times V_{x,i}$.

WLOG we may assume that $x \in U_{x,i}$ for all $i = 1, 2, \ldots, n_x$. (If $x \notin U_{x,i}$ then $U_{x,i} \times V_{x,i} \cap \{x\} \times Y = \emptyset$ and so $U_{x,i} \times V_{x,i}$ can be removed from the finite subcover).

So $\quad U_x = \bigcap_{i=1}^{n_x} U_{x,i}$ is an open set in X containing x.

Moreover, $\bigcup_{i=1}^{n_x} U_{x,i} \times V_{x,i} \supset U_x \times Y$.

(Indeed, given $z \in U_x$, $y \in Y$ we have $(x, y) \in U_{x,i} \times V_{x,i}$ for some $1 \leq i \leq n_x$ so $(z, y) \in U_{x,i} \times V_{x,i}$).

We carry out the above for each $x \in X$ and obtain the open cover $\{ U_x : x \in X \}$ for X. Since X is compact, there is a finite subset $F \subset X$ s.t. $X = \bigcup_{x \in F} U_x$.

It follows that

$$X \times Y = \left(\bigcup_{x \in F} U_x \right) \times Y \subset \bigcup_{x \in F} \bigcup_{i=1}^{n_x} (U_{x,i} \times V_{x,i})$$

and hence $\{ U_{x,i} \times V_{x,i} : x \in F, 1 \leq i \leq n_x \}$ is a finite subcover of $\tilde{U}$.

$\square$

Remark Converse of Thm 8 is true: if $X \times Y$ are compact and X, Y nonempty then X, Y are compact (cts images under coordinate projections q_X, q_Y).

Theorem 9 (Heine-Borel Theorem) A subset K of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof $\Rightarrow$ Function $\mathbb{R}^n \to \mathbb{R}$, $x \mapsto \|x\|$ is cts.
Hence image of the compact set K is bounded: $\exists M \geq 0$ s.t. $\|x\| \leq M$ $\forall x \in K$. So $K \subset B_M(0)$ and K is bounded.
As a compact subset of the Hausdorff space $\mathbb{R}^n$, the set K is closed in $\mathbb{R}^n$ (Prop 4(b)). ✓

$\Leftarrow$ Fix $M \geq 0$ s.t. $\|x\| \leq M$ $\forall x \in K$. Then $K \subset [-M, M]^n$. By Tychonov's theorem, $[-M,M]^n$ is compact in the product topology (recall this is the Euclidean topology of $\mathbb{R}^n$ restricted to $[-M,M]^n$).

So K is a closed subset of the compact space $[-M,M]^n$ and hence compact by Prop 4(a). $\square$

Another application of TIFT:

$Q = [0,1]^2$ (unit square), let R be the "torus" E.R (as defined on slide 124). Then the quotient space Q/R is homeomorphic to the torus in $\mathbb{R}^3$:

$$T^2 = \{((2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta) : \theta, \phi \in [0, 2\pi]\}$$

Note first: $\{(2+\cos\theta, 0, \sin\theta) : \theta \in [0, 2\pi]\}$ is the circle of radius 1, centre $(2,0,0)$ in xz-plane.

Applying the matrix $\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\phi \in [0, 2\pi]$

rotates it about the z-axis, sweeping out the torus surface.

Proof Define $f: Q \to T^2$,

$$f(s,t) = ((2+\cos 2\pi t)\cos 2\pi s, (2+\cos 2\pi t)\sin 2\pi s, \sin 2\pi t)$$

Can check f fully respects the relation R and it is cts and surjective.

PTO

So $\exists!$ map $\tilde{f} : Q/R \to T^2$ s.t. $f = \tilde{f} \circ q$, $\tilde{f}$ cts bijection

And by Heine-Borel, $Q = [0,1]^2$ is compact so the cts image Q/R ($q : Q \to Q/R$) is also compact. Also $T^2 \subset \mathbb{R}^3$ is Hausdorff, so by Thm 7 $\tilde{f}$ is a homeomorphism. $\square$

Definition Let f_n ($n \in \mathbb{N}$) and f be scalar functions on a topological space X. Then (f_n) converges to f locally uniformly if $\forall x \in X$, $\exists$ ngbd U of x s.t. $f_n \to f$ uniformly on U.

Remark If (f_n), f are on an open subset U of $\mathbb{R}^d$ then $f_n \to f$ locally uniformly on U iff $f_n \to f$ uniformly on every compact set $K \subset U$.

Proof $\Rightarrow$ $K \subset U$, K compact: for each $x \in K$ fix open ngbd U_x of x s.t. $U_x \subset U$ and $f_n \to f$ uniformly on U_x. Then $\{U_x : x \in K\}$ is a family of open subsets of U covering K. Since K is compact, $\exists$ finite subset $F \subset K$ s.t. $K \subset \bigcup_{x \in F} U_x$. It follows that $f_n \to f$ uniformly on K. ✓

$\Leftarrow$ Since U is open, for each $x \in U$ there exists $r > 0$ s.t. $B_r(x) \subset U$. The set $B_r(x)$ is closed and bounded so compact by Heine-Borel.

By assumption $f_n \to f$ uniformly on $B_r(x)$ which is a ngbd of x. ✓

$\square$

Analysis and Topology – Lecture 18 22/7/21

Sequential Compactness

A topological space is sequentially compact if every sequence in X has a convergent subsequence.

Examples

1. Every closed, bounded subset of $\mathbb{R}$ is sequentially compact (follows from Bolzano–Weierstrass).

2. Every closed, bounded subset X of $\mathbb{R}^n$ is sequentially compact. Indeed, given a sequence $\underline{x}_m$ in $\mathbb{R}^n$, write each term $\underline{x}_m = (x_{m,1}, x_{m,2}, \dots, x_{m,n})$.

Since X is bounded, $(x_{m,1})_{m \in \mathbb{N}}$ is bounded in $\mathbb{R}$ so by B–W $\exists$ infinite $M_1 \subset \mathbb{N}$ s.t. $(x_{m,1})_{m \in M_1}$ converges. Now similarly $(x_{m,2})_{m \in M_1}$ is bounded so can choose $M_2 \subset M_1$ s.t. $(x_{m,2})_{m \in M_2}$ converges, and so on.

Remark

It follows from the above and from Heine–Borel that a subset X of $\mathbb{R}^n$ is compact if and only if it is sequentially compact. This turns out to be true if X is any metric space.

PTO

<u>Definitions</u> Fix a metric space (M, d) for this chapter.

For $\varepsilon > 0$, a subset $F \subset M$ is called an <u>ε-net</u> for M if $\forall x \in M$, $\exists y \in F$ s.t. $d(y, x) \leq \varepsilon$ (or equivalently $M = \bigcup_{y \in F} B_\varepsilon(y)$).

We say M is <u>totally bounded</u> if $\forall \varepsilon > 0$, $\exists$ <u>finite</u> ε-net for M.

<u>Example</u>

Given $\varepsilon > 0$, choose $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. Then $\left\{ \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n} \right\}$ is a finite ε-net for $(0,1)$. So $(0,1)$ is totally bounded.

<u>Definition</u> For a non-empty subset $A \subset M$, define the <u>diameter</u> of A (diam A) by

$$\operatorname{diam} A = \sup \{ d(x,y) : x, y \in A \}.$$

<u>Lemma 10</u> Assume M is totally bounded, A is a nonempty closed subset of M. Then for any $\varepsilon > 0$, $\exists K \in \mathbb{N}$ and nonempty closed subsets $B_1, B_2, \ldots, B_K$ of A such that $A = \bigcup_{k=1}^{K} B_k$ and $\operatorname{diam} B_k \leq \varepsilon$ for $1 \leq k \leq K$.

PTO

Proof Let $F \subset M$ be a finite $\frac{\varepsilon}{2}$-net for M. Then $M = \bigcup_{x \in F} B_{\varepsilon/2}(x)$ and hence $A = \bigcup_{x \in F} (A \cap B_{\varepsilon/2}(x))$.

For $x \in F$ set $B_x = A \cap B_{\varepsilon/2}(x)$ and let $G = \{x \in F : B_x \neq \emptyset\}$. Then $A = \bigcup_{x \in G} B_x$ and for each $x \in G$, the set B_x is nonempty, closed and $\operatorname{diam}(B_x) \leq \operatorname{diam} B_{\varepsilon/2}(x) \leq \varepsilon$. $\square$

Theorem 11 For a metric (M, d) the following are equivalent:

(a) M is compact

(b) M is sequentially compact

(c) M is complete and totally bounded

Proof (a) $\Rightarrow$ (b): Let (x_n) be a sequence in M. For each $n \in \mathbb{N}$ set $T_n = \{x_k : k > n\}$. (Note that if a subseq. of (x_n) converges to some $x \in M$ then $x \in \bigcap_{n \in \mathbb{N}} \mathrm{Cl}(T_n)$).

We first show that $\bigcap_{n \in \mathbb{N}} \mathrm{Cl}(T_n)$ is nonempty. Assume not. Then $\bigcup_{n \in \mathbb{N}} M \setminus \mathrm{Cl}(T_n) = M$ so $\{M \setminus \mathrm{Cl}(T_n) : n \in \mathbb{N}\}$ is an open cover for M. Since M is compact, there is a finite subcover. Since $\mathrm{Cl}(T_m) \supset \mathrm{Cl}(T_n)$ $\forall\, m \leq n$, we have $M = M \setminus \mathrm{Cl}(T_n)$ for some $n \in \mathbb{N}$: absurd. ※

Now fix $x \in \bigcap_{n \in \mathbb{N}} Cl(T_n)$. We show that some subseq of (x_n) converges to x.

Since $x \in Cl(T_1)$, have $D_1(x) \cap T_1 \neq \emptyset$ so $\exists k_1 > 1$ s.t. $x_{k_1} \in D_1(x)$. Since $x \in Cl(T_{k_1})$ we have $D_{1/2}(x) \cap T_{k_1} \neq \emptyset$ so $\exists k_2 > k_1$ s.t. $x_{k_2} \in D_{1/2}(x)$

In general, assume we have found $k_1 < k_2 < \dots < k_n$ s.t. $x_{k_j} \in D_{1/j}(x)$ for $j = 1, 2, \dots, n$.

Since $x \in Cl(T_{k_n})$ have $D_{1/n+1}(x) \cap T_{k_n} \neq \emptyset$, so $\exists k_{n+1} > k_n$ s.t. $x_{k_{n+1}} \in D_{1/n+1}(x)$. By induction, we have constructed a subsequence (x_{k_n}) of (x_n) s.t. $\underline{d(x_{k_n}, x) < \frac{1}{n}}$ $\forall n$. So $\underline{x_{k_n} \to x}$. ✓

(b) $\Rightarrow$ (c) We first show that M is complete. Let (x_n) be a Cauchy sequence in M. Since M is sequentially compact, there is a subsequence (x_{k_n}) of (x_n) that converges to some $x \in M$. We show that $x_n \to x$.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ s.t. $d(x_m, x_n) < \varepsilon$ for all $m, n \geqslant N$. Since $x_{k_n} \to x$, we have $d(x_{k_n}, x) < \varepsilon$ for all sufficiently large n. In particular we can fix $n_0 \in \mathbb{N}$ s.t. $n_0 \geqslant N$ and $d(x_{k_{n_0}}, x) < \varepsilon$.

PTO

Then $k_{n_0} \geq n_0 \geq N$ and hence for all $n \geq N$, we have

$$d(x_n, x) \leq d(x_n, x_{k_{n_0}}) + d(x_{k_{n_0}}, x) < 2\varepsilon$$

We next show that M is <u>totally bounded</u>. If not, then for some $\varepsilon > 0$, there is no finite ε-net for M. Then we can construct a sequence (x_n) inductively as follows. Pick $x_1 \in M$ arbitrarily, and for $n \geq 2$, we choose $x_n \in M \setminus \bigcup_{k=1}^{n-1} B_\varepsilon(x_k)$ (nonempty, else $\{x_1, \ldots, x_{n-1}\}$ would be an ε-net for M). The sequence (x_n) satisfies $d(x_m, x_n) \geq \varepsilon \ \forall m \neq n$ so has no Cauchy subsequence, contradicting sequential compactness. ✓

(c) $\Rightarrow$ (a) Follow the proof of compactness of $[0,1]$.
Let $\tilde{U}$ be an open cover for M and assume M is <u>not</u> finitely covered by $\tilde{U}$. We inductively construct a <u>nested</u> sequence $A_0 \supset A_1 \supset A_2 \supset \ldots$ of non-empty closed subsets of M s.t. $\operatorname{diam}(A_n) \to 0$ and A_n cannot be finitely covered by $\tilde{U}$ for any n.

We let $A_0 = M$. Assume $n \geq 1$ and A_{n-1} has already been constructed. By Lemma 10, there exist $K \in \mathbb{N}$ and nonempty closed subsets $B_1, B_2, \ldots, B_K$ of A_{n-1} s.t. $A_{n-1} = \bigcup_{k=1}^{K} B_k$ and $\operatorname{diam} B_k \leq \frac{1}{n}$ for $1 \leq k \leq K$.

PTO

Since A_{n-1} cannot be finitely covered by $\tilde{U}$, there must exist k with $1 \leq k \leq K$ s.t. B_k cannot be finitely covered by $\tilde{U}$. We then set $A_n = B_k$. This completes the inductive construction.

For each $n \in \mathbb{N}$, choose $x_n \in A_n$. For all $N \in \mathbb{N}$, $m, n \geq N$ we have $x_m, x_n \in A_N$ so $d(x_m, x_n) \leq \operatorname{diam}(A_N)$. So (x_n) is Cauchy, and M is complete so $x_n \to x \in M$. Since $x_m \in A_n$ for $m \geq n$, and A_n is closed, have $x \in A_n$ for all $n \in \mathbb{N}$. Since $\tilde{U}$ is an open cover for M, there exists $U \in \tilde{U}$ with $x \in U$. As U is open, there is an $r > 0$ s.t. $D_r(x) \subset U$. Choose $n \in \mathbb{N}$ with $\operatorname{diam} A_n < r$. Then for all $y \in A_n$, we have $d(y, x) < r$ and thus $\underline{A_n \subset D_r(x) \subset U}$. This <u>contradicts</u> the assumption that A_n cannot be finitely covered by $\tilde{U}$. $\square$

<u>Remarks</u>

1. In $\mathbb{R}^n$, Heine-Borel and Bolzano-Weierstrass can be deduced from each other.

2. We now have another proof that the product of compact metric spaces is compact in the product topology. (Replace compactness with sequential compactness).

3. There are topological spaces that are compact but not sequentially compact, and topological spaces that are sequentially compact but not compact.

7 Differentiation and the Inverse Function Theorem

We write $L(\mathbb{R}^m, \mathbb{R}^n) = \{T : \mathbb{R}^m \to \mathbb{R}^n : T \text{ linear}\}$

$$\cong \underset{\uparrow}{M_{n,m}} \cong \mathbb{R}^{mn}$$

$n \times m$ real matrices

$\underline{e}_i$ standard basis for $\mathbb{R}^m$, $\underline{e}_i'$ for $\mathbb{R}^n$

For $\underline{x} = \sum_{j=1}^{n} x_j \underline{e}_j'$, $\underline{y} = \sum_{j=1}^{n} y_j \underline{e}_j'$, $\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j$

Cauchy-Schwarz: $|\langle x, y \rangle| \leq \|x\| \|y\|$

Linear map $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ is identified with the $n \times m$ matrix $(T_{j,i})_{1 \leq j \leq n,\, 1 \leq i \leq m} \in M_{n,m}$

where $T_{j,i} = \langle T\underline{e}_i, \underline{e}_j' \rangle$

We can arrange the entries $T_{j,i}$ into a column vector of size mn in $\mathbb{R}^{mn}$ so can view $L(\mathbb{R}^m, \mathbb{R}^n)$ as a real mn-dimensional Euclidean space with Euclidean norm

$$\|T\| = \Big(\sum_{\substack{1 \leq j \leq n \\ 1 \leq i \leq m}} T_{j,i}^2 \Big)^{1/2} = \Big(\sum_{i=1}^{m} \|T\underline{e}_i\|^2 \Big)^{1/2}$$

Then $L(\mathbb{R}^m, \mathbb{R}^n)$ becomes a metric space with the Euclidean distance $d(S, T) = \|S - T\|$

Lemma 1

(a) For $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $x \in \mathbb{R}^m$, we have $\|Tx\| \leq \|T\|\|x\|$. It follows that T is Lipschitz and hence continuous.

(b) For $S \in L(\mathbb{R}^n, \mathbb{R}^p)$ and $T \in L(\mathbb{R}^m, \mathbb{R}^n)$, we have $\|ST\| \leq \|S\|\|T\|$.

Proof (a) Writing $x = \sum_{i=1}^m x_i e_i$, we have

or just write $|x_i|$

$$\|Tx\| = \left\| \sum_{i=1}^m x_i Te_i \right\| \overset{\Delta \text{ ineq}}{\leq} \sum_{i=1}^m \|x_i\| \|Te_i\|$$

$$\leq \left(\sum_{i=1}^m |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^m \|Te_i\|^2 \right)^{1/2} \quad \text{(by C-S)}$$

$$= \|T\| \|x\|$$

So for any $x, y \in \mathbb{R}^m$, we have

$$d(Tx, Ty) = \|Tx - Ty\| = \|T(x-y)\| \leq \|T\|\|x-y\|$$
$$= \underline{\|T\| \, d(x,y)}.$$

(b) Using (a) we obtain

$$\|ST\| = \left(\sum_{i=1}^m \|STe_i\|^2 \right)^{1/2} \leq \left(\sum_{i=1}^m \|S\|^2 \|Te_i\|^2 \right)^{1/2}$$

$$\leq \|S\| \|T\| \qquad \square$$

Let f be differentiable at a $(f: \mathbb{R} \to \mathbb{R},\ a \in \mathbb{R})$.

Define $\varepsilon: \mathbb{R} \to \mathbb{R}$, $\quad \varepsilon(h) = \begin{cases} \frac{f(a+h)-f(a)}{h} - f'(a) & (h \neq 0) \\ 0 & (h=0) \end{cases}$

Then $\varepsilon(h) \to 0$ as $h \to 0$ (ε is cts at 0) and

$$f(a+h) = f(a) + f'(a)h + \varepsilon(h)h \qquad \forall h \in \mathbb{R}.$$

In Analysis I we proved one half of the following equivalence:

- f is differentiable at a
- $\exists \lambda \in \mathbb{R}$, $\exists \varepsilon: \mathbb{R} \to \mathbb{R}$ with $\varepsilon(0) = 0$ and ε cts at 0 s.t.

$$f(a+h) = f(a) + \lambda h + \varepsilon(h)h \qquad \forall h \in \mathbb{R}.$$

So f is approximated by a <u>linear</u> function and the error of approximation is <u>$o(h)$</u>.

Compare to <u>continuity</u>. We have that f is cts at a iff there is a function $\eta: \mathbb{R} \to \mathbb{R}$ with $\eta(0) = 0$ and η cts at 0, s.t. $f(a+h) = f(a) + \eta(h) \quad \forall h \in \mathbb{R}$.

Here f is approximated by a constant function with error of approximation $o(1)$.

More generally: if f is <u>n-times</u> differentiable at a, then f is approximated by a polynomial of degree <u>n</u> and the error of approximation is <u>$o(h^n)$</u>.

<u>Definition</u> $f : \mathbb{R}^m \to \mathbb{R}^n$ and a point $a \in \mathbb{R}^m$

We say f is <u>differentiable</u> at a if there exists $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ and a function $\varepsilon : \mathbb{R}^m \to \mathbb{R}^n$ with $\varepsilon(0) = 0$ and ε continuous at 0 such that

$$f(a+h) = f(a) + T(h) + \|h\| \varepsilon(h) \qquad (\dagger)$$

for all $h \in \mathbb{R}^m$. So ε is given by

$$\varepsilon(h) = \begin{cases} \dfrac{f(a+h) - f(a) - T(h)}{\|h\|} & \text{if } h \neq 0 \\ 0 & \text{if } h = 0. \end{cases}$$

<u>Note:</u> if there exists $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ such that $\dfrac{f(a+h) - f(a) - T(h)}{\|h\|} \to 0$ as $h \to 0$, then ε above is cts at 0 and $(\dagger)$ holds $\forall\, h \in \mathbb{R}^m$. So f is diffble at a.

<u>Note:</u> $(\dagger)$ can be written $f(a+h) = f(a) + T(h) + o(\|h\|)$

<u>Note:</u> Suppose $S, T \in L(\mathbb{R}^m, \mathbb{R}^n)$ satisfy

$$\frac{f(a+h) - f(a) - S(h)}{\|h\|} \to 0, \quad \frac{f(a+h) - f(a) - T(h)}{\|h\|} \to 0 \text{ as } h \to 0$$

Then $\dfrac{S(h) - T(h)}{\|h\|} \to 0$ as $h \to 0$. So for any nonzero $x \in \mathbb{R}^m$, we have

$$\frac{S(x) - T(x)}{\|x\|} = \frac{S(x/k) - T(x/k)}{\|x/k\|} \to 0 \text{ as } k \to \infty$$

so $S(x) = T(x)$. Thus <u>$S = T$</u>. So the map is unique.

__Definition__ If f is differentiable at a then the unique linear map $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ satisfying $\frac{f(a+h)-f(a)-T(h)}{\|h\|} \to 0$ is called the __derivative__ of f at a, denoted $f'(a)$.

__Remark__ on the case $m=1$.

First consider $T \in L(\mathbb{R}, \mathbb{R}^n)$. Set $v = T(1)$. Then $T(h) = T(h \cdot 1) = h \cdot v$ for all $h \in \mathbb{R}$.

So $L(\mathbb{R}, \mathbb{R}^n) \cong \mathbb{R}^n$, isomorphism given by $T \mapsto T(1)$.

Now we are given a function $f : \mathbb{R} \to \mathbb{R}^n$ and a point $a \in \mathbb{R}$. Then f is differentiable at a iff $\exists\, v \in \mathbb{R}^n$ and $\varepsilon : \mathbb{R} \to \mathbb{R}^n$, $\varepsilon(0) = 0$, ε cts at 0 s.t.

$f(a+h) = f(a) + hv + h\varepsilon(h)$ for all $h \in \mathbb{R}$.

This $v \in \mathbb{R}^n$ is then unique, $v = f'(a)$. Note that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{and so}$$

$$f(a+h) = f(a) + hf'(a) + o(h).$$

Conversely if the above limit exists, then letting $v \in \mathbb{R}^n$ be the limit, we have $\frac{f(a+h)-f(a)-hv}{h} \to 0$ as $h \to 0$.

It follows that f is differentiable at a.

Definitions A function $f: \mathbb{R}^m \to \mathbb{R}^n$ is <u>differentiable</u> on $\mathbb{R}^m$ if f is differentiable at a for every $a \in \mathbb{R}^m$. In this case, the derivative of f is the function

$$f': \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n), \quad a \mapsto f'(a).$$

Examples

1. Constant functions. Let $b \in \mathbb{R}^n$, $f: \mathbb{R}^m \to \mathbb{R}^n$ with $\underline{f(x) = b}$ $\forall x \in \mathbb{R}^m$. Then for any $a \in \mathbb{R}^m$, have

$$f(a+h) = b = f(a) + 0 + 0 \qquad \forall h \in \mathbb{R}^m.$$

So f is diff'ble at a with $\underline{f'(a) = 0}$. So f is diff'ble on $\mathbb{R}^m$.

2. Linear functions. Let $f \in L(\mathbb{R}^m, \mathbb{R}^n)$. Then for any $a \in \mathbb{R}^m$, we have $f(a+h) = f(a) + f(h) + 0$ $\forall h \in \mathbb{R}^m$. Thus f is diff'ble at a with $f'(a) = f$. Thus f is diff'ble on $\mathbb{R}^m$ with <u>constant</u> derivative $\mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n)$, value f.

3. Consider $f: \mathbb{R}^m \to \mathbb{R}$, $f(x) = \|x\|^2$. Fix $a \in \mathbb{R}^m$. Then we have

$$\begin{aligned} f(a+h) = \|a+h\|^2 &= \|a\|^2 + 2\langle h, a\rangle + \|h\|^2 \\ &= f(a) + 2\langle h, a\rangle + \|h\|^2 \end{aligned}$$

Since $\|h\|^2 = o(\|h\|)$ it follows that f is diff'ble at a with $f'(a)(h) = 2\langle h, a\rangle$.

4. Let M_n denote the space of $n \times n$ real matrices (can think of as $\mathbb{R}^{n^2}$). Consider $f : M_n \to M_n$, $f(A) = A^2$. For fixed $A \in M_n$, we have

$$f(A+H) = (A+H)^2 = f(A) + (AH + HA) + H^2$$

$\forall H \in M_n$.

Since $\|H^2\| \leq \|H\|^2$ (Lemma 1), it follows that f is <u>differentiable</u> at A with $f'(A)(H) = AH + HA$.

Examples (continued)

5. Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^p$ be a bilinear map. Fix $(a, b) \in \mathbb{R}^m \times \mathbb{R}^n$. Then

$$f((a,b) + (h,k)) = f(a+h, b+k) = f(a,b) + f(a,k) + f(h,b) + f(h,k)$$

Note that the map $T : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^p$ with $T(h,k) = f(a,k) + f(h,b)$ is linear. We show that $f(h,k) = o(\|(h,k)\|)$. Write $h = \sum_{i=1}^m h_i e_i$ and $k = \sum_{j=1}^n k_j e_j'$. We then have

$$\|f(h,k)\| = \left\| \sum_{i=1}^m \sum_{j=1}^n h_i k_j f(e_i, e_j') \right\|$$

$$\leq \sum_{i=1}^m \sum_{j=1}^n |h_i| |k_j| \, \|f(e_i, e_j')\| \leq C \|(h,k)\|^2$$

where $C = \sum_{i=1}^m \sum_{j=1}^n \|f(e_i, e_j')\|$ and we used

$|h_i| \leq \|(h,k)\|$, $|k_j| \leq \|(h,k)\|$ for all i, j.

So f is differentiable at (a, b) with derivative

$$f'(a,b)(h,k) = f(a,k) + f(h,b).$$

(Note $f' : \mathbb{R}^m \times \mathbb{R}^n \to L(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^p)$ is linear).

Definition Given open set $U \subset \mathbb{R}^m$, $f: U \to \mathbb{R}^n$, $a \in U$:
f is differentiable at a if there exists $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ and a function $\varepsilon : \{h \in \mathbb{R}^m \mid a+h \in U\} \to \mathbb{R}^n$ with $\varepsilon(0) = 0$ and ε continuous at 0 such that

$$f(a+h) = f(a) + T(h) + \|h\| \, \varepsilon(h) \qquad (+)$$

for all $h \in \mathbb{R}^m$ s.t. $a+h \in U$. It follows that

$$\varepsilon(h) = \begin{cases} \dfrac{f(a+h) - f(a) - T(h)}{\|h\|} & \text{if } h \neq 0 \text{ and } a+h \in U \\ 0 & \text{if } h = 0 \end{cases}$$

Since U is open, $D_r(a) \subset U$ for some $r > 0$ and hence ε is defined on $D_r(0)$.

Note f is diffble at a if and only if

$$\exists \, T \in L(\mathbb{R}^m, \mathbb{R}^n) \text{ s.t. } \frac{f(a+h) - f(a) - T(h)}{\|h\|} \to 0 \text{ as } h \to 0.$$

To see the "if" part, define ε as above and observe that $\varepsilon(0) = 0$, ε is cts at 0 and $(+)$ holds.

Proposition 2 We are given an open subset U of $\mathbb{R}^m$, a function $f : U \to \mathbb{R}^n$ and a point $a \in U$. If f is differentiable at a, then f is continuous at a.

Proof By assumption we have

$$f(a+h) = f(a) + f'(a)(h) + \|h\| \varepsilon(h)$$

where $\varepsilon(0) = 0$ and ε is continuous at 0. Given $x \in U$, putting $h = x - a$, we get

$$f(x) = f(a) + f'(a)(x-a) + \|x-a\| \varepsilon(x-a)$$

By Lemma 1 the linear map $f'(a)$ is continuous, and $\|\cdot\|$ is cts by Δ ineq and ε is cts at 0 by assumption. Using results about continuity of sums, products, composites, we deduce that f is cts at a. $\square$

Proposition 3 (Chain Rule) U open subset of $\mathbb{R}^m$, V open subset of $\mathbb{R}^n$. Given $f: U \to \mathbb{R}^n$ with $f(U) \subset V$ and $g: V \to \mathbb{R}^p$: let $a \in U$ and $b = f(a)$. If f is diff^{ble} at a and g is diff^{ble} at $b = f(a)$ then $\underline{g \circ f}$ is diff^{ble} at a, and moreover $(g \circ f)'(a) = g'(f(a)) \circ f'(a)$.

Proof Set $S = f'(a)$, $T = g'(b)$. Then

$$f(a+h) = f(a) + S(h) + \|h\| \varepsilon(h) \quad \text{and}$$

$$g(b+k) = g(b) + T(k) + \|k\| \zeta(k) \quad \text{for suitable}$$

error functions ε and ζ. Then

$$(g \circ f)(a+h) = g(f(a+h)) = g(f(a) + \underbrace{S(h) + \|h\| \varepsilon(h)}_{k = k(h)})$$

PTO

$$= g(f(a)) + T(S(h) + \|h\|\,\varepsilon(h)) + (\|k\|\,\zeta(k))$$

$$= (g\circ f)(a) + (T\circ S)(h) + \|h\|\,T(\varepsilon(h)) + \|k\|\,\zeta(k)$$

$$= (g\circ f)(a) + g'(f(a))\circ f'(a)\,(h) + \|h\|\,\eta(h)$$

where $\eta(h) = \begin{cases} T(\varepsilon(h)) + \dfrac{\|k\|}{\|h\|}\zeta(k) & \text{if } h \neq 0 \\ 0 & \text{if } h = 0. \end{cases}$

Need to show η is continuous at 0. Recall that $k = S(h) + \|h\|\,\varepsilon(h)$. Using Lemma 1, note that

$$\|k\| = \|S(h) + \|h\|\,\varepsilon(h)\| \leqslant \|S(h)\| + \|h\|\,\|\varepsilon(h)\|$$
$$\leqslant \|h\|\,(\|S\| + \|\varepsilon(h)\|)$$

Choose $\delta > 0$ s.t. if $\|h\| < \delta$ then $\|\varepsilon(h)\| < 1$ and hence $\dfrac{\|k\|}{\|h\|} < \|S\| + 1$.

Note that $k(0) = 0$ and k is cts at 0 by composition. Then $\zeta(k(0)) = 0$ and $\zeta(k)$ is also cts at 0.

Since T is cts everywhere, $T(\varepsilon)$ is cts at 0. So η is cts at 0. We spell this out:

Given $\theta > 0$, choose $\delta' > 0$ s.t. if $\|h\| < \delta'$ then $\|T(\varepsilon(h))\| < \theta$ and $\|\zeta(k)\| < \theta$. Hence if $\|h\| < \min\{\delta, \delta'\}$ then

$$\|\eta(h)\| \leqslant \|T(\varepsilon(h))\| + \frac{\|k\|}{\|h\|}\|\zeta(k)\| \leqslant (\|S\| + 2)\,\theta. \qquad \square$$

<u>Proposition 4</u> Given an open subset U of $\mathbb{R}^m$, a function $f: U \to \mathbb{R}^n$ and a point $a \in U$. For each $j = 1, \dots, n$ let $f_j : U \to \mathbb{R}$ be the j^{th} component of f. Then f is diff^{ble} at a $\iff$ each f_j is diff^{ble} at a in which case we have $f'(a)(h) = \sum_{j=1}^{n} f_j'(a)(h)\, e_j'$ for all $h \in \mathbb{R}^m$.

<u>Proof</u> Let $q_j : \mathbb{R}^n \to \mathbb{R}$ be the j^{th} coordinate projection: $q_j(y) = \langle y, e_j' \rangle$ for $y \in \mathbb{R}^n$. Then $f_j = q_j \circ f$ for each j, and so $f(x) = (f_1(x), \dots, f_n(x))$ for $x \in \mathbb{R}^m$.

$\Rightarrow$ If f is diff^{ble} at a, so is each f_j by chain rule:

$$f_j'(a) = q_j'(f(a)) \circ f'(a) = q_j \circ f'(a)$$

by linearity of q_j.

$\Leftarrow$ Conversely assume each f_j is diff^{ble} at a. Then for suitable functions ε_j, we have

$$f(a+h) = \sum_{j=1}^{n} f_j(a+h)\, e_j' = \sum_{j=1}^{n} \left(f_j(a) + f_j'(a)(h) + \|h\|\varepsilon_j(h)\right) e_j'$$

$$= f(a) + \sum_{j=1}^{n} f_j'(a)(h)\, e_j' + \|h\| \sum_{j=1}^{n} \varepsilon_j(h)\, e_j'$$

Since $h \mapsto \sum_{j=1}^{n} f_j'(a)(h)\, e_j'$ is linear and since $\varepsilon(h) = \sum_{j=1}^{n} \varepsilon_j(h)\, e_j'$ is <u>cts</u> at 0 with $\varepsilon(0) = 0$, the result follows. $\square$

Corollary 5 We are given an open subset U of $\mathbb{R}^m$, functions $f, g : U \to \mathbb{R}^n$ and $\lambda : U \to \mathbb{R}$ and a point $a \in U$. If f and g are differentiable at a, then so is $f+g$ with $(f+g)'(a) = f'(a) + g'(a)$. If f and λ are diff^{ble} at a, then so is λf with

$$(\lambda f)'(a)(h) = \lambda'(a)(h)\, f(a) + \lambda(a)\, f'(a)(h) \quad \forall h \in \mathbb{R}^m.$$

Proof We define functions

$$H : U \to \mathbb{R}^n \times \mathbb{R}^n, \quad H(x) = (f(x), g(x))$$
$$K : U \to \mathbb{R} \times \mathbb{R}^n, \quad K(x) = (\lambda(x), f(x))$$
$$A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad A(y,z) = y+z$$
$$S : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad S(t,y) = ty$$

Observe that $f+g = A \circ H$ and $\lambda f = S \circ K$.

It follows from Prop. 4 that H, K are diff^{ble} at a with

$$H'(a)(h) = (f'(a)(h), g'(a)(h))$$
$$K'(a)(h) = (\lambda'(a)(h), f'(a)(h))$$

As A is linear and S is bilinear, they are diff^{ble} with

$$A'(y,z) = A$$
$$S'(t,y)(u,h) = S(t,h) + S(u,y) = th + uy$$

$\forall\, y, z, h \in \mathbb{R}^n, \quad t, u \in \mathbb{R}$.

Proof cont.

Finally it follows by the Chain Rule that $f+g = A \circ H$ and $\lambda f = S \circ K$ are diff^ble at a with

$$(f+g)'(a)(h) = [A'(H(a)) \circ H'(a)](h)$$
$$= A'(H(a))[H'(a)(h)] = A[(f'(a)(h), g'(a)(h))]$$
$$= f'(a)(h) + g'(a)(h)$$

and similarly

$$(\lambda f)'(a)(h) = [S'(K(a)) \circ K'(a)](h)$$
$$= S'(K(a))[K'(a)(h)]$$
$$= S'(\lambda(a), f(a))[(\lambda'(a)(h), f'(a)(h))]$$
$$= \lambda(a) f'(a)(h) + \lambda'(a)(h) f(a) \qquad \square$$

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Alternative Proof of Corollary 5

We begin with λf. By defn of differentiability, we have

$$(\lambda f)(a+h) = \lambda(a+h) f(a+h)$$
$$= \left[\lambda(a) + \lambda'(a)(h) + \|h\|\varepsilon(h)\right]\left[f(a) + f'(a)(h) + \|h\|\zeta(h)\right]$$
$$= (\lambda f)(a) + \lambda(a) f'(a)(h) + \lambda'(a)(h) f(a) + \|h\|\eta(h)$$

where

$$\eta(h) = \begin{cases} \lambda(a)\zeta(h) + \dfrac{\lambda'(a)(h)}{\|h\|} f'(a)(h) + \lambda'(a)(h)\zeta(h) + \varepsilon(h)\left[f(a) + f'(a)(h) + \|h\|\zeta(h)\right] & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

By Lemma 1 we have $\|\lambda'(a)(h)\| \leq \|\lambda'(a)\| \, \|h\|$ and hence $\dfrac{\lambda'(a)(h)}{\|h\|}$ is bounded.

Then by composition, η is <u>cts</u> at 0.

A similar and simpler argument shows that $(f+g)'(a) = f'(a) + g'(a)$.

$\square$

Partial Derivatives

We are given an open set U of $\mathbb{R}^m$, a function $f : U \to \mathbb{R}^n$ and a point $a \in U$. We fix a direction u in $\mathbb{R}^m$, with $u \in \mathbb{R}^m$, $u \neq 0$. If the limit

$$\lim_{t \to 0} \frac{f(a+tu) - f(a)}{t}$$ exists, we call it $D_u f(a)$, the <u>directional</u> derivative of f in the direction u at a.

Note

1. $D_u f(a) \in \mathbb{R}^n$ and $f(a+tu) = f(a) + t D_u f(a) + o(t)$

2. Define $\gamma : \mathbb{R} \to \mathbb{R}^m$ by $\gamma(t) = a + tu$. Then $f \circ \gamma$ is defined on $\gamma^{-1}(U)$ which is open (since U is open and γ is continuous) and contains 0. Then

$$\frac{f(a+tu) - f(a)}{t} = \frac{(f\circ\gamma)(t) - (f\circ\gamma)(0)}{t}$$

and hence $D_u f(a)$ exists iff $f \circ \gamma$ is differentiable at 0, and in that case $D_u f(a) = (f\circ\gamma)'(0)$.

Special case: $u = e_i$ for some $1 \leqslant i \leqslant m$

When $D_{e_i} f(a)$ exists, it is called the i^{th} partial derivative of f at a, denoted $D_i f(a)$.

Proposition 6 Let U, f, a be as above. If f is differentiable at a, then $D_u f(a)$ exists for all $u \in \mathbb{R}^m \setminus \{0\}$ and moreover $D_u f(a) = f'(a)(u)$. It follows that

$$f'(a)(h) = \sum_{i=1}^{m} h_i D_i f(a) \qquad \forall h = \sum_{i=1}^{m} h_i e_i \in \mathbb{R}^m.$$

Proof Fix $u \in \mathbb{R}^m \setminus \{0\}$. By assumption, for suitable error function ε, we have

$$f(a+h) = f(a) + f'(a)(h) + \|h\| \varepsilon(h)$$

Put $h = tu$ and use linearity of $f'(a)$ to get

$$f(a+tu) = f(a) + t f'(a)(u) + |t| \|u\| \varepsilon(tu)$$

Hence

$$\frac{f(a+tu)-f(a)}{t} = f'(a)(u) + \frac{|t|}{t}\|u\|\varepsilon(tu) \to f'(a)(u)$$
as $t \to 0$

For the last part, observe

$$f'(a)(h) = \sum_{i=1}^{m} h_i f'(a)(e_i) = \sum_{i=1}^{m} h_i D_i f(a) \qquad \square$$

The Jacobian matrix

We are given an open set U of $\mathbb{R}^m$, function $f: U \to \mathbb{R}^n$ and a point $a \in U$. Assume f is differentiable at a. The Jacobian matrix of f at a, denoted $Jf(a)$, is the $n \times m$ matrix representing $f'(a)$ wrt the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$. Thus, for $1 \leq i \leq m$, the i^{th} column of $Jf(a)$ is $f'(a)(e_i) = D_i f(a)$
and for $1 \leq j \leq n$ the (j, i)-entry of $Jf(a)$ is

$$[Jf(a)]_{j,i} = \langle D_i f(a), e_j' \rangle = q_j(f'(a)(e_i)) = f_j'(a)(e_i)$$
$$= D_i f_j(a) = \frac{\partial f_j}{\partial x_i}(a)$$

Here $q_j : \mathbb{R}^n \to \mathbb{R}$ is the j^{th} coordinate projection given by $q_j(y) = \langle y, e_j' \rangle$.
Moreover, $f_j = q_j \circ f$ is the j^{th} component of f, and we are using Prop 4 and Prop 6.

Remarks

1. See slides for an alternative proof of Prop. 6.

2. If for some $u \in \mathbb{R}^m \setminus \{0\}$, $D_u f(a)$ exists, then $D_u f_j(a)$ exists for all $1 \leq j \leq n$. Indeed

$$\frac{f_j(a+tu) - f_j(a)}{t} = q_j\left(\frac{f(a+tu) - f(a)}{t}\right) \to q_j(D_u f(a))$$

by linearity / continuity of q_j. Note that we do not assume that f is differentiable at a.

3. Converse of Prop. 6 is false in general (see ES4).

<u>Theorem 7</u> Given an open set U of $\mathbb{R}^m$, a function $f: U \to \mathbb{R}^n$ and a point $a \in U$. Assume that there exists $r > 0$ such that $D_r(a) \subset U$ and for each $1 \leq i \leq m$, the partial derivative $D_i f(x)$ exists for all $x \in D_r(a)$ and the function $x \mapsto D_i f(x) : D_r(a) \to \mathbb{R}^n$ is continuous at a. Then f is differentiable at a.

<u>Proof</u> By Prop. 4 and by Remark 2, we can assume WLOG that $n = 1$. We will prove for $m = 2$ (general proof analogous).

Let $a = (p, q)$. If f is diff^ble^ at a, then

$$f'(a)(h, k) = h D_1 f(a) + k D_2 f(a).$$

So it is necessary (and sufficient) to show that

$$f((p,q) + (h,k)) = f(p,q) + h D_1 f(p,q) + k D_2 f(p,q) + o(\|(h,k)\|)$$

So, for all $(h,k) \in D_r(0)$, we have

$$f((p,q)+(h,k)) - f(p,q) - hD_1 f(p,q) - kD_2 f(p,q)$$
$$= f(p+h, q+k) - f(p+h, q) - kD_2 f(p,q)$$
$$+ f(p+h, q) - f(p,q) - hD_1 f(p,q)$$

The <u>second</u> expression $f(p+h,q) - f(p,q) - hD_1 f(p,q)$ is <u>$o(h)$</u> by definition of $D_1 f(p,q)$ and hence

$$f(p+h, q) - f(p,q) - hD_1 f(p,q) = o(\|(h,k)\|).$$

To deal with the <u>first</u> expression, fix $(h,k) \in D_r(0)$ and define

$$\phi : [0,1] \to \mathbb{R}, \quad \phi(t) = f(p+h, q+tk)$$

Note that ϕ is <u>cts</u> on $[0,1]$ and <u>diff'ble</u> on $(0,1)$.

Indeed for $t \in (0,1)$ have $\phi'(t) = kD_2 f(p+h, q+tk)$.

Then by <u>MVT</u>, for some $t = t(h,k) \in (0,1)$, have

$$f(p+h, q+k) - f(p+h, q) - kD_2 f(p,q)$$
$$= \phi(1) - \phi(0) - kD_2 f(p,q)$$
$$= \phi'(t) - kD_2 f(p,q)$$
$$= k\left[D_2 f(p+h, q+tk) - D_2 f(p,q)\right]$$

Since $D_2 f$ is <u>cts</u> at $a = (p,q)$ it follows that

$$\frac{|f(p+h,q+k) - f(p+h,q) - kD_2 f(p,q)|}{\|(h,k)\|} \leq |D_2 f(p+h, q+tk) - D_2 f(p,q)|$$
$$\to 0 \text{ as } (h,k) \to (0,0).$$

Remarks

1. For $D_1 f$, we only needed existence at a. For general m, it is enough to assume that for all but one value of i, the partial derivative $D_i f(x)$ exists for all x in a ngbd of a and it is continuous at a, whereas for the remaining value of i, it is enough to assume the existence of $D_i f$ at a.

2. The MVT works for functions defined on intervals with values in $\mathbb{R}$. There is also a version for functions with values in $\mathbb{R}^2$, but not with values in $\mathbb{R}^n$ for $n \geq 3$. We saw one remedy in the previous proof; we considered components f_j of f and restricted f_j to a line: for fixed $a \in U$ and for a fixed direction u, we considered $t \mapsto f_j(a+tu)$. We also have the following result (see L22).

Theorem 8 (Mean Value Inequality)

Let U be an open subset of $\mathbb{R}^m$, let $f: U \to \mathbb{R}^n$ be a function that is differentiable at every $z \in U$ and let $a, b \in U$. Assume that the line segment $[a,b]$ joining a, b is contained in U i.e.

$$[a,b] = \{(1-t)a + tb : 0 \leqslant t \leqslant 1\} \subset U$$

and that for some M, we have

$$\forall z \in [a,b], \quad \|f'(z)\| \leqslant M.$$

Then $$\|f(b) - f(a)\| \leqslant M \|b-a\|.$$

Proof Assume $a \neq b$. Set $u = b-a$ and $v = f(b) - f(a)$. For $t \in \mathbb{R}$ define $\gamma(t) = a + tu$. Then $f \circ \gamma$ is defined on the open set $\gamma^{-1}(U)$ which contains the closed interval $[0,1]$ since $\gamma([0,1]) = [a,b]$. On $\gamma^{-1}(U)$, by the Chain Rule we have

$$(f \circ \gamma)'(t) = f'(\gamma(t))(\gamma'(t)) = f'(a+tu)(u) = D_u f(a+tu)$$

For $\underline{t \in \gamma^{-1}(U)}$ define

$$\phi(t) = \langle f(a+tu), v \rangle = \langle (f \circ \gamma)(t), v \rangle.$$ Then

$$\phi(1) - \phi(0) = \langle f(b) - f(a), v \rangle = \|f(b) - f(a)\|^2$$

PTO

Moreover, since the map $\mathbb{R}^n \to \mathbb{R}$, $y \mapsto \langle y, v \rangle$ is linear, so diff'ble, it follows by the Chain Rule that ϕ is differentiable with

$$\phi'(t) = \langle (f \circ \gamma)'(t), v \rangle = \langle f'(a+tu)(u), v \rangle$$

Now by the MVT, there exists $\theta \in (0,1)$ s.t.

$$\begin{aligned} \phi(1) - \phi(0) &= \phi'(\theta) = \langle f'(a+\theta u)(u), v \rangle \\ &\leq \| f'(a+\theta u)(u) \| \, \|v\| \qquad (\text{C-S}) \\ &\leq \| f'(a+\theta u) \| \, \|u\| \, \|v\| \qquad (\text{Lemma } 1) \\ &\leq M \|b-a\| \, \|f(b) - f(a)\| \end{aligned}$$

Hence $\|f(b) - f(a)\|^2 \leq M \|b-a\| \, \|f(b) - f(a)\|$ and the result follows. $\square$

<u>Corollary 9</u> Let U be an open connected subset of $\mathbb{R}^m$ and $f : U \to \mathbb{R}^n$ be a function that is differentiable on U with $f'(a) = 0$ $\forall a \in U$. Then f is <u>constant</u> on U.

<u>Proof</u> For $a, b \in U$, if the line segment $[a,b] \subset U$, then $f(a) = f(b)$ by the MVI. Given $a \in U$, there exists $r > 0$ s.t. $D_r(a) \subset U$. Then for all $x \in D_r(a)$, we have $[a, x] \subset U$ and hence $\underline{f(a) = f(x)}$. So f is <u>locally</u> constant. Since U is connected, f is constant. $\square$

__Remark__ Suppose $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^n$ are open sets and $f : V \to W$ is a __bijection__. Then let $a \in V$ and assume that f is differentiable at a and f^{-1} is diff'ble at $f(a)$. Put $S = f'(a)$ and $T = (f^{-1})'(f(a))$. Then by chain rule:

$$TS = (f^{-1} \circ f)'(a) = I_m \, , \quad ST = (f \circ f^{-1})'(f(a)) = I_n$$

It follows that $m = \operatorname{tr}(TS) = \operatorname{tr}(ST) = n$ and hence $f'(a)$ is __invertible__.

__Definition__ U open, $U \subset \mathbb{R}^m$, $f : U \to \mathbb{R}^n$

Say f is a __C^1-function__ on U if it's __continuously diff'ble__ on U.

__Theorem 10__ (Inverse Function Theorem)

Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be a C^1-function on U. Let $a \in U$ and assume that $f'(a)$ is invertible. Then there are open subsets V, W of $\mathbb{R}^n$ such that $a \in V \subset U$ and $f\restriction_V : V \to W$ is a bijection whose inverse $g : W \to V$ is a C^1-function with $g'(y) = f'(g(y))^{-1}$ for every $y \in W$.

__Proof__ __STEP 1__ We show that we can assume $a = f(a) = 0$ and $f'(a) = I$.

To see this, let $T = f'(a)$ and define

$$h(x) = T^{-1}(f(x+a) - f(a)).$$

The domain of h is $U-a$ and h is a C^1-function with $h'(x) = T^{-1} \circ f'(x+a)$ by chain rule. Note that by Lemma 1 we have

$$\| h'(x) - h'(y) \| = \| T^{-1} \circ (f'(x+a) - f'(y+a)) \|$$
$$\leq \| T^{-1} \| \, \| f'(x+a) - f'(y+a) \|$$

Note also that $h(0) = 0$ and $h'(0) = T^{-1} \circ f'(a) = I$. Now if we can prove the result for h, then the result follows for f since $f(x) = T(h(x-a)) + f(a) \quad \forall x \in U$.

STEP 2 We now assume $f(0) = 0$ and $f'(0) = I$. Since f' is continuous, we choose $r > 0$ s.t. $B_r(0) \subset U$ and $\| f'(x) - I \| < \frac{1}{2}$ for all $x \in B_r(0)$. We show that for all $x, y \in B_r(0)$, we have

$$\| f(x) - f(y) \| \geq \frac{1}{2} \| x - y \|.$$

To see this, define $p : U \to \mathbb{R}^n$ by $p(x) = f(x) - x$. Then p is diff^{ble} with $p'(x) = f'(x) - I \quad \forall x \in U$.

It follows that $\| p'(x) \| < \frac{1}{2} \quad \forall x \in B_r(0)$.

Now given $x, y \in B_r(0)$, the line segment $[x,y] \subset B_r(0)$:

$$\| (1-t)x + ty \| \leq (1-t)\|x\| + t\|y\| \leq (1-t)r + tr = r$$

for any $0 \leq t \leq 1$. So by Thm 8 $\| p(x) - p(y) \| \leq \frac{1}{2} \| x - y \|$.

PTO

It follows that

$$\|f(x)-f(y)\| = \|p(x)+x-(p(y)+y)\| \geqslant \|x-y\| - \|p(x)-p(y)\|$$
$$\geqslant \frac{1}{2}\|x-y\| \quad \text{as required.}$$

<u>STEP 3</u> Put $s = \frac{r}{2}$. We show $f(D_r(0)) \supset D_s(0)$.

Fix $w \in D_s(0)$. For $x \in B_r(0)$, let

$q(x) = w - p(x) = w - f(x) + x$. Since

$p(0) = f(0) - 0 = f(0) = 0$ it follows that

$$\|q(x)\| \leqslant \|w\| + \|p(x)-p(0)\| \leqslant \|w\| + \frac{1}{2}\|x-0\| < 2s = r$$

Thus q maps $B_r(0)$ into $B_r(0)$. Next for $x, y \in B_r(0)$:

$$\|q(x)-q(y)\| = \|p(x)-p(y)\| \leqslant \frac{1}{2}\|x-y\|$$

Hence q is a contraction mapping on the non-empty complete metric space $B_r(0) \subset \mathbb{R}^n$. By the <u>CMT</u>, (Thm 3.20) q has a unique fixed point, so there is a unique $x \in B_r(0)$ s.t. $f(x) = w$. Note that $\|x\| = \|q(x)\| < r$ from above.

<u>STEP 4</u> Set $W = D_s(0)$ and $V = f^{-1}(D_s(0)) \cap D_r(0)$.

We show that V and W satisfy the conclusions of the theorem.

W is open and $f(0) \in W$. Since f is cts, V is open with $0 \in V \subset U$. From STEP 3 it follows that $f\restriction_V : V \to W$ is a bijection.

Let $g: W \to V$ be its inverse. Given $u, v \in W$, set $x = g(u)$, $y = g(v)$. It follows from Step 2 that

$$\| g(u) - g(v) \| = \| x - y \| \leq 2 \| f(x) - f(y) \| = 2 \| u - v \|$$

Thus g is a Lipschitz map, so g is <u>cts</u>. The proof of STEP 5 is non-examinable. As a preliminary of STEP 5, observe that for $x \in B_r(0)$, we have $\| f'(x) - I \| < \frac{1}{2}$ and hence

$$\| f'(x)(h) \| \geqslant \| h \| - \| h - f'(x)(h) \| \geqslant \frac{1}{2} \| h \|$$

It follows that $f'(x)$ is injective, so hence <u>invertible</u>.

<u>STEP 5</u> (proof non-examinable)

$g: W \to V$ is a C^1-function and $g'(y) = f'(g(y))^{-1}$ for every $y \in W$. Fix $y \in W$. Set $x = g(y)$ and $T = f'(x)$. Then for a suitable error function ε we have:

$$f(x+h) = f(x) + T(h) + \| h \| \varepsilon(h)$$

Choose $\delta > 0$ s.t. $D_\delta(y) \subset W$. For $k \in D_\delta(0)$, define $h = h(k) = g(y+k) - g(y)$. Then $g(y+k) = g(y) + h = x + h$ and hence $y + k = f(x+h)$ and $k = f(x+h) - f(x)$.

From above, it follows that

$$k = T(h) + \|h\|\,\varepsilon(h) \quad \text{so} \quad h = T^{-1}(k) - \|h\|\,T^{-1}(\varepsilon(h)).$$

We then obtain

$$g(y+k) = g(y) + h = g(y) + T^{-1}(k) - \|h\|\,T^{-1}(\varepsilon(h))$$

The composite $T^{-1} \circ \varepsilon \circ h$ is 0 at 0 and cts at 0, whereas

$$\|h\| = \|g(y+k) - g(y)\| \leqslant 2\|k\| \qquad \text{by STEP 4.}$$

Hence we get

$$g(y+k) = g(y) + T^{-1}(k) + o(\|k\|).$$

So g is diff'ble at y and

$$g'(y) = T^{-1} = f'(g(y))^{-1}. \qquad \square$$

Second derivative

We are given an open set $U \subset \mathbb{R}^m$, a function $f: U \to \mathbb{R}^n$ and a point $a \in U$. We say f is twice differentiable at a if f is differentiable on some open set V with $a \in V \subset U$ and $f': V \to L(\mathbb{R}^m, \mathbb{R}^n)$ is differentiable at a. We let $f''(a) = (f')'(a)$ and call $f''(a)$ the second derivative. (Note $f''(a) \in L(\mathbb{R}^m, L(\mathbb{R}^m, \mathbb{R}^n))$).

Second derivative as a bilinear map

Given $T \in L(\mathbb{R}^m, L(\mathbb{R}^m, \mathbb{R}^n))$ the map $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ given by $(h, k) \mapsto T(h)(k)$ is bilinear. Conversely, given a bilinear map $T: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$, for each $h \in \mathbb{R}^m$, the map $T(h): \mathbb{R}^m \to \mathbb{R}^n$ defined by $T(h)(k) = T(h, k)$ is linear, and moreover, the map $\mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n)$ given by $h \mapsto T(h)$ is linear. This proves that

$L(\mathbb{R}^m, L(\mathbb{R}^m, \mathbb{R}^n)) \cong Bil(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$ (all bilinear maps).

Under this identification, for a member T of either of these two spaces we write $T(h)(k) = T(h,k)$ for $h, k \in \mathbb{R}^m$.

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<u>Proposition 11</u> Open set $U \subset \mathbb{R}^m$, $f : U \to \mathbb{R}^n$, $a \in U$

Suppose f is diff'ble on some open set V with $a \in V \subset U$. Then f is <u>twice</u> differentiable at a <u>iff</u> there is a bilinear map $T \in \mathrm{Bil}(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$ s.t. for every fixed $k \in \mathbb{R}^m$, we have

$$f'(a+h)(k) = f'(a)(k) + T(h,k) + o(\|h\|)$$

<u>Proof</u> $\Rightarrow$: Let $T = f''(a)$. Then

$$f'(a+h) = f'(a) + T(h) + \|h\| \varepsilon(h),$$

where $\varepsilon : V - a \to L(\mathbb{R}^m, \mathbb{R}^n)$ is cts at 0, $\varepsilon(0) = 0$.

Fix $k \in \mathbb{R}^m$, evaluate at k:

$$f'(a+h)(k) = f'(a)(k) + T(h,k) + \|h\| \varepsilon(h)(k)$$

Thinking of T as a <u>bilinear</u> map, by Lemma 1

$$\|\varepsilon(h)(k)\| \leq \|\varepsilon(h)\| \, \|k\| \to 0 \quad \text{as } h \to 0$$

so $\|h\| \varepsilon(h)(k) = o(\|h\|)$. ✓

$\Leftarrow$: Thinking of the given $T \in \mathrm{Bil}(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$ as a member of $L(\mathbb{R}^m, L(\mathbb{R}^m, \mathbb{R}^n))$ define

$$\varepsilon : V - a \to L(\mathbb{R}^m, \mathbb{R}^n), \quad \varepsilon(h) = \begin{cases} \dfrac{f'(a+h) - f'(a) - T(h)}{\|h\|} & h \neq 0 \\ 0 & h = 0 \end{cases}$$

Then $f'(a+h) = f'(a) + T(h) + \|h\| \varepsilon(h)$ for all h.

We just need ε cts at 0. By assumption, for fixed $k \in \mathbb{R}^m$,

$$\varepsilon(h)(k) = \frac{f'(a+h)(k) - f'(a)(k) - T(h,k)}{\|h\|} \to 0 \quad \text{as } h \to 0$$

It follows that

$$\|\varepsilon(h)\| = \left(\sum_{i=1}^{m} \|\varepsilon(h)(e_i)\|^2 \right)^{1/2} \to 0 \quad \text{as } h \to 0. \qquad \square$$

<u>Examples</u>

1. Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is linear. Then f is diffble on $\mathbb{R}^m$ and $f'(x) = f \quad \forall x \in \mathbb{R}^m$. Thus $f' : \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n)$ is the <u>constant</u> function with value f. It follows that f is <u>twice</u> diffble on $\mathbb{R}^m$, $f''(x) = 0 \quad \forall x \in \mathbb{R}^m$.

2. Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^p$ be <u>bilinear</u>. Then f is diffble on $\mathbb{R}^m \times \mathbb{R}^n$ and $f' : \mathbb{R}^m \times \mathbb{R}^n \to L(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^p)$ is given by $$f'(a,b)(h,k) = f(a,k) + f(h,b)$$ This is linear in (a,b) with (h,k) fixed. So f' is a <u>linear</u> map, so diffble with $\underline{f''(a,b) = f'}$.

3. Consider $f : M_n \to M_n$, $f(A) = A^3$. Then

$f(A+H) = (A+H)^3$ (expanding gives the following:)

$= f(A) + (A^2H + AHA + HA^2) + o(\|H\|)$

Indeed, for example $\| AH^2 \| \leq \|A\| \, \|H\|^2$ by Lemma 1 and thus $AH^2 = o(\|H\|)$. The other terms can be dealt with similarly. So f is diff^ble at A with

$f'(A)(H) = A^2H + AHA + HA^2$. We now consider

$$f'(A+H)(K) = (A+H)^2K + (A+H)K(A+H) + K(A+H)^2$$

Expanding:

$$= f'(A)(K) + T(H,K) + H^2K + HKH + KH^2$$

The map $T : M_n \times M_n \to M_n$ given by

$$T(H,K) = AHK + HAK + AKH + HKA + KAH + KHA$$

is bilinear. The remainder $H^2K + HKH + KH^2$ is $o(\|H\|)$, (with K fixed).

So by Prop 11, f is <u>twice diff^ble</u> at A with <u>$f''(A) = T$</u>. ✓

<u>Partial derivatives</u>

We are given an open set $U \subset \mathbb{R}^m$, a function $f : U \to \mathbb{R}^n$ and a point $a \in U$. Assume f is twice diff^ble at a. Then for each fixed $k \in \mathbb{R}^m$, we have

$$f'(a+h)(k) = f'(a)(k) + f''(a)(h,k) + o(\|h\|)$$

Now fix directions $u, v \in \mathbb{R}^m \setminus \{0\}$. Putting $k = v$ above gives $D_v f(a+h) = D_v f(a) + f''(a)(h,v) + o(\|h\|)$

It follows that $D_v f : V \to \mathbb{R}^n$ is diffble at a and

$(D_v f)'(a)(h) = f''(a)(h, v)$

Hence $D_u D_v f(a) = D_u (D_v f)(a) = (D_v f)'(a)(u)$

$= f''(a)(u, v)$

In particular, $D_i D_j f(a) = f''(a)(e_i, e_j)$ for $1 \leq i, j \leq m$.

<u>Theorem 12</u> (Symmetry of mixed partial derivatives)

We are given an open set $U \subset \mathbb{R}^m$, a function $f : U \to \mathbb{R}^n$ and a point $a \in U$.

Assume f is <u>twice diffble</u> on an open set V with $a \in V \subset U$ (i.e. f is diffble on V and $f' : V \to L(\mathbb{R}^m, \mathbb{R}^n)$ is diffble on V). Assume that $f'' : V \to Bil(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$ is <u>continuous</u> at a. Then $\underline{D_u D_v f(a) = D_v D_u f(a)}$, $\forall$ directions $u, v \in \mathbb{R}^m \setminus \{0\}$. Thus $f''(a)$ is symmetric, bilinear.

<u>Proof</u> First reduce to $n=1$ case. For $x \in V$ we have

$(D_u f)_j(x) = [D_u f(x)]_j = [f'(x)(u)]_j \underset{(\text{Prop } 4)}{=} f_j'(x)(u) = D_u f_j(x)$

Hence $(D_u f)_j = D_u f_j$ which implies that

$(D_u D_v f)_j = D_u (D_v f)_j = D_u D_v f_j$. So it is <u>enough</u> to show $\underline{D_u D_v f_j(a) = D_v D_u f_j(a)}$ for each $1 \leq j \leq n$.

So WLOG can take $n=1$.

Define ϕ on a suitable ngbd of $(0,0) \in \mathbb{R}^2$ as follows:

$$\phi(s,t) = f(a) + f(a+su+tv) - f(a+tv) - f(a+su)$$

Fix s, t and consider $\psi(y) = f(a+yu+tv) - f(a+yu)$.

Note that $\phi(s,t) = \psi(s) - \psi(0)$. By MVT we find

$\alpha = \alpha(s,t) \in (0,1)$ s.t.

$$\phi(s,t) = \psi(s) - \psi(0) = s\psi'(\alpha s) = s\big(D_u f(a+\alpha su+tv) - D_u f(a+\alpha su)\big)$$

Applying MVT to $D_u f(a+\alpha su+yv)$ we find

$\beta = \beta(s,t) \in (0,1)$ s.t.

$$\begin{aligned} \psi(s,t) &= st\, D_v D_u f(a+\alpha su+\beta tv) \\ &= st\, f''(a+\alpha su+\beta tv)(v,u) \end{aligned}$$

We do this for every (s,t) and use continuity of f'' at a to get

$$\frac{\phi(s,t)}{st} = f''(a+\alpha su+\beta tv)(v,u) \to f''(a)(v,u) = \underline{D_v D_u f(a)}$$

as $(s,t) \to (0,0)$.

Now repeat the above, starting with

$\psi(y) = f(a+su+yv) - f(a+yv)$, ending up with

$$\frac{\phi(s,t)}{st} \to f''(a)(u,v) = \underline{D_u D_v f(a)} \quad \text{as } (s,t) \to (0,0).$$

$\square$

Definitions Given an open set $U \subset \mathbb{R}^m$, a function $f: U \to \mathbb{R}$ and a point $a \in U$:

We say f has a local maximum at a if there is an $r > 0$ s.t. $D_r(a) \subset U$ and $f(x) \leq f(a)$ $\forall x \in D_r(a)$.

We say f has a local minimum at a if there is an $r > 0$ s.t. $D_r(a) \subset U$ and $f(x) \geq f(a)$ $\forall x \in D_r(a)$.

We say a is a stationary point of f if f is differentiable at a and $f'(a) = 0$.

Proposition 13 Given open set $U \subset \mathbb{R}^m$, $f: U \to \mathbb{R}$, $a \in U$:
If f is differentiable at a and f has a local extremum (local min/max) at a, then $f'(a) = 0$.

Proof Replacing f with $-f$ we may assume that f has a local maximum at a. Now assume that $f'(a) \neq 0$. Then there exists $u \in \mathbb{R}^m$ s.t. $f'(a)(u) \neq 0$. By rescaling, we may assume $f'(a)(u) > 0$, $\|u\| = 1$. By defn of differentiability we have

$$f(a+h) = f(a) + f'(a)(h) + \|h\| \varepsilon(h) \qquad (\varepsilon(0) = 0, \ \varepsilon \text{ cts at } 0).$$

Choose $\delta > 0$ s.t. if $\|h\| < \delta$ then $|\varepsilon(h)| < f'(a)(u)$ and $f(a+h) \leq f(a)$. Putting $h = \delta u$ we have

$$0 \geq f(a+\delta u) - f(a) = \delta\left(f'(a)(u) + \varepsilon(\delta u)\right) > 0$$

$\divideontimes$ $\square$

Remark Converse of Prop 13 generally false

Lemma 14 Given open set $U \subset \mathbb{R}^m$, $f : U \to \mathbb{R}^n$, $a \in U$: if f is twice diffble at a, then

$$f(a+h) = f(a) + f'(a)(h) + \tfrac{1}{2} f''(a)(h,h) + o(\|h\|^2)$$

Proof By considering components of f, may assume $n=1$. By defn of 2nd derivative, $\underline{f'(a+h) = f'(a) + f''(a)(h) + \|h\|\varepsilon(h)}$ ($\varepsilon(0) = 0$, ε cts at 0). We next define

$$g(h) = f(a+h) - f(a) - f'(a)(h) - \tfrac{1}{2} f''(a)(h,h)$$

which is defined on some open neighbourhood of 0. We need to show $g(h) = o(\|h\|^2)$. Fix h and define $\phi : [0,1] \to \mathbb{R}$ by $\phi(t) = g(th)$. Note that

$$\phi(t) = f(a+th) - f(a) - tf'(a)(h) - \frac{t^2}{2} f''(a)(h,h)$$

So ϕ is $\underline{\text{cts}}$ on $[0,1]$ and $\underline{\text{diffble}}$ on $(0,1)$ with

$$\begin{aligned}
\phi'(t) &= f'(a+th)(h) - f'(a)(h) - tf''(a)(h,h) \\
&= f'(a+th)(h) - f'(a)(h) - f''(a)(th,h) \\
&= [f'(a+th) - f'(a) - f''(a)(th)](h) \\
&= \|th\| \, \varepsilon(th)(h)
\end{aligned}$$

By MVT, $\exists\ t = t(h) \in (0,1)$ s.t.

$$g(h) = \phi(1) - \phi(0) = \phi'(t) = \|th\| \, \varepsilon(th)(h)$$

So by Lemma 1, $|\varepsilon(th)(h)| \leq \|\varepsilon(th)\| \, \|h\|$.

Hence $|g(h)| \leq \|h\|^2 \|\varepsilon(th)\|$ and result follows. $\square$

<u>From Linear Algebra</u>: A symmetric bilinear map $T: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is <u>positive definite</u> if $\underline{T(x,x) > 0}$ $\forall x \in \mathbb{R}^m \setminus \{0\}$ (and negative definite if $T(x,x) < 0$ $\forall x \in \mathbb{R}^m \setminus \{0\}$).

<u>Theorem 15</u> Given open set $U \subset \mathbb{R}^m$, $f: U \to \mathbb{R}$, $a \in U$:
Assume f is <u>twice</u> diff'ble on U, f'' <u>cts</u> at a.
If a is a stationary pt of f and $f''(a)$ is <u>positive</u> definite then f has a local <u>minimum</u> at a.
(Also if negative definite, then have local maximum).

<u>Note</u> By Thm 12 if f'' exists on a ngbd of a and is cts at a, then $f''(a)$ is <u>symmetric</u> and <u>bilinear</u>.
The $m \times m$ matrix H defined by

$$H_{i,j} = f''(a)(e_i, e_j) = D_i D_j f(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \quad \text{is}$$

called the <u>Hessian</u> of f at a.

<u>Proof</u> Assume $f'(a) = 0$ and $f''(a)$ is positive definite.
Let $u_1, \ldots, u_m$ be an <u>orthonormal</u> basis of $\mathbb{R}^m$ s.t. $f''(a)(u_i, u_j) = 0$ if $i \neq j$. Set

$$\mu = \min\{ f''(a)(u_i, u_i) : 1 \leq i \leq m \}$$

PTO

Then $\mu > 0$ and for $h = \sum_{i=1}^{m} h_i u_i \in \mathbb{R}^m$ we have

$$f''(a)(h,h) = \sum_{i,j=1}^{m} h_i h_j f''(a)(u_i, u_j)$$

$$= \sum_{i=1}^{m} h_i^2 f''(a)(u_i, u_i) \geq \mu \sum_{i=1}^{m} h_i^2 = \mu \|h\|^2$$

Now by Lemma 14, we have

$$f(a+h) = f(a) + \frac{1}{2} f''(a)(h,h) + \|h\|^2 \varepsilon(h)$$

where $\varepsilon(0) = 0$, ε is cts at 0.

Choose $\delta > 0$ s.t. $|\varepsilon(h)| < \frac{\mu}{2}$ whenever $\|h\| < \delta$.

Then $$f(a+h) - f(a) \geq \|h\|^2 \left(\frac{\mu}{2} + \varepsilon(h)\right) \geq 0$$

whenever $h \in D_\delta(0)$.

Thus f has a local <u>minimum</u> at a. The proof of the second statement is similar.

$\square$