Variational Principles – Lecture 1

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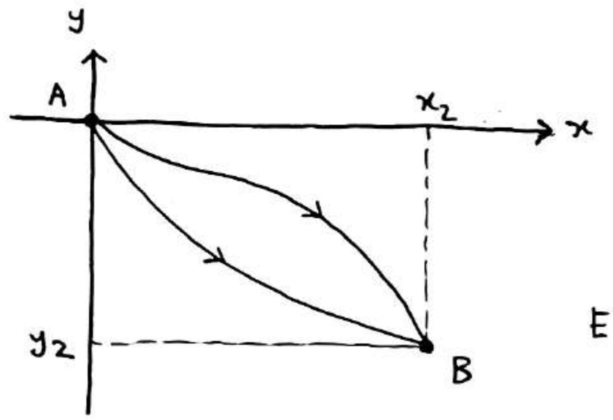
Section 0 Motivation

Example 0.1: Brachistochrone problem (Johann Bernoulli, 1696)

Particle slides on wire under influence of gravity, between 2 fixed points, A and B. What shape should the wire be for the shortest travel time, starting from rest?



Travel time

$$T = \int dt = \int_A^B \frac{dl}{v(x,y)}$$

E conserved: $T + V = \text{const.}$

at A: $y_1 = 0$

$$\frac{1}{2} m v^2 + mgy = mgy_1 = 0 \qquad \text{so} \qquad v = \sqrt{2g}\sqrt{-y}$$

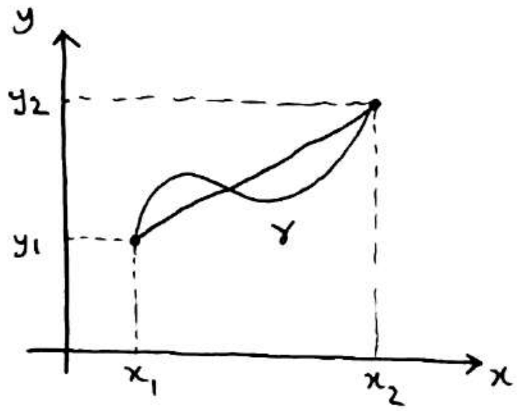
So minimise $$T(y) = \frac{1}{\sqrt{2g}} \int_0^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{-y}}\, dx$$

(line elt. dl)

subject to $y(0) = 0, \quad y(x_2) = y_2$.

Example 0.2: Geodesics (shortest path γ between 2 points on a surface Σ, if they exist)

Take $\Sigma = \mathbb{R}^2$ (a plane, Pythagorean thm holds)



Distance along γ:

$$D(y) = \int_A^B dl = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$$

Minimise D by varying path.

In general we want to minimise / maximise some function

$$F(y) = \int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx \qquad (0.1)$$

among all functions with $y(x_1) = y_1$, $y(x_2) = y_2$
Note $y = y(x)$.

(0.1) is a functional (a function on the space of functions).

Functions: numbers $\to$ numbers
Functionals: functions $\to$ numbers (example: area under graph)

Calculus of variations: finding extrema of functionals on spaces of functions.

Notation $C(\mathbb{R})$ space of continuous functions on $\mathbb{R}$

$C^k(\mathbb{R})$ space of functions on R with continuous k^{th}-derivatives

$C^k_{(\alpha,\beta)}(\mathbb{R})$ space of functions on ^ R with continuous k^{th}-derivatives s.t. $f(\alpha) = f(\beta)$

Need to specify the function space beforehand.

Example 0.3 Fermat's principle

Light between two points travels along paths which require the least time.

Example 0.4 Principle of least (stationary) action

$$S(\gamma) = \int_{t_1}^{t_2} (T - V)\, dt \qquad \text{for motion of particle}$$

(e.g. $m\ddot{\underline{x}} = -\nabla V$ so Newton's eqns should follow)

Variational Principles - Lecture 2

Section 1 Calculus for Functions on $\mathbb{R}^n$

$f \in C^2(\mathbb{R}^n)$, $\quad f : \mathbb{R}^n \to \mathbb{R}$ (cts 2nd derivatives)

The point $\underline{a} \in \mathbb{R}^n$ is stationary if $\nabla f(\underline{a}) = \underline{0}$

Expand near $\underline{x} = \underline{a}$:

$$f(\underline{x}) = f(\underline{a}) + \underbrace{(\underline{x} - \underline{a}) \cdot \nabla f|_{\underline{a}}}_{\substack{0 \text{ as } \underline{a} \text{ stationary} \\ \nabla f|_{\underline{a}} = \underline{0}}} + \frac{1}{2}(x_i - a_i)(x_j - a_j)\,\partial^2_{ij} f|_{\underline{a}} + o(|\underline{x} - \underline{a}|^2)$$

The Hessian matrix is given by $H_{ij} = \partial_i \partial_j f = H_{ji}$

Shift origin to set $\underline{a} = \underline{0}$. Diagonalise $H(\underline{0})$ by an orthogonal transformation.

$$H' = R^T H(\underline{0}) R = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$f(\underline{x}') - f(\underline{0}) = \frac{1}{2} \sum_i \lambda_i (x_i')^2 + o(|\underline{x}'|^2)$$

(note 1st order term is 0 as $\underline{0}$ is stationary)

(i) If all $\lambda_i > 0$ then $f(\underline{x}') > f(\underline{0})$ in all directions
local minimum

(ii) all $\lambda_i < 0$: local maximum

(iii) some $\lambda_i > 0$, some $\lambda_i < 0$: saddle point

(iv) some $\lambda_i = 0$: need higher order terms.

Special case $n=2$ (may be easier than finding evals)

$\det(H) = \lambda_1 \lambda_2 \qquad tr(H) = \lambda_1 + \lambda_2$

- $\det H > 0$ and $tr\, H > 0$: local minimum
- $\det H > 0$ and $tr\, H < 0$: local maximum
- $\det H < 0$: saddle point
- $\det H = 0$: higher order derivatives

Remarks If we have $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$ then may have extremum on boundary of domain

Here we cannot find the global max/min via derivatives.

Remark 2 If f is harmonic on $\mathbb{R}^2$: $f_{xx} + f_{yy} = 0$ with f defined on $D \subseteq \mathbb{R}^2$

Then anywhere in D, $tr\, H = 0$ so critical points must be saddle points and the min/max is on the boundary (see reason above).

Example 1.1 $f(x,y) = x^3 + y^3 - 3xy$

$\nabla f = (3x^2 - 3y,\ 3y^2 - 3x) = (0,0)$ at critical pt

$x^2 - y = 0, \quad y^2 - x = 0 \quad \Rightarrow \quad y^4 = y$

So critical points are $(0,0)$ and $(1,1)$.

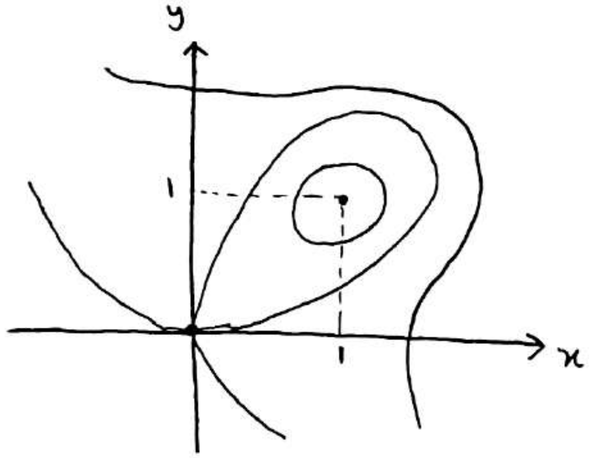
$$H = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

At $(0,0)$ $\det H = -9 < 0$ $\operatorname{tr} H = 0$

so have a saddle point, where $f = 0$.

At $(1,1)$ $\det H = 27 > 0$ $\operatorname{tr} H = 12 > 0$

so have a local minimum, where $f = -1$.

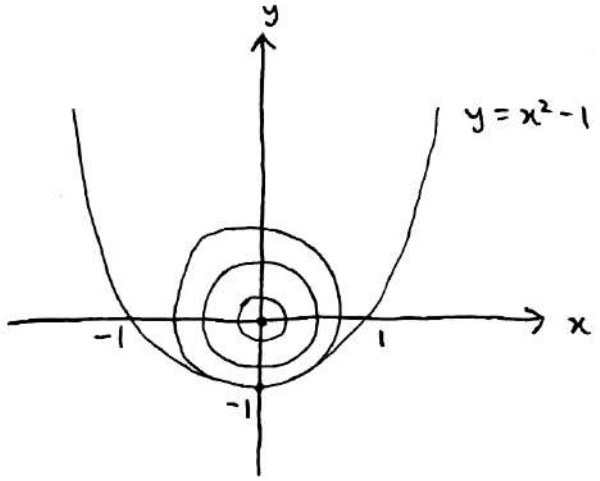


Near $(0,0)$ $f \approx -3xy$
as x^3, y^3 small
so decreases on $y = x$
and increases on $y = -x$
(no global min / max)

(only a rough sketch)

<u>Section 1.1</u> Constraints and Lagrange multipliers

<u>Example 1.2</u> Find the circle centred at $(0,0)$ with smallest radius, which intersects the parabola $y = x^2 - 1$.



<u>Direct method</u>: solve the constraints

Minimise $x^2 + y^2$ subject to $y = x^2 - 1$

so minimise

$$x^2 + (x^2-1)^2 = x^4 - x^2 + 1$$

$$\Rightarrow 4x^3 - 2x = 0$$

2 solns: $x = \pm \frac{1}{\sqrt{2}}$, $y = -\frac{1}{2}$ $\Rightarrow$ radius is $\frac{\sqrt{3}}{2}$

$x = 0$, $y = -1$ $\Rightarrow$ radius is 1

so $\underline{\sqrt{3}/2}$ is smallest.

But what if we can't solve analytically?

Lagrange Multipliers

Define new function $\underline{h(x, y, \lambda) = f(x,y) - \lambda g(x, y)}$

where f is the function we want to min / max

and $\underline{g(x, y) = 0}$ is the constraint. λ is Lagrange multiplier.

Here $h = x^2 + y^2 - \lambda(y - x^2 + 1)$

Then extremise over $\underline{3}$ variables with no constraint.

$$\frac{\partial h}{\partial x} = 2x + 2\lambda x = 0 \qquad \frac{\partial h}{\partial y} = 2y - \lambda = 0 \qquad \frac{\partial h}{\partial \lambda} = y - x^2 + 1 = 0$$

(constraint)

Combining first 2: $2x + 4xy = 0 \Rightarrow x = 0$ or $y = -\frac{1}{2}$

so using $\frac{\partial h}{\partial \lambda}$ have either $x = 0, \ y = -1$

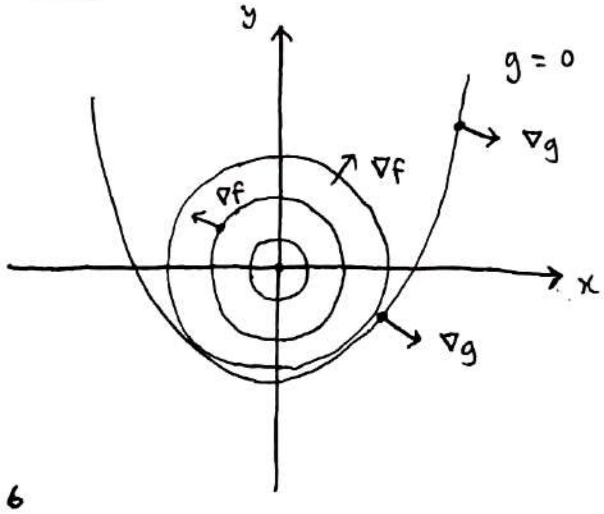
or $x = \pm\frac{1}{\sqrt{2}}, \ y = -\frac{1}{2}$ as before.

$(0, -1) \rightarrow f = 1 \quad (\lambda = -2)$

$\left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) \rightarrow f = \frac{3}{4} \quad (\lambda = -1)$

Finding the critical points without solving analytically.

Why does this work?



∇g perpendicular to "surface" $g = 0$.

Representing circles by $f =$ constant,

have also ∇f perpendicular to $f =$ constant.

At the extremum, ∇f and ∇g are <u>parallel</u>

i.e. $\nabla f = \lambda \nabla g$ i.e. $\underline{\nabla(f - \lambda g) = \underline{0}}$

so we consider the extrema of

$\underline{h = f - \lambda g}$ to find the solution(s).

For multiple constraints:

Extremise $f : \mathbb{R}^n \to \mathbb{R}$ subject to $g_\alpha(\underline{x}) = 0$

$(g_\alpha : \mathbb{R}^n \to \mathbb{R}, \ \alpha = 1, \dots, k)$

Then define $\underline{h(x_1, \dots, x_n, \lambda_1, \dots, \lambda_k) = f - \sum_{\alpha=1}^{k} \lambda_\alpha g_\alpha}$

a function of $n + k$ vars, with k Lagrange multipliers.

So we work with

$$\frac{\partial h}{\partial x_i} = 0, \quad \frac{\partial h}{\partial \lambda_\alpha} = 0,$$ eliminate λ_α and solve for $\underline{x}$.

The method works also if the constraints cannot be eliminated algebraically.

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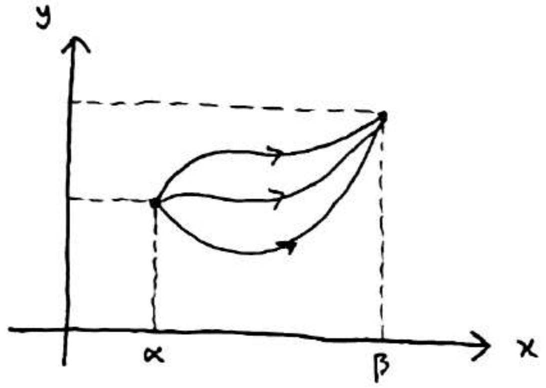
Section 2 Euler - Lagrange equations

Extremise functional 0.1
$$F(y) = \int_\alpha^\beta f(x, y, y')\, dx \qquad (2.1)$$



f is given, α, β are fixed
Functional depends on y.

Small perturbation $y \to y + \varepsilon \eta(x)$ in (2.1)
Compute $\underline{F(y + \varepsilon\eta(x))}$ with $\eta(\alpha) = \eta(\beta) = 0$
(so perturbed function goes through <u>same</u> fixed pts)

<u>Lemma</u> If $g : [\alpha, \beta] \to \mathbb{R}$ is continuous on $[\alpha, \beta]$ <u>and</u>
$\int_\alpha^\beta g(x)\eta(x)\, dx = 0$ for <u>all</u> continuous $\eta(x)$ on $[\alpha, \beta]$
s.t. $\eta(\alpha) = \eta(\beta) = 0$, then $g(x) = 0 \quad \forall x \in [\alpha, \beta]$.

<u>Proof</u> Suppose $\exists\ \bar{x} \in (\alpha, \beta)$ s.t. $g(\bar{x}) \neq 0$. WLOG suppose that $g(\bar{x}) > 0$. Then $\exists$ interval $[x_1, x_2] \subseteq (\alpha, \beta)$ s.t. $g(x) > c$ on $[x_1, x_2]$ for some $c > 0$.

Set
$$\eta(x) = \begin{cases} (x - x_1)(x_2 - x) & x \in [x_1, x_2] \\ 0 & \text{otherwise} \end{cases} \qquad (2.2)$$

Then $\eta(x)$ is continuous and we see $\int_{x_1}^{x_2} g(x)\eta(x)\, dx > 0$
so $\int_\alpha^\beta g(x)\eta(x)\, dx > 0$ (as η is 0 elsewhere). ⋇

(This was on IA Analysis I ES4).

<u>Remark</u> η is a bump function (in 2.2)

A general form for C^k bump functions ($x \in [x_1, x_2]$ bump interval)

is
$$\eta = \begin{cases} \left((x-x_1)(x_2-x)\right)^{k+1} & x \in [x_1, x_2] \\ 0 & \text{otherwise} \end{cases}$$

Now back to 2.1. Expand in ε:

$$F(y+\varepsilon\eta) = \int_\alpha^\beta f(x, y+\varepsilon\eta, y'+\varepsilon\eta')\, dx$$

$$= F(y) + \varepsilon \int_\alpha^\beta \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' \right) dx + O(\varepsilon^2)$$

(Note: we get this by expanding the $y+\varepsilon\eta$ and $y'+\varepsilon\eta'$ as these are the only dependencies of f that depend on ε. Can check this works and the $F(y)$ comes from the 1st order term)

At extremum, have $F(y+\varepsilon\eta) = F(y) + O(\varepsilon^2)$

i.e. $\underline{\left.\dfrac{df}{d\varepsilon}\right|_{\varepsilon=0} = 0}$. (1st derivative term must be 0 at extremum).

Integrating by parts on the ε-term, we want

$$0 = \int_\alpha^\beta \left(\frac{df}{dy}\eta - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta \right) dx + \cancelto{0}{\left[\frac{\partial f}{\partial y'}\eta\right]_\alpha^\beta}$$

$(\eta(\alpha) = \eta(\beta) = 0)$

$$= \int_\alpha^\beta \underbrace{\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)}_{\text{apply lemma with this } \underline{0}}\eta\, dx$$

apply lemma with this $\underline{0}$, as $\eta(\alpha) = \eta(\beta) = 0$.

So $$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}} \quad (2.3)$$

(necessary condition for extremum)

<u>Euler-Lagrange equation</u>

Remarks

- (2.3) is a 2nd order ODE for $y(x)$ with boundary conditions $y(\alpha) = y_1$, $y(\beta) = y_2$.

- $\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) - \dfrac{\partial f}{\partial y}$ is called a functional derivative

  denoted $\dfrac{\delta F(y)}{\delta y(x)}$

- Sometimes $\delta y = \varepsilon \eta(x)$ is written, called a "small variation" and they write $F(y + \delta y) = F(y) + \delta F(y)$ where

  $$\delta F = \int_\alpha^\beta \left(\frac{\delta F(y)}{\delta y(x)} \delta y(x) \right) dx .$$

- Other boundary conditions are possible e.g. $\left.\dfrac{\partial f}{\partial y'}\right|_{\alpha, \beta} = 0$

- We consider x, y, y' to be independent vars when taking partial derivatives.

  Total derivative: $\dfrac{dh}{dx} = \dfrac{\partial h}{\partial x} + \dfrac{\partial h}{\partial y}\dfrac{dy}{dx} + \dfrac{\partial h}{\partial y'}\dfrac{d^2 y}{dx^2}$

Section 2.1 First integrals of the Euler-Lagrange equation

In some cases 2.3 (E-L eqn) can be integrated once to get a 1st order ODE - "first integral"

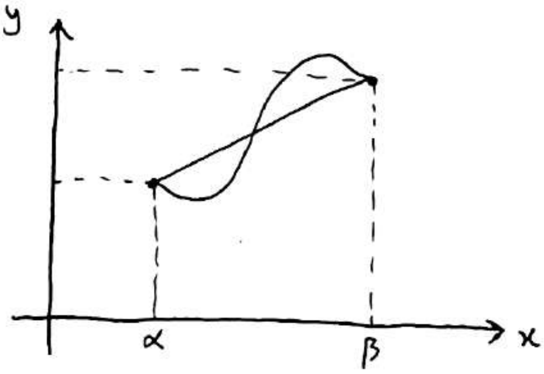
Cases:

(a) f does not explicitly depend on y, so $\dfrac{\partial f}{\partial y} = 0$

Then we get $\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ i.e. $\dfrac{\partial f}{\partial y'} = c$ (constant)

(2.4)

Example 2.1 Geodesics on Euclidean plane



$$F(y) = \int_\alpha^\beta \sqrt{dx^2 + dy^2}$$

arc length functional

so $$F(y) = \int_\alpha^\beta \underbrace{\sqrt{1+(y')^2}}\, dx$$

$f(y')$ doesn't depend on y explicitly

So use 2.4 : $\frac{\partial f}{\partial y} = 0$ so $\frac{y'}{\sqrt{1+(y')^2}}$ = constant

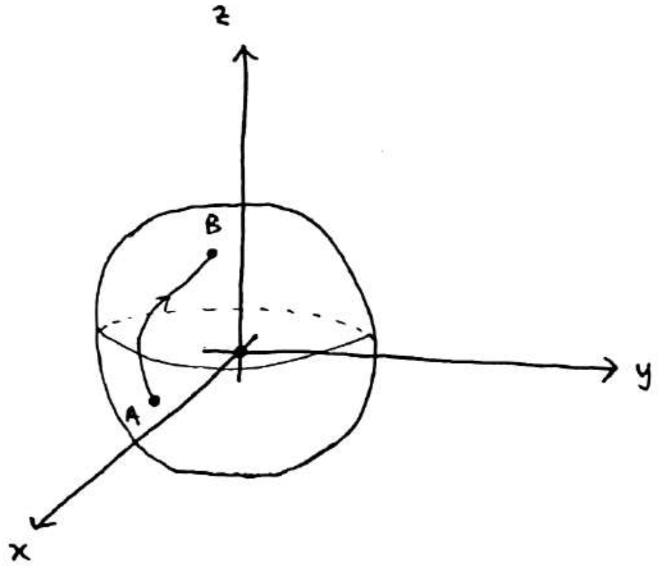
so $(y')^2 = c + c(y')^2$ for some c

$\Rightarrow \quad (y')^2 = \frac{c}{1-c} \quad \Rightarrow \quad y' = \pm\sqrt{\frac{c}{1-c}}$

So y' must be <u>constant</u> so have $y' = m$ for some m

$\Rightarrow$ <u>$y = mx + k$</u> a straight line.

Example 2.2 Geodesics on a sphere $S^2 \subseteq \mathbb{R}^3$



Parametrise sphere:

$$x = \sin\theta \sin\phi$$
$$y = \sin\theta \cos\phi$$
$$z = \cos\theta$$

$0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi$

take as unit sphere

$$ds^2 = dx^2 + dy^2 + dz^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

Then we parametrise as $\underline{\phi = \phi(\theta)}$ and can write

$$ds = \sqrt{1 + \sin^2\theta (\phi')^2} \, d\theta$$

Then
$$F(\phi) = \int_{\theta_1 = \alpha}^{\theta_2 = \beta} \sqrt{1 + \sin^2\theta \, (\phi')^2} \, d\theta$$

Integrand $f = f(\theta, \phi')$ doesn't depend on ϕ itself

$$\frac{\partial f}{\partial \phi} = 0 \implies \frac{\partial f}{\partial \phi'} = K \quad (\text{constant})$$

Evaluating $\frac{\partial f}{\partial \phi'}$ we have

$$\frac{\phi' \sin^2\theta}{\sqrt{1 + \sin^2\theta (\phi')^2}} = K$$

We square and solve for $(\phi')^2$:

$$(\phi')^2 = \frac{K^2}{\sin^2\theta (\sin^2\theta - K^2)}$$

So $\phi = \pm \int \frac{K}{\sin\theta\sqrt{\sin^2\theta - K^2}} \, d\theta$ (2 solns, each going one way around sphere)

Make substitution $u = \cot\theta$

Then get
$$\pm \frac{\sqrt{1-K^2}}{K} \cos(\phi - \phi_0) = \cot\theta$$

a <u>great circle</u>. (Geodesics are segments of great circles)

(b) Consider for general $f(x, y, y')$:

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \cancel{y''\frac{\partial f}{\partial y'}} - \cancel{y''\frac{\partial f}{\partial y'}} - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$= y'\left(\cancelto{0}{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}}\right) + \frac{\partial f}{\partial x} = \underline{\frac{\partial f}{\partial x}}$$

(by E-L)

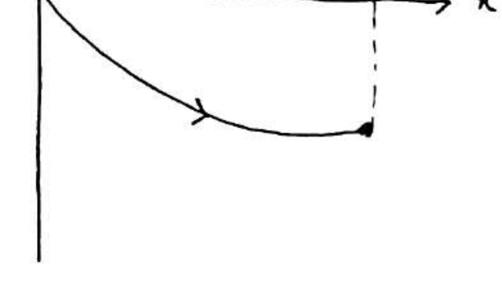
So if f does <u>not</u> explicitly depend on $\underline{x}$, then $\underline{\frac{\partial f}{\partial x} = 0}$

so we have $\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$

so
$$\underline{f - y'\frac{\partial f}{\partial y'} = c} \qquad (2.5)$$

and we can reduce the order to get a 1st order ODE.

<u>Example 2.3</u> Brachistochrone problem



Recall from section 0 that the functional is

$$F(y) = \frac{1}{\sqrt{2g}} \int_0^{\beta} \underbrace{\frac{\sqrt{1+(y')^2}}{\sqrt{-y}}}_{\text{depends on } y,\, y' \text{ but not } x \,:\, \text{use } 2.5.} \, dx$$

From $f - y' \dfrac{\partial f}{\partial y'} = c$ we get

$$\frac{\sqrt{1+(y')^2}}{\sqrt{-y}} - y' \frac{y'}{\sqrt{1+(y')^2}\,\sqrt{-y}} = c$$

Rearranging gives $\dfrac{1}{\sqrt{1+(y')^2}} = c\sqrt{-y}$

$$\Rightarrow \quad y' = \pm \frac{\sqrt{1+c^2y^2}}{c\sqrt{-y}} \quad \Rightarrow \quad x = \pm c \int \frac{\sqrt{-y}}{\sqrt{1+c^2y}} \, dy$$

Set $y = -\dfrac{1}{c^2}\sin^2\dfrac{\theta}{2}$, $\quad dy = -\dfrac{1}{c^2}\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

Then have

$$x = \pm c \int (-1)\frac{1}{c^2} \frac{\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}}{\sqrt{1-\sin^2\frac{\theta}{2}}} \, d\theta$$

$$= \mp \frac{1}{2c^2} \int (1-\cos\theta)\, d\theta$$

$$= \mp \frac{1}{2c^2}(\theta - \sin\theta)$$ (curve goes through $(0,0)$ so constant of integration is 0)

Then we get $\quad x = \dfrac{\theta - \sin\theta}{2c^2}, \quad y = -\dfrac{1}{c^2}\sin^2\dfrac{\theta}{2}$

The parametrisation of a <u>cycloid</u> (curve traced out by a point on a circle as it rolls without slipping)

Section 2.2 Fermat's principle

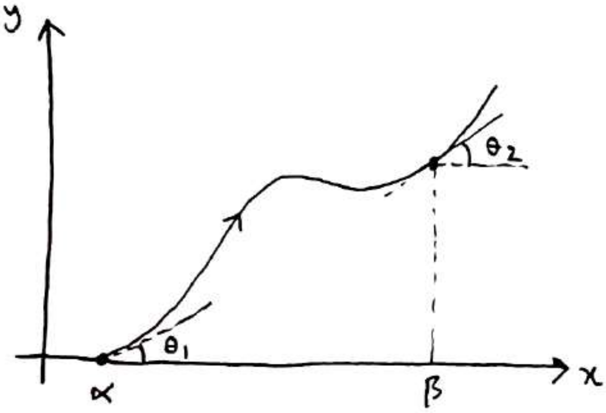
Light / sound travels along the path between two points that takes the least time.

Ray : path $y = y(x)$ speed $c(x,y)$

$$F(y) = \int \frac{dl}{c} = \int_{\alpha}^{\beta} \frac{\sqrt{1+(y')^2}}{c(x,y)}\, dx$$

Assume $\underline{c = c(x)}$ only. Then $\frac{\partial f}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y'} = \text{constant}$

$$\frac{y'}{\sqrt{1+(y')^2}\, c(x)} = \text{constant}$$

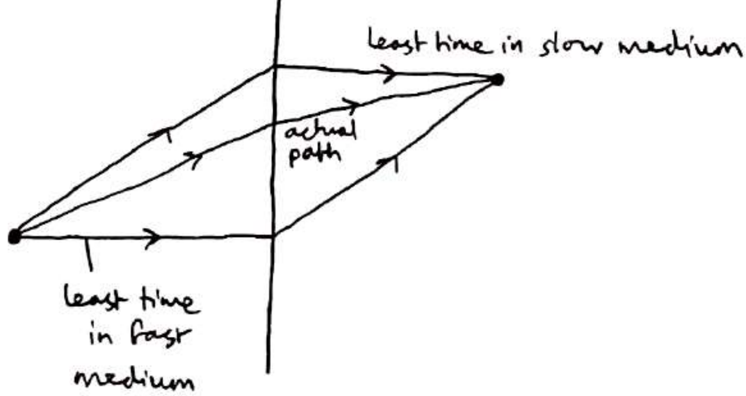


$\tan\theta = y'$ so making this sub, we get

$$\frac{\sin\theta_1}{c(x_1)} = \frac{\sin\theta}{c(x)} \qquad (2.6)$$

Snell's law in optics

Fast (c_F constant) Slow (c_S constant) $c_S < c_F$



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Variational Principles - Lecture 5

Section 3 Extensions of the Euler-Lagrange equations

3.1 Euler-Lagrange equations with constraints

Extremise $F(y) = \int_\alpha^\beta f(x, y, y')\, dx$ subject to

$$G(y) = \int_\alpha^\beta g(x, y, y')\, dx = k \quad (\text{constant}).$$

Use Lagrange multipliers: extremise

$$\Phi(y; \lambda) = F(y) - \lambda G(y)$$

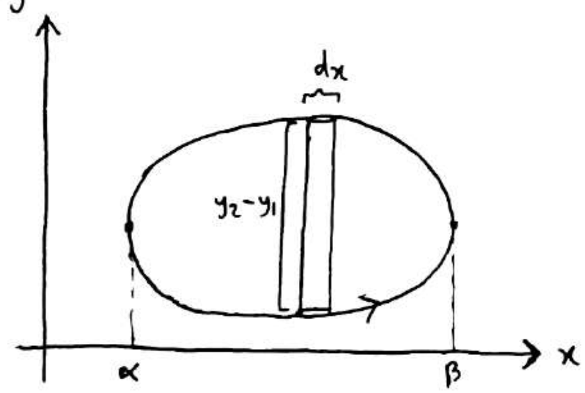
Replace f in E-L eqn by $f - \lambda g$:

$$\frac{d}{dx}\left(\frac{\partial}{\partial y'}(f - \lambda g)\right) = \frac{\partial}{\partial y}(f - \lambda g) = 0 \qquad (3.1)$$

Example 3.1 Dido problem / Isoparametric problem

What simple, closed plane curve of fixed length L maximises the enclosed area?

WLOG we assume convexity.



x monotonically increases from α to β, decreases from β to α.

(not coords)

Given x, there exists (y_1, y_2) on the curve with $y_1(x) = y_2(x)$ and $y_2 > y_1$

so $dA = \left[y(x)\right]_{x_1}^{x_2} dx$

Find area functional:

Area functional

$$A(y) = \int_{\alpha}^{\beta} \left(y_2(x) - y_1(x)\right)\, dx = \oint_C y(x)\, dx$$

Constraint

$$L(y) = \oint dl = \oint_C \sqrt{1+(y')^2}\, dx = L$$

Let $\underline{h = y - \lambda\sqrt{1+(y')^2}}$ (Lagrange multiplier function)

Using $\frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = \frac{\partial h}{\partial y}$ note $\frac{\partial h}{\partial x} = 0$ so can use <u>first integral</u> of E-L. (see L4)

So $h - y'\frac{\partial h}{\partial y'} = k$ for some constant k

Writing this out <u>explicitly</u>, we get

$$k = y - \frac{\lambda}{\sqrt{1+(y')^2}} \quad \Rightarrow \quad \underline{(y')^2 = \frac{\lambda^2}{(y-k)^2} - 1}$$

Solving this gives $(x-x_0)^2 + (y-y_0)^2 = \lambda^2$

And circumference $2\pi\lambda = L \Rightarrow \lambda = \frac{L}{2\pi}$

This gives a <u>circle</u> of radius $\frac{L}{2\pi}$.

<u>Example 3.2</u> The Sturm-Liouville problem

Let function $\rho(x) > 0$ for $x \in [\alpha, \beta]$. Define also $\sigma = \sigma(x)$.

Then define

$$F(y) = \int_{\alpha}^{\beta} \left(\rho \cdot (y')^2 + \sigma \cdot (y^2)\right)\, dx$$

and <u>extremise</u> F subject to

$$G(y) = \int_\alpha^\beta y^2 \, dx = 1 .$$

$G(y) - 1$
rewrite 1 using an integrand :

$$\int_\alpha^\beta \frac{1}{\beta - \alpha} \, dx = 1$$

$$\Phi(y; \lambda) = F(y) - \lambda\left(G(y) - 1\right)$$

Write $h = \rho \cdot (y')^2 + \sigma \cdot (y^2) - \lambda\left(y^2 - \frac{1}{\beta-\alpha}\right)$

$$\frac{\partial h}{\partial y'} = 2\rho y' , \quad \frac{\partial h}{\partial y} = 2\sigma y - 2\lambda y$$

Applying E-L, rearranging gives

$$\underbrace{-\frac{d}{dx}(\rho y') + \sigma y}_{\mathcal{L}(y)} = \lambda y \qquad (3.2)$$

$\mathcal{L}$ is called the Sturm - Liouville operator

Then $\underline{\mathcal{L}(y) = \lambda y}$ is an eigenvalue problem
(note: similar form to TISE)

Note : if $\sigma > 0$ then $\underline{F(y) > 0}$.

<u>Claim</u> The smallest ~~positive~~ eigenvalue is equal to the positive minimum.

<u>Proof</u> Take (3.2) $\times$ y and IBP from α to β :

$$F(y) - \underbrace{\left[y \cdot y' \rho \right]_\alpha^\beta}_{0} = \lambda \underbrace{G(y)}_{1} = \lambda$$

(boundary term is 0 - fixed end problem)

So lowest eigenvalue = minimum of $\underline{\dfrac{F(y)}{G(y)}}$. $\square$

3.2 Several dependent variables

$$\underline{y}(x) = (y_1(x), y_2(x), \ldots, y_n(x)) \qquad \text{Extremise}$$

$$F(\underline{y}) = \int_\alpha^\beta f(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\, dx$$

Perturbation $\quad y_i \to y_i(x) + \varepsilon \eta_i(x)$

with $i = 1, \ldots, n$ and $\eta_i(\alpha) = \eta_i(\beta) = 0$.

Using same derivation as that of E-L eqn, we get

$$F(\underline{y} + \varepsilon \underline{\eta}) - F(\underline{y}) = \int_\alpha^\beta \sum_{i=1}^{n} \eta_i \left(\frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) - \frac{\partial f}{\partial y_i} \right) dx$$

$$+ \text{ boundary terms } + O(\varepsilon^2)$$

Recall the "fundamental lemma" from L3: by setting all the η_i's but one to zero in turn, we get again

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = \frac{\partial f}{\partial y_i} \qquad (3.3)$$

a system of n 2nd order ODEs.

First integrals of 3.3

* if $\frac{\partial f}{\partial y_j} = 0$ for some $1 \leq j \leq n$ then $\frac{\partial f}{\partial y_j'}$ = constant
* if $\frac{\partial f}{\partial x} = 0$ then $f - \sum_i y_i' \frac{\partial f}{\partial y_i'}$ = constant

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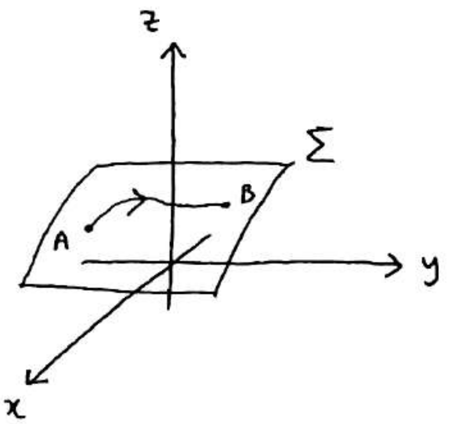
Example Geodesics on surfaces



$\Sigma \subseteq \mathbb{R}^3$ is a surface given by $g(x, y, z) = 0$.

Find shortest path on surface between 2 points, if one exists.

Take t to be a parameter on the curve:

$A = \underline{x}(0)$, $B = \underline{x}(1)$ $\underline{x} = (x, y, z) = \underline{x}(t)$

$$\Phi(\underline{x}, \lambda) = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} - \lambda(t)\, g(x, y, z) \; dt$$

(Note $\lambda = \lambda(t)$ as we need the curve to lie on the surface i.e. $g(x,y,z) = 0$ everywhere - if don't have $\lambda = \lambda(t)$ then we are just saying $g(x,y,z)$ integrates to 0 over curve, not necessarily $g(x,y,z) = 0$)

Write integrand $\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} - \lambda(t)\, g = \underline{h(x, y, z, \dot{x}, \dot{y}, \dot{z}, \lambda)}$

Use E-L with h

- Variation wrt λ: $\underbrace{\frac{d}{dt}\left(\frac{\partial h}{\partial \dot{\lambda}}\right)}_{0 \text{ (h does not depend on } \dot{\lambda})} = \underbrace{\frac{\partial h}{\partial \lambda}}_{g(x,y,z)} \Rightarrow \underline{g(x,y,z) = 0} \quad \forall t$

Variation wrt x_i (x, y, z) :

$$\frac{d}{dt}\left(\frac{\dot{x}_i}{\sqrt{\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2}}\right) + \lambda \frac{\partial g}{\partial x_i} = 0 \qquad i = 1, 2, 3$$

a system of ODEs to be solved.

Alternatively, solve the constraint $\underline{g = 0}$, as we did in Ex 2.2 (geodesics on sphere).

3.3 Several independent variables

In general with $\Phi : \mathbb{R}^n \to \mathbb{R}^m$: for $n > 1$, E-L eqns become PDEs.

Consider case where $n = 3$, $m = 1$ so $\Phi : \mathbb{R}^3 \to \mathbb{R}$

$$F(\phi) = \iiint_D f(x, y, z, \phi, \phi_x, \phi_y, \phi_z)\, dx\, dy\, dz$$

(note $\phi_x := \frac{\partial \phi}{\partial x}$)

This is a volume integral over a domain $D \subseteq \mathbb{R}^3$

Assume there is an extremum, consider perturbation

$\phi \to \phi(x, y, z) + \varepsilon \eta(x, y, z)$ where $\underline{\eta = 0 \text{ on } \partial D}$

$$\underset{\text{(expanding)}}{F(\phi + \varepsilon\eta) - F(\phi)} = \varepsilon \int_D \left(\eta \frac{\partial f}{\partial \phi} + \eta_x \frac{\partial f}{\partial \phi_x} + \eta_y \frac{\partial f}{\partial \phi_y} + \eta_z \frac{\partial f}{\partial \phi_z}\right) dx\,dy\,dz + o(\varepsilon^2)$$

$$= \varepsilon \int_D \left(\eta \frac{\partial f}{\partial \phi} + \underbrace{\nabla \cdot \left(\eta \left(\frac{\partial f}{\partial \phi_x}, \frac{\partial f}{\partial \phi_y}, \frac{\partial f}{\partial \phi_z}\right)\right)}_{\substack{\text{use div. thm: } \eta = 0 \text{ on } \partial D \\ \text{so this becomes } 0}} - \eta \nabla \cdot \left(\frac{\partial f}{\partial \phi_x}, \frac{\partial f}{\partial \phi_y}, \frac{\partial f}{\partial \phi_z}\right)\right) dx\,dy\,dz + o(\varepsilon^2)$$

Then we get

$$F(\phi + \varepsilon\eta) - F(\phi) = \varepsilon \int \eta \underbrace{\left(\frac{\partial f}{\partial \phi} - \nabla \cdot \left(\frac{\partial f}{\partial \phi_x}, \frac{\partial f}{\partial \phi_y}, \frac{\partial f}{\partial \phi_z} \right)\right)}_{\text{then apply lemma:}} dx\,dy\,dz + o(\varepsilon^2)$$

$$\frac{\partial f}{\partial \phi} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial(\partial_i \phi)} \right) = 0 \qquad (3.4)$$

In general we have this for n rather than 3 specifically:

$$\underline{\frac{\partial f}{\partial \phi} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial(\partial_i \phi)} \right) = 0} \qquad (3.4.1)$$

<u>Example</u> Extremise "potential energy" $n = 2$

$$F(\phi) = \iint_{D \subseteq \mathbb{R}^2} \frac{1}{2}\left(\phi_x^2 + \phi_y^2 \right) dx\,dy \qquad f = \text{integrand}$$

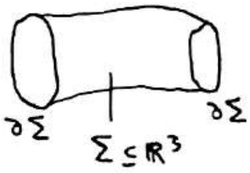
$\frac{\partial f}{\partial \phi} = 0$, $\frac{\partial f}{\partial \phi_x} = \phi_x$, $\frac{\partial f}{\partial \phi_y} = \phi_y$ so (3.4.1) gives

$$\frac{\partial}{\partial x}\phi_x + \frac{\partial}{\partial y}\phi_y = 0 \quad \text{so} \quad \underline{\phi_{xx} + \phi_{yy} = 0}$$

Laplace equation

<u>Example</u> Minimal surfaces

Minimise area of $\Sigma \subseteq \mathbb{R}^3$ subject to boundary conditions of 2 curves <u>given</u> as the boundary of the surface.



Take $\Sigma = \{ \underline{x} \in \mathbb{R}^3 \text{ s.t. } g(x,y,z) = 0 \}$

Assume (note: can use implicit function theorem) that we solved $g = 0$ to give $\underline{z = \phi(x, y)}$

$$ds^2 = dx^2 + dy^2 + dz^2, \qquad dz = \phi_x \, dx + \phi_y \, dy$$

so $$ds^2 = (1 + \phi_x^2) \, dx^2 + (1 + \phi_y^2) \, dy^2 + 2\phi_x \phi_y \, dx \, dy$$

This is called the <u>first fundamental form</u> or <u>Riemannian metric</u>

$$ds^2 = \sum_{i,j=1}^{2} \bar{g}_{ij}(x, y) \, dx_i \, dx_j \qquad \bar{g} = \begin{pmatrix} 1 + \phi_x^2 & \phi_x \phi_y \\ \phi_x \phi_y & 1 + \phi_y^2 \end{pmatrix}$$

Area element $\sqrt{\det \bar{g}} \, dx \, dy$

Then we get the area functional

$$A(\phi) = \int_D \underbrace{\sqrt{1 + \phi_x^2 + \phi_y^2}}_{h} \, dx \, dy \qquad \text{apply E-L to } h \; (3.4.1)$$

$$\frac{\partial h}{\partial \phi_x} = \frac{\phi_x}{\sqrt{1 + \phi_x^2 + \phi_y^2}} \qquad \frac{\partial h}{\partial \phi_y} = \frac{\phi_y}{\sqrt{1 + \phi_x^2 + \phi_y^2}}$$

Since $\frac{\partial h}{\partial \phi} = 0$, applying 3.4.1 gives

$$\partial_x \left(\frac{\partial h}{\partial \phi_x} \right) + \partial_y \left(\frac{\partial h}{\partial \phi_y} \right) = 0$$ and expanding derivatives

gives $$\underline{(1 + \phi_y^2) \phi_{xx} + (1 + \phi_x^2) \phi_{yy} - 2 \phi_x \phi_y \phi_{xy} = 0} \qquad (3.5)$$

the minimal surface equation.

Assume <u>circular symmetry</u> $\underline{z = \phi(r)}$, $r = \sqrt{x^2 + y^2}$

$\Rightarrow$ 3.5 becomes an <u>ODE</u>

$$\underline{r z'' + z' + (z')^3 = 0}$$ (check by chain rule - find derivatives of ϕ).