Analysis - Lecture 1 19/1/21

Section 1 - Limits and Convergence (6 lectures)

Review from N&S

Definition We say $a_n \to a$ as $n \to \infty$ if given $\varepsilon > 0$, $\exists N$ with $|a_n - a| < \varepsilon$ for all $n \geqslant N$.

Note $N = N(\varepsilon)$

Increasing sequence: $a_n \leqslant a_{n+1}$, decreasing $a_n \geqslant a_{n+1}$ $\Big\}$ monotone

Strictly increasing: $a_n < a_{n+1}$, strictly decreasing $a_n > a_{n+1}$

Fundamental Axiom of the Real Numbers

If $a_n \in \mathbb{R}$, $\forall n \geqslant 1$, $A \in \mathbb{R}$
and $a_1 \leqslant a_2 \leqslant a_3 \leqslant \dots$ and $a_n \leqslant A$ for all n, then $\exists a \in \mathbb{R}$ with $a_n \to a$ as $n \to \infty$.

(i.e. an increasing sequence of reals bounded above converges)

Equivalently - a decreasing sequence of reals bounded below converges

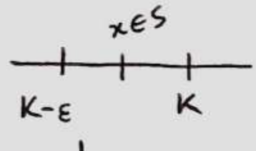
Equivalently: every nonempty set of real numbers bounded above has a supremum ("least upper bound")

This is the same as the least upper bound axiom (recall from N&S).

Recall: **Definition** (Supremum) Have $S \subset \mathbb{R}$, $S \neq \emptyset$.

$\sup S = K$ if (i) $x \leqslant K$ $\forall x \in S$

and (ii) given $\varepsilon > 0$, $\exists x \in S$ s.t. $x > K - \varepsilon$



Note: supremum is unique.

Also see infimum defined similarly - greatest lower bound.

Lemma 1.1 (i) The limit of a sequence is unique:
$a_n \to a$ and $a_n \to b \Rightarrow a = b$

(ii) If $a_n \to a$ as $n \to \infty$ and $n_1 < n_2 < n_3 < \dots$ ($\in \mathbb{N}$) then $a_{n_j} \to a$ as $j \to \infty$. (i.e. subsequences converge to the same limit).

(iii) If $a_n = c \ \forall n$ then $a_n \to c$ as $n \to \infty$

(iv) If $a_n \to a$ and $b_n \to b$ then $a_n + b_n \to a + b$

(v) If $a_n \to a$ and $b_n \to b$ then $a_n b_n \to ab$

(vi) If $a_n \to a$, $a_n \neq 0 \ \forall n$ and $a \neq 0$ then $\frac{1}{a_n} \to \frac{1}{a}$

(vii) If $a_n \leq A \ \forall n$ and $a_n \to a$, then $a \leqslant A$.

<u>Proof</u> We will do (i), (ii) and (v) and others as exercises.

(i) suppose $a_n \to a$ and $a_n \to b$.

Given $\varepsilon > 0$, $\exists n_1$ s.t. $|a_n - a| < \varepsilon \quad \forall n \geqslant n_1$

$\exists n_2$ s.t. $|a_n - a| < \varepsilon \quad \forall n \geqslant n_2$

$N := \max \{n_1, n_2\}$. Then $\forall n \geqslant N$

$|a - b| \leq |a_n - a| + |a_n - b| < \underset{\uparrow \atop \text{arbitrary}}{2\varepsilon}$ (first ineq. by Δ ineq)

so $|a - b| \to 0$.

Alternatively: $a \neq b$ lets us take $\varepsilon = \frac{|a-b|}{3}$

$\Rightarrow |a - b| < \frac{2}{3}|a - b|$ ※.

(ii) Given $\varepsilon > 0$, $\exists N$ s.t. $|a_n - a| < \varepsilon \quad \forall n \geqslant N$

Since $n_j \geqslant j$ (can prove by induction if you want...)

$\Rightarrow |a_{n_j} - a| < \varepsilon \quad \forall j \geqslant N$ i.e. $a_{n_j} \to a$ as $j \to \infty$.

(v) $|a_n b_n - ab| \underset{(\Delta \text{ ineq})}{\leq} |a_n b_n - a_n b| + |a_n b - ab| = |a_n||b_n - b| + |b||a_n - a|$

$a_n \to a$: given $\varepsilon > 0$, $\exists N_1$ s.t. $|a_n - a| < \varepsilon \quad \forall n \geqslant N_1$ (*)

$b_n \to b$: given $\varepsilon > 0$, $\exists N_2$ s.t. $|b_n - b| < \varepsilon \quad \forall n \geqslant N_2$

(*): If $n \geqslant \underset{\uparrow \atop \text{function of } \varepsilon}{N_1}(1)$ then $|a_n - a| < 1$ so $|a_n| \leq |a| + 1$ (again Δ ineq)

$\Rightarrow |a_n b_n - ab| \leq |a_n||b_n - b| + |b||a_n - a| \leq |a_n|\varepsilon + |b|\varepsilon$

$\forall n \geqslant \max(N_1, N_2)$

and $\forall n \geq N_1(1)$, $|a_n| \leq |a| + 1$ so $|a_n|\varepsilon + |b|\varepsilon \leq \varepsilon(|a|+1+|b|)$

hence $|a_n b_n - ab| \leq \varepsilon(|a|+1+|b|)$

constant multiple of ε - fine

$\forall n \geq N_3(\varepsilon) = \max\{N_1(1), N_1(\varepsilon), N_2(\varepsilon)\}$. $\square$

<u>Lemma 1.2</u> $\frac{1}{n} \to 0$ as $n \to \infty$.

<u>Proof</u> $\frac{1}{n}$ is a decreasing sequence bounded below by 0. By Fundamental Axiom, it has a limit a.

<u>Claim</u> $a = 0$.

<u>Proof of claim</u> $\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \to \frac{a}{2}$ by Lemma 1.1 (v) and (iii)

But $\frac{1}{2n}$ is a subsequence, so by Lemma 1.1 (ii)

$\frac{1}{2n} \to a$. By uniqueness $a = \frac{a}{2} \Rightarrow \underline{a = 0}$. $\square$

<u>Remark</u> The definition of limit of a sequence makes perfect sense for $a_n \in \mathbb{C}$.

<u>Definition</u> $a_n \to a$ if given $\varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$.

(disc in complex plane) $\longrightarrow$

Can check Lemma 1.1 for $\mathbb{C}$.

The last one (vii) does not really make sense over $\mathbb{C}$ as it uses the <u>order</u> of the real numbers.

Analysis - Lecture 2

19/1/21

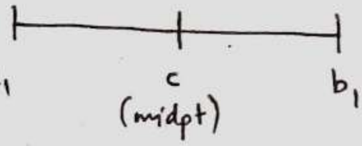
Theorem 1.3 (Bolzano - Weierstrass Theorem) If $x_n \in \mathbb{R}$ and there exists K s.t. $|x_n| \leq K \ \forall n$, then we can find $n_1 < n_2 < n_3 < \ldots$ and $x \in \mathbb{R}$ s.t. $x_{n_j} \to x$ as $j \to \infty$.

"Every bounded sequence has a convergent subsequence"

Remark we say nothing about uniqueness of limit e.g. $x_n = (-1)^n$

$x_{2n+1} \to -1, \quad x_{2n} \to 1$

Proof Set $[a_1, b_1] = [-K, K]$



Consider the following alternatives:

(1) $x_n \in [a_1, c]$ for infinitely many n
(2) $x_n \in [c, b_1]$ for infinitely many n

could hold at same time

If (1) holds: set $a_2 = a_1$ and $b_2 = c$. If (1) fails, we have that (2) must hold. Then we set $a_2 = c$, $b_2 = b_1$.

Proceed inductively to construct sequences a_n, b_n s.t. $x_m \in [a_n, b_n]$ for infinitely many values of m. (for all n).

By construction: $a_{n-1} \leq a_n \leq b_n \leq b_{n-1}$

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} \quad (*)$$ (as we did bisection) "lion hunting"

a_n is increasing, b_n is decreasing

both are bounded above and below

So by fundamental axiom, a_n and b_n are convergent sequences

$a_n \to a \in [a_1, b_1]$

$b_n \to b \in [a_1, b_1]$

by $(*)$: $b - a = \frac{b-a}{2} \Rightarrow b - a = 0 \Rightarrow \underline{a = b}$

so these are actually the same sequence.

Since $x_m \in [a_n, b_n]$ for infinitely many values of m, having chosen n_j s.t. $x_{n_j} \in [a_j, b_j]$, there is $n_{j+1} > n_j$ s.t. $x_{n_{j+1}} \in [a_{j+1}, b_{j+1}]$

(have an "unlimited supply")

Hence $a_j \leq x_{n_j} \leq b_j \Rightarrow x_{n_j} \to a$. $\square$

Cauchy Sequences

Definition (Cauchy sequence) $a_n \in \mathbb{R}$ is a Cauchy sequence if given $\varepsilon > 0$, $\exists N > 0$ s.t. $|a_n - a_m| < \varepsilon \quad \forall n, m \geq N$.

Lemma 1.4 A convergent sequence is a Cauchy sequence. note: (iff)

Proof If $a_n \to a$, given $\varepsilon > 0$ $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$

Take $m, n \geq N$:

$$|a_n - a_m| \underset{(\Delta \text{ ineq})}{\leq} |a_n - a| + |a_m - a| < 2\varepsilon \qquad \square$$

Now prove opposite direction: prove Cauchy $\Rightarrow$ convergent.

Theorem 1.5 Every Cauchy sequence is convergent.

Proof First note that if a_n is Cauchy, then it is bounded.

Take $\varepsilon = 1$: $N = N(1)$, use Cauchy property

$|a_n - a_m| < 1 \quad \forall n, m \geq N(1)$

$\therefore$ Use Δ ineq: $|a_m| \leq |a_m - a_N| + |a_N| < 1 + |a_N| \quad \forall m \geq N$

Let $K = \max \{1 + |a_N|, |a_n| \text{ with } n = 1, 2, \ldots, N-1\}$

we have $|a_n| \leq K \quad \forall n$. ✓

By Bolzano-Weierstrass theorem, $a_{n_j} \to a$ (convergent subseq.)

Claim $a_n \to a$.

PTO

Proof of Claim Given $\varepsilon > 0$, $\exists j_0$ s.t. $\forall j \geq j_0$, $|a_{n_j} - a| < \varepsilon$.
(as a_{n_j} converges)

Also, $\exists N(\varepsilon)$ s.t. $|a_m - a_n| < \varepsilon \quad \forall m, n \geq N(\varepsilon)$ (Cauchy).

Take j s.t. $n_j \geq \max\{N(\varepsilon), n_{j_0}\}$ then if $n \geq N(\varepsilon)$:

$$|a_n - a| \underset{(\Delta \text{ ineq})}{\leq} \underbrace{|a_n - a_{n_j}|}_{<\varepsilon} + \underbrace{|a_{n_j} - a|}_{<\varepsilon} < 2\varepsilon. \qquad \square$$

Thus on $\mathbb{R}$, a sequence is convergent if and only if it is Cauchy.
Sometimes called "general principle of convergence"

Useful property: since we don't need to know what the limit is.

Series

Definition have $a_n \in \mathbb{R}$ or $\mathbb{C}$. We say that $\sum_{j=1}^{\infty} a_j$ converges to S if the sequence of partial sums

$S_N = \sum_{j=1}^{N} a_j \to S$ as $N \to \infty$. We write $\sum_{j=1}^{\infty} a_j = S$.

If S_N does not converge, then we say $\sum_{j=1}^{\infty} a_j$ diverges.

Remark Any problem on series can be turned into a problem on sequences by considering partial sums.

Lemma 1.6 (i) If $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ converge, then so does

$$\sum_{j=1}^{\infty} (\lambda a_j + \mu b_j) \quad \text{with } \lambda, \mu \in \mathbb{C}.$$

(ii) Suppose $\exists N$ s.t. $a_j = b_j \quad \forall j \geq N$. Then either $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ both converge, or they both diverge.

<u>Proof</u> (i) $$S_N = \sum_{j=1}^{N} (\lambda a_j + \mu b_j) = \lambda \underbrace{\sum_{j=1}^{N} a_j}_{c_N} + \mu \underbrace{\sum_{j=1}^{N} b_j}_{d_n}$$

$$= \lambda c_N + \mu d_N$$

We know $c_N \to c$, $d_N \to d$ so by Lemma 1.1,

$\lambda c_N + \mu d_N \to \lambda c + \mu d.$ $\square$

(ii) $n \geqslant N$
$$s_n = \sum_{j=1}^{n} a_j = \sum_{j=1}^{N-1} a_j + \sum_{j=N}^{n} a_j$$
$$d_n = \sum_{j=1}^{n} b_j = \sum_{j=1}^{N-1} b_j + \sum_{j=N}^{n} b_j$$

so $$s_n - d_n = \sum_{j=1}^{N-1} a_j - \sum_{j=1}^{N-1} b_j \quad (a_j = b_j \text{ for } j \geqslant N)$$

so s_n converges iff d_n does. $\square$

Analysis - Lecture 3

An important series: the geometric series

Set $a_n = x^{n-1} \quad n \geq 1$

$$S_n = \sum_{j=1}^{n} a_j = 1 + x + x^2 + \dots + x^{n-1}$$

Then $$S_n = \begin{cases} \frac{1-x^n}{1-x} & \text{for } x \neq 1 \\ n & \text{for } x = 1 \end{cases}$$

$$x S_n = x + x^2 + \dots + x^n = S_n - 1 + x^n \Rightarrow \text{result.}$$

If $|x| < 1$, then $x^n \to 0$ so $S_n \to \frac{1}{1-x}$

If $x > 1$, $x^n \to \infty$ and $S_n \to \infty$

<u>Definition</u> $S_n \to \infty$ if given A, $\exists N$ s.t. $S_n > A \ \forall n \geq N$

$S_n \to -\infty$ if given A, $\exists N$ s.t. $S_n < -A \ \forall n \geq N$

If S_n does not converge and does not tend to $\pm\infty$, then we say S_n <u>oscillates</u>.

If we have $x < -1$ then S_n oscillates.

If $x = -1$: $S_n = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$

Thus the geometric series converges if and only if <u>$|x| < 1$</u>.

To see that $x^n \to 0$ as $n \to \infty$ if $|x| < 1$, consider $0 < x < 1$ (exercise). Write $\frac{1}{x} = 1 + \delta \quad (\delta > 0)$

$$x^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+\delta n} \quad \text{because } (1+\delta)^n \geq 1 + n\delta \text{ (binomial exp.)}$$

and $\frac{1}{1+\delta n} \to 0$ as $n \to \infty$. (can fill in gaps / do $-1 < x < 0$ as exercise)

Observation:

<u>Lemma 1.7</u> If $\sum_{j=1}^{\infty} a_n$ converges, then $\lim_{j \to \infty} a_j = 0$.

Proof $S_n = \sum_{j=1}^{n} a_j$ so $a_n = S_n - S_{n-1}$. If $S_n \to a$

then $S_{n-1} \to a \Rightarrow a_n \to a - a$ so $\underline{a_n \to 0}$. $\square$

Remark The converse of 1.7 is False e.g. $\sum_{j=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

$S_n = \sum_{j=1}^{n} \frac{1}{j}$, write $S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$

As $\frac{1}{n+k} \geqslant \frac{1}{2n}$ for $k = 1, 2, \dots, n$, we have

$S_{2n} = S_n + \frac{1}{n+1} + \dots + \frac{1}{n+n} > S_n + \frac{1}{2} \quad \forall n$

If $S_n \to a$, then $S_{2n} \to a$ also and hence $a \geqslant a + \frac{1}{2}$ $\#$

So the harmonic series diverges.

Series of non-negative terms (work with $a_n \in \mathbb{R}$, $a_n \geqslant 0$)

The following is a basic result:

Theorem 1.8 (Comparison test) Suppose $0 \le b_n \le a_n \ \forall n$.
Then if $\sum_{j=1}^{\infty} a_n$ converges, so does $\sum_{j=1}^{\infty} b_n$.

Proof Let $S_N = \sum_{n=1}^{N} a_n$, $d_n = \sum_{n=1}^{N} b_n$

$b_n \le a_n \Rightarrow d_N < S_N$ but $S_N \to S$, so

$d_N \le S_N \le S \quad \forall N$ and d_N is increasing, so d_N converges. (as monotone)

d_N bounded above $\square$

Example $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can write $\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} = a_n$ $(n \geqslant 2)$

$$\sum_{2}^{N} a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{N-1} - \frac{1}{N} = 1 - \frac{1}{N}$$

so $\frac{1}{n^2}$ sum converges by comparison test.

In fact $\sum_1^\infty \frac{1}{n^2} \leq 1+1 = 2$ (actual value $\frac{\pi^2}{6}$)

Theorem 1.9 (Root test / Cauchy's test for convergence)

Assume $a_n \geq 0$ and $a_n^{1/n} \to a$ as $n \to \infty$. Then if $a > 1$, $\sum a_n$ diverges and if $a < 1$ then $\sum a_n$ converges.

Remark If $a = 1$, nothing can be said (examples later).

Proof If $a < 1$, choose $a < r < 1$. By definition of limit, $\exists N$ s.t. $\forall n \geq N$, $a_n^{1/n} < r$. (like taking ε as r)

(Note: "first few terms" don't affect convergence)

$a_n^{1/n} < r \Rightarrow a_n < r^n$ but $r < 1$ so geometric series converges. So by comparison (1.8), $\sum a_n$ converges.

If $a > 1$ then for $n \geq N$, $a_n^{1/n} > 1 \Rightarrow a_n > 1$, thus $\sum a_n$ diverges (since $a_n \not\to 0$). □

Theorem 1.10 (Ratio test / D'Alembert's test) Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to L$. If $L < 1$, $\sum a_n$ converges. If $L > 1$, then $\sum a_n$ diverges. (can't say anything if $L = 1$)

Proof Suppose $L < 1$ and choose r with $L < r < 1$. Then $\exists N$ s.t. $\forall n \geq N$, $\frac{a_{n+1}}{a_n} < r$ (by limit def; $\frac{a_{n+1}}{a_n} \to L$)

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}, \quad n > N$$

(by applying $\frac{a_{n+1}}{a_n} < r$ several times) $\Rightarrow$ $a_n < Kr^n$ with K constant

then by comparison, $\sum a_n$ converges.

Similarly if $L > 1$ then choose $1 < r < L$. Then $\frac{a_{n+1}}{a_n} > r$ $\forall n \geq N$ for some N. As before: $a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N}$

bounded below by divergent series: $\sum a_n$ diverges. □

Analysis - Lecture 4

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Recall from last time: ratio/root tests.

Examples

1. $\sum_1^\infty \frac{n}{2^n}$ - try ratio test. $\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n}$

$\frac{n+1}{2n} \to \frac{1}{2} < 1$ so it converges.

2. $\sum_1^\infty \frac{1}{n}$ diverges. Ratio test gives limit 1: $\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \to 1$

$\sum_1^\infty \frac{1}{n^2}$ converges. Ratio test also gives limit 1 : inconclusive.

Also: since $n^{1/n} \to 1$ as $n \to \infty$ (will see later), the root test is also inconclusive when limit $= 1$.

Write $n^{1/n} = 1 + \delta_n$, $\delta_n > 0$

$\Rightarrow n = (1+\delta_n)^n > \frac{n(n-1)}{2}\delta_n^2$ (by binomial expansion)

$\Rightarrow \delta_n^2 < \frac{2}{n-1} \Rightarrow \delta_n \to 0 \Rightarrow n^{1/n} \to 1$.

3. $\sum_1^\infty \left(\frac{n+1}{3n+5}\right)^n$ root test

Have $\frac{n+1}{3n+5} \to \frac{1}{3} < 1$ so it converges.

Another useful test:

Theorem 1.11 (Cauchy's condensation test). Let a_n be a decreasing sequence of positive terms. Then $\sum_1^\infty a_n$ is convergent iff $\sum_1^\infty 2^n a_{2^n}$ converges.

Proof First observe that if a_n is decreasing:

$$\underset{(*_1)}{a_{2^k} \leq a_{2^{k-1}+i}} \underset{(*_2)}{\leq a_{2^{k-1}}} \qquad 1 \leq i \leq 2^{k-1},\ k \geq 1$$

PTO

Assume that the sequence converges $\left(\sum_1^{\infty} a_n\right)$, to A.

Then $2^{n-1} a_{2^n} = \underbrace{a_{2^n} + \dots + a_{2^n}}_{2^{n-1} \text{ times}}$

$(*_1)$: $\underbrace{a_{2^n} + \dots + a_{2^n}}_{2^{n-1} \text{ times}} \leq a_{2^{n-1}+1} + a_{2^{n-1}+2} + \dots + a_{2^n}$

$$= \sum_{m=2^{n-1}+1}^{2^n} a_m$$

Thus $\displaystyle\sum_{n=1}^{N} 2^{n-1} a_{2^n} \leq \sum_{n=1}^{N} \sum_{m=2^{n-1}+1}^{2^n} a_m = \sum_{m=2}^{2^N} a_m$

$$\Rightarrow \underbrace{\sum_{n=1}^{N} 2^n a_{2^n}} \leq 2 \sum_{m=2}^{2^N} a_m \leq 2(A - a_1)$$

bounded above and increasing: converges. Proves $\Rightarrow$ direction.

$\Leftarrow$: Assume $\displaystyle\sum_{n=1}^{N} 2^n a_{2^n}$ converges.

$$\sum_{m=2^{n-1}+1}^{2^n} a_m = a_{2^{n-1}+1} + a_{2^{n-1}+2} + \dots + a_{2^n}$$

$$\leq a_{2^{n-1}} + a_{2^{n-1}} + \dots + a_{2^{n-1}} = 2^{n-1} a_{2^{n-1}}$$

(by $(*_2)$)

$$\Rightarrow \sum_{m=2}^{2^N} a_m = \sum_{n=1}^{N} \sum_{m=2^{n-1}+1}^{2^n} a_m \leq \sum_{n=1}^{N} 2^{n-1} a_{2^{n-1}} \leq B$$

for some B, since this series converges (assumed)

so $\displaystyle\sum_{m=1}^{2^N} a_m$ is bounded above and increasing: converges.

Proves $\Leftarrow$ direction. $\square$

Example (condensation test)

$\sum_1^\infty \frac{1}{n^k}$ converges iff $k > 1$ $(k > 0)$ (Riemann zeta function of k)

The sequence $a_n = \frac{1}{n^k}$ is decreasing, consists of positive terms.

$$\frac{1}{(n+1)^k} < \frac{1}{n^k} \iff \left(\frac{n}{n+1}\right)^k < 1 \iff \frac{n}{n+1} < 1$$

$$2^n a_{2^n} = 2^n \left(\frac{1}{2^n}\right)^k = 2^{n-nk} = (\underbrace{2^{1-k}}_{=r})^n$$ converges iff $|r| < 1$

$\Rightarrow \sum_1^\infty \frac{1}{n^k}$ converges iff $2^{1-k} < 1$, iff $\underline{k > 1}$. □

Alternating Series

Theorem 1.12 (Alternating series test) If a_n decreases and tends to 0 as $n \to \infty$, then $\underline{\sum_1^\infty (-1)^{n+1} a_n}$ converges.

Example $\sum_1^\infty \frac{(-1)^{n+1}}{n}$ converges.

Proof $s_n = a_1 - a_2 + \dots + (-1)^{n+1} a_n$ ~~with~~

$$s_{2n} = \underbrace{(a_1 - a_2) + (a_3 - a_4) + \dots}_{s_{2n-2}} + \underbrace{(a_{2n-1} - a_{2n})}_{\substack{\geq 0 \\ (a_n \text{ decreasing})}} \geq s_{2n-2}$$

Also $s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2n-2} - a_{2n-1}) - a_{2n}$
$\leq \underline{a_1}$ as all the $a_i - a_{i+1}$ are ≥ 0

so s_{2n} is increasing and bounded above - <u>converges</u>.

$s_{2n+1} = s_{2n} + a_{2n+1} \to s + \underset{\uparrow \, a_n \to 0}{0} = s$ even and odd terms converge to <u>same</u> limit

Hence s_n converges to s:
given $\varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N_1$, $|s_{2n} - s| < \varepsilon$ and $\exists N_2$ s.t. $\forall n \geq N_2$, $|s_{2n+1} - s| < \varepsilon$. Take $N = 2\max\{N_1, N_2\} + 1$: $k \geq N \Rightarrow$
$|s_k - s| < \varepsilon$ so $\underline{s_k \to s}$. □

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Analysis - Lecture 5

Absolute Convergence

Definition Take $a_n \in \mathbb{C}$. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series is absolutely convergent.

Note: since $|a_n| \geqslant 0$ can use previous tests to check absolute convergence.

Theorem 1.13 If $\sum a_n$ is absolutely convergent, then it is convergent.

Proof Suppose first that $a_n \in \mathbb{R}$.

Let $v_n = \begin{cases} a_n & \text{if } a_n \geqslant 0 \\ 0 & \text{if } a_n < 0 \end{cases}$ $\qquad w_n = \begin{cases} 0 & \text{if } a_n \geqslant 0 \\ -a_n & \text{if } a_n > 0 \end{cases}$

so $v_n = \dfrac{|a_n| + a_n}{2}$, $\quad w_n = \dfrac{|a_n| - a_n}{2}$

Clearly $v_n, w_n \geqslant 0 \ \forall n$, $\quad a_n = v_n - w_n$, $\quad |a_n| = v_n + w_n \geqslant v_n, w_n$

If $\sum |a_n|$ converges, by comparison $\sum v_n$, $\sum w_n$ converge.

$\Rightarrow \sum a_n$ converges as we can write a_n in terms of v_n, w_n.

If $a_n \in \mathbb{C}$: $\quad a_n = x_n + iy_n \Rightarrow |x_n|, |y_n| \leqslant |a_n|$

$\Rightarrow \sum x_n$, $\sum y_n$ are absolutely convergent $\Rightarrow \sum x_n$, $\sum y_n$ converge.

Since $a_n = x_n + iy_n$, $\sum a_n$ converges. $\square$

Examples 1) $\sum \frac{(-1)^n}{n}$ convergent but not absolutely convergent

2) $\sum_{n=1}^{\infty} \frac{z^n}{2^n}$, $\quad$ consider $\sum \left(\frac{|z|}{2}\right)^n$ converges when $|z| < 2$ $\quad (*)$

and hence have absolute convergence

If $|z| > 2$, then general term $\not\to 0$ so it diverges.

Terminology: If $\sum a_n$ converges but not absolutely, then the series is conditionally convergent.

"conditional": sum to which it converges depends on order of terms.

(can be made to be anything - see Riemann series thm)

<u>Example sheet</u>: (sheet 1 Q7)

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ (I) s_n partial sum

$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots$ (II) t_n partial sum

Have $s_n \to s > 0$, $t_n \to 3s/2$

<u>Rearrangement</u> <u>Def</u>: let σ be a bijection of positive integers, then $a_n' = a_{\sigma(n)}$ is a rearrangement.

<u>Theorem 1.14</u> If $\sum_1^\infty a_n$ is absolutely convergent, then every series consisting of the same terms in any order has the same sum.

<u>Proof</u> Take $a_n \in \mathbb{R}$, a_n' be a rearrangement.

$s_n = \sum_1^n a_n$, $t_n = \sum_1^n a_n'$. Suppose first that $a_n \geqslant 0$ $\forall n$.

Given n, can find q s.t. s_q contains every term of t_n. Since $a_n \geqslant 0$, $t_n \leqslant s_q \leqslant s$ so as $n \to \infty$, $t_n \to t$ (MTC thm) so $t \leqslant s$. But argument is symmetric so $s \leqslant t \Rightarrow \underline{s = t}$.

If a_n has any sign, consider $v_n = \frac{|a_n| + a_n}{2}$, $w_n = \frac{|a_n| - a_n}{2}$

Consider $\sum a_n'$, $\sum v_n'$, $\sum w_n'$

Since $\sum |a_n|$ converges, $\sum v_n$, $\sum w_n$ converge (see prev. page)

now can use $v_n, w_n \geqslant 0 \Rightarrow \sum v_n', \sum w_n'$ converge with

$\sum v_n' = \sum v_n$, $\sum w_n' = \sum w_n$

and the claim follows since $\underline{a_n = v_n - w_n}$.

For $a_n \in \mathbb{C}$, consider $a_n = x_n + i y_n$ as before. $\square$

Analysis - Lecture 6

31/1/21

Section 2: Continuity [3]

Take $E \subseteq \mathbb{C}$ non-empty, $f : E \to \mathbb{C}$ any function, $a \in E$.
(This includes the case in which f is real-valued and $E \subseteq \mathbb{R}$)

Definition 1 f is continuous at $a \in E$ if for every sequence $z_n \in E$ with $z_n \to a$, then we have $f(z_n) \to f(a)$.

Definition 2 f is continuous at $a \in E$ if given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if $|z-a| < \delta$, then $|f(z) - f(a)| < \varepsilon$. ($\varepsilon$-$\delta$ definition)

We prove right away that these are equivalent.

Def 2 $\Rightarrow$ def 1: we know that given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|z-a| < \delta$, $z \in E$ then $|f(z) - f(a)| < \varepsilon$. Let $z_n \to a$, then $\exists\, n_0$ s.t. $\forall n \geq n_0$, $|z_n - a| < \delta \Rightarrow |f(z_n) - f(a)| < \varepsilon$ i.e. $f(z_n) \to f(a)$. ✓

Def 1 $\Rightarrow$ Def 2: Assume $f(z_n) \to f(a)$ whenever $z_n \to a$

Suppose f is not continuous at a according to Def 2, i.e.

(*) $\exists\, \varepsilon > 0$ s.t $\forall \delta > 0$, $\exists\, z \in E$ s.t. $|z-a| < \delta$ but $|f(z) - f(a)| \geq \varepsilon$

Let $\delta = \frac{1}{n}$, then from (*) get z_n s.t. $|z_n - a| < \frac{1}{n}$ and $|f(z_n) - f(a)| \geq \varepsilon$. Certainly $z_n \to a$ but $f(z_n) \not\to f(a)$ ⨳ ✓

Proposition 2.1 $a \in E$, $g, f : E \to \mathbb{C}$ with g, f continuous at a. Then $f(z) + g(z)$, $f(z) g(z)$, $\lambda f(z)$ are continuous, and if $f(z) \neq 0$ $\forall z \in E$ then $\frac{1}{f}$ is continuous at a.

Proof Using Def 1 this is obvious using analogous results for sequences (Lemma 1.1). $\square$

Note: $f(z) = z$ continuous $\Rightarrow$ all polynomials continuous everywhere in $\mathbb{C}$.
f is continuous on E if it's continuous at every point in E.

Exercise prove Prop 2.1 using Def 2.

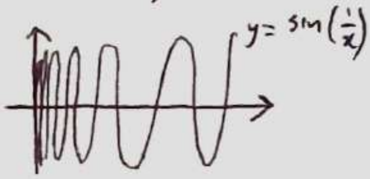
Next look at compositions.

__Theorem 2.2__ Let ~~FEA~~ $f: A \to \mathbb{C}$, $g: B \to \mathbb{C}$, $f(A) \subset B$. If f is continuous at $a \in A$ and g is continuous at $f(a)$ then $g \circ f : A \to \mathbb{C}$ is __continuous__ at a.

__Proof__ Take any sequence $z_n \to a$. By assumption, $f(z_n) \to f(a)$. Set $w_n = f(z_n)$. Then $w_n \in B$ and $w_n \to f(a)$. Hence $g(w_n) \to g(f(a))$. $\square$

__Examples__

(1) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ (assume $\sin x$ continuous)



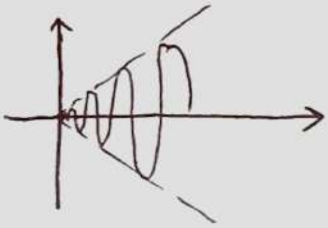
so $\sin\left(\frac{1}{x}\right)$ is continuous at all $x \neq 0$.

Discontinuous at 0:

$\frac{1}{x_n} = (2n + \frac{1}{2})\pi$ gives $f(x_n) = 1$ but $x_n \to 0$ as $n \to \infty$.

But $f(0) = 0$ so it's not continuous. ($x_n \to 0$ but $f(x_n) \not\to f(0)$)

(2) $f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$



f is continuous at 0: take $x_n \to 0$, then

$|f(x_n)| \leq |x_n|$ because $|\sin \frac{1}{x}| \leq 1$ $\Rightarrow f(x_n) \to 0 = \underline{f(0)}$.

(3) $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Discontinuous at __every__ point: if $x \in \mathbb{Q}$, take sequence $x_n \to x$ with ~~x_n~~ $x_n \notin \mathbb{Q}$, then $f(x_n) = 0 \not\to f(x) = 1$

if $x \notin \mathbb{Q}$, take $x_n \to x$ with $x_n \in \mathbb{Q}$ then $f(x_n) = 1 \not\to f(x) = 0$.

Not continuous anywhere.

Analysis - Lecture 7

31/1/21

Limit of a function

Definition Let $E \subseteq \mathbb{C}$, $a \in \mathbb{C}$. Say a is a limit point of E if for any $\delta > 0$, $\exists\, z \in E$ s.t. $0 < |z-a| < \delta$ (i.e. can get arbitrarily close to a while in E)

Remark a is a limit point iff $\exists$ sequence $z_n \in E$ s.t. $z_n \to a$ and $z_n \neq a$ $\forall n$ (check equivalence).

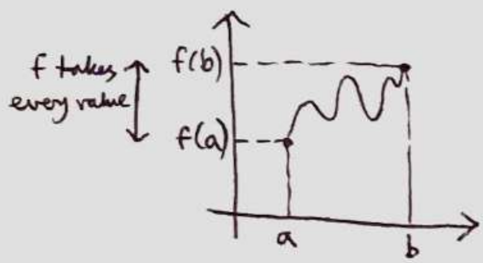
Definition Let $f: E \subseteq \mathbb{C} \to \mathbb{C}$ and $a \in \mathbb{C}$ be a limit point of E. We say that $\lim_{z \to a} f(z) = L$ if given $\varepsilon > 0$, $\exists \delta > 0$ s.t. whenever $0 < |z-a| < \delta$ and $z \in E$, have $|f(z) - L| < \varepsilon$.
(equivalently: $f(z_n) \to L$ for any $(z_n) \in E$, $z_n \neq a$ and $z_n \to a$).
Immediately: if $a \in E$ is a limit point then $\lim_{z \to a} f(z) = f(a) \Leftrightarrow$ continuous at a.

If $a \in E$ is isolated (not a limit point): continuity of f at a always holds.
Limit of function has similar properties to sequences: uniqueness etc.

1) $f(z) \to A$, $f(z) \to B$ $\Rightarrow |A-B| \leq |A - f(z)| + |f(z) - B|$ (Δ ineq)
if $z \in E$ is s.t. $0 < |z-a| < \delta_1, \delta_2$ then $|A-B| < 2\varepsilon \Rightarrow \underline{A = B}$.
(a is a limit point $\Rightarrow$ such a z exists)

2) $f(z) + g(z) \to A+B$, 3) $f(z)g(z) \to AB$, 4) $\frac{f(z)}{g(z)} \to \frac{A}{B}$ $(B \neq 0)$

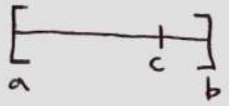
Theorem 2.3 (Intermediate Value Theorem) If $f: [a,b] \to \mathbb{R}$ with f continuous and $f(a) \neq f(b)$, then f takes every value between $f(a)$ and $f(b)$.



Proof WLOG suppose $f(a) < f(b)$.
Take $f(a) < \eta < f(b)$.
Let $S = \{x \in [a,b] : f(x) < \eta\}$
$a \in S$ so S is nonempty (and bounded above by b) so there is a supremum C where $c \leq b$. By defn, given n, $\exists\, x_n \in S$ s.t. $c - \frac{1}{n} < x_n \leq c$.

So $x_n \to c$ and since $x_n \in S$, $f(x_n) < \eta$. By continuity of f, $f(x_n) \to f(c)$. Thus have $f(c) \leq \eta$ $(*)$

Now observe that $c \neq b$, for if $c = b$ then $f(b) \leq \eta$ by $(*)$ but we chose $\eta < f(b)$ earlier.



Then for n large, $c + \frac{1}{n} \in [a,b]$ and $c + \frac{1}{n} \to c$ so again by continuity $f(c + \frac{1}{n}) \to f(c)$. But since $c + \frac{1}{n} > c$, $f(c + \frac{1}{n}) \geq \eta$. Thus $f(c) \geq \eta$, so must have $f(c) = \eta$. $\square$

Remark theorem very useful for finding zeros of fixed points.

Example Existence of the n^{th} root of positive real number

$f(x) = x^N$ $x \geq 0$. Let $y > 0$. f is continuous on $[0, 1+y]$ (and $\mathbb{R}$)

$0 = f(0) < y < (1+y)^N = f(1+y)$

By IVT, $\exists\, c \in (0, 1+y)$ s.t. $f(c) = y$ i.e. $c^N = y$

(c is positive N^{th} root of y).

Uniqueness: if $d^N = y$ with $d > 0$ and $d \neq c$: WLOG suppose $d < c \Rightarrow d^N < c^N$ (induction on N) $\Rightarrow y < y$ ※

Analysis - Lecture 8 31/1/21

Bounds of a continuous function

Theorem 2.4 Let $f: [a,b] \to \mathbb{R}$ be continuous. Then $\exists K$ s.t. $|f(x)| \leq K \quad \forall x \in [a,b]$.

Proof By contradiction: suppose false. Then given $n \geq 1$, $n \in \mathbb{Z}$, $\exists \; x_n \in [a,b]$ s.t. $|f(x_n)| > n$.

By Bolzano-Weierstrass, x_n has a convergent subsequence $x_{n_j} \to x$.

Since $a \leq x_{n_j} \leq b$, have $x \in [a,b]$.

By continuity of f, $f(x_{n_j}) \to f(x)$ but $|f(x_{n_j})| > n_j \to \infty$ # $\square$

Theorem 2.5 $f: [a,b] \to \mathbb{R}$ continuous. Then $\exists x_1, x_2 \in [a,b]$ s.t. $f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a,b]$. (i.e. global max/min)

"A continuous function on a closed bounded interval is bounded and attains its bounds"

Proof Let $A = \{f(x) : x \in [a,b]\}$. By Thm 2.4, A is bounded. Since it's nonempty, $\exists \sup A = M$. By defn, given $n \in \mathbb{N}$, $\exists x_n \in [a,b]$ s.t. $M - \frac{1}{n} < f(x_n) \leq M$. (*)

By Bolzano-Weierstrass, $\exists \; x_{n_j} \to x \in [a,b]$ and since $f(x_{n_j}) \to M$ (by (*)) and f is continuous, we have $f(x) = M$ so $x_2 := x$.

$\square$

Similarly for minimum.

Second proof $A = f([a,b])$, $M = \sup A$ as before. Suppose $\nexists x_2 : f(x_2) = M$. Let $g(x) = \frac{1}{M - f(x)}$ $x \in [a,b]$ is defined and continuous. By (2.4) applied to g, $\exists K > 0$ s.t. $g(x) \leq K \quad \forall x \in [a,b]$.

Hence $f(x) \leq M - \frac{1}{K}$ on $[a,b]$: absurd as M is a L.U.B. # $\square$

Note 2.4 and 2.5 false for ~~open~~ non-closed interval e.g. $(0,1]$ $f(x) = \frac{1}{x}$

Now consider inverse functions.

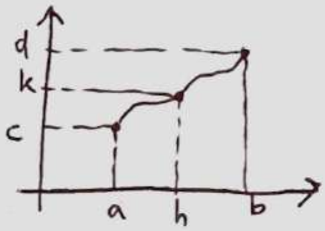
<u>Definition</u> f is increasing for $x \in [a,b]$ if $f(x_1) \leq f(x_2)$ for all x_1, x_2 s.t. $a \leq x_1 \leq x_2 \leq b$.

<u>Theorem 2.6</u> $f : [a,b] \to \mathbb{R}$ continuous and strictly increasing for $x \in [a,b]$. Let $c = f(a)$, $d = f(b)$ then $f : [a,b] \to [c,d]$ is bijective and the inverse $g := f^{-1} : [c,d] \to [a,b]$ is continuous and str. increasing. (similar holds for str. decreasing)

<u>Proof</u> Take $c < k < d$. From IVT, $\exists h$ s.t. $f(h) = k$.

Define $g(k) := h$. (h is unique: f str. increasing)

This gives an inverse $g : [c,d] \to [a,b]$ for f.



And g is strictly increasing: $y_1 < y_2$ $\quad y_1 = f(x_1)$, $y_2 = f(x_2)$

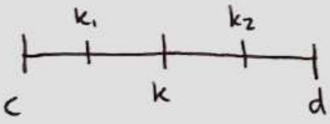
if $x_2 \leq x_1$, since f is increasing $\Rightarrow f(x_2) \leq f(x_1) \Rightarrow y_2 \leq y_1$ ⋇

so $\underline{x_1 < x_2}$. (as required).

g is continuous: let $\varepsilon > 0$. Let $k_1 = f(h - \varepsilon)$, $k_2 = f(h + \varepsilon)$

f str. incr. $\Rightarrow k_1 < k < k_2$

If $k_1 < y < k_2 \Rightarrow h - \varepsilon < g(y) < h + \varepsilon$



choose $\delta = \min\{k_2 - k, k - k_1\}$

if we choose y within δ of k then this inequality holds. Here $k \in (c,d)$. So have g continuous.

(Similar argument establishes continuity at endpoints). $\square$

Analysis - Lecture 9 10/2/21

3 Differentiability (5)

Let $f : E \subseteq \mathbb{C} \to \mathbb{C}$, most of the time $E \subseteq \mathbb{R}$ (interval)

Definition (recall) Let $x \in E$ be a point s.t. $\exists x_n \in E$ with $x_n \neq x$ $\forall n$ and $x_n \to x$. Then x is a limit point.

Definition f is differentiable at x with derivative $f'(x)$ if

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'(x)$$

If f is differentiable at each $x \in E$, we say f is differentiable on E.
(Think of E as an interval or a disc in case of $\mathbb{C}$)

Important remarks

1) Other common notations : $\frac{dy}{dx}$, $\frac{df}{dx}$

2) often written $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $(y = x+h)$

3) "another important look at defn":

Let $\varepsilon(h) := f(x+h) - f(x) - hf'(x)$

then $\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$ then write

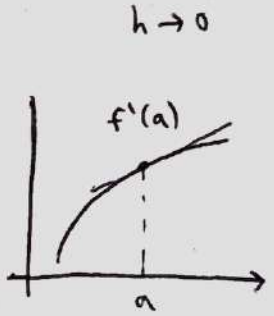
$$f(x+h) = f(x) + \underbrace{hf'(x)}_{\text{linear: } h \mapsto hf'(x)} + \varepsilon(h)$$

Alternative definition : f is differentiable at x if $\exists A$ and ε s.t.

$f(x+h) + f(x) + hA + \varepsilon(h)$ where $\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$ and

if A exists then it's unique, since

$$A = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(unique)}.$$



think of as first linear approximation

4) If f is differentiable at x, then f is <u>continuous</u> at x:

since $\varepsilon(h) \to 0$, then $f(x+h) \to f(x)$ as $h \to 0$

so f must be continuous.

5) Alternatively:

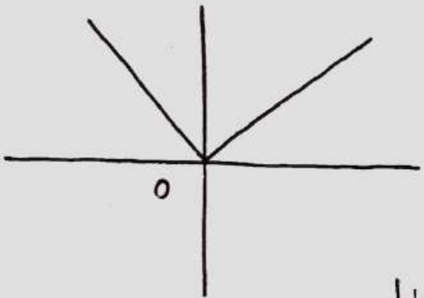
$$f(x+h) = f(x) + hf'(x) + h\varepsilon_f(h)$$

with $\varepsilon_f(h) \to 0$ as $h \to 0$.

or $f(x) = f(a) + (x-a)f'(a) + (x-a)\varepsilon_f(x)$

where $\lim_{x \to a} \varepsilon_f(x) = 0$.

<u>Example</u> $f(x) = |x|$, $f: \mathbb{R} \to \mathbb{R}$



$f'(x) = 1$ if $x > 0$

$f'(x) = -1$ if $x < 0$

Take h_n decreasing, $\to 0$:

$$\lim_{n \to \infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n \to \infty} \frac{h_n}{h_n} = 1$$

take $h_n \to 0$ increasing: $\lim_{n \to \infty} \frac{f(h_n - f(0)}{h_n} = \lim_{n \to \infty} \frac{-h_n}{h_n} = -1$

so <u>not</u> differentiable, at $x = 0$.

<u>Differentiation of sums, products, etc</u>

<u>Proposition 3.1</u> (i) $f(x) = c \quad \forall x \in E \Rightarrow f$ differentiable, $f'(x) = 0$

(ii) f, g differentiable at x: so is $f+g$ and $(f+g)'(x) = f'(x) + g'(x)$

(iii) f, g differentiable at x: so is fg and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

(iv) f differentiable at x, $f(x) \neq 0 \quad \forall x \in E$, then $\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f(x))^2}$
($\frac{1}{f}$ differentiable).

Proof (i) $\lim_{h\to 0} \frac{c-c}{h} = \underline{0}$.

(ii) $\lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$

$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f'(x)+g'(x)$ by limit properties

(iii) $\phi(x) = f(x)\,g(x)$

$$\frac{\phi(x+h)-\phi(x)}{h} = \frac{f(x+h)\,g(x+h)-f(x)\,g(x)}{h}$$

$$= \underbrace{f(x+h)}_{\substack{\downarrow \\ \to f(x) \\ \text{(continuous)}}} \underbrace{\frac{(g(x+h)-g(x))}{h}}_{\substack{\downarrow \\ \to g'(x)}} + \underbrace{\frac{g(x)\,(f(x+h)-f(x))}{h}}_{\substack{\downarrow \\ \to g(x)\,f'(x)}}$$

by standard properties: let $h\to 0$ to give $f(x)\,g'(x)+g(x)\,f'(x)$

(iv) $\phi(x) = \frac{1}{f(x)}$

$$\frac{\phi(x+h)-\phi(x)}{h} = \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \frac{f(x)-f(x+h)}{h\,f(x)\,f(x+h)}$$

$\to \frac{-f'(x)}{f(x)^2}$ by properties of limits and continuity as claimed. $\square$

<u>Remark</u> from (iii) and (iv) we get

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x)-f(x)\,g'(x)}{g(x)^2}$$

Analysis - Lecture 10

11/2/21

Example $f(x) = x^n, \quad n \in \mathbb{Z}, n > 0$

$n = 1, \quad f(x) = x, \quad f'(x) = 1$

Claim $f'(x) = nx^{n-1}$

Proof Induction : true for $n = 1$

Suppose true : $f(x) = x x^n \Rightarrow f'(x) = x^n + x(nx^{n-1}$

$= \underline{(n+1)x^n}$ □

$f(x) = x^{-n} = \dfrac{1}{x^n} \quad n \in \mathbb{Z}, n > 0$

Use Prop 3.1 (iv) : $f'(x) = \dfrac{-(x^n)'}{x^{2n}} = \underline{-nx^{n-1}}$

Theorem 3.2 (Chain rule) Let $f : U \to \mathbb{C}$ s.t. $f(x) \in V \ \forall x \in U$. If f is differentiable at $a \in U$ and $g : V \to \mathbb{C}$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = f'(a) g'(f(a))$$

Proof $f(x) = f(a) + (x-a) f'(a) + \varepsilon_f(x)(x-a) \quad \left(\lim_{x \to a} \varepsilon_f(a) = 0\right)$

$g(y) = g(b) + (y-b) g'(b) + \varepsilon_g(y)(y-b) \quad \left(\lim_{y \to b} \varepsilon_g(y) = 0\right)$

$b = f(a)$

Set $\varepsilon_f(a) = 0, \quad \varepsilon_g(b) = 0$ to make them continuous at $x = a$, $y = b$.

Now $y = f(x)$ gives $g(f(x)) = g(b) + (f(x) - b) g'(b) + \varepsilon_g(f(x))(f(x) - b)$

$= g(f(a)) + \left[(x-a) f'(a) + \varepsilon_f(x)(x-a)\right]\left(g'(b) + \varepsilon_g(f(x))\right)$

$= g(f(a)) + (x-a) f'(a) g'(b) + (x-a)\underbrace{\left[\varepsilon_f(x) g'(b) + \varepsilon_g(f(x))\left(f'(a) + \varepsilon_f(x)\right)\right]}_{\sigma(x) \text{ error term}}$

$\sigma(x) = \underset{\to 0}{\varepsilon_f(x)} g'(b) + \underset{\to 0 \text{ (continuity)}}{\varepsilon_g(f(x))} \underset{\to f'(a)}{\left(f'(a) + \varepsilon_f(x)\right)}$

want $\sigma(x) \to 0$ as $x \to a$ so $\to 0$ as required.

then certainly $(g(f(x)))'$ evaluated at $x = a$ gives $f'(a) g'(f(a))$. □

Examples 1) $f(x) = \sin(x^2)$

(will later prove $(\sin x)' = \cos x$)

$f'(x) = 2x\cos(x^2)$

2) $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ — (continuous at every x)

Differentiable at every $x \neq 0$ by previous theorem

At $x = 0$, $\dfrac{f(x) - f(0)}{x - 0} = \dfrac{x\sin\left(\frac{1}{x}\right)}{x} = \sin\left(\frac{1}{x}\right)$.

but $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist, so not differentiable at $x = 0$.

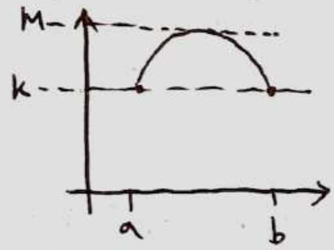
The Mean Value Theorem

Theorem 3.3 (Rolle's theorem)

$f : [a,b] \to \mathbb{R}$ continuous and differentiable on $[a,b]$.

If $f(a) = f(b)$, then $\exists\, c \in (a,b)$ s.t. $f'(c) = 0$.

Proof Let $M = \max_{x \in [a,b]} f(x)$, $m = \min_{x \in [a,b]} f(x)$



Recall (Thm 2.5) these exist.

Let $k = f(a)$. If $M = m = k$ then f is constant and hence $f'(c) = 0$ $\forall c \in (a,b)$.

Otherwise, either $M > k$ or $m < k$.

Suppose $M > k$. By thm 2.5, $\exists\, c \in (a,b)$ s.t. $f(c) = M$

If $f'(c) > 0$, then there are values to the right of c for which $f(x) > f(c)$ since ~~f(c+h) = hf~~

$f(h+c) - f(c) = h(f'(c) + \varepsilon(h)) > 0$ since $\varepsilon(h) \to 0$ as $h \to 0$ and thus $f'(c) + \varepsilon(h) > 0$ for h small. $\#$ M being max.

Similarly if $f'(c) < 0$, $\exists x$ to the left of c with $f(x) > f(c)$

$\Rightarrow \underline{f'(c) = 0}$ as required.

(A similar argument holds for a minimum). $\square$

A simple tweak gives

<u>Theorem 3.4</u> (Mean value theorem) Let $f: [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a, b). Then $\exists c \in (a,b)$ with $f(b) - f(a) = f'(c)(b-a)$.

<u>Proof</u> Write $\phi(x) = f(x) - kx$ with k chosen s.t. $\phi(a) = \phi(b)$.

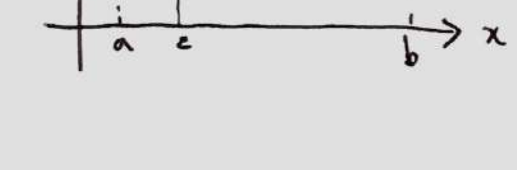
$$\text{Then } f(b) - bk = f(a) - ak \Rightarrow k = \frac{f(b) - f(a)}{b-a}$$

by Rolle's theorem on ϕ (differentiable)

$\exists c \in (a,b)$ s.t. $\phi'(c) = 0$ i.e. $f'(c) = k$

i.e. $f(b) - f(a) = f'(c)(b-a)$. $\square$

(Additional note: what this is saying is that if we draw a line joining $(a, f(a))$ and $(b, f(b))$, then there is some point in (a,b) at which the slope is the same as the line's slope)



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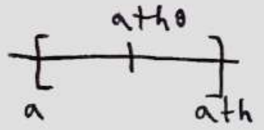
Analysis - Lecture 11

Remark on MVT: often write MVT in form $f(a+h) = f(a) + hf'(a+\theta h)$



$\theta \in (0,1)$

Careful: $\theta = \theta(h)$

Corollary 3.5 $f:[a,b] \to \mathbb{R}$ continuous, differentiable on (a,b).

Then (i) If $f'(x) > 0 \ \forall x \in (a,b)$ then f is strictly increasing on $[a,b]$.

(ii) If $f'(x) \geq 0 \ \forall x \in (a,b)$ then f is increasing on $[a,b]$

(iii) If $f'(x) = 0 \ \forall x \in (a,b)$ then f is constant on $[a,b]$

Proof (i) $f(y) - f(x) = f'(c)(y-x)$ $\quad c \in (x,y)$

Have $f'(c) > 0 \Rightarrow f(y) > f(x)$

(ii) proof same as above but $f'(c) \geq 0 \Rightarrow f(y) \geq f(x)$

(iii) Take $x \in [a,b]$ then use MVT in $[a,x]$ to get $c \in (a,x)$ with $f(x) - f(a) = f'(c)(x-a) = 0 \Rightarrow \underline{f(x) = f(a)}$ $\forall x \in [a,b]$. □

Inverse rule / inverse function theorem (see also Analysis & Topology, IB)

Theorem 3.6 $f:[a,b] \to \mathbb{R}$ continuous, differentiable on (a,b) with $f'(x) > 0 \quad \forall x \in (a,b)$. Then let $f(a) = c$ and $f(b) = d$.

Then $f: [a,b] \to [c,d]$ is bijective and f^{-1} is differentiable on (c,d) with $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

Proof By Corollary 3.5, f is strictly increasing on $[a,b]$. By Theorem 2.6 $\exists g: [c,d] \to [a,b]$ continuous, str. increasing inverse of f.

Need $g'(y) = \dfrac{1}{f'(x)}$, $y = f(x)$

$k \neq 0$ given: let h be given by $y + k = f(x+h)$ or $g(y+k) = x+h$

Then $\dfrac{g(y+k) - g(y)}{k} = \dfrac{\cancel{x} + h - \cancel{x}}{f(x+h) - f(x)}$ $\quad$ Let $k \to 0$, then $h \to 0$ (continuity)

Then $g'(y) = \lim\limits_{h \to 0} \dfrac{h}{f(x+h) - f(x)} = \underline{\dfrac{1}{f'(x)}}$ as required. □

<u>Example</u> $g(x) = x^{1/q}$ $(x>0,\ q \in \mathbb{N})$

$f(x) = x^q$ $\big(g(f(x)) = x\big)$

$f'(x) = qx^{q-1}$. Since f is differentiable, so is g: $g'(x) = \dfrac{1}{q(x^{1/q})^{q-1}}$

$= \dfrac{1}{q} x^{(1/q)-1}$ (as expected!)

Generally if $g(x) = x^{p/q}$ $p \in \mathbb{Z}$, $q \in \mathbb{Z}^+$ can find g' by chain rule: $g(x) = (x^p)^{1/q} = (x^{1/q})^p$.

So if $g(x) = x^r$ $r \in \mathbb{Q}$: have $\underline{g'(x) = rx^{r-1}}$.

Suppose $f, g : [a,b] \to \mathbb{R}$: continuous, and differentiable on (a,b) and $g(a) \neq g(b)$. Then MVT on each gives $s, t \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{(b-a)f'(s)}{(b-a)g'(t)} = \frac{f'(s)}{g'(t)}$$

Cauchy showed one can take $s = t$.

<u>Theorem 3.7</u> (Cauchy's mean value theorem) Let $f, g : [a,b] \to \mathbb{R}$ be continuous, differentiable on (a,b). Then $\exists\, t \in (a,b)$ s.t.

$$\big(f(b)-f(a)\big)\, g'(t) = f'(t)\big(g(b)-g(a)\big)$$

(We recover the MVT if we take $g(x) = x$).

Proved next lecture (12).

Analysis - Lecture 12 17/2/21

Theorem 3.7 (Cauchy's MVT) Let $f, g : [a,b] \to \mathbb{R}$ be continuous, differentiable on (a,b). Then $\exists\, t \in (a,b)$ s.t.

$$(f(b)-f(a))\, g'(t) = f'(t)(g(b)-g(a)) \qquad (*)$$

Proof Let
$$\phi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}$$
ϕ is continuous on $[a,b]$, differentiable on (a,b).

Also $\phi(a) = \phi(b) = 0$

By Rolle's theorem $\exists\, t \in (a,b)$ s.t. $\phi'(t) = 0$

Expanding $\phi(x)$ gives $(*)$ immediately. $\square$

Application L'Hôpital's Rule

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x} = ? \qquad \text{Have } \frac{e^x - e^0}{\sin x - \sin 0} = \frac{e^t}{\cos t}$$ for some $t \in (0,x)$ by 3.7

As $x \to 0$, $t \to 0$ so $\frac{e^t}{\cos t} \to 1$.

Want to extend MVT to higher order derivatives

Theorem 3.8 (Taylor's theorem with Lagrange's remainder)
Suppose f and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exists for $x \in (a, a+h)$. Then

$$\cancel{f(a+h) +} \; f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \theta h) \quad \text{where } \theta \in (0,1).$$

Note: 1) For $n=1$ get back the MVT so this is an "n^{th} order MVT"

2) $R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$ is known as Lagrange's form of the remainder

Proof Define for $0 \le t \le h$

$$\phi(t) = f(a+t) - f(a) - tf'(a) - \dots - \frac{t^{n-1}}{(n-1)!} f^{(n-1)}(a) - \frac{t^n}{n!} B,$$

where B is a constant s.t. $\phi(h) = 0$. (similar idea to MVT proof)

We see: $\phi(0) = \phi'(0) = \dots = \phi^{(n-1)}(0) = 0$

Use Rolle's theorem n times: $\phi(0) = \phi(h) = 0 \Rightarrow \phi'(h_1) = 0 \quad 0 < h_1 < h$

$\phi'(0) = \phi'(h_1) = 0 \Rightarrow \phi''(h_2) = 0, \quad 0 < h_2 < h_1$

Finally $\phi^{(n-1)}(0) = \phi^{(n-1)}(h_{n-1}) = 0 \Rightarrow \phi^{(n)}(h_n) = 0.$

$0 < h_n < h_{n-1} < \dots < h$ so $h_n = \theta h \quad (0 < \theta < 1)$

Now $\phi^{(n)}(t) = f^{(n)}(a+t) - B \Rightarrow B = f^{(n)}(a + \theta h)$

Set $t = h$, $\phi(h) = 0$ and put this value of B in the second line. This gives the result. $\square$

Theorem 3.9 (Taylor's theorem with Cauchy's form of remainder)

With same hypothesis as Thm 3.8, $a = 0$ we have

$$f(h) = f(0) + hf'(0) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n \quad \text{with}$$

$$R_n = \frac{(1-\theta)^{n-1} f^{(n)}(\theta h)}{(n-1)!} \qquad \theta \in (0,1)$$

Proof $F(t) = f(h) - f(t) - (h-t) f'(t) - \dots - \frac{(h-t)^{n-1} f^{(n-1)}(t)}{(n-1)!} \quad t \in [0,h]$ ← not meant to be crossed

$$F'(t) = -f'(t) + \cancel{f'(t)} - \cancel{(h-t) f''(t)} + \cancel{(h-t) f''(t)} + \dots - \frac{(h-t)^{n-1}}{(n-1)!} f^{(n)}(t)$$

$$\Rightarrow F'(t) = -\frac{(h-t)^{n-1}}{(n-1)!} f^{(n)}(t)$$

Set $\phi(t) = F(t) - \left[\frac{h-t}{h}\right]^p F(0) \quad p \in \mathbb{Z}, \ \underline{1 \le p \le n}$

then $\phi(0) = \phi(h) = 0$. By Rolle's theorem $\exists\, \theta \in (0,1)$ s.t. $\phi'(\theta h) = 0$.

But $\phi'(\theta h) = F'(\theta h) + \frac{p(1-\theta)^{p-1}}{h} F(0) = 0$

$$\Rightarrow 0 = -\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta h) + \frac{p(1-\theta)^{p-1}}{h}\left[f(h) - f(0) - hf'(0) - \dots - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0)\right]$$

$$\Rightarrow f(h) = f(0) + hf'(0) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{h^n (1-\theta)^{n-1}}{(n-1)!\, p(1-\theta)^{p-1}} f^{(n)}(\theta h)$$

If $p = n$ we get Lagrange, if $p = 1$ we get Cauchy (have choice of p)

To get a Taylor series for f, need to show $R_n \to 0$ as $n \to \infty$.
This requires "estimates" and "effort"

$$R_n = \frac{h^n f^{(n)}(\theta h)}{n!} \text{ Lagrange}, \quad R_n = \frac{(1-\theta)^{n-1} h^n f^{(n)}(\theta h)}{(n-1)!} \text{ Cauchy}$$

Remark Theorems 3.8, 3.9 work equally well in an interval $[a+h, a]$ with $h<0$.

Example Binomial series

$$f(x) = (1+x)^r \qquad r \in \mathbb{Q}$$

Claim If $|x|<1$ then

$$(1+x)^r = 1 + \binom{r}{1}x + \dots + \binom{r}{n}x^n + \dots \text{ where}$$
$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}$$

Proof Clearly: $f^{(n)}(x) = r(r-1)\dots(r-n+1)(1+x)^{r-n}$

If $r \in \mathbb{Z}$, $r \geq 0$ then $f^{(r+1)} \equiv 0$ - we have a deg. r polynomial

In general (Lagrange) $R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = \binom{r}{n} \frac{x^n}{(1+\theta x)^{n-r}} \quad \theta \in (0,1)$

Note: in principle θ depends on both x and n.

For $0<x<1$, $(1+\theta x)^{n-r} > 1$ for $n>r$

Now observe that the series $\sum \binom{r}{n} x^n$ is absolutely convergent for $|x|<1$. By ratio test: $a_n = \binom{r}{n}x^n$ then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{r(r-1)\cdots(r-n+1)(r-n)x^{n+1}}{(n+1)!}\right| \left|\frac{n!}{r(r-1)\dots(r-n+1)x^n}\right|$$

$$= \left|\frac{(r-n)x}{n+1}\right| \to |x| \text{ as } n \to \infty$$

as required, so converges absolutely for $|x|<1$. Then $\binom{r}{n}x^n \to 0$ as $n \to \infty$.

Hence for $n>r$, $0<x<1$, have

$|R_n| \leq \left|\binom{r}{n}x^n\right| = |a_n| \to 0$ as $n \to \infty$ so proved for $0 \leq x < 1$.

Breaks down for $-1<x<0$, but Cauchy's form of R_n works:

$$R_n = \frac{(1-\theta)^{n-1} r(r-1)\dots(r-n+1)(1+\theta x)^{r-n} x^n}{(n-1)!}$$

$$= \frac{r(r-1)\dots(r-n+1)}{(n-1)!} \frac{(1-\theta)^{n-1}}{(1+\theta x)^{n-r}} x^n = r\binom{r-1}{n-1} x^n (1+\theta x)^{r-1} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$$

<1 for $x \in (-1,1)$

$$\Rightarrow |R_n| \leq \left| r\binom{r-1}{n-1} x^n \right| (1+\theta x)^{r-1}.$$

Check!

$(1+\theta x)^{r-1} < \max\{1, (1+x)^{r-1}\}$

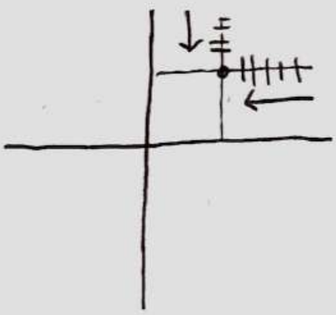
$K_r = r\max\{1, (1+x)^{r-1}\}$ independent of n

so $|R_n| \leq K_r \left| \binom{r-1}{n-1} x^n \right| \to 0$ as $a_n \to 0$. Thus $R_n \to 0$. $\square$

Remarks on complex differentiation

Formally we have the properties regarding sums, products, chain rule etc. But it is much more restrictive than differentiability of functions on the real ~~the~~ line.

Example $f(z) = \bar{z}$ is nowhere $\mathbb{C}$-differentiable.



$$z_n = z + \frac{1}{n} \to z$$

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} + \frac{1}{n} - \bar{z}}{z + \frac{1}{n} - z} = 1$$

different

$$z_n = z + \frac{i}{n} \to z, \quad \frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} - \frac{i}{n} - \bar{z}}{z + \frac{i}{n} - z} = -1$$

so $\lim\limits_{w \to z} \frac{f(w) - f(z)}{w - z}$ does not exist.

On the other hand $f(x,y) = z = x + iy$ "super-differentiable"

$(x, -y)$

see IB Complex Analysis

Analysis - Lecture 14

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4 Power series (4-5)

We want to look at $\sum_{n=0}^{\infty} a_n z^n \quad z \in \mathbb{C} \quad a_n \in \mathbb{C}$

(The case $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, z_0 fixed follows from this by translation)

Lemma 4.1 If $\sum_{0}^{\infty} a_n z_1^n$ converges and $|z| < |z_1|$ then $\sum_{0}^{\infty} a_n z^n$ converges absolutely.

Proof Since $\sum_{0}^{\infty} a_n z_1^n$ converges, $a_n z_1^n \to 0$. Thus $\exists K > 0$ s.t. $|a_n z_1^n| < K \ \forall n$. Then $|a_n z^n| \leq K \left|\frac{z}{z_1}\right|^n$

geometric series converges

so follows by comparison. $\square$

Using this lemma, we'll prove that every power series has a radius of convergence.

Theorem 4.2 A power series either (1) converges absolutely $\forall z$ or (2) converges absolutely $\forall z$ inside circle $|z| = R$ and diverges for all z outside it, or (3) converges for $R = 0$ only.

Definition The circle $|z| = R$ is called the circle of convergence and R is the radius of convergence. (1) : $R = \infty$, (3): $R = 0$

Proof Let $S = \{ x \in \mathbb{R} : x \geq 0$ and $\sum a_n x^n$ converges $\}$. Have $0 \in S$.

By 4.1: $x_1 \in S \Rightarrow [0, x_1] \in S$.

If $S = [0, \infty)$ then have case (1)

If not, there exists a finite supremum $R \geq 0$: $R = \sup S < \infty$

If $R > 0$, we'll prove that if $|z_1| < R$, then $\sum a_n z_1^n$ converges absolutely:

PTO

Choose R_0 s.t. $|z_1| < R_0 < R$. Then $R_0 \in S$ and series converges for $z = R_0$. By 4.1, $\sum |a_n z_1^n|$ converges.

Finally show:

$|z_2| > R \geq 0 \Rightarrow$ series does not converge for z_2. Take R_0 s.t. $R < R_0 < |z_2|$. If $\sum a_n z_2^n$ converges, by 4.1, $\sum a_n R_0^n$ would be convergent, contradicting $R = \sup S$. $\square$

The following lemma is useful:

<u>Lemma 4.3</u> If $\left|\frac{a_{n+1}}{a_n}\right| \to l$ as $n \to \infty$, then $\underline{R = \frac{1}{l}}$.

<u>Proof</u> By ratio test: have absolute convergence if

$$\lim \left|\frac{a_{n+1}}{a_n} \frac{z^{n+1}}{z^n}\right| < 1 \Rightarrow \text{if } |z| < \frac{1}{l} \text{ we have } \underline{\text{absolute}} \text{ convergence.}$$

If $|z| > \frac{1}{l}$ the series diverges, again by ratio test. $\square$

<u>Remark / Exercise</u> One can also use root test to get that if $|a_n|^{1/n} \to l$, then $R = \frac{1}{l}$.

<u>Examples</u> 1) $\sum_0^\infty \frac{z^n}{n!}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 = l$
$\Rightarrow \underline{R = \infty}$

2) Geometric series $\sum_0^\infty z^n$

$R = 1$. Note that at $|z| = 1$ have divergence

3) $\sum_0^\infty n!\, z^n$ has $R = 0$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{n!} = n+1 \to \infty$

Only converges at $z = 0$

4) $\sum_1^\infty \frac{z^n}{n}$ has $R = 1$ (ratio test) but diverges for $z = 1$ (HS)

What if $|z| = 1$ and $z \neq 1$?

Consider $\sum_1^\infty \frac{z^n}{n}(1-z)$ (see ES 1)

$$S_N = \sum_1^N \left[\frac{z^n - z^{n+1}}{n}\right] = \sum_1^N \frac{z^n}{n} - \sum_1^N \frac{z^{n+1}}{n}$$

$$= \sum_1^N \frac{z^n}{n} - \sum_2^{N+1} \frac{z^n}{n-1} = z - \frac{z^{N+1}}{N} + \sum_2^{N+1} \frac{-z^n}{n(n-1)}$$

If $|z| = 1$, $\frac{z^{N+1}}{N} \to 0$ as $N \to \infty$ and $\sum_2^\infty \frac{1}{n(n-1)}$ converges,

so S_N converges $\Rightarrow$ $\sum_1 \frac{z^n}{n}$ converges for all z with $|z| = 1$, $z \neq 1$.

5) $\sum_1^\infty \frac{z^n}{n^2}$, $R = 1$ and converges $\forall z$ with $|z| = 1$

6) $\sum_0^\infty n z^n$, $R = 1$ but diverges for all $|z| = 1$

<u>Conclusion</u> In principle nothing can be said about $|z| = R$ and each case must be discussed separately.

Within the radius of convergence, "life is great": power series behave as if they were polynomials.

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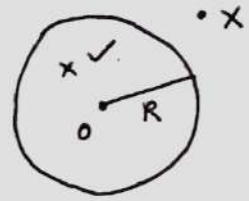
Analysis – Lecture 15

From last time $\sum_{0}^{\infty} a_n z^n$

Radius of convergence R



Theorem 4.4 $f(z) = \sum_{0}^{\infty} a_n z^n$ has radius of convergence R.
Then f is differentiable at all points with $|z| < R$ with

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \quad \text{(i.e. can differentiate term by term).}$$

Proof (non-examinable)

Lemma 4.5 If $\sum_{0}^{\infty} a_n z^n$ has radius of convergence R, then so do $\sum_{1}^{\infty} n a_n z^{n-1}$ and $\sum_{2}^{\infty} n(n-1) a_n z^{n-2}$

Lemma 4.6 (i) $\binom{n}{r} \leq n(n-1)\binom{n-2}{r-2} \quad \forall\, 2 \leq r \leq n$

(ii) $|(z+h)^n - z^n - nhz^{n-1}| \leq n(n-1)(|z|+|h|)^{n-2}|h|^2 \quad \forall z, h \in \mathbb{C}$

Proof of 4.4

By Lemma 4.5 may define $f'(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}, \quad |z| < R$

RTP: $\lim_{h \to 0} \dfrac{f(z+h) - f(z) - hf'(z)}{h} = 0$.

Define $I := \dfrac{f(z+h) - f(z) - hf'(z)}{h} = \dfrac{1}{h} \sum_{0}^{\infty} a_n\left((z+h)^n - z^n - hnz^{n-1}\right)$

$$|I| = \frac{1}{|h|} \left| \lim_{N \to \infty} \sum_{0}^{N} a_n\left((z+h)^n - z^n - nhz^{n-1}\right) \right|$$

continuity $:= \dfrac{1}{|h|} \lim_{N \to \infty} \left| \sum_{0}^{N} a_n\left((z+h)^n - z^n - nhz^{n-1}\right) \right|$

$$\leq \frac{1}{|h|} \sum_{0}^{\infty} |a_n| \, |(z+h)^n - z^n - hnz^{n-1}|$$

by 4.6 (ii)

$$\leqslant \frac{1}{|h|} \sum_{2}^{\infty} |a_n| n(n-1) \left(|z| + |h|\right)^{n-1} |h|^2$$

$$= |h| \sum_{2}^{\infty} |a_n| n(n-1) \left(|z| + |h|\right)^{n-2}$$

By Lemma 4.5, for $|h|$ small enough, $\sum_{2}^{\infty} |a_n| n(n-1)(|z|+|h|)^{n-2}$ converges to $A(h)$, but $A(h) \leqslant A_r$ for $|h| < r$ and $|z| + r < R$

$$\Rightarrow |I| \leqslant |h| A(h) \leqslant |h| A(r) \to 0 \text{ as } h \to 0. \qquad \square$$

<u>Proof of 4.5</u> Take z and R_0 s.t. $0 < |z| < R_0 < R$. Since $a_n R_0^n \to 0$, $\exists K$ s.t. $|a_n R_0^n| \leqslant K \quad \forall n \geqslant 0$

Thus $$|a_n n z^{n-1}| = \frac{n}{|z|} |a_n R_0^n| \left|\frac{z}{R_0}\right|^n \leqslant \frac{kn}{|z|} \left|\frac{z}{R_0}\right|^n$$

But $\sum n \left|\frac{z}{R_0}\right|^n$ converges by ratio test:

$$\frac{n+1}{n} \left|\frac{z}{R_0}\right|^{n+1} \left|\frac{R_0}{z}\right|^n = \frac{n+1}{n} \left|\frac{z}{R_0}\right| \to |z/R_0| < 1$$

If $|z| > R$, series diverges as $|a_n z^n|$ is unbounded, hence so is $n|a_n z^n|$.

The same proof applies to $\sum_{2}^{\infty} n(n-1) a_n z^{n-2}$. $\square$

<u>Proof of 4.6</u>

(i) $$\binom{n}{r} \Big/ \binom{n-2}{r-2} = \frac{n!}{r!(n-r)!} \frac{(r-2)!(n-r)!}{(n-2)!} = \frac{n(n-1)}{r(r-1)} \leqslant n(n-1) \checkmark$$

(ii) $(z+h)^n - z^n - nhz^{n-1} = \sum_{r=2}^{n} \binom{n}{r} z^{n-r} h^r$

Thus $\left|(z+h)^n - z^n - nhz^{n-1}\right| \leq \sum_{r=2}^{n} \binom{n}{r} |z|^{n-r} |h|^r$

$$\overset{(i)}{\leq} n(n-1) \underbrace{\left(\sum_{r=2}^{n} \binom{n-2}{r-2} |z|^{n-r} |h|^{r-2} \right)}_{(|z|+|h|)^{n-2}} |h|^2$$

$$= \cancel{n(n-1)(|z|+|h|)^{n-2}}$$

$$= n(n-1)(|z|+|h|)^{n-2} \qquad \square$$

The standard functions

We have already seen that $\sum_0^\infty \frac{z^n}{n!}$ has $R = \infty$

Define $e : \mathbb{C} \to \mathbb{C}$ by $e(z) = \sum_0^\infty \frac{z^n}{n!}$

Straight from 4.4: e is differentiable and $e'(z) = e(z)$.

Observation: If $F : \mathbb{C} \to \mathbb{C}$ has $F'(z) = 0 \ \forall z \in \mathbb{C}$ then F is const.

Proof in $\mathbb{C}$ consider $g(t) = F(tz) = u(t) + iv(t)$

By chain rule: $g'(t) = F'(tz)\, z = 0$

$= u'(t) + iv'(t)$ (check)

$\Rightarrow u' = v' = 0$ real-valued so by Corollary 3.5 it's constant. $\square$

Now let $a, b \in \mathbb{C}$ and consider $F(z) = e(a+b-z)\, e(z)$

$F'(z) = -e(a+b-z)\, e(z) + e(a+b-z)\, e(z) = 0$

$\Rightarrow F$ is constant $\Rightarrow e(a+b-z)\, e(z) = F(0) = e(a+b)$

Set $z = b$: $e(a)\, e(b) = e(a+b)$.

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$e(z) = \sum_{0}^{\infty} \frac{z^n}{n!}$, $R = \infty$, $e : \mathbb{C} \to \mathbb{C}$

Last time: 1) $e'(z) = e(z)$ 2) $e(a+b) = e(a)e(b)$

Now restrict our function $e : \mathbb{R} \to \mathbb{R}$

Theorem 4.7 (i) $e : \mathbb{R} \to \mathbb{R}$ everywhere differentiable

(ii) $e(x+y) = e(x)e(y)$ (iii) $e(x) > 0 \quad \forall x \in \mathbb{R}$

(iv) e is strictly increasing (v) $e(x) \to \infty$ as $x \to \infty$, $e(x) \to 0$ as $x \to -\infty$

(vi) $e : \mathbb{R} \to (0, \infty)$ is a bijection

Proof (i) and (ii) done last time

(iii) Clearly $e(x) > 0 \quad \forall x \geq 0$ and $e(0) = 1$

Also $e(0) = e(x - x)$ so by (2) $e(0) = 1 = e(x)e(-x)$

$\Rightarrow e(-x) = \frac{1}{e(x)} \Rightarrow e(x) \geq 0 \quad \forall x < 0$.

(iv) $e'(x) = e(x) > 0 \quad \forall x$

(v) $e(x) > 1 + x$ for $x > 0$, so if $x \to \infty$ then $e(x) \to \infty$

For $x > 0$, since $e(-x) = \frac{1}{e(x)}$, $e(x) \to 0$ as $x \to -\infty$

(vi) Strictly increasing $\Rightarrow$ injective

Surjectivity: Take $y \in (0, \infty)$.

Since $e(x) \to \infty$ as $x \to \infty$ and $e(x) \to 0$ as $x \to -\infty$

$\Rightarrow \exists a, b \in \mathbb{R}$ s.t. $e(a) < y < e(b)$

so by IVT ($e(x)$ continuous) $\exists c \in (a, b)$ with $e(c) = y$.

So e is bijective.

$\square$

Remark $e : (\mathbb{R}, +) \to ((0, \infty), \times)$ is a group isomorphism

e is a bijection $\therefore$ consider inverse function $l : (0, \infty) \to \mathbb{R}$

Theorem 4.8 (i) $l : (0, \infty) \to \mathbb{R}$ is a bijection and

$l(e(x)) = x \quad \forall x \in \mathbb{R}, \quad e(l(t)) = t \quad \forall t \in (0, \infty)$

(ii) l is differentiable and $l'(t) = \frac{1}{t}$

(iii) $l(xy) = l(x) + l(y) \quad \forall x, y \in (0, \infty)$

Proof (i) obvious from definition

(ii) Inverse rule (thm 3.6): l is differentiable with

$$l'(t) = \frac{1}{e(l(t))} = \frac{1}{t}$$

(iii) From IA groups: if e is an isomorphism, so is its inverse $\square$

Now define for $\alpha \in \mathbb{R}$, $x > 0$:

$$r_\alpha(x) \stackrel{\text{def}}{=} e(\alpha\, l(x))$$

Theorem 4.9 Suppose $x, y > 0$, $\alpha, \beta \in \mathbb{R}$. Then

(i) $r_\alpha(xy) = r_\alpha(x)\, r_\alpha(y)$

(ii) $r_{\alpha+\beta}(x) = r_\alpha(x) \cdot r_\beta(x)$

(iii) $r_\alpha(r_\beta(x)) = r_{\alpha\beta}(x)$

(iv) $r_1(x) = x, \quad r_0(x) = 1$

Proof (i) $r_\alpha(xy) = e(\alpha\, l(xy)) = e(\alpha l(x) + \alpha l(y))$

$$= e(\alpha l(x))\, e(\alpha l(y)) = r_\alpha(x)\, r_\alpha(y)$$

(ii) $r_{\alpha+\beta}(x) = e((\alpha+\beta) l(x)) = e(\alpha l(x))\, e(\beta l(x))$

$$= r_\alpha(x)\, r_\beta(x)$$

(iii) $r_\alpha(r_\beta(x)) = r_\alpha(e(\beta l(x))) = e(\alpha\, l(e(\beta l(x))))$

$$= e(\alpha\beta\, l(x)) = r_{\alpha\beta}(x)$$

(iv) $r_1(x) = e(l(x)) = x$

$r_0(x) = e(0) = 1$ $\square$

$r_n(x)$ $n \geq 1,\ n \in \mathbb{Z}$:

$$r_n(x) = r_{\underbrace{1+1+\dots+1}_{n}}(x) = \underbrace{x \cdot x \cdot x \cdots x}_{n} = x^n$$

$$r_1(x)\, r_{-1}(x) = r_0(x) = 1, \quad r_{-1}(x) = \frac{1}{x}, \quad r_{-n}(x) = \frac{1}{x^n}$$

$q \in \mathbb{Z},\ q \geq 1$:

$$\left(r_{1/q}(x)\right)^q = r_1(x) = x \Rightarrow r_{1/q}(x) = x^{1/q}$$

Similarly $r_{p/q}(x) = \left(r_{1/q}(x)\right)^p = \underline{x^{p/q}}$

Then $r_\alpha(x)$ agrees with x^α, $\alpha \in \mathbb{Q}$ as previously defined.

$\exp(x) = e(x)$ $\qquad$ $\log(x) = l(x)$ $\qquad$ $x^\alpha = r_\alpha(x)$

$x \in \mathbb{R}$ $\qquad$ $x \in (0, \infty)$ $\qquad$ $\alpha \in \mathbb{R},\ x \in (0, \infty)$

$e(x) = e(x \log e) = r_x(e) = e^x$ if we define

$e := \sum_0^\infty \frac{1}{n!} = e(1)$ so $\exp(x)$ is also a power. (e^x)

Finally we compute $(x^\alpha)'$

$$(x^\alpha)' = \left(e^{\alpha \log x}\right)' = e^{\alpha \log x} \cdot \frac{\alpha}{x} = x^\alpha \frac{\alpha}{x} = \underline{\alpha x^{\alpha - 1}}$$

Also note: if $f(x) = a^x$, $a > 0$ then

$$f'(x) = \left(e^{x \log a}\right)' = e^{x \log a} \log a = \underline{a^x \log a}$$

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Remark "Exponentials beat polynomials"

$$\lim_{x\to\infty} \frac{e^x}{x^k} \quad (k>0) \; : \; \to \infty$$

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} > \frac{x^n}{n!} \quad \text{for } x>0 \text{ and pick } n>k$$

Then $\frac{e^x}{x^k} > \frac{x^{n-k}}{n!}$ $((n-k)>0) \to \infty$ as $x\to\infty$.

Trigonometric Functions

Define: $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_0^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_0^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$

Use ratio test: check both have infinite radius of convergence

By Thm 4.4 they're differentiable: $(\sin z)' = \cos z$, $(\cos z)' = -\sin z$

By power series: $\underline{e^{iz} = \cos z + i\sin z}$, similarly $\underline{e^{-iz} = \cos z - i\sin z}$

so $\underline{\cos z = \frac{1}{2}(e^{iz} + e^{-iz})}$, $\underline{\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})}$ $(*)$

(Addition formulas follow from these - use $e^a e^b = e^{a+b}$)
as does $\cos^2 z + \sin^2 z = 1$

Now if $x \in \mathbb{R}$ then $\sin x, \cos x \in \mathbb{R}$

and $(*) \Rightarrow |\sin x|, |\cos x| \leq 1$ <u>only always true in $\mathbb{R}$</u>

Periodicity

Proposition 4.10 There is a smallest $\omega > 0$ $\left(\sqrt{2} < \frac{\omega}{2} < \sqrt{3}\right)$

s.t. $\cos\left(\frac{\omega}{2}\right) = 0$

Proof If $0 < x < 2$: $\sin x = \underbrace{\left(x - \frac{x^3}{3!}\right)}_{>0} + \underbrace{\left(\frac{x^5}{5!} - \frac{x^7}{7!}\right)}_{>0} + \dots > 0$

So for $0 < x < 2$, $(\cos x)' = -\sin x < 0 \Rightarrow \cos x$ is strictly decreasing

We'll show $\cos\sqrt{2} > 0$, $\cos\sqrt{3} < 0$: sufficient by IVT.

$$\cos(\sqrt{2}) = \underbrace{\left(\frac{(\sqrt{2})^4}{4!} - \frac{(\sqrt{2})^6}{6!}\right)}_{>0} + \underbrace{(\quad)}_{>0} + \underbrace{(\quad)}_{>0} + \dots > 0$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \underbrace{\left(\frac{x^6}{6!} - \frac{x^8}{8!}\right)}_{>0} - \dots$$

$$x = \sqrt{3} \qquad \cos(\sqrt{3}) = 1 - \frac{3}{2} + \frac{9}{4 \times 3 \times 2} + \dots = 1 - \frac{3}{2} + \frac{3}{8} + \dots$$

$$= -\frac{1}{8} - (>0) < 0$$

□

<u>Corollary 4.11</u> $\sin\frac{w}{2} = 1$

<u>Proof</u> $\sin^2\frac{w}{2} + \underbrace{\cos^2\frac{w}{2}}_{0} = 1$, $\sin\frac{w}{2} > 0$ so $\sin\frac{w}{2} = 1$. □

Define <u>$\pi = w$</u>

<u>Theorem 4.12</u> (1) $\sin\left(z + \frac{\pi}{2}\right) = \cos z$, $\cos\left(z + \frac{\pi}{2}\right) = -\sin z$

(2) $\sin(z + \pi) = -\sin z$, $\cos(z + \pi) = -\cos z$

(3) $\sin(z + 2\pi) = \sin z$, $\cos(z + 2\pi) = \cos z$

Immediate from addition formulas. □

This implies $e^{iz + 2\pi i} = \cos(z + 2\pi) + i\sin(z + 2\pi)$

$$= \cos(z) + i\sin z = e^{iz}$$

so e^z is periodic with period $2\pi i$.

Analysis - Lecture 18 1/3/21

Remark Given two vectors $x, y \in \mathbb{R}^2$, define $x \cdot y$ as in V&M:

$$x \cdot y = x_1 y_1 + x_2 y_2 \quad \text{where } x = (x_1, x_2),\ y = (y_1, y_2)$$

By Cauchy - Schwarz $|x \cdot y| \leq \|x\| \|y\|$ where $\|x\|^2 = x_1^2 + x_2^2$

Hence for $x \neq 0$, $y \neq 0$, $-1 \leq \frac{x \cdot y}{\|x\| \|y\|} \leq 1$ so we define angle between x, y as the unique θ s.t. $\cos\theta = \frac{x \cdot y}{\|x\| \|y\|}$

Definition $\cosh z = \frac{1}{2}(e^z + e^{-z})$, $\sinh z = \frac{1}{2}(e^z - e^{-z})$

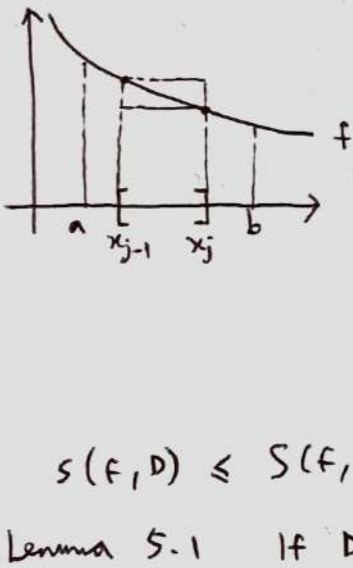
$\Rightarrow \cosh z = \cos(iz)$, $\sinh z = -i \sin(iz)$ check std. properties

5 Integration

$f : [a,b] \to \mathbb{R}$ bounded : $\exists k$ s.t. $|f(x)| \leq k$, $\forall x \in [a,b]$

Definition A dissection or partition of $[a,b]$, D, is a finite subset of $[a,b]$ containing endpoints a, b. Write

$D = \{x_0, x_1, \dots, x_n\}$ with $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$



Definition Define upper sum and lower sum associated with D by

$$S(f, D) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x) \quad \text{(upper)}$$

$$s(f, D) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x) \quad \text{(lower)}$$

$s(f, D) \leq S(f, D) \quad \forall D$

Lemma 5.1 If D and D' are dissections with $D' \supseteq D$, then

$S(f, D) \geq S(f, D') \geq s(f, D') \geq s(f, D)$

Proof $S(f, D') \geq s(f(D')$ is obvious

Suppose D' contains an extra point than D, say $y \in (x_{r-1}, x_r)$

Clearly : $\sup_{x \in [x_{r-1}, y]} f(x)$, $\sup_{x \in [y, x_r]} f(x) \leq \sup_{x \in [x_{r-1}, x_r]} f(x)$

$$\Rightarrow (x_r - x_{r-1}) \sup_{x\in[x_{r-1},x_r]} f(x) \geq (y - x_{r-1}) \sup_{x\in[x_{r-1},y]} f(x) + (x_r - y) \sup_{x\in[y,x_r]} f(x)$$

$$\Rightarrow S(f, D) \geq s(f, D')$$

The same for s and the same if D' has more extra points. $\square$

<u>Lemma 5.2</u> D_1, D_2 two arbitrary dissections - then

$$S(f, D_1) \geq S(f, D_1 \cup D_2) \geq s(f, D_1 \cup D_2) \geq s(f, D_2)$$

so $S(f, D_1) \geq s(f, D_2)$

<u>Proof</u> Take $D' = D_1 \cup D_2$, apply Lemma 5.1 $\square$

<u>Definition</u> The <u>upper integral</u> of f is

$$I^*(f) = \inf_D S(f, D)$$

(always exists - bounded below by lower sum)

and the <u>lower integral</u> of f is

$$I_*(f) = \sup_D s(f, D)$$

By Lemma 5.2, $I^*(f) \geq I_*(f)$ as $S(f, D_1) \geq s(f, D_2)$

$I^*(f) = \inf_{D_1} S(f, D_1) \geq s(f(D_2))$, $I^*(f) \geq \sup_{D_2} s(f, D_2) = I_*(f)$

<u>Definition</u> A bounded function $f: [a, b] \to \mathbb{R}$ is said to be <u>(Riemann) integrable</u> if $I^*(f) = I_*(f)$ and we set

$$\int_a^b f(x)\, dx = I^*(f) = I_*(f) = \int_a^b f$$

<u>Example</u> $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0, 1] \\ 0 & x \notin \mathbb{Q} \cap [0, 1] \end{cases}$

$f: [0, 1] \to \mathbb{R}$; f is <u>not</u> Riemann integrable

$\sup_{[x_{j-1}, x_j]} f(x) = 1$, $\inf_{[x_{j-1}, x_j]} f(x) = 0$

$\Rightarrow S(f, D) = 1$, $s(f, D) = 0$ $\forall D$ so $I^*(f) = 1$, $I_*(f) = 0$

<u>not</u> Riemann integrable.

Analysis - Lecture 19

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A useful criterion for integrability:

Theorem 5.3 A bounded function $f: [a,b] \to \mathbb{R}$ is Riemann integrable iff given $\varepsilon > 0$, $\exists D$ s.t. $S(f, D) - s(f, D) < \varepsilon$.

Proof For every dissection D we have $0 \leq I^*(f) - I_*(f) \leq S(f, D) - s(f, D)$ so if given condition holds, then

$$0 \leq I^*(f) - I_*(f) \leq S(f, D) - s(f, D) < \varepsilon \quad \forall \varepsilon > 0 \implies I^*(f) = I_*(f).$$

Conversely if f is integrable, by defn of sup, inf, there are partitions D_1 and D_2 s.t $\int_a^b f - \frac{\varepsilon}{2} = I_*(f) - \frac{\varepsilon}{2} < s(f, D_1)$

and $S(f, D_2) < I^*(f) + \frac{\varepsilon}{2} = \int_a^b f + \frac{\varepsilon}{2}$

By Lemma 5.1, $S(f, D_1 \cup D_2) - s(f, D_1 \cup D_2) \leq S(f, D_2) - s(f, D_1)$

$$< \int_a^b f + \frac{\varepsilon}{2} - \int_a^b f + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

We use this to show that monotonic or continuous functions are R. integrable.

Theorem 5.4 $f: [a,b] \to \mathbb{R}$ monotonic. Then f is integrable.

Proof Suppose WLOG that f is increasing.

Then $\sup_{x \in [x_{j-1}, x_j]} f(x) = f(x_j) \qquad \inf_{x \in [x_{j-1}, x_j)} f(x) = f(x_{j-1})$

Thus $$S(f, D) - s(f, D) = \sum_{j=1}^{n} (x_j - x_{j-1}) \left[f(x_j) - f(x_{j-1}) \right] \qquad (*)$$

Now choose $D = \left\{ a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \dots, b \right\}$

$x_j = a + \frac{(b-a)j}{n}, \quad 0 \leq j \leq n$

$S(f, D) - s(f, D) = \frac{b-a}{n} (f(b) - f(a))$ by $(*)$: $x_j - x_{j-1} = \frac{b-a}{n} \; \forall j$

Take n large enough that $\frac{(b-a)}{n} (f(b) - f(a)) < \varepsilon$

then use Thm 5.3. $\square$

Continuous functions

Lemma 5.5 $f : [a,b] \to \mathbb{R}$ continuous. Given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad (\underline{\forall x} \in [a,b])$ uniform continuity

Point is: this δ works $\underline{\forall} x, y$ with $|x-y| < \delta$ (not same as regular continuity)

Proof Suppose false. Then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, we can find $x, y \in [a,b]$ s.t. $|x-y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Take $\delta = \frac{1}{n}$, to get $(x_n), (y_n)$ with $|x_n - y_n| < \frac{1}{n}$ but

$|f(x_n) - f(y_n)| \geq \varepsilon$

By Bolzano-Weierstrass, $\exists x_{n_k} \to c$

$|y_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| \to 0$

But $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$. Let $k \to \infty$, by continuity of f

~~|f(c) - f~~ $|f(c) - f(c)| \geq \varepsilon \Rightarrow 0 \geq \varepsilon$ # □

Theorem 5.6 If $f : [a,b] \to \mathbb{R}$ is continuous, then f is Riemann integrable.

Proof Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Let $D = \left\{ a + \frac{(b-a)j}{n}, \ j = 0, 1, \ldots, n \right\}$

Choose n large enough that $\frac{b-a}{n} < \delta$. Then for any $x, y \in [x_{j-1}, x_j]$,

$|f(x) - f(y)| < \varepsilon$ since $|x-y| \leq |x_j - x_{j-1}| = \frac{b-a}{n} < \delta$

This means: $\max\limits_{x \in [x_{j-1}, x_j]} f(x) - \min\limits_{x \in [x_{j-1}, x_j]} f(x) = f(p) - f(q) \quad p, q \in [x_{j-1}, x_j]$

max/min exist (cts)

$$\Rightarrow S(f, D) - s(f, D) = \sum_{j=1}^{n} (x_j - x_{j-1}) \left[\max_{x \in [x_{j-1}, x_j]} f(x) - \min_{x \in [x_{j-1}, x_j]} f(x) \right]$$

$$= \sum_{j=1}^{n} \frac{(b-a)}{n} \underbrace{(f(p_j) - f(q_j))}_{< \varepsilon} < \varepsilon (b-a)$$ and use Thm 5.3. □

More complicated functions can be Riemann integrable.

Example $f: [0,1] \to \mathbb{R}$ $\quad f(x) = \begin{cases} 1/q & x = p/q \in (0,1] \text{ lowest form} \\ 0 & \text{otherwise} \end{cases}$

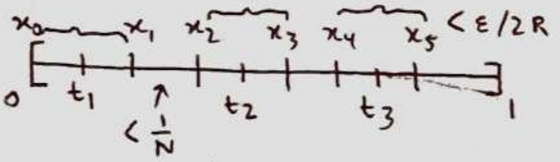
Clearly $s(f,D) = 0 \quad \forall D$. We'll show that given $\varepsilon > 0$, $\exists D$ s.t.
$S(f,D) < \varepsilon \implies f$ integrable with $\int_0^1 f = 0$.

Take $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{\varepsilon}{2}$. Consider set $\{x \in [0,1] : f(x) \geq \frac{1}{N}\}$
$= \{p/q : 1 \leq q \leq N \text{ and } 1 \leq p \leq q\}$ finite with $0 < t_1 < t_2 < \dots < t_R = 1$

Consider dissection D of $[a,b]$ s.t.

(1) each t_k, $1 \leq k \leq R$ is in some $[x_{j-1}, x_j]$

(2) $\forall k$, the unique interval containing t_k has length at most $\frac{\varepsilon}{2R}$



R such intervals

Note $f \leq 1$ everywhere

$S(f,D) \leq \frac{1}{N} + \frac{\varepsilon}{2} < \varepsilon$ $\square$

Elementary properties of integral

f, g bounded, integrable on $[a,b]$

(1) If $f \leq g$ on $[a,b]$ then $\int_a^b f \leq \int_a^b g$

(2) $f+g$ integrable on $[a,b]$: $\int_a^b f+g = \int_a^b f + \int_a^b g$

(3) kf integrable for any k constant: $k\int_a^b f = \int_a^b kf$

(4) $|f|$ is integrable and $\left| \int_a^b f \right| \leq \int_a^b |f|$

(5) product fg is integrable

Proofs (1) if $f \leq g$, then $\int_a^b f = I^*(f) \leq S(f,D) \leq S(g,D)$
hence $\int_a^b f = I^*(f) \leq I^*(g) = \int_a^b g$

(2) $\sup_{[x_{j-1},x_j]} (f+g) \leq \sup_{[x_{j-1},x_j]} f + \sup_{[x_{j-1},x_j]} g \implies S(f+g, D) \leq S(f,D) + S(g,D)$

Take 2 dissections D_1, D_2:
$I^*(f+g) \leq S(f+g, D_1 \cup D_2) \leq S(f, D_1 \cup D_2) + S(g, D_1 \cup D_2)$

Lemma 5.1

$= \quad S(f, D_1) + S(g, D_2)$

Fix D_1 and inf over D_2 to get $I^*(f+g) \le S(f, D_1) + I^*(g)$

Now take inf over all D_1 to get $\underline{I^*(f+g) \le I^*(f) + I^*(g)}$

Similarly $\int_a^b f + \int_a^b g \le I_*(f+g) \Rightarrow f+g$ is integrable with integral equal to the sum of the integrals.

(3) exercise

(4) Consider $f_+(x) = \max(f(x), 0)$. $\sup\limits_{[x_{j-1}, x_j]} f_+ - \inf\limits_{[x_{j-1}, x_j]} f_+ \le \sup\limits_{[x_{j-1}, x_j]} f - \inf\limits_{[x_{j-1}, x_j]} f$ check!

We know: given $\varepsilon > 0$, $\exists D$ s.t. $S(f, D) - s(f, D) < \varepsilon$

$\Rightarrow S(f_+, D) - s(f_+, D) < \varepsilon \Rightarrow f_+$ is integrable.

But $|f| = 2f_+ - f$ so by linearity results follows.

Since $-|f| \le f \le |f|$, property (1) $\Rightarrow \left| \int_a^b f \right| \le \int_a^b |f|$. $\square$

(5) Take f integrable and ≥ 0

Then $\sup\limits_{[x_{j-1}, x_j]} f^2 = \Big(\underbrace{\sup\limits_{[x_{j-1}, x_j]} f}_{M_j}\Big)^2$, $\inf\limits_{[x_{j-1}, x_j]} f^2 = \Big(\underbrace{\inf\limits_{[x_{j-1}, x_j]} f}_{m_j}\Big)^2$

Thus $S(f^2, D) - s(f^2, D)$

$$= \sum_{j=1}^{n} (x_j - x_{j-1})(M_j^2 - m_j^2) = \sum_{j=1}^{n} (x_j - x_{j-1})(M_j + m_j)(M_j - m_j)$$

~~so $\le 2K(S(f,D)$,~~

bounded:
$|f(x)| \le K \quad \forall x \in [a, b]$

so $\le 2K\,(S(f, D) - s(f, D))$

Using 5.3, f^2 is integrable.

Now take any f, then $|f| \geq 0$, integrable. Since $f^2 = |f|^2$, have f^2 integrable $\forall f$.

Finally for fg, note: $\underline{4fg = (f+g)^2 - (f-g)^2}$

$\Rightarrow fg$ integrable. $\square$

Analysis - Lecture 21 8/3/21

<u>Property (6)</u> f integrable on $[a,b]$. If $a<c<b$, then f is integrable on $[a,c]$ and $[c,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

Conversely if f is integrable over $[a,c]$ and $[c,b]$, then f is integrable over $[a,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$

<u>Proof</u> First observe: (1) If D_1 is a dissection of $[a,c]$ and D_2 is a dissection of $[b,c]$ then $D = D_1 \cup D_2$ is a dissection of $[a,b]$ and $S(f, D_1 \cup D_2) = S(f|_{[a,c]}, D_1) + S(f|_{[a,b]}, D_2)$ $(*_1)$

(2) If D is a dissection of $[a,b]$, then

$S(f,D) \geqslant S(f, D \cup \{c\}) = S(f|_{[a,c]}, D_1) + S(f|_{[c,b]}, D_2)$ $(*_2)$

where D_1 dissects $[a,c]$, D_2 dissects $[c,b]$

$(*_1) \Rightarrow I^*(f) \leqslant I^*(f|_{[a,c]}) + I^*(f|_{[c,b]})$

$(*_2) \Rightarrow I^*(f) \geqslant I^*(f|_{[a,c]}) + I^*(f|_{[c,b]})$

so combining, we have equality.

Similarly: $I_*(f) = I_*(f|_{[a,c]}) + I_*(f|_{[a,b]})$

Thus $0 \leqslant I^*(f) - I_*(f) = \underbrace{I^*(f|_{[a,c]}) - I_*(f|_{[a,c]})}_{\geqslant 0} + \underbrace{I^*(f|_{[c,b]}) - I_*(f|_{[a,b]})}_{\geqslant 0}$

from which result follows. $\square$

<u>Fundamental Theorem of Calculus</u>

$f: [a,b] \to \mathbb{R}$ bounded and integrable. Write $F(x) = \int_a^x f(t)\,dt$

<u>Theorem 5.7</u> F is continuous.

<u>Proof</u> $F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$

$|F(x+h) - F(x)| = \left| \int_x^{x+h} f(t)\,dt \right| \leqslant k|h|$: let $\underline{h \to 0}$. $\square$

bound for f on this interval

by (+) on next page

<u>Theorem 5.8</u> (FTC) If f is continuous at x, then F is differentiable at x and $F'(x) = f(x)$.

<u>Proof</u> Consider $\left| \dfrac{F(x+h) - F(x)}{h} - f(x) \right|$ with $x+h \in [a,b]$, $h \neq 0$

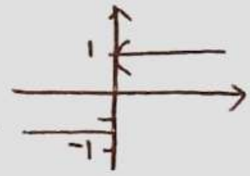
$$= \frac{1}{|h|} \left| \int_x^{x+h} f(t)\,dt - h f(x) \right| = \frac{1}{|h|} \left| \int_x^{x+h} [f(t) - f(x)]\,dt \right|$$

f continuous at x: given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if $|t - x| < \delta$ then $|f(t) - f(x)| < \varepsilon$. If $|h| < \delta$, we can write

$$\frac{1}{|h|} \left| \int_x^{x+h} [f(t) - f(x)]\,dt \right| \leq \frac{1}{|h|} \varepsilon |h| = \varepsilon$$

so $\lim_{h \to 0} \dfrac{F(x+h) - F(x)}{h} = f(x)$. $\square$

<u>Example</u> $f(x) = \begin{cases} -1 & x \in [-1, 0] \\ 1 & x \in [0, 1] \end{cases}$



monotone $\Rightarrow$ integrable

Then $F(x) = -1 + |x|$ continuous : $F(x) = \begin{cases} -x-1 & x \leq 0 \\ x - 1 & x \geq 0 \end{cases}$

<u>Corollary 5.9</u> "integration is inverse of differentiation":

$f = g'$ continuous on $[a, b]$, then $\int_a^x f(t)\,dt = g(x) - g(a)$ $\forall x \in [a, b]$

<u>Proof</u> From 5.8, $F - g$ has zero derivative in $[a, b]$

$\Rightarrow$ $F - g$ is constant and since $F(a) = 0$, $F(x) = g(x) - g(a)$. $\square$

Every continuous function has an antiderivative $\int f(x)\,dx$ determined up to a constant.

<u>Remark</u> Have solved ODE $y'(x) = f(x)$, $y(a) = y_0$ uniquely

<u>Important property</u>: $(\dagger)$

If $|f| \leq K$ on $[a, b]$, then $\left| \int_a^b f \right| \leq K(|b - a|)$

using property (4) from last lecture (20).

Analysis - Lecture 22

Corollary 5.10 (Integration by parts)

Suppose f', g' exist, continuous on $[a,b]$. Then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$$

Proof By product rule, $(fg)' = f'g + fg'$. By 5.9,

$$f(b)g(b) - f(a)g(a) = \int_a^b f'g + \int_a^b fg' \qquad \square$$

Corollary 5.11 (Integration by substitution)

Let $g : [\alpha, \beta] \to [a,b]$ with $g(\alpha) = a$ and $g(\beta) = b$, g' exists and continuous on $[\alpha, \beta]$. Let $f : [a,b] \to \mathbb{R}$ be continuous.

Then $\displaystyle\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))\,g'(t)\,dt$

Proof $F(x) = \displaystyle\int_a^x f(t)\,dt$ as before. Let $h(t) = F(g(t))$

- defined as g takes values in $[a,b]$

Then $\displaystyle\int_\alpha^\beta f(g(t))\,g'(t)\,dt \underset{FTC}{=} \int_\alpha^\beta F'(g(t))\,g'(t)\,dt$

$$\overset{\text{chain rule}}{=} \int_\alpha^\beta h'(t)\,dt = h(\beta) - h(\alpha) = F(b) - F(a) = \int_a^b f(x)\,dx. \qquad \square$$

Theorem 5.12 (Taylor's theorem, integral remainder)

$f^{(n)}(x)$ continuous for $x \in [0,h]$

Then $f(h) = f(0) + \dots + \dfrac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} + R_n$, $\quad R_n = \dfrac{h^n}{(n-1)!}\displaystyle\int_0^1 (1-t)^{n-1} f^{(n)}(th)\,dt$

Proof $u = th$: $R_n = \dfrac{1}{(n-1)!}\displaystyle\int_0^h (h-u)^{n-1} f^{(n)}(u)\,du$

IBP: $R_n = -\dfrac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} + \underbrace{\dfrac{1}{(n-2)!}\displaystyle\int_0^h (h-u)^{n-2} f^{(n-1)}(u)\,du}_{R_{n-1}}$ recurrence relation

IBP $n-1$ times: $R_n = -\dfrac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} - \dots - hf'(0) + \underbrace{\displaystyle\int_0^h f'(u)\,du}_{f(h)-f(0)} \qquad \square$

Can get Cauchy and Lagrange form from this.

However, note that above proof uses <u>continuity</u> of $f^{(n)}$.

<u>Theorem 5.13</u> $f, g : [a,b] \to \mathbb{R}$ continuous, $g(x) \neq 0 \ \forall x \in (a,b)$

Then $\exists \, c \in (a,b)$ s.t. $\displaystyle\int_a^b f(x) g(x) \, dx = f(c) \int_a^b g(x) \, dx$

Note: $g(x) = 1 \Rightarrow \displaystyle\int_a^b f(x) \, dx = f(c)(b-a)$ (like MVT for integrals)

<u>Proof</u> Let $F(x) = \displaystyle\int_a^x fg$, $G(x) = \displaystyle\int_a^x g$ satisfy CMVT (3.7)

So $\exists \, c \in (a,b)$ s.t. $(F(b) - F(a)) G'(c) = F'(c)(G(b) - G(a))$

$$\left(\int_a^b fg \right) g(c) = f(c) g(c) \left(\int_a^b g \right) \text{ by FTC}$$

If $g(c) \neq 0$ we simplify and we're done. $\square$

$R_n = \dfrac{h^n}{(n-1)!} \displaystyle\int_0^1 (1-t)^{n-1} f^{(n)}(th) \, dt$ - use Thm 5.13 with $g \equiv 1$

to get $R_n = \dfrac{h^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta h)$, $\theta \in (0,1)$

precisely Cauchy's form of remainder.

To get Lagrange, use Thm 5.13, $g(t) = (1-t)^{n-1}$ which is > 0

for $(0,1) \Rightarrow \exists \, \theta \in (0,1)$ s.t.

$$R_n = \frac{h^n}{(n-1)!} f^{(n)}(\theta h) \left[\int_0^1 (1-t)^{n-1} \, dt \right]$$

$$\int_0^1 (1-t)^{n-1} \, dt = - \frac{(1-t)^n}{n} \Big|_0^1 = \frac{1}{n}$$

so $R_n = \underline{\dfrac{h^n}{n!} f^{(n)}(\theta h)}$ - Lagrange's form of remainder.

Analysis - Lecture 23

Improper Integrals

Definition Suppose $f : [a, \infty] \to \mathbb{R}$ is integrable (and bounded) on every interval $[a, R]$ and as $R \to \infty$, $\int_a^R f(x)\,dx \to l$

Then we say that $\int_a^\infty f(x)\,dx$ exists/converges and has value l.

If the limit does not exist, we say the integral diverges.

If $\int_a^\infty f = l_1$, $\int_{-\infty}^a f = l_2$ then say $\int_{-\infty}^\infty f = l_1 + l_2$ (indep. of a).

Careful: not the same as $\lim_{R\to\infty} \int_{-R}^R f(x)\,dx$ - it is stronger e.g. $\int_{-R}^R x\,dx = 0$

Example $\int_1^\infty \frac{dx}{x^k}$ converges iff $k > 1$. If $k \neq 1$, $\int_1^R \frac{dx}{x^k} = \left.\frac{x^{1-k}}{1-k}\right]_1^R$

$= \frac{R^{1-k} - 1}{k-1}$ and as $R \to \infty$, this is finite iff $k > 1$.

If $k = 1$, $\int_1^R \frac{dx}{x} = \log R \to \infty$.

Remarks 1) $1/\sqrt{x}$ continuous on $[\delta, 1]$ for any $\delta > 0$ and have $\int_\delta^1 \frac{dx}{\sqrt{x}} = \left. 2\sqrt{x}\right]_\delta^1 = 2 - 2\sqrt{\delta} \to 2$ as $\delta \to 0$

but $\frac{1}{\sqrt{x}}$ is unbounded on $(0, 1]$: $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\delta\to 0} \int_\delta^1 \frac{dx}{\sqrt{x}} = 2$

2) If $f \geq 0$ and $g \geq 0$ for $x \geq a$

and $f(x) \leq K g(x)$, K constant, $x \geq a$,

then if $\int_a^\infty g$ converges then $\int_a^\infty f$ converges and

$\int_a^\infty f \leq K \int_a^\infty g$. Just note $\int_a^R f \leq K \int_a^R g$ and the function is increasing and bounded above - take $l = \sup_{R \geq a} \int_a^R f < \infty$ and check $\lim_{R\to\infty} \int_a^R f = l$.

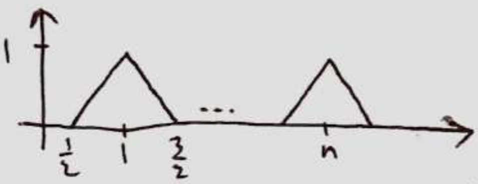
<u>Example</u> $\displaystyle\int_0^\infty e^{-x^2/2}\,dx = I$

$e^{-x^2/2} \leqslant e^{-x/2}$, $x \geqslant 1$ and $\displaystyle\int_1^R e^{-x/2}\,dx \to \frac{1}{2} e^{-1/2}$ as $R \to \infty$

$\Rightarrow I$ converges

3) We know if $\sum a_n$ converges, then $a_n \to 0$. But

$\displaystyle\int_a^\infty f$ converges $\not\Rightarrow$ $f \to 0$



$f(n) = 1$

$n \in \mathbb{N}$

Area of $\Delta = \dfrac{2}{(n+1)^2}$ <u>Counterexample</u>?

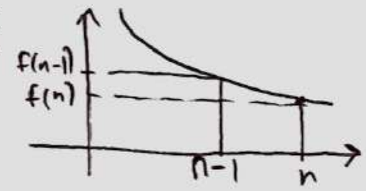
The integral test

<u>Theorem 5.14</u> Let $f(x)$ be a positive decreasing function, $x \geqslant 1$. Then

(1) $\displaystyle\int_1^\infty f(x)\,dx$ and $\displaystyle\sum_1^\infty f(n)$ <u>both</u> either converge or diverge.

(2) As $n \to \infty$, $\displaystyle\sum_{r=1}^n f(r) - \int_1^n f(x)\,dx \to l$ with $0 \le l \le f(1)$

<u>Proof</u>



If $n-1 \le x \le n$, then $f(n-1) \geqslant f(x) \geqslant f(n)$

$\displaystyle\Rightarrow f(n-1) \geqslant \int_{n-1}^n f(x)\,dx \geqslant f(n) \quad (*)$

Adding: $\displaystyle\sum_1^{n-1} f(r) \geqslant \int_1^n f(x)\,dx \geqslant \sum_2^n f(r) \quad (**)$

From this (1) follows immediately.

For proof of (2) set $\displaystyle\phi(n) = \sum_1^n f(r) - \int_1^n f(x)\,dx$

then $\displaystyle\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^n f(x)\,dx \le 0$ by $(*)$

Also from $(**)$, $0 \le \phi(n) \le f(1)$ so by ~~MAT~~ MCT (2) follows.

Analysis - Lecture 24

Examples 1) $\sum_{1}^{\infty} \frac{1}{n^k}$ converges iff $k > 1$. In L23, saw $\int_1^{\infty} \frac{1}{x^k}$ converges iff $k > 1$. Use integral test

2) $\sum_{2}^{\infty} \frac{1}{n \log n}$ $\quad f(x) = \frac{1}{x \log x}$, $x \geq 2$ and $\int_2^R \frac{dx}{x \log x} = \log(\log x)\Big]_2^R$

$= \log(\log R) - \log(\log 2) \to \infty$ as $R \to \infty$ - series diverges.

Corollary 5.15 (Euler's constant) - as $n \to \infty$, $1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n \to \gamma$ where $0 \leq \gamma \leq 1$:

Proof $f(x) = \frac{1}{x}$, use Thm 5.14. $\square$

Definition A function $f : [a,b] \to \mathbb{R}$ is <u>piecewise continuous</u> if there is a dissection $D = \{x_0 = a, x_1, \dots, x_n = b\}$ s.t.

1) f is continuous on (x_{j-1}, x_j) $\forall j$

(2) one-sided limits $\lim_{x \to x_{j-1}^+} f(x)$, $\lim_{x \to x_{j-1}^-}$ exist.

Exercise Check that piecewise continuous functions are Riemann integrable.

Q: How large can discontinuity set be while f is still Riemann integrable?

Recall $f(x) = \begin{cases} 1/q & x = p/q \\ 0 & \text{otherwise} \end{cases}$

Non-examinable

f bounded, $f : [a,b] \to \mathbb{R}$. f is Riemann integrable iff the set of discontinuity points has <u>measure zero</u>.

Definition $l(I) :=$ length of interval I

$A \subset \mathbb{R}$ has measure zero if $\forall \varepsilon > 0$, $\exists$ a countable family of intervals s.t. ~~$A \subset \Sigma$~~ $A \subset \bigcup_{j=1}^{\infty} I_j$ and $\sum_j l(I_j) < \varepsilon$

Lemma (1) Every countable set has measure zero.

(2) If B has measure zero and $A \subset B$, then A has measure zero.

(3) If A_k has measure zero $\forall k \in \mathbb{N}$, then $\bigcup_{k \in \mathbb{N}} A_k$ has measure zero.

Proof of criterion uses the concept of oscillation of f:

I interval: $\omega_f(I) = \sup_I f - \inf_I f$

Oscillation at a point: $\omega_f(x) = \lim_{\varepsilon \to 0} \omega_f(x-\varepsilon, x+\varepsilon)$

<u>Lemma</u> f is continuous at x iff $\omega_f(x) = 0$. (Check)

<u>Sketch proof of Lebesgue criterion</u> $D = \{x \in [a,b] : f \text{ discontinuous at } x\}$
$= \{x : \omega_f(x) > 0\}$

$\Rightarrow$ RTP: D has measure zero.

$N(\alpha) = \{x : \omega_f(x) \geqslant \alpha\}$ and $D = \bigcup_1^\infty N(1/k)$

We'll show that for fixed α, $N(\alpha)$ has measure zero.

Let $\varepsilon > 0$. Then $\exists D$ s.t. $S(f,D) - s(f,D < \varepsilon$

$$S(f,D) - s(f,D) = \sum_{j=1}^n \omega_f([x_{j-1}, x_j])(x_j - x_{j-1})$$

$F = \{j : (x_{j-1}, x_j) \cap N(\alpha) \neq \underset{\substack{\uparrow \\ \text{empty set}}}{\phi}\}$ then for each $j \in F$, $\omega_f([x_{j-1}, x_j]) \geqslant \alpha$

$$\alpha \sum_{j \in F} (x_j - x_{j-1}) \leqslant \sum_{j \in F} \omega_f([x_{j-1}, x_j])(x_j - x_{j-1}) < \frac{\varepsilon\alpha}{2}$$

so $\sum_{j \in F} (x_j - x_{j-1}) < \frac{\varepsilon}{2}$. These cover $N(\alpha)$ except perhaps at finitely many endpoints. ✓

$\Leftarrow$ Let $\varepsilon > 0$ be given. $N(\varepsilon) \subset D$ so $N(\varepsilon)$ has measure zero. It is closed and bounded, so can be covered with finitely many open sets of total length $< \varepsilon$: $N(\varepsilon) \subset \bigcup_{i=1}^m U_i$, let $I_i = \overline{U_i}$ (closure = adding endpts)

WLOG I_i do not overlap.

The complement $K = [a,b] \setminus \bigcup_{i=1}^m U_i$ is compact so it can be covered by finitely many disjoint closed intervals J_j s.t. $\omega_f(J_j) < \varepsilon$

Now the I_i's and J_j's give a dissection of $[a,b]$ s.t.

$$\sum_1^n \omega_f([x_{j-1}, x_j])(x_j - x_{j-1}) = \sum_{i=1}^m \omega_f(I_i) l(I_i) + \sum_{j=1}^k \underbrace{\omega_f(J_j)}_{< \varepsilon} l(J_j)$$

$$\leqslant \underset{\substack{\uparrow \\ |f| \leqslant K}}{2K} \sum_1^m l(I_i) + \varepsilon(b-a)$$

$$\leqslant 2K\varepsilon + \varepsilon(b-a). \quad \square$$