Differential Equations - Lecture 1

8/10/20

Mathematics of change

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See SIR model for spread of pandemics

Course: basic calculus, 1st order linear DEs, nonlinear 1st order DEs, higher-order DEs, multivariate functions and applications

Definition A differential equation is an equation involving derivatives of function(s).

Informal definition (Limit)

If $\lim_{x \to x_0} f(x) = A$ then $f(x)$ can be made arbitrarily close to A by making x sufficiently close to x_0. (Note $f(x_0) = A$ isn't necessarily true)

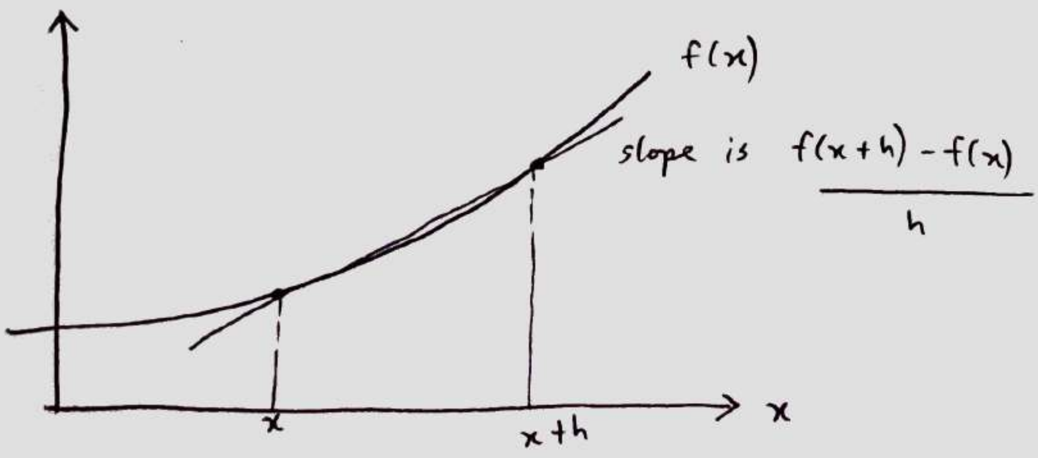
There are one-sided limits

From left: $\lim_{x \to x_0^-} f(x) = A$ (require $x < x_0$)

From right: $\lim_{x \to x_0^+} f(x) = A$ (require $x > x_0$)

Definition (Derivative) The derivative of a function $f(x)$ with respect to its argument x is

$$\frac{df}{dx} =: \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (1.1)$$



For the derivative to exist at point x, we require

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}.$$

e.g. $|x|$ is <u>not</u> differentiable at $x = 0$ but it is elsewhere.

Notation for derivatives:

$$\frac{df}{dx} = f'(x) = \dot{f}(x)$$

(Leibnitz) (Lagrange) (Newton)
usually time

For sufficiently smooth functions (derivative exists at each step) we can define derivatives recursively

e.g. $$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2} = f''(x) = \ddot{f}(x)$$

n^{th} derivative is $\frac{d^n f}{dx^n} = f^{(n)}(x)$

<u>Rules for differentiation</u>

<u>Chain rule</u> Consider $f(x) = F(g(x))$

$$\frac{df}{dx} = F'(g(x)) \frac{dg}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

<u>Product rule</u> Consider $f(x) = u(x)\,v(x)$

$$\frac{df}{dx} = u'(x)\,v(x) + u(x)\,v'(x)$$
$$= u'v + vu'$$

(prove by definition, see example sheets)

Leibnitz' rule (generalisation of product rule)

Consider $f(x) = u(x)\, v(x)$

$$f' = u'v + uv'$$

$$f'' = u''v + 2u'v' + uv''$$

$$f''' = u'''v + 3u''v' + 3u'v'' + uv'''$$

Rule: $$f^{(n)}(x) = u^{(n)}v + nu^{(n-1)}v' + \dots + \frac{n!}{m!(n-m)!}u^{(n-m)}v^{(m)} + \dots + uv^{(n)}$$

so $$f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$ (prove by induction)

Order of magnitude

Goal is to compare the sizes of functions, in the vicinity of specific points.

1. "little oh" : $\underline{o}$
2. "big oh" : $\mathcal{O}$

1. "little oh" : Defined as $f(x) = \underline{o}(g(x))$ as $x \to x_0$

if $$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \qquad (1.5)$$

Informally: "$f(x)$ is much smaller than $g(x)$" locally

Example $x^2 = \underline{o}(x)$ as $x \to 0$: here $f(x) = x^2$, $g(x) = x$, $x_0 = 0$

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0.$$

2. "Big oh"

$f(x) = O(g(x))$ as $x \to x_0$

Informally: "Can $f(x)$ be bounded by $g(x)$?"

<u>Case A</u> x_0 is finite

$f(x) = O(g(x))$ as $x \to x_0$ iff there exist $\underline{M, \delta > 0}$ such that $\forall x$ where $|x - x_0| < \delta$, we have $|f(x)| \leq M|g(x)|$.

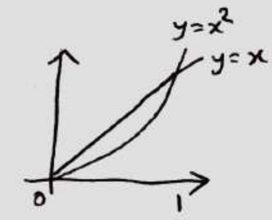
Loosely this means "around x_0, we can bound $f(x)$ by (a multiple/scaling of) $g(x)$".

<u>Case B</u> x_0 is infinite $(x \to \infty)$

$f(x) = O(g(x))$ as $x \to \infty$ if there exist $M, x_1 > 0$ such that for all $x > x_1$, $|f(x)| \leq M|g(x)|$.

<u>Examples</u>

- $x^2 = O(x)$ as $x \to 0$.
  We can choose $\delta = 1$ and $M = 1$:
  $\forall x : |x| < 1$ we have $|x^2| \leq |x|$.



- $x \neq O(x^2)$ as $x \to 0$.
  Suppose $x = O(x^2)$ as $x \to 0$. Then there are M, δ such that for $|x - x_0| < \delta$ we have $|f(x)| \leq M|g(x)|$, that is for $\underline{|x| < \delta}$ we have $\underline{|x| \leq Mx^2}$.

  By symmetry about $x = 0$, consider $x \leq Mx^2$
  $\Leftrightarrow Mx^2 - x \geq 0 \Leftrightarrow x(Mx - 1) \geq 0$
  $\Leftrightarrow x \leq 0$ or $x \geq \frac{1}{M}$

  but for $0 \leq x \leq \frac{1}{M}$, $Mx^2 - x < 0 \Rightarrow \underline{x > Mx^2}$.
  So no matter the choice of M, we still have $x > Mx^2$ for sufficiently small x so $\underline{x \neq O(x^2)}$ for $x \to 0$.

- $x^2 = O(x^2), \quad x \to 0$

  Just choose $M = 1$ and $\delta > 0$.

- $2x^3 + 4x + 12 = O(x^3)$ as $x \to \infty$

  that is $\exists\, M, x_1$ s.t. $\forall x > x_1$, $|2x^3 + 4x + 12| \leqslant M|x^3|$.

<u>Now a link to the equation of a tangent line:</u>

$$\left.\frac{df}{dx}\right|_{x=x_0} = \frac{f(x_0+h) - f(x_0)}{h} \quad \text{as } h \to 0 \quad \text{by definition.}$$

$$f(x) = \underline{o}(g(x)) \quad \text{iff} \quad \lim_{x \to 0} \frac{f(x)}{g(x)} = 0, \quad \text{so}$$

by definition $\quad \dfrac{\underline{o}(g(x))}{g(x)} \to 0 \quad$ as $x \to 0$.

So we have $\quad \left.\dfrac{df}{dx}\right|_{x=x_0} \quad \dfrac{f(x_0+h) - f(x_0)}{h} + \dfrac{\underline{o}(h)}{h} \quad$ as $h \to 0$.

Rearranging:

$$f(x_0+h) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} h + \underline{o}(h) \quad \text{as } h \to 0$$

first 2 terms of Taylor series

Let $x = x_0 + h$, $\quad y = f(x)$, $\quad m = \left.\dfrac{df}{dx}\right|_{x=x_0}$

giving a more familiar form

$$f(x) = y = y_0 + m(x - x_0) + \underline{o}(h)$$

– similar form to eqn of line.

"Little oh" – function locally much smaller than $g(x)$

"Big oh" – function locally bounded by $g(x)$

Differential Equations - Lecture 2

10/10/20

Taylor Series

Suppose we want to approximate $f(x)$ with a polynomial of order n.

i.e. $f(x) \simeq a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = P_n(x)$.

Note $f'(x) \approx a_1 + 2a_2 x + \dots + n a_n x^{n-1}$

$f''(x) \approx 2a_2 + \dots + n(n-1) a_n x^{n-2}$ etc.

Evaluate all at $x = 0$:

$$f(0) = a_0, \quad f'(0) = a_1, \quad f''(0) = 2a_2, \dots, f^{(n)}(0) = n! a_n$$

Hence $$\underline{P_n(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0).}$$

For $x = x_0$ we "shift back to 0":

$$\underline{P_n(x) \approx f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)}$$

$P_n(x)$ is the Taylor polynomial of degree n.

Alternatively we can write

$$\underline{f(x) = P_n(x) + E_n} \qquad (2.1)$$

$E_n =$ error (or remainder)

Recall from Lecture 1: $\quad f(x+h) = f(x) + h f'(x) + \underline{o}(h)$ as $h \to 0$.

This can be generalised, provided that the first n derivatives of $f(x)$ exist.

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x) + \underline{o}(h^n)$$

(show by recursively using derivative definition) $\qquad (2.2)$

Comparing 2.1 and 2.2:

$E_n = \underline{o}(h^n)$

<u>Taylor's Theorem</u> $E_n = O(h^{n+1})$ as $h \to 0$ provided that $f^{(n+1)}(x)$ exists.

<u>Proof</u> will be done in IA Analysis

Note this is a <u>stronger</u> statement than $E_n = \underline{o}(h^n)$

e.g. $h^{n+a} = \underline{o}(h^n)$ as $h \to 0$ $\forall a \in (0,1)$

since $\displaystyle\lim_{h \to 0} \frac{h^{n+a}}{h^n} = \lim_{h \to 0} h^a = 0$

But $h^{n+a} \neq O(h^{n+1})$ as $h \to 0$ for $a \in (0,1)$.
We can't keep h^{n+a} smaller than h^{n+1} everywhere in the interval near $h = 0$ for arbitrarily small h.

<u>L'Hôpital's Rule</u>

Let $f(x)$, $g(x)$ be differentiable at $x = x_0$ and

$\displaystyle\lim_{x \to x_0} f(x) = f(x_0) = 0, \quad \lim_{x \to x_0} g(x) = g(x_0) = 0.$

(i.e. $\displaystyle\lim_{x \to x_0} \frac{f(x)}{g(x)}$ is a " $\frac{0}{0}$ " limit.)

Then $\displaystyle\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ if $g'(x_0) \neq 0$.

<u>Proof</u>

As $x \to x_0$

$f(x) = f(x_0) + (x - x_0) f'(x_0) + \underline{o}(x - x_0)$

$g(x) = g(x_0) + (x - x_0) g'(x_0) + \underline{o}(x - x_0)$

$f(x_0), g(x_0) = 0$

So as $x \to x_0$ $(x - x_0 \to 0)$

$$\frac{f(x)}{g(x)} = \frac{f'(x_0) + \frac{o(x-x_0)}{x-x_0}}{g'(x_0) + \frac{o(x-x_0)}{x-x_0}}$$

dividing through by $x - x_0$

From definition of o, we have $\lim_{x - x_0 \to 0} \frac{o(x-x_0)}{x-x_0} = 0$

so $\frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$ as $x \to x_0$.

So $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$. $\square$

This generalises to higher order derivatives

e.g. $f(x) = 3\sin x - \sin 3x$ (Exercise)

$g(x) = 2x - \sin 2x$

Done 10/10/20

Show $\lim_{x \to 0} \frac{f(x)}{g(x)} = 3$.

Differential Equations – Lecture 3

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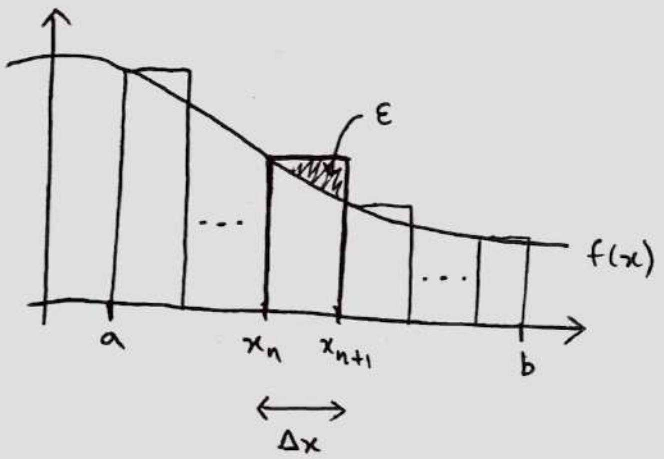
Integration

Note: here we assume functions are sufficiently well-behaved for the integral to exist.

Consider $\sum_{n=0}^{N-1} f(x_n)\,\Delta x$ with $\Delta x = \frac{b-a}{N}$, $x_n = a + n\Delta x$

(3.1)



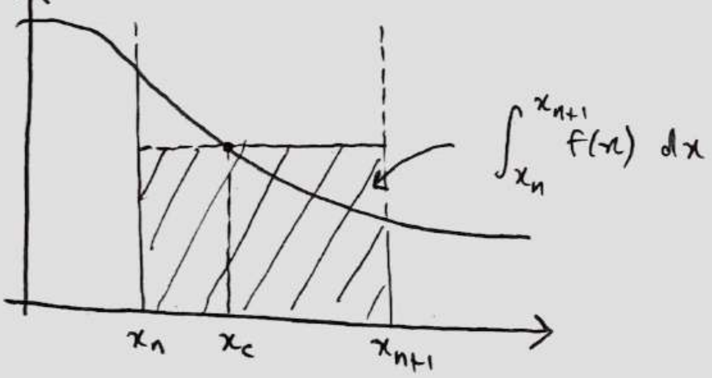
How close is (3.1) to the area under the curve for some large N?

How big is ε?

Mean Value Theorem (MVT) (for def. integrals)

For a continuous $f(x)$: $\int_{x_n}^{x_{n+1}} f(x)\,dx = f(x_c)(x_{n+1} - x_n)$

for some $x_c \in (x_n, x_{n+1})$



Expand $f(x)$ in a Taylor series about $x = x_n$, then evaluate at $x = x_c$.

$f(x_c) = f(x_n) + O(x_c - x_n)$ by Taylor's theorem.

(as $x_c - x_n \to 0$ as Δx reduced)

PTO

$$\underline{f(x_c) = f(x_n) + \mathcal{O}(x_{n+1} - x_n)} \quad \text{because } |x_{n+1} - x_n| > |x_c - x_n|$$

so if the error can be bounded by some constant multiple of $x_c - x_n$, then it can also be bounded by $x_{n+1} - x_n$ (larger).

~~f(x_n)~~ Hence $\displaystyle\int_{x_n}^{x_{n+1}} f(x)\,dx = [\underbrace{f(x_n) + \mathcal{O}(x_{n+1} - x_n)}_{f(x_c)}](x_{n+1} - x_n)$

by (3.2). Or if $\Delta x = x_{n+1} - x_n$ then this becomes

$$\int_{x_n}^{x_{n+1}} f(x)\,dx = \Delta x\, f(x_n) + \mathcal{O}(\Delta x^2) \qquad (3.3)$$

Think: why can you do this?

$\Delta x f(x_n)$ is rectangle area

$\mathcal{O}(\Delta x^2)$ is the error, giving how it is bounded.

$\underline{\varepsilon = \mathcal{O}(\Delta x^2)}$

from interpreting graphically

Hence $\displaystyle\int_a^b f(x)\,dx = \lim_{\substack{\Delta x \to 0 \\ (N \to \infty)}} \left\{ \left[\sum_{n=0}^{N-1} f(x_n)\Delta x \right] + \mathcal{O}(N\Delta x^2) \right\}$

Note $\mathcal{O}(N\Delta x^2) = \mathcal{O}\left(\frac{(b-a)^2}{N}\right)$ as $\Delta x = \frac{b-a}{N}$

Letting $\Delta x \to 0$ is letting $N \to \infty$ (as a, b fixed)

So $\mathcal{O}\left(\frac{(b-a)^2}{N}\right) = 0$ as $N \to \infty$

Error is <u>bounded</u> by constant multiple of a function that goes to 0 as $N \to \infty$. So <u>error $\to 0$</u> as well.

Hence
$$\boxed{\int_a^b f(x)\,dx = \lim_{N \to \infty} \sum_{n=0}^{N-1} f(x_n)\Delta x} \qquad (3.4)$$

Can think of this as the definition of a definite integral.

The Fundamental Theorem of Calculus (FTC)

Let $F(x) = \int_a^x f(t)\,dt$

From derivative definition

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{1}{h}\left\{ \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right\}$$

$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} \frac{1}{h}\left[f(x)h + o(h^2) \right] \quad \text{by (3.3) with } h = \Delta x$$

$$= \lim_{h\to 0} \left(f(x) + \frac{o(h^2)}{h} \right)$$

$$= \lim_{h\to 0} \left(f(x) + o(h) \right)$$

$h \to 0$ and $o(h)$ is bounded by h that goes to 0,

so $o(h) \to 0$:

$$= \lim_{h\to 0} f(x) = \underline{f(x)}.$$

So
$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x).} \qquad \text{(FTC)} \quad (3.5)$$

(Note FTC is really a DE: $F(x)$ is a solution).

<u>Corrollaries to FTC:</u>

$$\frac{d}{dx}\int_x^b f(t)\,dt = -f(x) \qquad (3.6)$$

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \frac{d}{dx}F(g(x)) = g'(x)F'(g(x))$$
$$= f(g(x))\frac{dg}{dx}$$

PTO

Indefinite integrals

$$\int f(x)\,dx \equiv \int_{x_0}^{x} f(t)\,dt$$ (for some arbitrary x_0 that determines the constant of integration).

Differential Equations - Lecture 4

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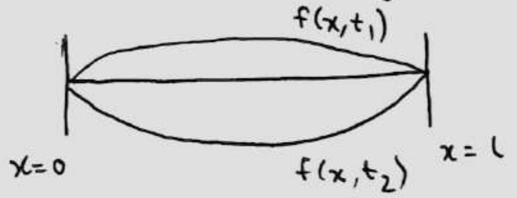
Introduction to Multivariate Functions

Functions may involve more than one independent variable.

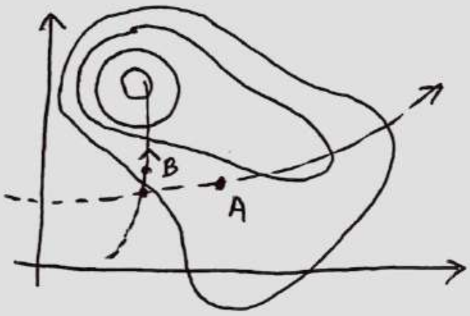
Example waves on a string



where $f(x,t)$ is height displacement of string

How do we define a derivative for such functions?

Suppose $f(x,y)$ is elevation of the terrain at a location (x,y)



"Gradient" or slope depends on where the path is and the direction taken.

Partial Derivatives

Partial derivative of $f(x,y)$ wrt x:

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} \qquad (4.1)$$

$$\left.\frac{\partial f}{\partial y}\right|_x = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} \qquad (4.2)$$

Example $f(x,y) = x^2 + y^3 + e^{xy^2}$

$$\left.\frac{\partial f}{\partial x}\right|_y = 2x + y^2 e^{xy^2}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_y = 2 + y^4 e^{xy^2}$$

We can also take cross-derivatives

e.g. $\quad f(x,y) = x^2 + y^3 + e^{xy^2}$

$$\frac{\partial}{\partial y}\left(\left.\frac{\partial f}{\partial x}\right|_y\right)\bigg|_x = 2ye^{xy^2} + 2xy^3 e^{xy^2}$$

Often we omit $|_x$ etc, and use of ∂ implies that all other variables are fixed. So above is $\frac{\partial^2 f}{\partial y \partial x}$.

<u>Note</u> $\quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ (can check easily) if all derivatives exist. $\quad$ (4.2)

e.g with $f(x,y,z)$

$$\frac{\partial f}{\partial x} \equiv \left.\frac{\partial f}{\partial x}\right|_{yz} \neq \left.\frac{\partial f}{\partial x}\right|_y$$: depends on the "path" in z

<u>Shorthand notation</u>

$$\frac{\partial f}{\partial x} = f_x \qquad \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \qquad \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

<u>Multivariate Chain Rule</u> (MV chain rule)

This is the chain rule applied to functions of more than one variable.

e.g. $\underline{f(x(t), y(t))}$

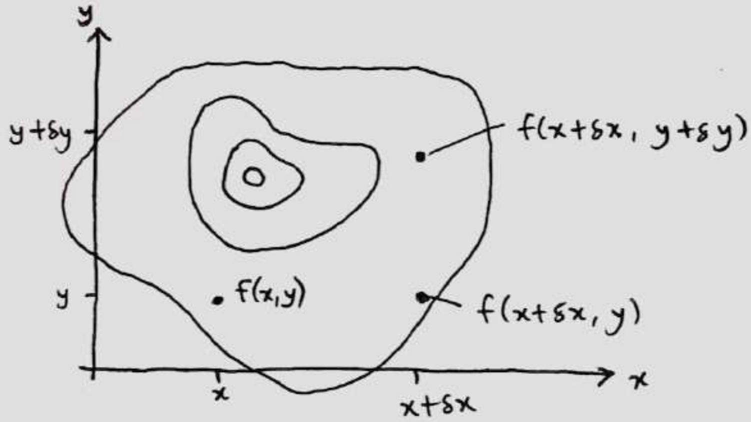
What is $\frac{df}{dt}$?

First define <u>differential</u> of a function.

<u>Definition</u> Differential of f is

$$\underline{\delta f = f(x + \delta x, y + \delta y) - f(x, y)} \qquad (4.3)$$

PTO



Add and subtract $f(x+\delta x, y)$ from the δf definition.

$$(*) \quad \delta f = f(x+\delta x, y+\delta y) - f(x+\delta x, y) + f(x+\delta x, y) - f(x,y)$$

Recall Taylor series: $$\boxed{f(x_0+h) = f(x_0) + \left.\frac{df}{dx}\right|_{x_0} h + \underline{o}(h)}$$

Then in $(*)$ we can <u>expand</u> $f(x+\delta x, y+\delta y)$ in y (note $y = x_0$, $\delta y = h$) and $f(x+\delta x, y)$ in x ($x = x_0$, $\delta x = h$).

$$\begin{aligned}\delta f &= \cancel{f(x+\delta x, y)} + \delta y\left(\frac{\partial f}{\partial y}(x+\delta x, y)\right) + \underline{o}(\delta y) - \cancel{f(x+\delta x, y)} \\ &\quad + \cancel{f(x,y)} + \delta x\left(\frac{\partial f}{\partial x}(x,y)\right) + \underline{o}(\delta x) - \cancel{f(x,y)} \qquad (\dagger)\end{aligned}$$

as $\delta x, \delta y \to 0$.

We can also expand the function $\frac{\partial f}{\partial y}(x+\delta x, y)$ with a TS in x:

$$\frac{\partial f}{\partial y}(x+\delta x, y) = \frac{\partial f}{\partial y}(x,y) + \delta x \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) + \underline{o}(\delta x)$$

Then from $(\dagger)$ we have

(also goes to 0 due to $\delta x \delta y$)

$$\begin{aligned}\delta f &= \delta y \frac{\partial f}{\partial y}(x,y) + \delta x \frac{\partial f}{\partial x}(x,y) + \delta x \delta y \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ &\quad + \underline{o}(\delta x) + \underline{o}(\delta y) + \underline{o}(\delta y \delta x)\end{aligned}$$

as $\delta x, \delta y \to 0$.

Then the $\underline{o}$ terms go to 0, leaving

$$\delta f = \delta x \frac{\partial f}{\partial x}(x,y) + \delta y \frac{\partial f}{\partial y}(x,y)$$

and letting $\delta x, \delta y \to 0$ we can write

$$(4.5) \quad \boxed{df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy}$$

writing "df" $= \lim_{\delta x, \delta y \to 0} \delta f$ etc.

This is the MV chain rule in differential form.

We can apply (4.5) by dividing by another differential (before taking the limit)

<u>Example</u> $\frac{d}{dt} f(x(t), y(t))$

$$(4.6) \quad \boxed{\frac{df}{dt} = \lim_{\delta x, \delta y, \delta t \to 0} \left[\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right]}$$

from "dividing" MV chain rule (4.5) by dt.

The multivariate chain rule

Standard way to write is

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

Similarly if $f = f(x, y(x))$ then

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

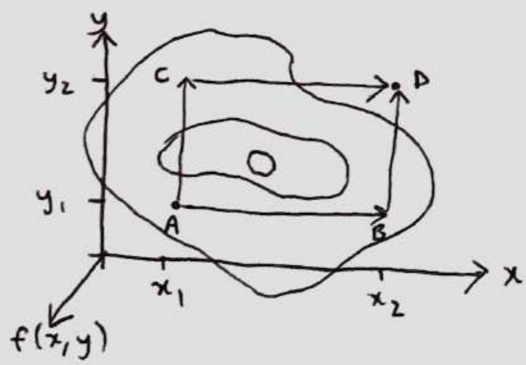
To get the integral form, integrate (4.5):

$$\boxed{\int df = \int \frac{\partial f}{\partial x}\, dx + \int \frac{\partial f}{\partial y}\, dy} \qquad (4.7)$$

MV chain rule in integral form

Note: need to integrate 4.7 along a <u>path</u>* from (x_1, y_1) to (x_2, y_2) (A to D)

e.g. $f(x, y)$:



$$\overbrace{f(x_2, y_2) - f(x_1, y_1)}^{\int_{x_1,y_1}^{x_2,y_2} df} = \int_{x_1}^{x_2} \frac{\partial f}{\partial x}(x, y_1)\, dx \quad \text{A to B, } y_1 \text{ fixed}$$

$$+ \int_{y_1}^{y_2} \frac{\partial f}{\partial y}(x_2, y)\, dy \quad \text{B to D, } x_2 \text{ fixed}$$

$$= \underbrace{\int_{y_1}^{y_2} \frac{\partial f}{\partial y}(x_1, y)\, dy}_{\text{A to C}} + \underbrace{\int_{x_1}^{x_2} \frac{\partial f}{\partial x}(x, y_2)\, dx}_{\text{C to D}}$$

$$\neq \int_{x_1}^{x_2} \frac{\partial f}{\partial x}(x, y_1)\, dx + \int_{y_1}^{y_2} \frac{\partial f}{\partial y}(x_1, y)\, dy \qquad \text{(A to B then A to C)}$$

* can think of as a "change in elevation"

Differential Equations - Lecture 5

Applications of multivariate chain rule

1) Change of variables

It's often useful to write a DE in a different coordinate system before solving (e.g. cartesian → polar)

Transform derivatives into new system

$$x = r\cos\theta$$
$$y = r\sin\theta$$

first write $f = f(x(r,\theta), y(r,\theta))$

$$\left.\frac{\partial f}{\partial r}\right|_\theta = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial r}\right|_\theta + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial r}\right|_\theta$$

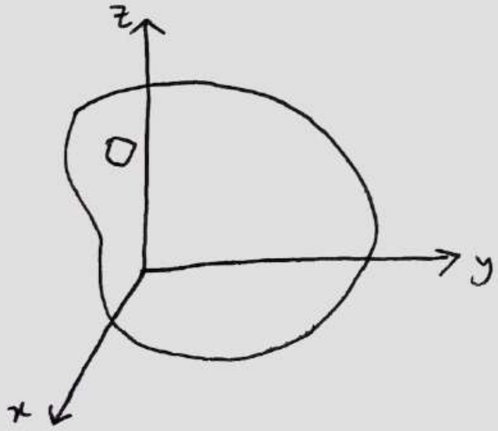
$$= \left.\frac{\partial f}{\partial x}\right|_y \cos\theta + \left.\frac{\partial f}{\partial y}\right|_x \sin\theta$$

Similar procedure for other derivatives.

2) Implicit differentiation

Consider $f(x,y,z) = c$ $\quad$ c constant

Describes a surface in 3D



Example $\quad xy + y^2 z + z^5 = 1 \quad (5.1)$

(5.1) implicitly defines

$x(y,z)$, $y(x,z)$, $z(x,y)$

We can't write $z(x,y)$ explicitly (quintic)

Can still find $\left.\frac{\partial z}{\partial x}\right|_y$ explicitly.

(Use implicit differentiation)

$$\frac{\partial}{\partial x}\Big|_y (5.1) \Rightarrow y + y^2 \frac{\partial z}{\partial x}\Big|_y + 5z^4 \frac{\partial z}{\partial x}\Big|_y = 0$$

In general, consider ~~f(x,y)~~ $\underbrace{f(x, y, z(x,y)) = c}_{\text{implicitly } z = z(x,y)}$

$\boxed{\text{Find } \frac{\partial z}{\partial x}\Big|_y}$. Use MV chain rule. (Differential form)

$$df = \frac{\partial f}{\partial x}\Big|_{yz} dx + \frac{\partial f}{\partial y}\Big|_{xz} dy + \frac{\partial f}{\partial z}\Big|_{xy} dz \qquad (5.2)$$

Find rate of change wrt x for fixed y along the surface $f = c$. *

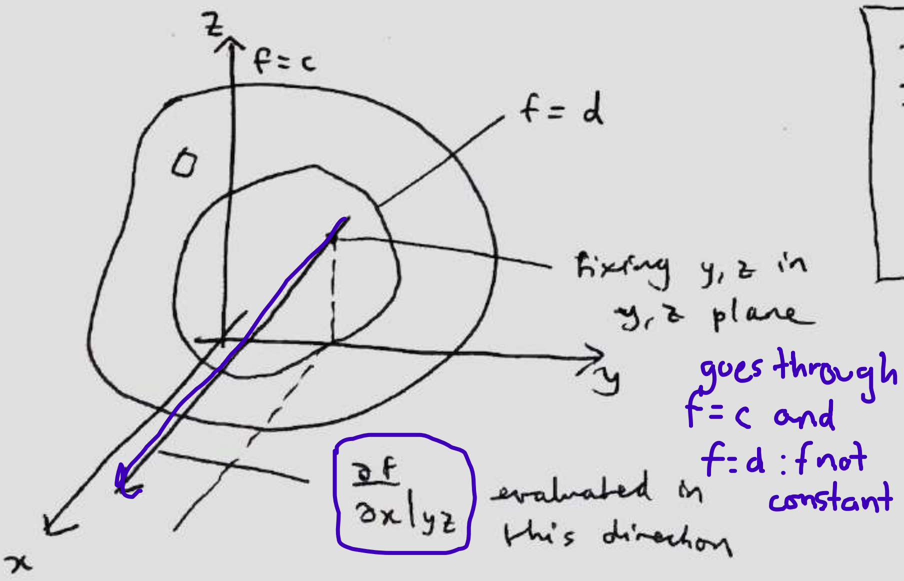
$$\frac{\partial f}{\partial x}\Big|_y = \frac{\partial f}{\partial x}\Big|_{yz} \overbrace{\frac{\partial x}{\partial x}\Big|_y}^{1} + \frac{\partial f}{\partial y}\Big|_{xz} \overbrace{\frac{\partial y}{\partial x}\Big|_y}^{\text{fixed } y:\ 0} + \frac{\partial f}{\partial z}\Big|_{xy} \frac{\partial z}{\partial x}\Big|_y$$

"divide" (5.2) by ∂x to turn into derivatives.

so
$$\underbrace{\frac{\partial f}{\partial x}\Big|_y}_{0\text{: moving along } f=c \text{ surface } z(x,y)} = \frac{\partial f}{\partial x}\Big|_{yz} + \frac{\partial f}{\partial z}\Big|_{xy} \frac{\partial z}{\partial x}\Big|_y \qquad (5.21)$$

Note $\frac{\partial f}{\partial x}\Big|_{yz} \neq 0$ in general so ~~too~~ from (5.21) we get

$$\boxed{\frac{\partial z}{\partial x}\Big|_y = -\frac{\frac{\partial f}{\partial x}\Big|_{yz}}{\frac{\partial f}{\partial z}\Big|_{xy}}}$$



* choose this so $\frac{\partial f}{\partial x}\Big|_y$ will become 0

Fixing just y can keep you "stuck" on the surface $f=c$: $\frac{\partial f}{\partial x}\Big|_y = 0$

Reciprocal rule

Applies to partial derivatives as long as <u>same</u> variables fixed e.g. $f(x(r,\theta), y(r,\theta))$

$$\left.\frac{\partial r}{\partial x}\right|_y = \frac{1}{\left.\partial x/\partial r\right|_y}$$

<u>but</u> $\underset{\text{fixed } y}{\frac{\partial r}{\partial x}} \neq \underset{\text{fixed } \theta}{\frac{1}{\partial x/\partial r}}$ ← Can't use: doesn't fix the same variables.

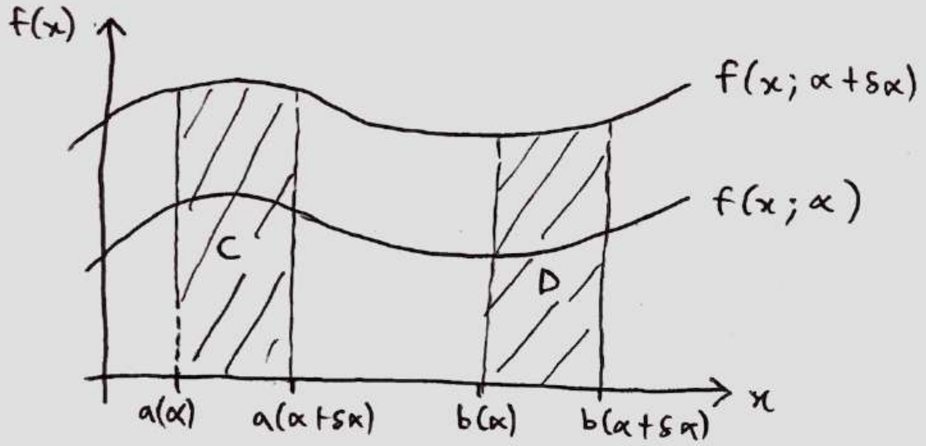
3) <u>Differentiate integral wrt its parameters</u>

Consider a family of functions $f(x;\alpha)$ where α is a parameter

e.g. $f(x;\alpha) = \log_\alpha x$

Define $I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)\, dx$. What is $\frac{dI}{d\alpha}$?

$$\frac{dI}{d\alpha} = \lim_{\delta\alpha \to 0} \frac{I(\alpha+\delta\alpha) - I(\alpha)}{\delta\alpha}$$



Exercise: divide up graph into areas and check that this works. ↓

$$\frac{dI}{d\alpha} = \lim_{\delta\alpha \to 0} \frac{1}{\delta\alpha}\left[\int_{a(\alpha+\delta\alpha)}^{b(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\, dx - \int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)\, dx\right]$$

$$= \lim_{\delta\alpha \to 0} \frac{1}{\delta\alpha}\left[\int_{a(\alpha)}^{b(\alpha)} f(x;\alpha+\delta\alpha) - f(x;\alpha)\, dx - \underbrace{\int_{a(\alpha)}^{a(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\, dx}_{C} + \underbrace{\int_{b(\alpha)}^{b(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\, dx}_{D}\right] \quad \leftarrow \text{(5.22)}$$

Can put $\frac{1}{\delta\alpha}$ inside the first term, giving $\frac{\partial f}{\partial \alpha}$ in the first integral.

Consider $\frac{1}{\delta\alpha}\int_{a(\alpha)}^{a(\alpha+\delta\alpha)} f(x;\, \alpha+\delta\alpha)\, dx$ as $\delta\alpha \to 0$.

Then $a(\alpha+\delta\alpha) \to a(\alpha)$ so the value of x gets very close to $\underline{a(\alpha)}$ so $f(x;\, a+\delta\alpha)$ effectively becomes $f(a;\alpha)$ giving

$$f(a;\alpha)\int_{a(\alpha)}^{a(\alpha+\delta\alpha)} \frac{1}{\delta\alpha}\, dx = f(a;\alpha)\,\frac{a(\alpha+\delta\alpha)-a(\alpha)}{\delta\alpha}$$

as $\delta\alpha \to 0$

$$= \underline{f(a;\alpha)\,\frac{da}{d\alpha}}.$$

Similarly for the 3rd term we get $f(b;\alpha)\,\frac{db}{d\alpha}$.

So from (5.22) we get

$$\boxed{\frac{dI}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}\, dx + f(b;\alpha)\frac{db}{d\alpha} - f(a;\alpha)\frac{da}{d\alpha}} \quad (5.3).$$

Notice that the last 2 terms disappear for constant a, b giving the standard DUTIS formula.

Differential Equations - Lecture 6

20/10/20

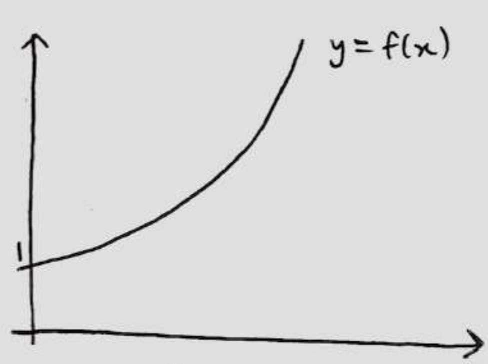
Section 2: First order linear ODEs

Terminology

- Ordinary DE (ODE): DE involving a function of one variable
- Partial DE (PDE): DE involving a function / functions of more than one variable (partial derivatives)
- n^{th} order DE: highest order derivative is n
- linear DE: dependent variable appears linearly

  e.g. $x^2 y + y' = 0$ is 1^{st} order and linear (y appears linearly)

Prelude Exponential function



Consider $f(x) = a^x, \quad a > 0$

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} a^x \frac{a^h - 1}{h}$$

$$= a^x \underbrace{\lim_{h\to 0} \frac{a^h - 1}{h}}_{\lambda \text{ (constant: no } x)} \quad \text{so} \quad \underline{\frac{df}{dx} = \lambda a^x} \qquad (6.1)$$

Define $\exp(x) = e^x$ as the solution to the DE

$$\frac{df}{dx} = f(x) \quad \text{with} \quad f(0) = 1. \qquad (6.2)$$

Then e is the value of a such that $\lambda = 1$:

$$\lim_{h\to 0} \frac{a^h - 1}{h} = 1$$

Numerically $e = 2.718...$

Natural logarithm $\ln x$ is the inverse of e^x such that $e^{\ln x} = x$

Consider (6.1). Let $y(x) = a^x = (e^{\ln a})^x = e^{x \ln a}$

Then $\frac{dy}{dx} = \ln a \, e^{x \ln a} = \underline{a^x \ln a}$

Hence in (6.1) $\lambda = \ln a$.

The exponential function plays a central role in DEs as it is the <u>eigenfunction</u> of the differential operator.

The <u>eigenfunction</u> of an operator is unchanged by the action of the operator, except for a multiplicative scaling by the <u>eigenvalue</u>.

Terminology from David Hilbert: "eigen" German for own

Consider $\frac{d}{dx}(e^{\lambda x}) = \lambda e^{\lambda x}$ so λ is the eigenvalue.

<u>Rules for linear ODEs</u> (n^{th} order)

1) Any linear homogeneous ODE with constant coefficients has solutions in the form of $\underline{e^{\lambda x}}$.

<u>Homogeneous</u>: all terms in equation involve the dependent variable (e.g. y) or its derivatives.

<u>Constant coefficients</u>: independent variable (e.g. x) doesn't appear explicitly

e.g. $5y' - 3y = 0$ (6.3)

Try $y = Ae^{\lambda x}$, $y' = A\lambda e^{\lambda x}$

(6.3) $\Rightarrow 5\cancel{A}\lambda \cancel{e^{\lambda x}} - 3\cancel{A}\cancel{e^{\lambda x}} = 0$ leaving <u>"characteristic eqn"</u>

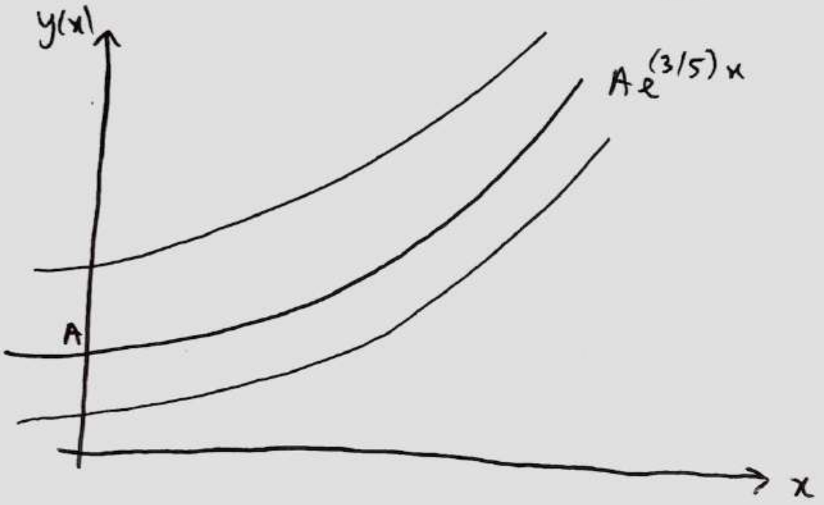
$\Rightarrow 5\lambda - 3 = 0 \Rightarrow \lambda = \frac{3}{5}$

so $y = Ae^{3x/5}$ (general solution)

2) For linear homogeneous ODEs, any constant multiple of a solution is also a solution.

3) An n^{th} order linear ODE has n independent solutions.
For constant coeff. ODEs, this follows from Fundamental Theorem of Calculus applied to characteristic polynomial eqn (order n).
So $y = Ae^{3x/5}$ is the general soln. to (6.3).

4) An n^{th} order ODE requires n initial / boundary conditions.



e.g. set $y = A$ at $x = 0$, the solutions is determined.

"family of solutions" depends on initial conditions.

Differential Equations – Lecture 7

22/10/20

Inhomogeneous (forced) 1st order ODEs with constant coefficients

1. Constant Forcing

e.g. $5y' - 3y = 10$

Solution steps

1. Write general solution $y = y_p + y_c$

   y_p = particular integral

   y_c = complementary function

2. Find y_p by setting $y' = 0$

   So $-3y_p = 10 \Rightarrow y_p = \frac{-10}{3}$ as $y' = 0 \Rightarrow y$ is constant

   and RHS term is constant so in this case $-\frac{10}{3}$ must be a solution.

3. Insert general soln into DE

   $y' = y_p' + y_c'$

   $5y_c' - 3y_c = 0$ (as y_p terms cancel RHS: $5y_p' - 3y_p = 10$).

   So y_c is a solution to the corresponding homogeneous eqn.

   $y_c = Ae^{3x/5}$ (cf last lecture)

   Then general solution is $y = Ae^{3x/5} - \frac{10}{3}$.

Case 2 : Eigenfunction forcing

Example Let n be the number of nuclei of isotope N.
$(n = a, b, c)$ $(N = A, B, C)$

In a sample of rock, isotope A decays into isotope B at a rate proportional to a, and isotope B decays into isotope C at a rate proportional to b.

Find $b(t)$.

rate:

$$\boxed{A} \xrightarrow{k_a a} \boxed{B} \xrightarrow{k_b b} \boxed{C}$$

$$\frac{da}{dt} = -k_a a \Rightarrow a = a_0 e^{-k_a t}$$

$$\frac{db}{dt} = k_a a - k_b b \Rightarrow \dot{b} + k_b b = k_a a = k_a a_0 e^{-k_a t}$$

$$\cancel{\text{so } \dot{b} + k_b b \Rightarrow}$$

$$\text{so} \quad \underline{\dot{b} + k_b b = k_a a_0 e^{-k_a t}} \qquad (7.1)$$

forcing term is eigenfunction of differential operator

Guess particular integral

$$b_p = c e^{-k_a t}$$

$$(7.1) \Rightarrow -k_a c + k_b c = k_a a_0 \quad (\text{cancelling } e^{-k_a t} > 0)$$

$$\Rightarrow c = \frac{k_a a_0}{k_b - k_a} \quad \text{for } k_b \neq k_a$$

General soln: $b = b_p + b_c$

(7.1) with b_p : we get $\dot{b}_c + k_b b_c = 0$ (b_p cancels RHS)

$$b_c = D e^{-k_b t}$$

so solution is $$b(t) = \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + D e^{-k_b t}$$

For initial conditions with $b = 0$ at $t = 0$

we get $D = -c = -\dfrac{k_a}{k_b - k_a} a_0$

so $b(t) = \dfrac{k_a}{k_b - k_a}\left(a_0 e^{-k_a t} - e^{-k_b t}\right)$

$\dfrac{b(t)}{a(t)}$ must be

$\dfrac{k_a}{k_b - k_a}\left[1 - e^{(k_a - k_b)t}\right]$

This result allows rocks / materials to be dated by measuring the ratio of isotopes.

First order ODEs with non-constant coefficients

General form $\quad a(x)\, y' + b(x)\, y = c(x)$

"standard form" $\quad \underline{y' + p(x)\, y = f(x)} \qquad (7.2)$

Solve using integrating factors (IF)

Multiply 7.2 by μ (IF) where we have

$$\underbrace{\mu y' + (\mu p)\, y}_{= (\mu y)'} = \mu f$$

$= (\mu y)'$ if $\mu p = \mu'$ (by product rule)

Hence we want $p = \dfrac{\mu'}{\mu} \Rightarrow \int p \, dx = \ln \mu$

$$\underline{\mu = e^{\int p(x)\, dx}} \qquad (7.3)$$

then (7.2) $\Rightarrow (\mu y)' = f(x)\mu \Rightarrow \underline{\mu y = \int \mu f \, dx}$

Integrate then solve for $y(x)$.

Differential Equations - Lecture 8

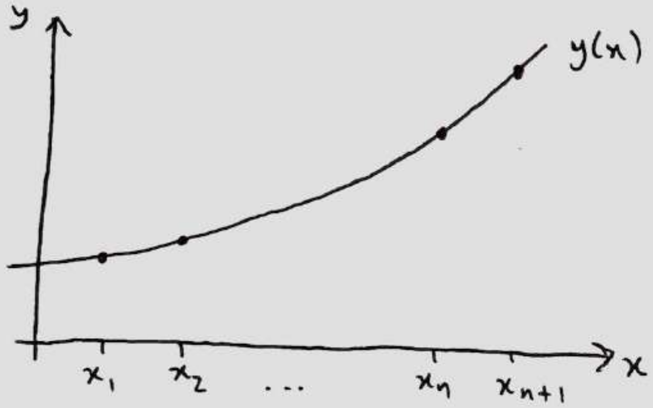
24/10/20

Discrete Equations

An equation involving a function evaluated at a discrete set of points.

1. Numerical Integration

Consider a discrete representation of $y(x)$



One approximation to $\frac{dy}{dx}$ is

$$\boxed{\left.\frac{dy}{dx}\right|_{x_n} \simeq \frac{y_{n+1}-y_n}{h}}$$

(uniform spacing: $h = \frac{x_n}{n}$)

(Forward Euler - not the best representation)

<u>Example</u> $5y' - 3y = 0 \qquad (8.1)$

$$5\frac{y_{n+1}-y_n}{h} - 3y_n = 0 \qquad \text{(difference equation)}$$

$$\Rightarrow \quad y_{n+1} = \left(1 + \frac{3h}{5}\right) y_n \quad , \text{ a recurrence relation.}$$

Apply recurrence relation repeatedly

$$y_n = \left(1 + \frac{3h}{5}\right) y_{n-1} = \left(1 + \frac{3h}{5}\right)^2 y_{n-2} = \dots$$

$$= \left(1 + \frac{3h}{5}\right)^n y_0 = \underline{\left(1 + \frac{3x_n}{5n}\right)^n y_0}$$

Euler defined the exponential function

$$\exp(x) := \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n \qquad \text{(see sheet 1)}$$

Hence $\displaystyle\lim_{n\to\infty} y_n = \underline{y_0 e^{3x/5}}$ as required.

Note for finite n, $y_n < y(x)$.

2. Series solutions

A powerful way to solve ODEs is to seek solutions in the form of an infinite power series.

Try $y(x) = \sum_{n=0}^{\infty} a_n x^n$ Plug form into DE, find a_n

Example (8-1) $5y' - 3y = 0$

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$

Multiply (8.1) by x:

$$xy' = \sum_{n=0}^{\infty} n a_n x^n = \sum_{n=1}^{\infty} n a_n x^n$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{m=1}^{\infty} a_{m-1} x^m$$

(same powers)

Then 8.1 $\Rightarrow$ $5\sum_{n=1}^{\infty} n a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = 0$

$$\Rightarrow \sum_{n=1}^{\infty} x^n (5n a_n - 3a_{n-1}) = 0 \qquad (8.2)$$

This must hold for all x, therefore $5n a_n - 3a_{n-1} = 0$

or $a_n = \frac{3}{5n} a_{n-1}$.

Iterating: $a_n = \left(\frac{3}{5}\right)^2 \frac{a_{n-2}}{n(n-1)} = \left(\frac{3}{5}\right)^3 \frac{a_{n-3}}{n(n-1)(n-2)}$

$$= \dots = \left(\frac{3}{5}\right)^n \frac{a_0}{n!}$$

Hence $y = a_0 \underbrace{\left(1 + \frac{3}{5}x + \frac{(3x/5)^2}{2!} + \dots\right)}_{\substack{\text{power series expansion} \\ \text{for } e^{3x/5} \text{ (converges } \forall x)}}$

so $\underline{y(x) = a_0 e^{3x/5}}$.

Nonlinear 1st order ODEs

General form $\quad Q(x,y)\dfrac{dy}{dx} + P(x,y) = 0 \qquad (8.3)$

1. Separable Equations

(8.3) is separable if it can be written in the form

$$q(y)\,dy = p(x)\,dx \qquad (8.4)$$

then solve for $y(x)$ by integrating both sides.

2. Exact Equations

(8.3) is an exact equation iff

$$Q(x,y)\,dy + P(x,y)\,dx \qquad (8.5)$$

is an <u>exact differential</u> of some function $f(x,y)$.

i.e. $\quad df = Q\,dy + P\,dx$

If this holds, then $(8.3) \Rightarrow df = 0$

and $f(x,y) = \text{constant}$ is ~~a~~ the solution.

To check (8.5) and find $f(x,y)$, use MV chain rule:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$df = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

Comparing with original (8.3), if (8.5) is an exact differential then $\exists\, f(x,y)$ s.t. $\dfrac{\partial f}{\partial x} = P(x,y)$, $\dfrac{\partial f}{\partial y} = Q(x,y) \qquad (8.6)$

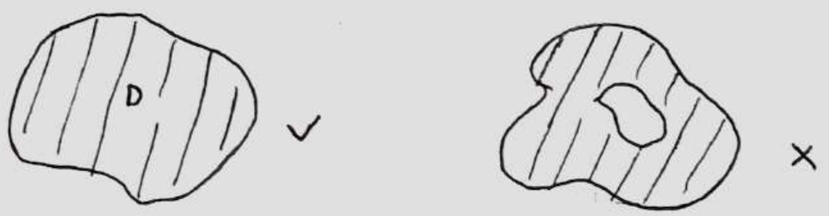
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}} \qquad (8.7)$$

If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ throughout a simply connected domain D,

then $P\,dx + Q\,dy$ is an exact differential of a single valued function $f(x, y)$ in D.

In 2D an S.C.D is a domain without "holes":



Use (8.7) to check for exact equations.
If 8.7 holds, find $f(x, y)$ by integrating (8.6)

<u>Example</u> $\quad 6y(y-x)\dfrac{dy}{dx} + (2x - 3y^2) = 0$

Here $P = 2x - 3y^2$, $\quad Q = 6y(y-x)$

$\dfrac{\partial P}{\partial y} = -6y \qquad \dfrac{\partial Q}{\partial x} = -6y \quad$ so they're the same, so it's <u>exact</u>.

$$\left.\frac{\partial f}{\partial x}\right|_y = P = 2x - 3y^2 \Rightarrow f = x^2 - 3xy^2 + h(y)$$

constant of integration (note y is fixed in $\left.\frac{\partial f}{\partial x}\right|_y$)

$$\left.\frac{\partial f}{\partial y}\right|_x = -6y + \frac{dh}{dy} = 6y(y-x) \quad \text{so}$$

$$\frac{dh}{dy} = 6y^2 \Rightarrow h = 2y^3 + C$$

So $f(x, y) = x^2 - 3xy^2 + 2y^3 + C$ so general solution is

$$\underline{x^2 - 3xy^2 + 2y^3 = C.}$$

Differential Equations - Lecture 9 27/10/20

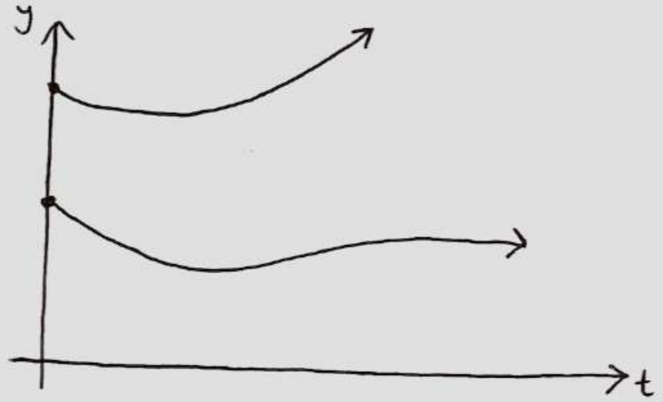
Isoclines and solution curves

Nonlinear DEs are not guaranteed to have simple closed form solutions. However we can analyse the behaviour of the system without actually solving the DE.

Consider an ODE of form

$$\frac{dy}{dt} = f(y,t).$$

Each initial condition (e.g. $y(0) = y_0$) gives a different solution curve.



For illustration, consider an equation that we can solve

e.g. $\frac{dy}{dt} = t(1-y^2)$

(9.1) $= f(y,t)$

Separable : $\frac{dy}{1-y^2} = t\,dt$

Integrate : we get $y = \frac{A - e^{-t^2}}{A + e^{-t^2}}$

This general solution produces a family of solution curves parameterised by A.

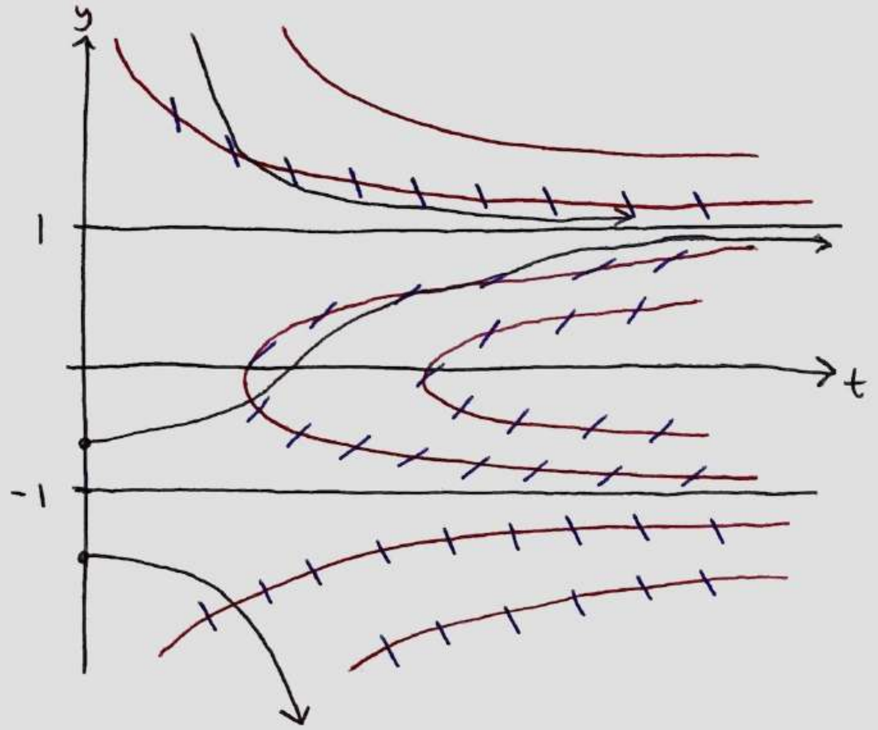
Can we sketch and describe the solution without actually solving it?

For (9.1) $\dot{y} = 0$ for $y = \pm 1$, $t = 0$

$f(y,t) = t(1-y^2)$ $\quad \dot{y} < 0$ for $|y| > 1$ (take time ≥ 0)

$\dot{y} > 0$ for $|y| < 1$

An isocline is a curve along which f is constant. ($\dot{y}$ is constant)



$$\frac{dy}{dt} = t(1-y^2)$$

$y=1$, $y=-1$ work

isoclines

$t(1-y^2) = c$

$\Rightarrow y^2 = 1 - \frac{c}{t}$

Given a fixed point (t, y), $\frac{dy}{dt} = t(1-y^2)$ takes precisely one value, then the solution curves cannot cross with different slopes

Fixed (equilibrium) points

These are points where $\dfrac{dy}{dt} = f(y,t) = 0 \quad \forall t$

Stable (unstable) fixed point is one where solution curves in a small neighbourhood of the fixed point converge (stable) or diverge (unstable) to or away from the fixed point.

Analyse stability by perturbation analysis

Let $y = a$ be a fixed point (F.P) of $\dfrac{dy}{dt} = f(y,t)$ such that $f(a,t) = 0 \quad \forall t$. $(*)$

Consider a small perturbation from the F.P.

Let $y = a + \varepsilon(t)$, ε small

MV Taylor series in y

keep t fixed as though constant

$$\text{so by DE,} \quad \frac{d\varepsilon}{dt} = f(a+\varepsilon, t)$$

$$= f(a,t) + \varepsilon \frac{\partial f}{\partial y}(a,t) + \mathcal{O}(\varepsilon^2)$$

0 by $(*)$

For sufficiently small ε can ignore $\mathcal{O}(\varepsilon^2)$

Then we get $\boxed{\dfrac{d\varepsilon}{dt} \approx \varepsilon \dfrac{\partial f}{\partial y}(a,t)}$ (9.2)

a linear ODE for $\varepsilon(t)$.

If $\lim_{t\to\infty} \varepsilon = 0$ then it's a <u>stable</u> fixed point

If $\lim_{t\to\infty} \varepsilon = \pm\infty$ then it's an <u>unstable</u> fixed point

If $\dfrac{\partial f}{\partial y}(a,t) = 0$, need higher order terms in Taylor series

<u>Example</u> $f(y,t) = t(1-y^2)$ (9.1) again

Fixed points: $y = \pm 1$

$$\frac{\partial f}{\partial y} = -2ty$$

<u>$y=1$</u> (9.1) $\Rightarrow$ $\dot{\varepsilon} = -2t\varepsilon$ by (9.2)

$$\Rightarrow \quad \varepsilon = \varepsilon_0 e^{-t^2}$$

$\lim_{t\to\infty} \varepsilon = 0$ so it's a stable fixed point.

<u>$y=-1$</u> (9.1) $\Rightarrow$ $\dot{\varepsilon} = 2t\varepsilon$

so $\varepsilon = \varepsilon_0 e^{t^2}$

as $t \to \infty$, $\varepsilon \to \pm\infty$ so it's an unstable fixed point.

Autonomous DEs

Special case where $\dot{y} = f(y)$

Near a FP where $y = a$,

$$y = a + \varepsilon(t)$$

$$\dot{\varepsilon} = \varepsilon \frac{df}{dy}(a) \quad \text{f(y) not f(y, t)} \quad = \varepsilon k, \ k \text{ constant}$$

($\frac{df}{dy}$ evaluated at a is k)

$$\Rightarrow \quad \varepsilon = \varepsilon_0 e^{kt}$$

Therefore, for autonomous DEs we have that

if $f'(a) < 0$ then stable FP

if $f'(a) > 0$ then unstable FP.

Differential Equations - Lecture 10 29/10/20

Phase Portraits

Another way to analyse solutions to a DE is using a geometrical representation of the solution called a phase portrait.

Example Chemical kinetics

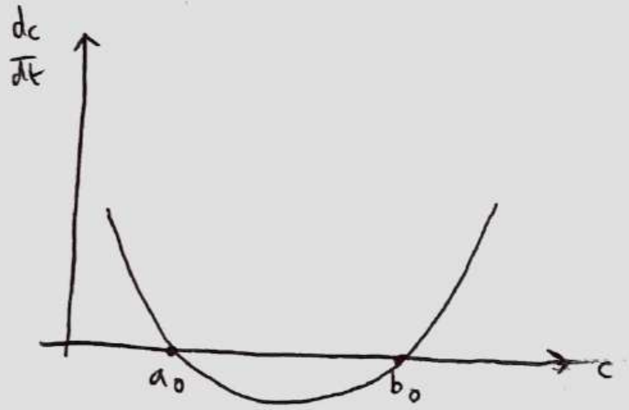
$$NaOH + HCl \longrightarrow H_2O + NaCl$$

# molecules	$a(t)$	$b(t)$	$c(t)$	$d(t)$
I.C.	a_0	b_0	0	0

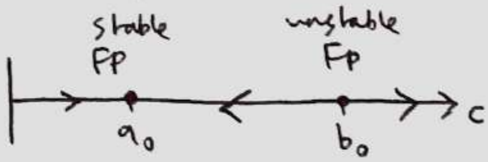
$$\frac{dc}{dt} = \lambda ab \qquad \left.\begin{array}{l} a = a_0 - c \\ b = b_0 - c \end{array}\right] \text{ atoms conserved}$$

$$\underline{\frac{dc}{dt} = \lambda(a_0 - c)(b_0 - c)}$$ autonomous nonlinear 1st order ODE

2D phase portrait



1D phase portrait



Example Population dynamics

Let $y(t)$ = population, αy = birth rate

βy = death rate

a) Linear model: $\frac{dy}{dt} = \alpha y - \beta y \Rightarrow y = y_0 e^{(\alpha - \beta)t}$

If $\alpha > \beta$ $\lim_{t \to \infty} y \to \infty$ unrealistic

b) Nonlinear model $\frac{dy}{dt} = (\alpha - \beta) y - \gamma y^2$

$$\frac{dy}{dt} = (\alpha - \beta) y - \gamma y^2$$

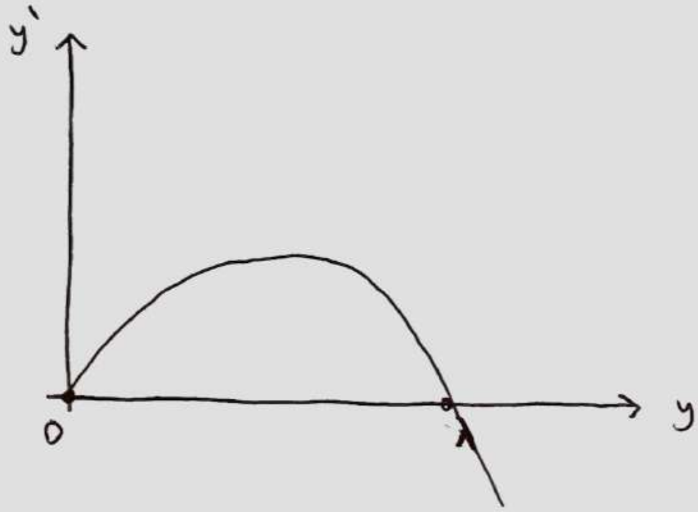
γy^2 models increased death rate at high populations

equivalently

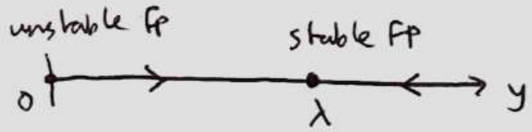
$$\dot{y} = r y \left(1 - \frac{y}{\lambda}\right) \qquad r = \alpha - \beta, \quad \lambda = \frac{\alpha - \beta}{\gamma}$$ "carrying capacity"

2D phase portrait

(Remember a fixed point y is where $y' = 0 \ \forall t$)



1D phase portrait



Fixed points in discrete equations

Consider a 1st order discrete equation of form

$$x_{n+1} = f(x_n)$$

Define fixed point as value of x_n where $x_{n+1} = x_n$

$$\Rightarrow f(x_n) = x_n$$

Stability of fixed points: perturbation analysis

Let x_f be a fixed point and perturb by ε (small)

Expand $f(x)$ in Taylor series about x_f

$$f(x_f + \varepsilon) = f(x_f) + \varepsilon \left.\frac{df}{dx}\right|_{x_f} + O(\varepsilon^2)$$

If we let $x_n = x_f + \varepsilon$

then $\cancel{x_{n+1} \approx f(x_f}$ $\quad x_{n+1} \approx x_f + \varepsilon \left.\frac{df}{dx}\right|_{x_f}$

$$x_f \text{ is } \begin{cases} \text{stable if } \left| \left.\frac{df}{dx}\right|_{x_f} \right| < 1 \\ \text{unstable if } \left| \left.\frac{df}{dx}\right|_{x_f} \right| > 1 \end{cases}$$

Example Logistic map

Nonlinear discrete population model

$$\frac{x_{n+1} - x_n}{\Delta t} = \lambda x_n - \gamma x_n^2$$

c.f nonlinear ODE earlier (example 2b)

or $$x_{n+1} = (\lambda \Delta t + 1) x_n - \gamma \Delta t \, x_n^2$$

cf. compound interest

A simpler version: $x_{n+1} = r x_n (1 - x_n) = f(x_n)$

"logistic map"

Fixed points: $f(x_n) = x_n$

so $r x_n (1 - x_n) = x_n \Rightarrow x_n = 0$ or $x_n = 1 - \frac{1}{r}$

Stability: $\frac{df}{dx} = r(1-2x)$ when $\underline{f(x) = rx(1-x)}$

$\underline{x_n = 0:}$ $\left.\frac{df}{dx}\right|_{x=0} = r$

For $0 < r < 1$: stable

For $r > 1$: unstable

$\underline{x_n = 1 - \frac{1}{r}}$: $\underline{\left.\frac{df}{dx}\right|_{x_n} = 2 - r}$

For $0 < r < 1$: unphysical (note pop. can't go negative)

$1 < r < 3$: stable $r > 3$: unstable

Differential Equations - Lecture 11

Part IV: Higher order linear ODEs

1. Linear 2nd order ODEs with constant coefficients

General form $ay'' + by' + cy = f(x)$ (11.1)

a, b, c constant

Exploit linearity and principle of superposition

From def. of derivative (1.2)

$$\frac{d}{dx}(y_1 + y_2) = \frac{dy_1}{dx} + \frac{dy_2}{dx}$$

$$\frac{d^2}{dx^2}(y_1 + y_2) = \frac{d^2 y_1}{dx^2} + \frac{d^2 y_2}{dx^2}$$

For a linear differential operator D built from a linear combination of derivatives

e.g. $D = \left[a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right]$

it follows that $D(y_1 + y_2) = D(y_1) + D(y_2)$

Exploit this by solving 11.1 in 3 steps:

1. Find complementary functions y_1, y_2 satisfying
   $ay'' + by' + cy = 0$ (11.2)

2. Find a particular integral y_p which solves (11.1)

3. If $y_1(x)$ and $y_2(x)$ are linearly independent, then $y_1 + y_p$ and $y_2 + y_p$ are linearly independent solutions to 11.1.

Follows since $D(y_1) = 0$, $D(y_p) = f(x)$

$\Rightarrow D(y_1 + y_p) = D(y_1) + D(y_p) = f(x)$

Linear Independence of Functions

Definition A set of functions are linearly dependent if

$$\sum_{i=1}^{N} c_i f_i(x) = 0 \qquad (11.3)$$

For a set of N functions, c_i constant and at least one c_i is nonzero

Equivalently, if any function can be written as a linear combination of the others: $f_1 = \alpha_2 f_2 + \alpha_3 f_3 + \dots \alpha_N f_N$ then are linearly dependent.

Eigenfunctions for 2nd order ODEs

Recall $e^{\lambda x}$ is eigenfunction of $\frac{d}{dx}$.
$e^{\lambda x}$ is also the eigenfunction of $\frac{d^n}{dx^n}$
In fact $e^{\lambda x}$ is eigenfunction of any linear differential operator D

Therefore solutions to 11.2 take the form $y_c = Ae^{\lambda x}$
plug into 11.2:

$$a\lambda^2 + b\lambda + c = 0 \qquad (11.3)$$

↑
characteristic (auxillary eqn)

From FTA there are 2 (possibly repeated) complex roots to (11.3). Let roots be λ_1, λ_2

<u>Case 1</u> $\lambda_1 \neq \lambda_2$

$$y_1 = Ae^{\lambda_1 x}, \quad y_2 = Be^{\lambda_2 x}$$

Here y_1, y_2 are linearly independent and <u>complete</u>: form a <u>basis</u> of <u>solution space</u>.

Any other solution to (11.3) is a linear combination of y_1, y_2

General form of y_c is $\underline{A e^{\lambda_1 x} + B e^{\lambda_2 x}}$.

<u>Case 2 $\lambda_1 = \lambda_2$</u> : degenerate

Here y_1 and y_2 are linearly dependent and not complete.

<u>Example</u> $y'' - 4y' + 4y = 0$ (11.4)

try $y_c = e^{\lambda x} \Rightarrow \lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$

2 roots $\lambda = 2, 2$ degenerate

<u>Detuning</u> Consider a slightly modified equation

$$y'' - 4y' + (4 - \varepsilon^2) y = 0 \qquad \varepsilon \text{ small}$$

Try $y_c = e^{\lambda x}$

$$\lambda^2 - 4\lambda + (4 - \varepsilon^2) = 0 \qquad \lambda = 2 \pm \varepsilon$$

$$\Rightarrow \quad y_c = A e^{(2+\varepsilon)x} + B e^{(2-\varepsilon)x}$$

$$= e^{2x} \left(A e^{\varepsilon x} + B e^{-\varepsilon x} \right)$$

Expand in Taylor series for small ε

(recall $e^{\varepsilon x} = 1 + \varepsilon x + O(\varepsilon^2)$)

$$y_c = e^{2x} \left[(A+B) + \varepsilon x (A-B) + O(\varepsilon^2) \right]$$

Letting $\varepsilon \to 0$ we get $e^{2x} \left[(A+B) + \varepsilon x (A-B) \right]$

Consider applying ICs to y_c at $x = 0$

$y_c|_{x=0} = C, \quad y_c'|_{x=0} = D$

$C = A + B \qquad D = 2C + \varepsilon (A - B)$

Hence $A + B = O(1), \quad A - B = O\left(\frac{1}{\varepsilon}\right)$ as $\varepsilon \to 0$

Let $\beta = \varepsilon(A - B) = \sigma(1)$

$\alpha = A + B = \sigma(1)$

hence $$\lim_{\varepsilon \to 0} y_c = e^{2x}\left[\alpha + \beta x\right]$$

<u>General rule</u> If $y_1(x)$ is a degenerate complementary function for linear ODE with constant coefficients Then $y_2 = x y_1$ is a linearly independent complementary function.

Differential Equations - Lecture 12 3/11/20

Homogeneous 2nd order linear ODEs with non-constant coefficients

General form:

$$\underline{y'' + p(x)\,y' + q(x)\,y = 0} \qquad (12.1)$$

Reduction of order : objective is given one solution to 12.1 $y_1(x)$, find 2nd solution $y_2(x)$ linearly independent of $y_1(x)$

Idea: look for a solution of the form $\underline{y_2(x) = v(x)\,y_1(x)}$ (12.2)

and find $v(x)$.

First note
$$y_2' = v'y_1 + vy_1'$$
$$y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

If y_2 satisfies (12.1)

$$y_2'' + p(x)\,y_2' + q(x)\,y_2 = 0$$

Use (12.2) and collect terms:

$$v(y_1'' + py_1' + qy_1) + v'(2y_1' + py_1) + v''y_1 = 0$$

y_1 solves (12.1) so this is 0

so $\underline{v'(2y_1' + py_1) + v''y_1 = 0}$, a 1st order equation for v'.

Let $u = v'$: $\underline{u'y_1 + u(2y_1' + py_1) = 0}$

Separable 1st order ODE for $u(x) = v'(x)$

Solve for $u(x)$ then integrate for $v(x)$.

Then $\underline{y_2(x) = v(x)\,y_1(x)}$.

Solution space

n^{th} order ODE (linear)

$$P(x)\, y^{(n)} + q(x)\, y^{(n-1)} + \dots + r(x)\, y = f(x)$$

This can be used to write $y^{(n)}$ in terms of $y, y', y'', \dots y^{(n-1)}$.

Example: damped oscillator $m\ddot{y} = -ky - L\dot{y}$

The state of the system can be described by an n-dimensional solution vector.

$$\underline{Y}(x) = \begin{pmatrix} y(x) \\ y'(x) \\ \dots \\ y^{(n-1)}(x) \end{pmatrix}$$

Example $y'' + 4y = 0$ (undamped oscillator)

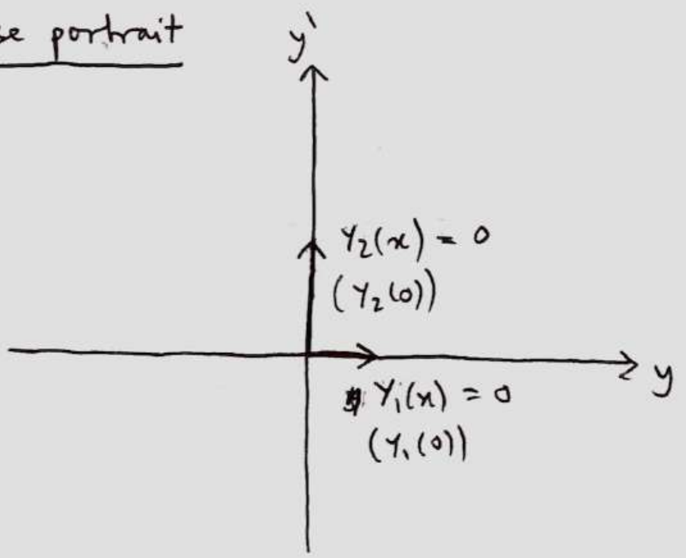
$$y_1 = \cos 2x \qquad y_2 = \sin 2x$$

$$y_1' = -2\sin 2x \qquad y_2' = 2\cos 2x$$

$$\underline{Y}_1(x) = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} \cos 2x \\ -2\sin 2x \end{pmatrix}$$

$$\underline{Y}_2(x) = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} \sin 2x \\ 2\cos 2x \end{pmatrix}$$

2D phase portrait



Since $\underline{Y}_1$ and $\underline{Y}_2$ are linearly independent for all x, any point in solution space (y, y') can be reached by a linear combination of Y_1 and Y_2.

$y_1, y_2, \ldots, y_n$ are linearly independent if their solution vectors $\underline{Y}_1, \underline{Y}_2, \ldots, \underline{Y}_n$ are linearly independent.

n linearly independent solution vectors form a basis for solution space of an n^{th} order ODE.

Initial Conditions

Consider ICs for a 2nd order homogeneous ODE:

$y(0) = a, \quad y'(0) = b$

If general solution is $y(x) = Ay_1(x) + By_2(x)$

then we have the following system

$$Ay_1(0) + By_2(0) = a$$
$$Ay_1'(0) + By_2'(0) = b$$

$$\underbrace{\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix}}_{\text{call this matrix } M} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Unique solutions for A, B exist if $\det M \neq 0$.

Wronskian $W(x)$

Definition: $W(x)$ is the determinant of the fundamental matrix, formed by placing solution vector $\underline{Y}_i$ in the i^{th} column.

$$W(x) = \begin{vmatrix} \vdots & \vdots & & \vdots \\ \underline{Y}_1 & \underline{Y}_2 & \cdots & \underline{Y}_n \\ \vdots & \vdots & & \vdots \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & \cdots & & \end{vmatrix}$$

For a second order ODE

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \qquad (12.4)$$

Solution vectors are linearly independent if $\underline{W(x) \neq 0}$.

e.g. $y'' + 4y = 0$

$$W(x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x$$
$$= 2(\cos^2 2x + \sin^2 2x)$$
$$= \underline{2 \neq 0}$$

so y_1, y_2 are lin. indep. $\forall x$.

Reverse implication: $\underline{y}_1, \underline{y}_2$ lin. dep $\Rightarrow W(x) = 0$

(contrapositive actually)

Suppose

$y(x)$ is a linear combination of $y_1(x), y_2(x)$

Then $\underline{y}(x), \underline{y}_1(x), \underline{y}_2(x)$ are a linearly dependent set

Hence $\begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = 0$ for $y_1 = \cos 2x$, $y_2 = \sin 2x$

$$\begin{vmatrix} y & \cos 2x & \sin 2x \\ y' & -2\sin 2x & 2\cos 2x \\ y'' & -4\cos 2x & -4\sin 2x \end{vmatrix} = 0$$

$$\Rightarrow y(8\sin^2 2x + 8\cos^2 2x) - y'(-4\cos 2x \sin 2x + 4\cos 2x \sin 2x)$$
$$+ y''(2\cos^2 2x + 2\sin^2 2x) = 0$$

$$\Rightarrow 8y + 2y'' = 0 \Rightarrow \underline{y'' + 4y = 0}$$

so we can actually reconstruct the original ODE this way.

Caution: $W(x) = 0 \not\Rightarrow$ linear dependence

We only have $W(x) \neq 0 \Rightarrow$ linear independence

Differential Equations - Lecture 13

Abel's Theorem

Consider a 2nd order ODE (homogeneous):

$$y'' + p(x)\, y' + q(x)\, y = 0 \qquad (13.1)$$

If $p(x)$ and $q(x)$ are continuous on an interval I, then the Wronskian $W(x)$ is either $W(x) = 0$, or $W(x) \neq 0 \;\; \forall x \in I$.

Sketch proof Let y_1 and y_2 be solutions to 13.1.

$$y_2 \left(y_1'' + p(x)\, y_1' + q(x)\, y_1 \right) = 0 \qquad (13.2)$$

$$y_1 \left(y_2'' + p(x)\, y_2' + q(x)\, y_2 \right) = 0 \qquad (13.3)$$

(13.2) - (13.3) gives

$$\left(y_2 y_1'' - y_1 y_2'' \right) + p(x) \left(y_2 y_1' - y_1 y_2' \right) = 0 \qquad (13.4)$$

Here $W(x) = y_1 y_2' - y_2 y_1'$

$$\frac{dW}{dx} = y_1 y_2'' + \cancel{y_1' y_2'} - \cancel{y_2' y_1'} - y_2 y_1''$$

Hence $(13.4) \Rightarrow \quad W' + pW = 0 \qquad (13.5)$

$$\frac{dW}{W} = -p(x)\, dx \qquad (13.6)$$

$$\Rightarrow \boxed{W(x) = W(x_0)\, e^{-\int_{x_0}^{x} p(u)\, du}} \qquad (13.7)$$

Abel's identity

Since $p(x)$ is continuous on I with $x \in I$, it is bounded and integrable.

Therefore $e^{-\int_{x_0}^{x} p(u)\, du} \neq 0$.

It follows that:

If $W(x_0) = 0$ then $W(x) = 0 \quad \forall x \in I$

$W(x_0) \neq 0 \Rightarrow W(x) \neq 0 \quad \forall x \in I$. $\square$

<u>Corollary</u> If $p(x) = 0$ then $W = W_0$ (constant).

Note that we can find $W(x)$ without solving the DE

<u>Example</u> Bessel's Equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

has no closed form solutions.

$$y'' + \frac{1}{x} y' + \frac{x^2 - n^2}{x^2} y = 0$$

$$p(x) = \frac{1}{x} \qquad \underline{W(x)} = W_0 e^{-\int^x p \, du}$$

$$= W_0 e^{-\int^x \frac{du}{u}} = W_0 e^{-\ln x}$$

Application of Abel's identity:

Find 2nd solution $\underset{y_2}{\wedge}$ given first solution y_1.

(13.7) For 2nd order ODE:

$$y_1 y_2' - y_2 y_1' = W_0 e^{-\int_{x_0}^{x} p(u) \, du} \qquad \text{gives a}$$

1st order ODE for y_2 that we can solve.

Abel's Theorem / Identity generalise to higher order ODEs.

Any linear n^{th} order ODE can be written

$$\underline{Y}' + \underline{\underline{A}}(x) \underline{Y} = \underline{0}$$

It can be shown that $\quad W' + \mathrm{Tr}(\underline{\underline{A}}) W = 0$

$$\Rightarrow W' = W_0 e^{-\int^x \mathrm{Tr}(\underline{\underline{A}}) \, du}$$

and Abel's theorem holds. (See ES 3, Q 7)

Equidimensional equations

An ODE is equidimensional if the differential operator is unaffected by a multiplicative scaling.

e.g. rescale from x to X with $X = \alpha x$, α constant

General form (2nd order equidimensional)

$$\boxed{a x^2 y'' + b x y' + c y = f(x)} \qquad (13.8)$$

a, b, c constant

Note $\dfrac{d}{dX} = \dfrac{1}{\alpha}\dfrac{d}{dx}$, $\quad \dfrac{d^2}{dX^2} = \dfrac{1}{\alpha^2}\dfrac{d^2}{dx^2}$

$$\therefore (13.8) \Rightarrow a X^2 \frac{d^2 y}{dX^2} + b X \frac{dy}{dX} + c y = f\left(\frac{X}{\alpha}\right)$$

$\boxed{\text{LHS unchanged}}$ ← required condition.

Solving

1. Note $y = x^k$ is an eigenfunction of the operator $x\dfrac{d}{dx}$

   since $x \dfrac{d}{dx} x^k = k x^{k-1} x = \underline{k x^k}$.

   To solve 13.8, try $y = x^k$:

   $(13.8) \Rightarrow \underline{ak(k-1) + bk + c = 0}$ (cancelling)

   Then solve quadratic for k_1, k_2

   If $k_1 \neq k_2$, $\quad y_c = A x^{k_1} + B x^{k_2} \qquad (13.9)$

2. Note $z = \ln x$ turns 13.8 into an equation with constant coefficients (exercise: double check)

   $$a \frac{d^2 y}{dx^2} + (b - a)\frac{dy}{dz} + c y = f(e^z)$$

Complementary function: $y = e^{\lambda z}$

$\Rightarrow \quad a\lambda^2 + (b-a)\lambda + c = 0$

solve for λ_1, λ_2

then $y_c = Ae^{\lambda_1 z} + Be^{\lambda_2 z}$ if $\lambda_1 \neq \lambda_2$

and note $z = \ln x \Rightarrow x = e^z$:

$\underline{y_c = Ax^{\lambda_1} + Bx^{\lambda_2}}$ (same form as 13.9)

If $\lambda_1 = \lambda_2$ then $y_c = Ae^{\lambda_1 z} + Bze^{\lambda_1 z}$

$(k_1 = k_2)$

$\Rightarrow \underline{y_c = Ax^{\lambda_1} + Bx^{\lambda_1} \ln x}$

Differential Equations - Lecture 14 8/11/20

Forced (inhomogeneous) 2nd order ODEs

Finding particular integrals $y_p(x)$

Method 1: Guesswork

Form of $f(x)$	Form of $y_p(x)$
e^{mx}	Ae^{mx}
$\sin kx, \cos kx$	$A\sin kx + B\cos kx$
$x^n, P_n(x)$	$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$

Steps
1. Insert guess into ODE
2. Equate coefficients of functions
3. Solve for coefficients

Method 2: Variation of Parameters

Finding y_p given y_c.

Given y_1, y_2 complementary functions (lin indep, sol^n vectors ⊥)

$$\underline{Y}_1 = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} \qquad \underline{Y}_2 = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$$

Suppose the sol^n vector for y_p satisfies

$$\underline{Y}_p = \begin{pmatrix} y_p \\ y_p' \end{pmatrix} = u(x)\,\underline{Y}_1 + v(x)\,\underline{Y}_2 \qquad (14.1)$$

Try finding 2 eqns for $u(x)$, $v(x)$

$$(14.1) \Rightarrow \quad y_p = uy_1 + vy_2 \qquad (a)$$
$$y_p' = uy_1' + vy_2' \qquad (b)$$

$\frac{d}{dx}(a) \Rightarrow u'y_1 + uy_1' + v'y_2 + vy_2'$ (c)

(c) − (b) : $\boxed{u'y_1 + v'y_2 = 0}$ (d)

$\frac{d}{dx}(b) \Rightarrow y_p'' = uy_1'' + u'y_1' + v'y_2' + vy_2''$ (e)

If $y_p(x)$ satisfies $y_p'' + p(x)\,y_p' + q(x)\,y_p = f(x)$

then (e) + p(x) (b) + q(x) (a) = f(x)

We also know $y_1'' + py_1' + qy_1 = 0$

$y_2'' + py_2' + qy_2 = 0$

Substitute in and simplify: (check this works)

gives $\boxed{u'y_1' + v'y_2' = f(x)}$ (f)

$$(d), (f) \Rightarrow \underbrace{\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

f(x)

Notice: This is the fundamental matrix ($w(x)$ = det)

$$\text{so} \quad \begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{w(x)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$\Rightarrow u' = -\frac{y_2 f}{w}, \quad v' = \frac{y_1 f}{w}$$

and $$\boxed{y_p = y_2 \int^x \frac{y_1(t)\,f(t)}{w(t)}\,dt - y_1 \int^x \frac{y_2(t)\,f(t)}{w(t)}\,dt}$$

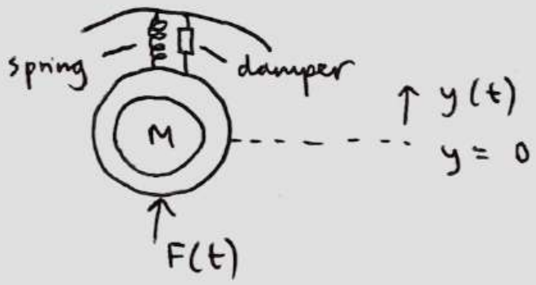
using (14.1).

Forced oscillating system

Transients and damping

Many physical systems have a restoring force and damping (e.g. friction)

Example: car suspension



Newton's 2nd Law

$$ma = \sum \text{Forces}$$

$$M\ddot{y} = F(t) - \underbrace{ky}_{\text{spring}} - \underbrace{L\dot{y}}_{\text{damper}} \qquad (14.2)$$

so $$\ddot{y} + \frac{L}{m}\dot{y} + \frac{k}{m}y = \frac{F(t)}{M}$$

Change vars: $\tau = \sqrt{\frac{k}{m}}\, t$

so $$y'' + 2Ky' + y = f(\tau)$$ with $y' = \frac{dy}{d\tau}$, $K = \frac{L}{2\sqrt{kM}}$, $f = \frac{F}{k}$

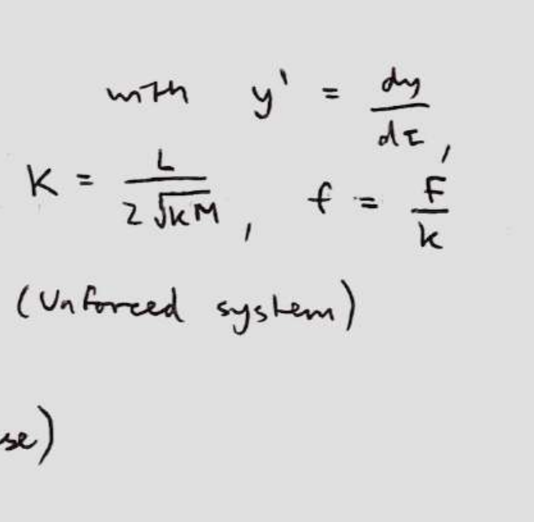
so only <u>one</u> parameter now, K. (Unforced system)

Unforced response (free / natural response)

$f = 0$: $$y'' + 2Ky' + y = 0$$

$$\lambda^2 + 2K\lambda + 1 = 0 \qquad \text{(char. eqn)}$$

$$\lambda = -K \pm \sqrt{K^2 - 1}$$



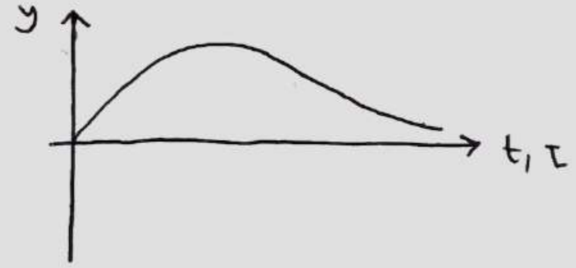
Case 1 $K < 1$ (underdamped)

$$\Rightarrow \quad y = e^{-K\tau}\left[A\sin\left(\sqrt{1-K^2}\,\tau\right) + B\cos\left(\sqrt{1-K^2}\,\tau\right)\right]$$

Note period $\to \infty$ as $K \to 1$.

Case 2 $K = 1$ Critical damping : $\lambda = -K$ (degenerate)

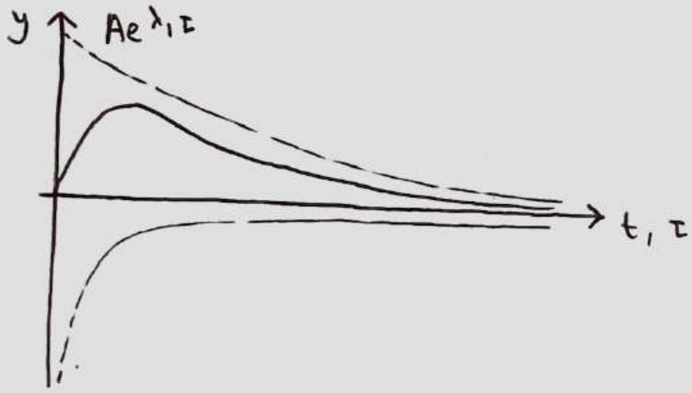
Use detuning : $y = (A + B\tau)\, e^{-K\tau}$ $(K = 1)$



Case 3 $K > 1$ Overdamped (prevents oscillations)

λ_1, λ_2 real, < 0

$y = A e^{\lambda_1 \tau} + B e^{\lambda_2 \tau}$ $(|\lambda_1| < |\lambda_2|$ wlog$)$



Note unforced response decays in all cases $(t \to \infty \Rightarrow y \to 0)$.

Differential Equations - Lecture 15

10/11/20

Damped oscillating systems: Forced response

e.g. $\ddot{y} + \mu\dot{y} + \omega_0^2 y = \sin\omega t$ (15.1)

Guess $y_p = A\sin\omega t + B\cos\omega t$

$\sin\omega t$: $-A\omega^2 - B\mu\omega + \omega_0^2 A = 1$ (a)

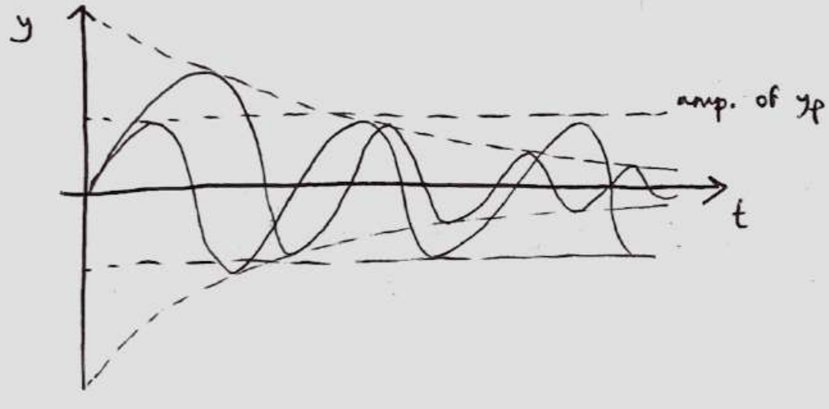
$\cos\omega t$: $-B\omega^2 + A\mu\omega + \omega_0^2 B = 0$ (b)

Solve simultaneously: gives $A = \dfrac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + \mu^2\omega^2}$

and $B = \dfrac{-\mu\omega}{(\omega_0^2-\omega^2)^2+\mu^2\omega^2}$,

so $$y_p = \frac{1}{(\omega_0^2-\omega^2)^2+\mu^2\omega^2}\left[(\omega_0^2-\omega^2)\sin\omega t - \mu\omega\cos\omega t\right]$$

$y = y_c + y_p$ (underdamped)



__Comments__ (for __damped__ systems)

Complementary functions give transient response to ICs
Particular integrals give long time response to forcing
(system "forgets" about ICs) as CF decays

Q: what if $\omega = \omega_0$?

If $\mu \neq 0$ (damped) $\displaystyle\lim_{\omega\to\omega_0} y_p = \frac{-\cos\omega_0 t}{\mu\omega_0}$ finite amplitude oscillation

Note amp. increases when $\mu = 0$.

Resonance in undamped systems

e.g. $\ddot{y} + \omega_0^2 y = \cancel{\sin} \sin \omega_0 t$ (5.2)

Use detuning : consider instead

$\ddot{y} + \omega_0^2 y = \sin \omega t$ (15.3) $\omega \neq \omega_0$

Guess $y_p = c \sin \omega t$ (no $\cos \omega t$ ∵ no $\dot{y}$ term)

$\Rightarrow c = (-\omega^2 + \omega_0^2) = 1$

$$y_p = \frac{1}{\omega_0^2 - \omega^2} \sin \omega t$$

$$y_p = \frac{1}{\omega_0^2 - \omega^2} \sin \omega t + A \sin \omega_0 t$$

as (15.2) linear so $y = y_p + A y_c$ also solves ODE

Pick $A = -\dfrac{1}{\omega_0^2 - \omega^2}$

$$\Rightarrow y_p = \frac{\sin \omega t - \sin \omega_0 t}{\omega_0^2 - \omega^2}$$

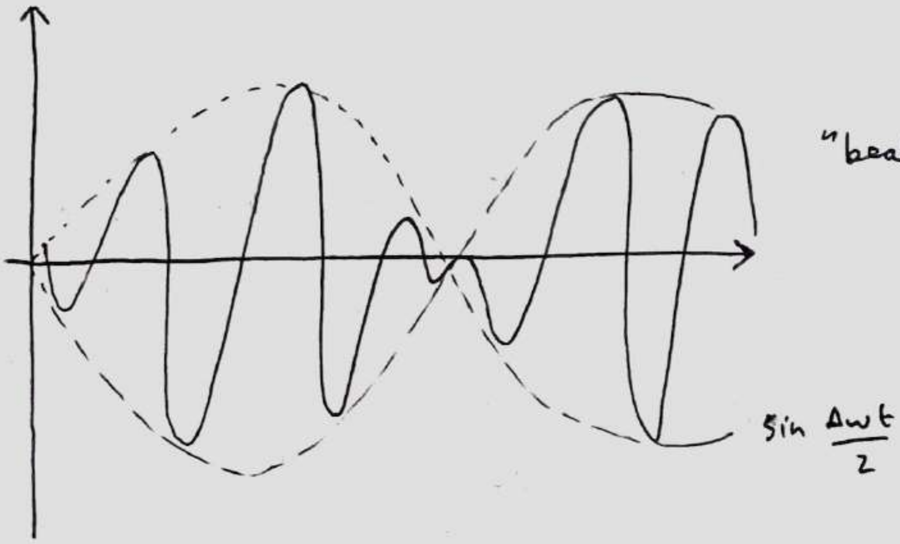
$$\Rightarrow y_p = \frac{2}{\omega_0^2 - \omega^2} \left[\cos\left(\frac{\omega + \omega_0}{2} t\right) \sin\left(\frac{\omega - \omega_0}{2} t\right) \right]$$ (double angle)

Let $\Delta \omega = \omega_0 - \omega$

$$\Rightarrow \frac{\omega + \omega_0}{2} = \omega_0 - \frac{\omega_0 - \omega}{2} = \omega_0 - \frac{\Delta \omega}{2}$$

$$y_p = \frac{-2}{\Delta \omega (\omega_0 + \omega)} \left[\cos\left(\left(\omega_0 - \frac{\Delta \omega}{2}\right) t \right) \sin \frac{\Delta \omega t}{2} \right]$$

↑ small frequency as $\Delta \omega$ small



For $\lim_{\Delta\omega \to 0} \sin\left(\frac{\Delta\omega}{2} t\right) \approx \frac{\Delta\omega}{2} t$

so $\lim_{\Delta\omega \to 0} \approx \frac{-2}{\omega_0 + \omega_0} \cos(\omega_0 t) \, \frac{t}{2}$

$\approx \frac{-t}{2\omega_0} \cos(\omega_0 t)$ linear growth in amplitude over time: unbounded

Note y_p takes form of (indep. var) $\times$ (CF)

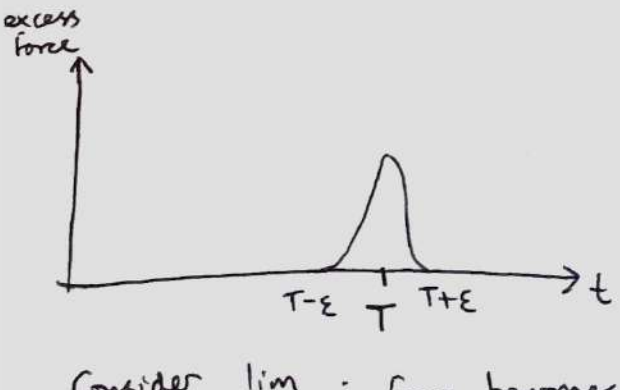
(Compare with degenerate case)

Impulses and point forces

Consider a system experiencing a sudden force e.g. car driving over bump

Forced, damped oscillator:

$$M\ddot{y} = F(t) - ky - L\dot{y} \quad (16.1)$$



Consider $\lim_{\varepsilon \to 0}$: force becomes sudden impulse

Integrate (16.1) in time from $T-\varepsilon$ to $T+\varepsilon$: take $\lim_{\varepsilon \to 0}$

$$\lim_{\varepsilon \to 0} \left[M[\dot{y}]_{T-\varepsilon}^{T+\varepsilon} = \int_{T-\varepsilon}^{T+\varepsilon} f(t)\,dt - \underbrace{k \int_{T-\varepsilon}^{T+\varepsilon} y\,dt}_{0 \text{ if } y \text{ finite}} - \underbrace{L[y]_{T-\varepsilon}^{T+\varepsilon}}_{0 \text{ if } y \text{ continuous}} \right]$$

(16.2)

Define Impulse I as $\lim_{\varepsilon \to 0} \int_{T-\varepsilon}^{T+\varepsilon} F(t)\,dt$

16.2 becomes $\quad I = \lim_{\varepsilon \to 0} M[\dot{y}]_{T-\varepsilon}^{T+\varepsilon}$

Velocity $\dot{y}$ experiences a sudden change (discontinuous) which depends on integral of force.

Differential Equations - Lecture 16

11/11/20

Dirac delta function

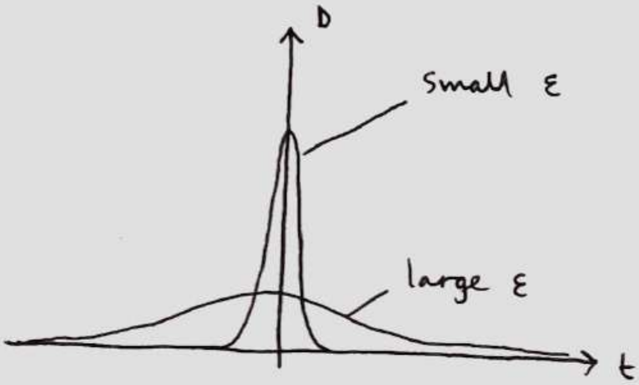
Consider family of functions $D(t;\varepsilon)$ with

1. $\lim_{\varepsilon \to 0} D(t;\varepsilon) = 0 \quad \forall\, t \neq 0$

2. $\int_{-\infty}^{\infty} D(t;\varepsilon)\, dt = 1$

Example

$$D(t;\varepsilon) = \frac{1}{\varepsilon\sqrt{\pi}} e^{-t^2/\varepsilon^2}$$



Dirac delta function

$$\delta(t) = \lim_{\varepsilon \to 0} D(t;\varepsilon)$$

Properties of $\delta(t)$

1. $\delta(t) = 0 \quad \forall\, t \neq 0$

2. $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

3. Sampling property: For continuous function $g(x)$,

$$\int_{-\infty}^{\infty} g(x)\,\delta(x)\, dx = g(0) \int_{-\infty}^{\infty} \delta(x)\, dx \quad \text{(as integrand} = 0 \ \forall x \neq 0)$$

$$= g(0) \times 1 = \underline{g(0)}.$$

More generally

$$\int_a^b g(x)\,\delta(x - x_0)\, dx = \begin{cases} g(x_0) & \text{if } a \leq x_0 \leq b \\ 0 & \text{otherwise} \end{cases} \qquad (16.3)$$

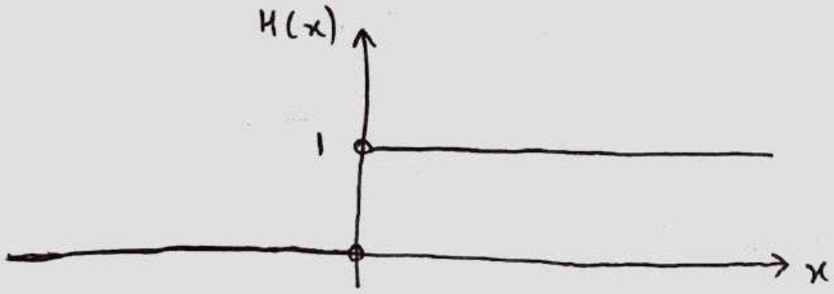
Heaviside step function

$$H(x) = \int_{-\infty}^{x} \delta(t)\, dt$$

(note: by FTC have $\dfrac{dH}{dx} = \delta(x)$)

Properties

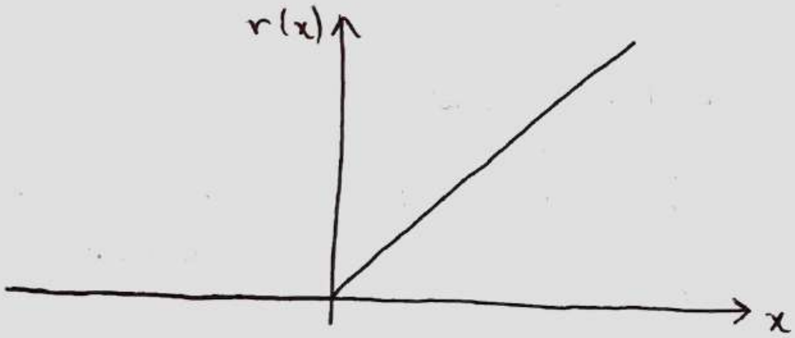
1. $H(x) = 0$ for $x < 0$
2. $H(x) = 1$ for $x > 0$
3. $H(x=0)$ is undefined



Ramp function

$$r(x) = \int_{-\infty}^{x} H(t)\, dt$$

Note that functions get "smoother" as we integrate.



Delta Function Forcing

Consider

$$y'' + p(x)\, y' + q(x)\, y = \delta(x) \qquad (16.5)$$

Note: highest order derivative "inherits discontinuity" from forcing

Since $\delta(x) = 0 \quad \forall x \neq 0$:

$y'' + py' + qy = 0 \quad \forall \ \underline{x < 0}$ and $\underline{x > 0}$

$y(x)$ satisfies "jump conditions"

1. $y(x)$ is continuous at $x = 0$:

$$\boxed{\lim_{\varepsilon \to 0} \Big[y \Big]_{x=-\varepsilon}^{x \pm \varepsilon} = 0}$$

2. $y'(x)$ has a jump of 1 at $x = 0$

Integrate (16.5) in a small window around $x = 0$:

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} (16.5) \, dx$$

$$\Rightarrow \boxed{\lim_{\varepsilon \to 0} \Big[y' \Big]_{-\varepsilon}^{\varepsilon} = 1}$$

<u>Method for solution</u>

- Solve (16.5) for $x < 0$ <u>and</u> $x > 0$, giving 4 unknown constants
- Use 2 jump conditions and 2 initial conditions to find constants.

<u>Example</u> $\quad y'' - y = 3\delta\left(x - \frac{\pi}{2}\right) \qquad (16.6)$

with $y = 0$ at $x = 0$, $x = \pi$.

First solve for $0 \leq x < \frac{\pi}{2}$: $\quad y'' - y = 0 \quad \Rightarrow \quad y = Ae^{x} + Be^{-x}$

or write as $y = A \sinh x + B \cosh x$,

changing constants

$y = 0$ at $x = 0 \quad \Rightarrow \quad y = A \sinh x \quad$ for $0 \leq x < \frac{\pi}{2}$

Equation is invariant under $x \to \pi - \tilde{x}$

so by symmetry for $\frac{\pi}{2} < x \leq \pi$, have $y = C \sinh(\pi - x)$

(Exercise: check by solving DE again)

PTO

Jump conditions

Integrate (16.6)

$$\lim_{\varepsilon \to 0} \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} (16.6)\, dx \quad : \quad y \text{ term} \to 0 \text{ (continuous)}$$

y'' integrates to $y' = \underbrace{\int 3\delta(x-\frac{\pi}{2})\,dx}_{3 \times 1 = 3}$

so get $\lim_{\varepsilon \to 0} \left[y' \right]_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} = 3.$

$\underline{y'}$ For $y = A\sinh x$, $y' = A\cosh x$ $\quad (0 \le x < \frac{\pi}{2})$

and y' at $\frac{\pi}{2}-\varepsilon$ is $\lim_{\varepsilon \to 0} A\cosh\left(\frac{\pi}{2}-\varepsilon\right) = \underline{A\cosh\frac{\pi}{2}}$

and for $\frac{\pi}{2} < x \le \pi$ we use $y = C\sinh(\pi - x)$ as

$\frac{\pi}{2} < x \le \pi$ is the interval for $\frac{\pi}{2}+\varepsilon$:

$y' = -C\cosh(\pi - x)$ so $y'\big|_{\frac{\pi}{2}+\varepsilon} = \lim_{\varepsilon \to 0} -C\cosh\left(\pi - \left(\frac{\pi}{2}+\varepsilon\right)\right)$

$= -C\cosh\frac{\pi}{2}.$

So $\lim_{\varepsilon \to 0} \left[y' \right]_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} = -A\cosh\frac{\pi}{2} - C\cosh\frac{\pi}{2} = 3.$

~~So A sinh~~

2nd jump condition $\quad \lim_{\varepsilon \to 0} \left[y \right]_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} = 0$

so $A\sinh\frac{\pi}{2} = C\sinh\frac{\pi}{2} \Rightarrow A = C$

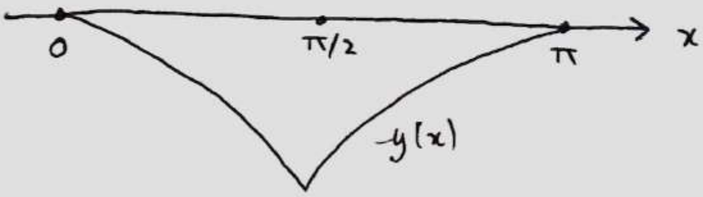
and $A = C = \dfrac{-3}{2\cosh\frac{\pi}{2}}$

so

$$y = \begin{cases} -\dfrac{3}{2}\dfrac{\sinh x}{\cosh\frac{\pi}{2}} & \text{for } 0 \le x < \frac{\pi}{2} \\[2ex] -\dfrac{3}{2}\dfrac{\sinh(\pi - x)}{\cosh\frac{\pi}{2}} & \text{for } \frac{\pi}{2} < x \le \pi \end{cases}$$

Sketch solution

(Note derivative is discontinuous at $\frac{\pi}{2}$)



Heaviside step function forcing

$$y'' + p(x)\, y' + q(x)\, y = H(x - x_0) \qquad (16.7)$$

$y(x)$ satisfies

$$\left.\begin{aligned} y'' + py' + qy &= 0 \quad \text{for } \cancel{x<0}\ x < x_0 \\ y'' + py' + qy &= 1 \quad \text{for } x > x_0 \end{aligned}\right] \quad (16.8)$$

Jump conditions

Evaluate (16.7) on either side of x_0

$$\left([y'']_{x_0^-}^{x_0^+} \right) \quad \underbrace{\lim_{\varepsilon \to 0} [y'']_{x_0-\varepsilon}^{x_0+\varepsilon} + p(x_0)[y']_{x_0^-}^{x_0^+} + q(x_0)[y]_{x_0^-}^{x_0^+} = 1}_{\text{this is our eqn}}$$

shorthand for

If $y'' \sim H(x)$ then $y' \sim r(x)$, $y \sim \int r(x)$ so

y' and y are both continuous. This gives 2 jump conditions

$$\boxed{[y']_{x_0^-}^{x_0^+} = 0, \qquad [y]_{x_0^-}^{x_0^+} = 0.}$$

Use ICs and jump conditions to find constants in solution to (16.8).

Differential Equations - Lecture 17

14/11/20

Higher order discrete / difference equations

m^{th} order linear eqns w/ constant coefficients

$$a_m y_{n+m} + a_{m-1} y_{n+m-1} + \dots + a_1 y_{n+1} + a_0 y_n = f_n \qquad (17.1)$$

~~Same~~ Same principles as ODEs

Eigenfunction

Difference operator $D[y_n] = y_{n+1}$ has eigenfunction

$\underline{y_n = k^n}$ since $D[k^n] = k^{n+1} = k(k^n) = k y_n$

Linearity (17.1) is linear in y, hence

$$y_n = \underbrace{y_n^{(c)}}_{\substack{\text{CF:} \\ \text{sol}^n \text{ to (17.1)} \\ \text{with } f_n = 0}} + \underbrace{y_n^{(p)}}_{\text{particular integral}}$$

Example $a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = f_n$

$f_n = 0$ (homogeneous):

$$a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = 0$$

try $y_n = \underline{k^n}$

$$a_2 k^2 + a_1 k + a_0 = 0$$

solve for $k_1, k_2 = k$

$$y_n^{(c)} = \begin{cases} A k_1^n + B k_2^n & \text{if } k_1 \neq k_2 \\ A k^n + B n k^n & \text{if } k_1 = k_2 = k \end{cases}$$

particular integral

f_n	$y_n^{(p)}$
k^n	$A k^n$ for $k \neq k_1, k_2$
k_1^n, k_2^n	$A n k_1^n + B n k_2^n$
n^p	$A n^p + B n^{p-1} + \dots + Cn + D$

Fibonacci Sequence

$1, 1, 2, 3, 5, 8, \ldots$

$y_n = y_{n-1} + y_{n-2}, \qquad y_0 = y_1 = 1$

$y_{n+2} - y_{n-1} - y_n = 0$ in standard form

Try $y_n = k^n$: $\quad k^2 - k - 1 = 0 \quad \Rightarrow \quad k = \dfrac{1 \pm \sqrt{5}}{2}$

$k_1 = \dfrac{1+\sqrt{5}}{2} = \phi \approx 1.618\ldots$ (golden ratio)

$k_2 = \dfrac{1-\sqrt{5}}{2} = -\dfrac{1}{\phi}$

$y_n = A k_1^n + B k_2^n$. Find A, B with ICs:

$n = 0$: $y_0 = 1 = A + B \qquad n = 1$: $y_1 = 1 \Rightarrow y_1 = 1 = Ak_1 + Bk_2$

so $A = \dfrac{k_1}{\sqrt{5}}, \quad B = \dfrac{-k_2}{\sqrt{5}}$

$$y_n = \frac{k_1^{n+1} - k_2^{n+1}}{\sqrt{5}}$$

gives a general term in the Fibonacci sequence.

$$y_n = \frac{\phi^{n+1} - \left(\frac{-1}{\phi}\right)^{n+1}}{\sqrt{5}}$$

Also $\displaystyle \lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{\phi^{n+2} - \left(-\frac{1}{\phi}\right)^{n+2}}{\phi^{n+1} - \left(-\frac{1}{\phi}\right)^{n+1}} = \underline{\phi}.$

Series solutions for higher order ODEs

Method of Frobenius: series solutions applied to linear, homogeneous 2nd order ODEs

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = 0 \qquad (17.2)$$

Seek power series expansion about $x = x_0$

First classify $x = x_0$

Ordinary point: $x = x_0$ is an ordinary point if Taylor series of $\frac{q}{p}$ and $\frac{r}{p}$ converge in some region around x_0:

i.e. $\frac{q}{p}, \frac{r}{p}$ are analytic.

Otherwise, $x = x_0$ is a singular point. 2 types:

If $x = x_0$ is singular but (17.2) can be written

$$\underline{P}(x)\,(x-x_0)^2\,y'' + Q(x)\,(x-x_0)\,y' + R(x)\,y = 0$$

and $\frac{Q}{\underline{P}}$ and $\frac{R}{\underline{P}}$ are analytic, then $x = x_0$ is a regular singular point.

Otherwise, $x = x_0$ is an irregular singular point.

Note $\frac{Q}{\underline{P}} = (x-x_0)\frac{q}{p}$, $\quad \frac{R}{\underline{P}} = (x-x_0)^2\,\frac{r}{p}$

Examples (classifying points)

1. $(1-x^2)\,y'' - 2xy' + 2y = 0$

$$\frac{q}{p} = -\frac{2x}{1-x^2}, \qquad x = \pm 1 \text{ singular points}$$

$$\frac{Q}{\underline{P}} = (x-1)\frac{q}{p} = \frac{-2x(x-1)}{(1-x)(1+x)} = \frac{2x}{1+x}$$

$\lim_{x\to 1} \frac{Q}{\underline{P}} = 1$ $\left(\text{and } \lim_{x\to 1}\frac{R}{\underline{P}} = 0\right)$ regular singular point at $x = \pm 1$

2) $y'' \sin x + y' \cos x + 2y = 0$

$$\frac{q}{p} = \frac{\cos x}{\sin x}, \quad \frac{r}{p} = \frac{2}{\sin x}$$

$x = n\pi,\ n \in \mathbb{Z}$: regular singular points

3) $(1+\sqrt{x})\, y'' - 2xy' + 2y = 0$

$$\frac{q}{p} = \frac{-2x}{1+\sqrt{x}}$$

$x = 0$ is an irregular singular point
(note: 2nd derivative is undefined).

Differential Equations - Lecture 18

17/11/20

Method of Frobenius

Fuch's theorem

1. If x_0 is an ordinary point, then there are 2 linearly independent solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$ convergent in some region around x_0.

2. If x_0 is a regular singular point, then there is at least one solution of form $$y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\sigma}$$

where σ is real and $a_0 \neq 0$.

Case 1 $$(1-x^2)y'' - 2xy' + 2y = 0 \qquad (18.1)$$

Find series solutions about $x = 0$

$x = 0$: ordinary point.

Try $$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$x^2 \cdot (18.1)$: $$(1-x^2)x^2 y'' - 2x^2 xy' + 2x^2 y = 0$$

or $$\sum_{n=2}^{\infty} a_n \left(n(n-1) - x^2 n(n-1)\right)x^n - 2\sum_{n=1}^{\infty} a_n (nx^2) x^n + 2\sum_{n=0}^{\infty} a_n x^2 x^n = 0$$

for all x, so equate x^n coefficients for $n \geq 2$

$$a_n(n(n-1)) - a_{n-2}(n-2)(n-3) - 2a_{n-2}(n-2) + 2a_{n-2} = 0$$

which is a discrete equation

$$n(n-1)\,a_n = (n^2-3n)\,a_{n-2}$$

or $$a_n = \frac{n-3}{n-1}\,a_{n-2} \qquad (18.2)$$

Note that a_0, a_1 not set by (18.2): are initial/boundary conditions. Note from (18.2): $a_3 = 0$, $a_5 = 0$, $a_n = 0$ for n odd. $(n \geqslant 3)$

For n even: $$a_n = \frac{n-3}{n-1}\,\frac{n-5}{n-3}\,a_{n-4} = \ldots = \frac{-1}{n-1}a_0$$

$$\therefore \quad y = a_1 x + a_0\left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \ldots\right]$$

Note: $$\ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \ldots$$

$$\therefore \quad \ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + 2\frac{x^5}{5} + \ldots$$

so $$y = a_1 x + a_0\left[1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right]$$

Case 2 $$4xy'' + 2(1-x^2)y' - xy = 0 \qquad (18.3)$$

Find series solutions about $x = 0$

$x = 0$: regular singular point

Try $$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma} \qquad a_0 \neq 0$$

$$y' = \sum_{n=0}^{\infty} a_n(n+\sigma)x^{n+\sigma-1} \qquad y'' = \sum_{n=0}^{\infty} a_n(n+\sigma)(n+\sigma-1)x^{n+\sigma-2}$$

hence can multiply $x \cdot (18.3)$:

$$4x^2 y'' + 2(1-x^2)xy' - x^2 y = 0$$

so
$$\sum_{n=0}^{\infty} a_n x^{n+\sigma} \left[4(n+\sigma)(n+\sigma-1) + 2(1-x^2)(n+\sigma) - x^2 \right] = 0 \qquad (18.4)$$

for all x.

Equate coeff. of $x^{n+\sigma}$ for $n \geq 2$

$$a_n \left[4(n+\sigma)(n+\sigma-1) + 2(n+\sigma) \right] + a_{n-2}\left[-2(n-2+\sigma) - 1 \right] = 0$$

or
$$2(n+\sigma)(2n+2\sigma-1)a_n = (2n+2\sigma-3)a_{n-2} \qquad (18.5)$$

giving a recurrence.

To find σ, equate coefficients of lowest power of x in (18.4)

Set $n=0$, equate coeff. of x^σ

$$a_0(4\sigma(\sigma-1)) + a_0 \cdot 2\sigma = 0$$

$a_0 \neq 0$ in Fuch's theorem so have $\sigma(2\sigma-1) = 0$

$$\Rightarrow \sigma = 0 \text{ or } \tfrac{1}{2}.$$

$\underline{\sigma = 0}$ Equate coeffs of lowest powers of x in 18.4

For $n=0$, $\sigma = 0$, coeff. of x^0 are

$$a_0(4(0)(-1)) + a_0(2)(0) = 0$$

$\Rightarrow a_0$ arbitrary constant

For $n=1$, $\sigma = 0$

$$a_1(4(1)(0)) + a_1(2)(1) = 0 \Rightarrow a_1 = 0$$

(18.5) with $\sigma = 0$

$$2n(2n-1)a_n = (2n-3)a_{n-2} \qquad (18.6)$$

Since $a_1 = 0$ $\therefore$ $a_3 = a_5 = \dots = 0$

n even: $a_n = \dfrac{2n-3}{2n(2n-1)} a_{n-2}$

$a_2 = \dfrac{a_0}{4 \cdot 3}$ $\quad a_4 = \dfrac{5}{8 \cdot 7} \dfrac{1}{4 \cdot 3} a_0$

so $y = a_0 + a_2 x^2 + a_4 x^4 + \dots$

so $y = a_0 \left(1 + \dfrac{x^2}{4 \cdot 3} + \dfrac{5x^4}{8 \cdot 7 \cdot 4 \cdot 3} + \dots \right)$

$\underline{\sigma = \frac{1}{2}}$ recurrence 18.5 with $\sigma = \frac{1}{2}$

$(2n+1)(2n) b_n = (2n-2) b_{n-2}$ (relabel $a_n = b_n$)

Equate coeffs of lowest powers of x

(18.4) with $\sigma = \frac{1}{2}$, $n = 0$, coeff. of $x^{1/2}$

$4\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) b_0 + 2\left(\frac{1}{2}\right) b_0 = 0 \Rightarrow -b_0 + b_0 = 0$

(arbitrary constant).

$n = 1$, $\sigma = \frac{1}{2}$, coeff. of $x^{3/2}$

$\left[4\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}-1\right) + 2\left(1+\frac{1}{2}\right)\right] b_1 = 0$

$\Rightarrow \left[2\left(\frac{3}{2}\right) + 2\left(\frac{3}{2}\right)\right] b_1 = 0 \Rightarrow b_1 = 0$ so again

$b_3 = b_5 = \dots = 0$.

Even: $y = b_0 x^{1/2} \left[1 + \dfrac{x^2}{2 \cdot 5} + \dfrac{3x^4}{2 \cdot 5 \cdot 4 \cdot 9} + \dots \right]$

Here, we found 2 linearly independent solutions.

Differential Equations - Lecture 19

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Special cases of the indicial equation

Consider expansion about $x = x_0$. Let σ_1, σ_2 be the roots of the indicial equation.

Case 1 $\sigma_1 - \sigma_2$ not an integer

2 linearly independent solutions

$$y = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n + (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

Note: as $x \to x_0$, $y \sim (x - x_0)^{\sigma_1}$ if $\sigma_1 < \sigma_2$

Case 2 $\sigma_1 - \sigma_2$ is an integer

There is one solution of form:

$$y_1 = (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

other solution: $$y_2 = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} b_n (x - x_0)^n + c y_1 \ln(x - x_0)$$

where c may or may not be 0.

Case 3 $\sigma_1 = \sigma_2$

Here $c \neq 0$

$$y_1 = (x - x_0)^{\sigma} \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$y_2 = (x - x_0)^{\sigma} \sum_{n=0}^{\infty} b_n (x - x_0)^n + y_1 \ln(x - x_0)$$

Example 1 Solve $x^2 y'' - xy = 0$ (19.1)

Seek series solutions about $x = 0$.

This is a regular singular point (check)

try $y = \sum_{n=0}^{\infty} a_n x^{n+\sigma}$ (Fuchs' Theorem)

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^{n+\sigma} \left[(n+\sigma)(n+\sigma-1) \cancel{-x} \right] = 0$$

$-x$

Equate coeffs of $x^{n+\sigma}$ for $n \geq 2$:

$$(n+\sigma)(n+\sigma-1)\, a_n = a_{n-1} \qquad (19.2)$$

Equate coeffs of lowest power of x here $n = 0$, equate coefficients of x^σ

$$\sigma(\sigma - 1)\, a_0 = 0 \qquad (19.3)$$

as $a_0 \neq 0 \Rightarrow \sigma = 0, 1$ (this is an example of case 2 on previous case)

$\underline{\sigma = 1}$ (19.2) $\Rightarrow$ $a_n = \frac{1}{n(n+1)} a_{n-1}$

$$= \frac{a_0}{(n+1)(n!)^2} \qquad \text{(check)}$$

$$y = a_0 x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \dots \right)$$

$\underline{\sigma = 0}$ (19.2) $\Rightarrow$ $n(n-1) b_n = b_{n-1}$

$n = 1$: equate coeffs of x

$$b_1 (1)(1-1) = 0 \qquad b_1 \text{ arbitrary}$$

$$b_2 = \frac{b_1}{2(1)} = \frac{b_1}{2} \qquad b_3 = \frac{b_2}{3(2)} = \frac{b_1}{12} \qquad \text{(same terms as before)}$$

$y(x)$ is linearly <u>dependent</u> on previous case. Therefore we will need soln of form $y_2 = y_1 \ln x + \sum_{x=0}^{\infty} b_n x^n$

Try reduction of order: $y_1 = a_0 x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \dots\right)$

Let $y_2 = v(x)\, y_1(x) \Rightarrow y_2' = vy_1' + v'y_1$, $y_2'' = vy_1'' + 2v'y_1' + v''y_1$

want $x^2 y_2'' - xy_2 = 0$

$$\Rightarrow x^2(v''y_1 + 2v'y_1') + v\underbrace{(x^2 y_1'' - xy_1)}_{0,\text{ as } y_1 \text{ solves eqn}} = 0$$

Let $u = v'$ then $u'y_1 + 2uy_1' = 0$

$$\Rightarrow \frac{u'}{u} = -2\frac{y_1'}{y_1} \Rightarrow \ln u = \ln(y_1^{-2}) + \ln B$$

$$\Rightarrow \underline{u = v' = \frac{B}{y_1^2}}$$

$$v' = \frac{B}{a_0^2 x^2}\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \dots\right)^{-2}$$

Note: constant of integration gives a constant multiple of y_1

Using binomial theorem:

$$v' = \frac{B}{a_0^2}\left(\frac{1}{x^2} - \frac{1}{x} + \sum_{n=0}^{\infty} B_n x^n\right)$$

Integrating:

$$v = \frac{-B}{a_0^2}\frac{1}{x} - \frac{B}{a_0^2}\ln x + \sum_{n=1} C_n x^n \quad \text{then } \underline{y_2 = vy_1}$$

$$= -\frac{B}{a_0} - \frac{B}{2a_0}x + \sum_{n=2}^{\infty} D_n x^n + Cy_1 \ln x$$

$$= \underline{\sum_{n=0}^{\infty} b_n x^n + Cy_1 \ln x}, \quad \text{the form stated earlier.}$$

Example 2

(18.1) : $(1-x^2)y'' - 2xy' + 2y = 0$

seek series solution expanded about $x = -1$

Re-define indep. variable: $z = 1 + x$

$$z(2-z)\, y'' - 2(z-1)\, y' + 2y = 0$$

Expand about $z = 0$: regular ~~singlet~~ singular point

Try $y = \sum_{n=0}^{\infty} a_n z^{n+\sigma}$, $a_0 \neq 0$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^{n+\sigma-1} \Big((n+\sigma)(n+\sigma-1)(2-z) - 2(n+\sigma)(z-1) + 2z \Big) = 0$$

Equate coeffs of lowest power of z

For $n = 0$, coeffs multiplying $z^{\sigma-1}$

Indicial eqn: $2\sigma(\sigma-1) + 2\sigma = 0$

$$\Rightarrow 2\sigma^2 = 0 \Rightarrow \underline{\sigma = 0} \text{ repeated}$$

Have <u>case 3</u> from first part of lecture. Note that we need term $y_1 \ln(x - x_0)$.

Differential Equations - Lecture 20

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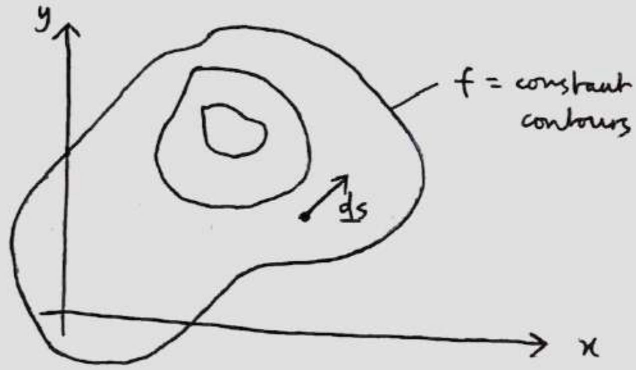
Part 5: Multivariate Functions and Applications

Functions of multiple independent variables

Directional derivatives, gradient vector

Consider $f(x, y)$



Consider a small displacement vector $\underline{ds}$ and find rate of change of f in this direction

Change in f, df, given change in x, y dx, dy given by MV chain rule:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (dx, dy)\cdot\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

(like a vector)

$$= \underline{ds}\cdot\nabla f \qquad (20.1)$$

where $\underline{ds} = (dx, dy)$, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ "grad f" gradient vector in Cartesian coordinates

If we let $\underline{ds} = ds\,\hat{\underline{s}}$ $\quad (|\hat{\underline{s}}| = 1)$

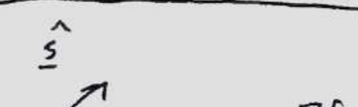
then $(20.1) \Rightarrow df = ds(\hat{\underline{s}})\cdot\nabla f$

so have $\boxed{df = ds(\hat{\underline{s}}\cdot\nabla f)}$

Directional derivatives: $\dfrac{df}{ds} = \hat{\underline{s}}\cdot\nabla f \qquad (20.2)$

rate of change in f in direction of $\hat{\underline{s}}$.

Properties of gradient vector



1. <u>Magnitude</u> of Δf is the maximum rate of change of $f(x, y)$

$$|\nabla f| = \max_{\forall \theta} \left(\frac{df}{ds} \right)$$

2. The <u>direction</u> of Δf is the direction in which f <u>increases</u> most rapidly.

$$\left| \frac{df}{ds} \right| = |\nabla f| \cos\theta \qquad \text{from } (20.1)$$

3. If $\underline{ds}$ (and $\hat{\underline{s}}$) are parallel to contours of f, then

$$\frac{df}{ds} = \hat{\underline{s}} \cdot \nabla f = \underline{0}.$$

Hence ∇f is <u>perpendicular</u> to contours of $f(x, y)$ and $|\nabla f|$ is the slope in the "uphill" direction.

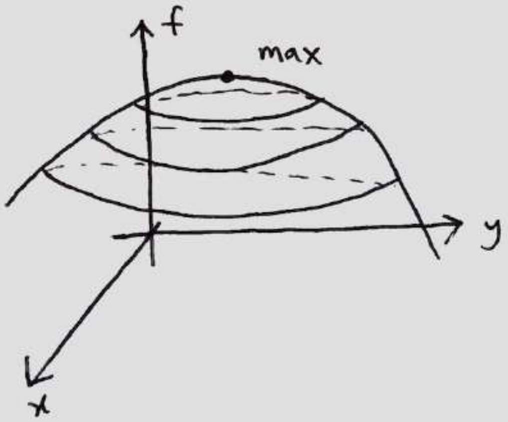
<u>Stationary Points</u>

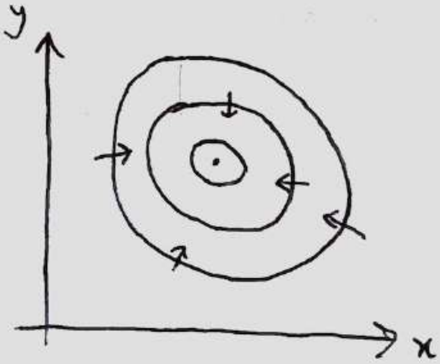
In general, there is always at <u>least one</u> direction in which $\frac{df}{ds} = 0$, parallel to contours of f.

Stationary points have $\frac{df}{ds} = 0 \quad \forall \hat{\underline{s}} \quad (\text{or } \forall \theta)$

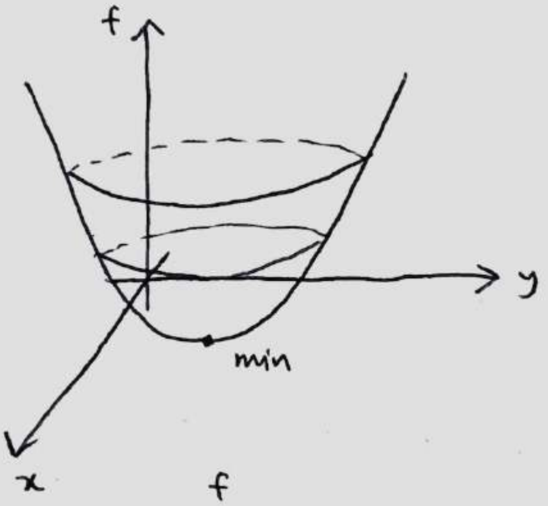
Since $\frac{df}{ds} = \hat{\underline{s}} \cdot \nabla f$, $\Rightarrow$ $\underline{\nabla f = \underline{0}}$ at stationary points.

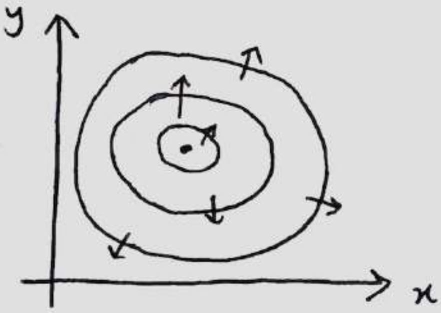
<u>Types of stationary points</u>



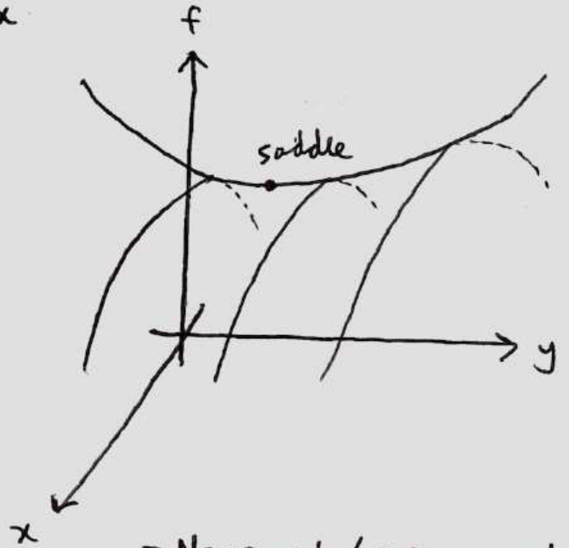


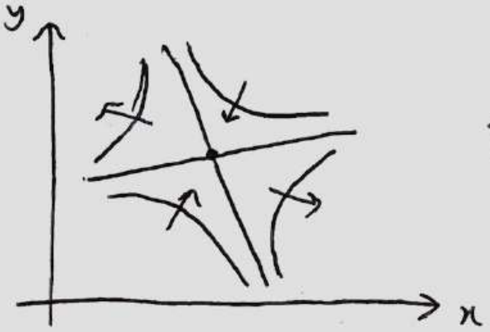
arrows ∇f





↑ ∇f
(uphill)

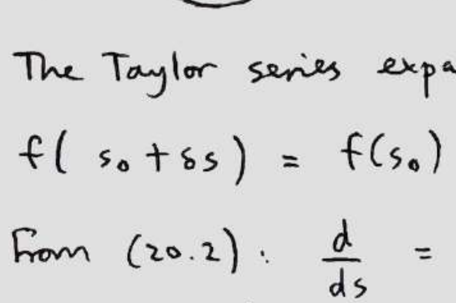




↑ ∇f

- Near min/max, contours of f are <u>elliptical</u>
- Near saddle, contours of f are <u>hyperbolic</u>
- Contours of f can only cross at saddle points.

Taylor series for multivariate functions



$$\underline{\delta s} = \delta s \, \hat{\underline{s}}$$

The Taylor series expansion in the direction of $\hat{\underline{s}}$ is:

$$f(s_0 + \delta s) = f(s_0) + \delta s \frac{df}{ds}\Big|_{s_0} + \frac{1}{2}(\delta s)^2 \frac{d^2 f}{ds^2}\Big|_{s_0} + \dots \quad (20.3)$$

From (20.2): $\dfrac{d}{ds} = \hat{\underline{s}} \cdot \nabla$ (operator)

hence $\delta s \dfrac{d}{ds} = \underline{\delta s} \cdot \nabla$

(20.3) $$f(s_0 + \delta s) = f(s_0) + \overbrace{\delta s \cdot \nabla f|_{s_0}}^{\text{term 1}} + \overbrace{\frac{1}{2}(\delta s)^2 (\hat{\underline{s}} \cdot \nabla)(\hat{\underline{s}} \cdot \nabla) f|_{s_0}}^{\text{term 2}} + \dots$$

In Cartesian coordinates $\quad \xi_0 = (x_0, y_0) \quad \underline{\delta s} = (\delta x, \delta y)$

$x = x_0 + \delta x, \quad y = y_0 + \delta y$

(1) : $\delta x \dfrac{\partial f}{\partial x}(x_0, y_0) + \delta y \dfrac{\partial f}{\partial y}(x_0, y_0)$

(2) : $\frac{1}{2}\left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y}\right)\left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y}\right) f \Big|_{x_0, y_0}$

$= \frac{1}{2}\left(\delta x^2 f_{xx} + \delta x \delta y f_{yx} + \delta y \delta x f_{xy} + \delta y^2 f_{yy}\right)\Big|_{x_0, y_0}$

$= \frac{1}{2}(\delta x, \delta y) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\Bigg|_{x_0, y_0} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$ (check)

<u>Hessian matrix</u>

$$\underline{\underline{H}} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \nabla(\nabla f)$$

so MV Taylor series in 2D Cartesian is

$$f(x, y) = f(x_0, y_0) + (x - x_0) f_x\big|_{x_0, y_0} + (y - y_0) f_y\big|_{x_0, y_0}$$
$$+ \frac{1}{2}\Big[(x - x_0)^2 f_{xx}\big|_{x_0, y_0} + (y - y_0)^2 f_{yy}\big|_{x_0, y_0}$$
$$+ 2(x - x_0)(y - y_0) f_{xy}\big|_{x_0, y_0}\Big] + \dots$$

(20.4)

In general :

$$f(\underline{x}) = f(\underline{x}_0) + \underline{\delta x} \cdot \nabla f(\underline{x_0}) + \frac{1}{2} \underline{\delta x} \cdot [\nabla(\nabla f)]\big|_{x_0} \cdot \delta x^T + \dots \quad (20.5)$$

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Classifying Stationary Points

$\nabla f = \underline{0}$ defines a SP so Taylor series about SP $\underline{x} = \underline{x}_s$ is

$$f(\underline{x}) \approx f(\underline{x}_s) + \frac{1}{2} \delta \underline{x} \cdot \underline{\underline{H}}\Big|_{x_s} \cdot \delta \underline{x}^T$$

(number of SP depends on H)

Consider function in n-dim. space

$$f = f(x_1, x_2, x_3, \ldots, x_n)$$

n-dim Hessian:

$$\underline{\underline{H}} = \begin{pmatrix} f_{x_1 x_1} & f_{x_1 x_2} & \cdots & f_{x_1 x_n} \\ \vdots & f_{x_2 x_2} & & \vdots \\ & & \ddots & \\ f_{x_n x_1} & \cdots & & f_{x_n x_n} \end{pmatrix}$$

IF derivatives defined then $f_{x_1 x_2} = f_{x_2 x_1}$ etc
Hence H is symmetric
so can be diagonalised wrt principle axes.

$\delta \underline{x} \cdot \underline{\underline{H}} \cdot \delta \underline{x}^T =$

$$(\delta x_1, \delta x_2, \ldots, \delta x_n) \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_n \end{pmatrix}$$

where λ_i are eigenvalues of $\underline{\underline{H}}$ and δx_i is the displacement along principal axis (eigenvector) i so

$$\delta \underline{x} \cdot \underline{\underline{H}} \cdot \delta \underline{x}^T = \lambda_1 \delta x_1^2 + \lambda_2 \delta x_2^2 + \ldots + \lambda_n \delta x_n^2 \quad (21.1)$$

Types of SPs

1. Minimum $\quad \delta \underline{x} \cdot \underline{\underline{H}} \cdot \delta \underline{x}^T > 0 \quad \forall \delta \underline{x}$

$$\Rightarrow \lambda_1, \lambda_2, \ldots, \lambda_n > 0, \quad \underline{\underline{H}} \text{ positive definite}$$

2. Maximum

$$\delta \underline{x} \cdot \underline{\underline{H}} \cdot \delta \underline{x}^T < 0 \qquad \forall \delta x$$

$\lambda_1, \lambda_2, \ldots, \lambda_n < 0$, $\underline{\underline{H}}$ negative definite

3. Saddle $\quad \underline{\underline{H}}$ indefinite

Signature of H is the pattern of signs of its subdeterminants.

e.g. for $f(x_1, x_2, \ldots, x_n)$

signs of $\underbrace{|f_{x_1x_1}|}_{|\underline{\underline{H}}_1|}$, $\underbrace{\begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{vmatrix}}_{|\underline{\underline{H}}_2|}$, $\ldots$, $\underbrace{\begin{vmatrix} f_{x_1x_1} & \cdots & f_{x_1x_n} \\ & \ddots & \vdots \\ & & f_{x_nx_n} \end{vmatrix}}_{|\underline{\underline{H}}|}$

From V&M: If $\underline{\underline{H}}$ is positive (or negative) definite then $\underline{\underline{H}}_1, \underline{\underline{H}}_2, \ldots, \underline{\underline{H}}_{n-1}$ are positive (or negative) definite.
(Sylvester's criterion)

i.e. a minimum (max) point in n-dim space is also a minimum (max) in any subspace that includes it.

Types of SP

- Minimum ($\lambda_i > 0$) $\quad +, +, +, \ldots$
- Maximum ($\lambda_i < 0$) $\quad -, +, -, \ldots$

Note that if $|\underline{\underline{H}}| = 0$, need higher order terms in Taylor series.

Contours of f near stationary point

Consider a coordinate system aligned with principal axes of $\underline{\underline{H}}$

i.e. $\underline{\underline{H}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Let $\delta \underline{x} = (\underline{x} - \underline{x}_s) = (\xi, \eta)$ $\qquad \underline{x}_s$ stationary point

In a small region near $\underline{x}_s$ contours of f satisfy

$$f = \text{constant} \approx f(\underline{x}_s) + \frac{1}{2}\, \delta \underline{x} \cdot \underline{\underline{H}} \cdot \delta \underline{x}^T$$

$$\Rightarrow \lambda_1 \xi^2 + \lambda_2 \eta^2 = \text{constant} \qquad (21.2)$$

Near min./max: λ_1, λ_2 same sign

(21.2) $\Rightarrow$ contours of f elliptical

Near saddle λ_1, λ_2 opposite sign $\therefore$ (21.2) $\Rightarrow$ contours of f hyperbolic

<u>Example</u> $f(x,y) = 4x^3 - 12xy + y^2 + 10y + 6$

SPs : $f_x = f_y = 0 \Rightarrow (x,y) = (1,1),\ (5,25)$

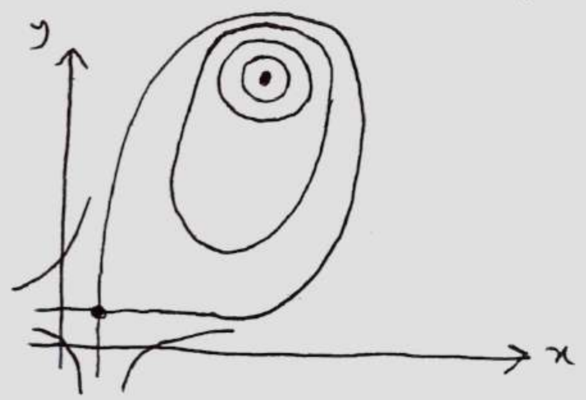
$f_{xx} = 24x, \quad f_{xy} = -12, \quad f_{yy} = 2$

<u>(1,1)</u> $\underline{\underline{H}} = \begin{pmatrix} 24 & -12 \\ -12 & 2 \end{pmatrix} \quad |\underline{\underline{H}}_1| = 24, \quad |\underline{\underline{H}}| = 48 - 144$

signature $+, - \Rightarrow$ saddle point

<u>(5, 25)</u> $\underline{\underline{H}} = \begin{pmatrix} 120 & -12 \\ -12 & 2 \end{pmatrix} \quad |\underline{\underline{H}}_1| = 120, \quad |\underline{\underline{H}}| = 240 - 144$

signature : $+, + \Rightarrow$ minimum



<u>Part 2</u> Systems of linear ODEs

Consider $y_1(t),\ y_2(t)$ which satisfy

$$\dot{y}_1 = a y_1 + b y_2 + f_1(t)$$
$$\dot{y}_2 = c y_1 + d y_2 + f_2(t) \qquad (21.3)$$

Vector form: $$\dot{\underline{Y}} = \underline{\underline{M}}\,\underline{Y} + \underline{F} \qquad (21.4)$$

with $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\underline{\underline{M}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\underline{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

Any n^{th} order DE can be written as a system of n first order ODEs.

e.g. standard form for 2nd order linear ODE,

$$\ddot{y} + a\dot{y} + by = f \qquad (21.5)$$

Let $y_1 = y$, $y_2 = \dot{y}$ so $Y = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$

hence $\dot{y}_1 = y_2$

and $(21.5) \Rightarrow \quad \dot{y}_2 = -ay_2 - by_1 + f$

or $$\dot{\underline{Y}} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \underline{Y} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

a system of two 1^{st} order ODEs.

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Matrix Methods

To solve a system of linear ODEs:

$$\dot{\underline{Y}} = M\underline{Y} + \underline{F} \qquad (22.1) \qquad M \text{ a matrix}$$

1. Write $\underline{Y} = \underline{Y}_c + \underline{Y}_p$ where $\dot{\underline{Y}}_c = M\underline{Y}_c$ (22.2)
2. Seek solutions of $\underline{Y}_c = \underline{v}e^{\lambda t}$

   (22.2) $\Rightarrow$ $\lambda\underline{v} = M\underline{v}$ (subbing in $\underline{Y}_c$)

   Note: λ and $\underline{v}$ are respectively the eigenvalues and eigenvectors of coefficient matrix M.

3. Find $\underline{Y}_p$ based on form of $\underline{F}$.

Example $\dot{\underline{Y}} - \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix}\underline{Y} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^t$ (22.3)

try $\underline{Y}_c = \underline{v}e^{\lambda t}$

eigenvectors and eigenvalues of M are:

$\lambda_1 = 2$, $\underline{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\lambda_2 = -8$, $\underline{v}_2 = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$

(check)

hence $\underline{Y}_c = A\begin{pmatrix} 4 \\ 1 \end{pmatrix}e^{2t} + B\begin{pmatrix} -6 \\ 1 \end{pmatrix}e^{-8t}$

Particular integral:

Try $\underline{Y}_p = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} e^t$ u_1, u_2 unknown

(22.3) plug in: $\Rightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 5 & -24 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Rightarrow u_1 = -4,\ u_2 = -1$

PTO

so general solution is

$$\underline{Y} = A\begin{pmatrix}4\\1\end{pmatrix}e^{2t} + B\begin{pmatrix}-6\\1\end{pmatrix}e^{-8t} - \begin{pmatrix}4\\1\end{pmatrix}e^{t}$$

Note: If forcing term on RHS matches $\underline{Y}_c$, then try

$\underline{Y}_p = ute^{\lambda t}$ instead.

From a ~~a~~ linear system of n first order ODEs, we can construct n uncoupled n^{th} order ODEs.

For (22.3) write as

$$\dot{y}_1 = -4y_1 + 24y_2 + 4e^t \qquad (22.4)$$

$$\dot{y}_2 = y_1 - 2y_2 + e^t \qquad (22.5)$$

$\frac{d}{dt}(22.4) \Rightarrow \quad \ddot{y}_1 = -4\dot{y}_1 + 24\dot{y}_2 + 4e^t$

Using (22.5): $\quad \ddot{y}_1 = -4\dot{y}_1 + 24y_1 - 48y_2 + 24e^t + 4e^t$

Using (22.4): $\quad 24y_2 = \dot{y}_1 + 4y_1 - 4e^t$

so get $\quad \ddot{y}_1 = -4\dot{y}_1 + 24y_1 - 2\dot{y}_1 - 8y_1 + 8e^t + 28e^t$

or $\quad \underline{\ddot{y}_1 + 6\dot{y}_1 - 16y_1 = 36e^t}$

General solution here is $\quad y_1 = Ae^{2t} + Be^{-8t} - 4e^t$

(see previous solution with matrix method).

<u>Phase Portraits</u> (revisited)

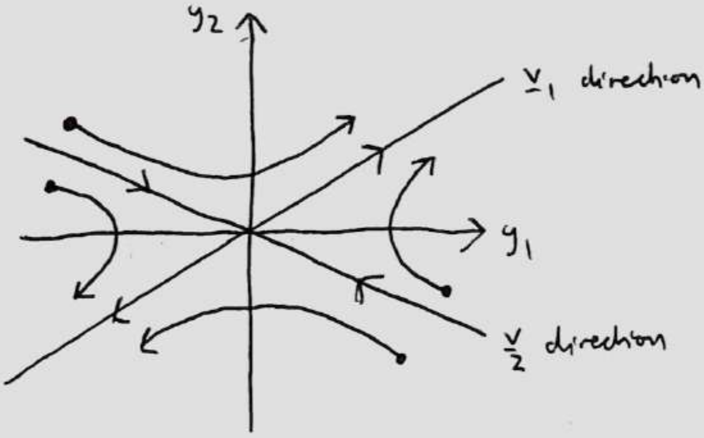
For complementary function $\underline{Y}_c$ satisfying $\quad \dot{\underline{Y}}_c = M\underline{Y}_c \quad (22.6)$

have $\underline{Y}_c = A\underline{v}_1 e^{\lambda_1 t} + B\underline{v}_2 e^{\lambda_2 t}$

PTO

3 cases

1. λ_1, λ_2 real and $\lambda_1 \lambda_2 < 0$ (WLOG: $\lambda_1 > 0$, $\lambda_2 < 0$)



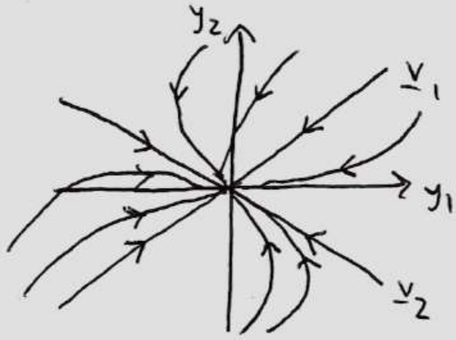
$(0, 0)$ is a saddle node

2. λ_1, λ_2 real $\lambda_1 \lambda_2 > 0$

a) $\lambda_1, \lambda_2 < 0$

wlog $|\lambda_1| > |\lambda_2|$

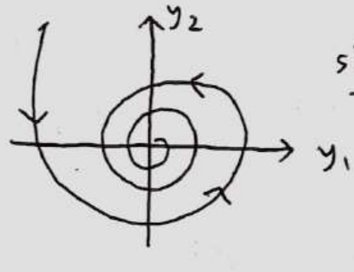
so "collapses faster" along $\underline{v}_1$ than $\underline{v}_2$. $(0, 0)$ is a stable node.



b) $\lambda_1, \lambda_2 > 0$: same with arrows reversed : unstable node

3. λ_1, λ_2 complex conjugate pair

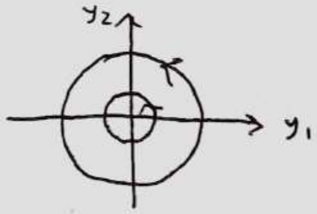
a) $Re(\lambda_1, \lambda_2) < 0$



stable spiral

b) $Re(\lambda_1, \lambda_2) > 0$ unstable spiral

c) $Re(\lambda_1, \lambda_2) = 0$



To find rotation direction, evaluate system of eqns at given point to find sign of $\dot{y}_1, \dot{y}_2$

e.g. $y_2 = 0$, $y_1 > 0$, find sign of $\dot{y}_2$

Nonlinear systems of ODEs

Consider an autonomous system of two nonlinear first order ODEs.

$$\dot{x} = f(x, y) \qquad (22.7)$$

$$\dot{y} = g(x, y) \qquad (22.8)$$

f, g nonlinear functions of x, y; autonomous: $\cancel{f(g)}$ f, g independent of t.

Equilibrium (fixed) points

Let (x_0, y_0) be a fixed point:

$$\dot{x}\big|_{x_0, y_0} = 0, \qquad \dot{y}\big|_{x_0, y_0} = 0 \quad \Rightarrow \quad \begin{aligned} f(x_0, y_0) &= 0 \\ g(x_0, y_0) &= 0 \end{aligned}$$

Solve system of eqns for (x_0, y_0)

Stability: perturbation analysis

MV Taylor series

$$(x, y) = (x_0 + a(t),\ y_0 + b(t))$$

$$(22.7) \Rightarrow \dot{a} = f(x_0 + a, y_0 + b) \approx f(x_0, y_0) + a f_x(x_0, y_0) + b f_y(x_0, y_0)$$

$$(22.8) \Rightarrow \dot{b} = g(x_0 + a, y_0 + b) \approx g(x_0, y_0) + a g_x(x_0, y_0) + b g_y(x_0, y_0)$$

but $f(x_0, y_0) = g(x_0, y_0) = 0$

Hence
$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \underbrace{\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}}_{M}\Bigg|_{x_0, y_0} \begin{pmatrix} a \\ b \end{pmatrix}$$
which is a homogeneous linear system of ODEs

Eigenvalues of M: λ_1, λ_2 determine the stability and behaviour: see phase portraits section earlier.

Differential Equations - Lecture 23

26/11/20

<u>Example</u> Predator-prey model (Lotka-Volterra equs)

Prey (phytoplankton) $\dot{x} = \alpha x - \beta xy = f(x,y)$

Predator (zooplankton) $\dot{y} = \delta xy - \gamma y = g(x,y)$

$\alpha, \beta, \gamma, \delta > 0$ constants

<u>Fixed points</u> $(\dot{x} = \dot{y} = 0)$

$\dot{x} = 0 \Rightarrow x = 0$ or $y = \frac{\alpha}{\beta}$

$\dot{y} = 0 \Rightarrow y = 0$ or $x = \frac{\gamma}{\delta}$

Fixed points: (x_0, y_0) are $\underline{(0,0)}$ or $\underline{\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)}$

trivial: no predators or prey

so $M = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix}$

<u>Stability</u> $(0,0)$

$$\underbrace{\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix}}_{\text{perturbation}} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

E.vals: $\lambda = \underset{>0}{\alpha}, \underset{<0}{-\gamma}$

so is a saddle node

$\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$: $\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

e.vals

$\lambda^2 + \alpha\gamma = 0$

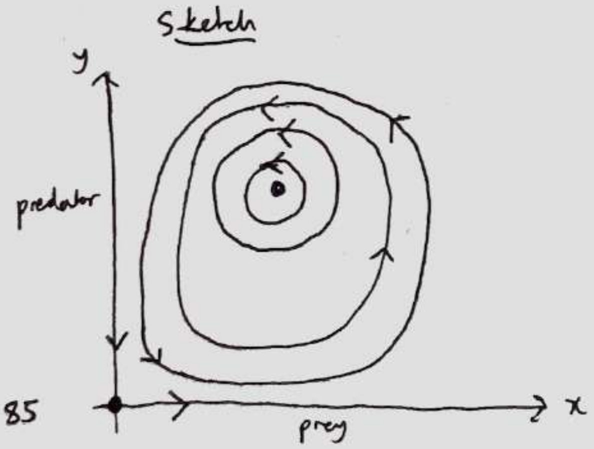
so $\lambda = \pm\sqrt{-\alpha\gamma} = \pm i\sqrt{\alpha\gamma}$

$Re(\lambda) = 0$ so is a centre.

$\dot{a} = \frac{-\beta\gamma}{\delta} b$ If $b > 0$, $\dot{a} < 0$

so anticlockwise

<u>Sketch</u>



Partial Differential Equations (PDEs)

This is a DE with multiple independent variables.

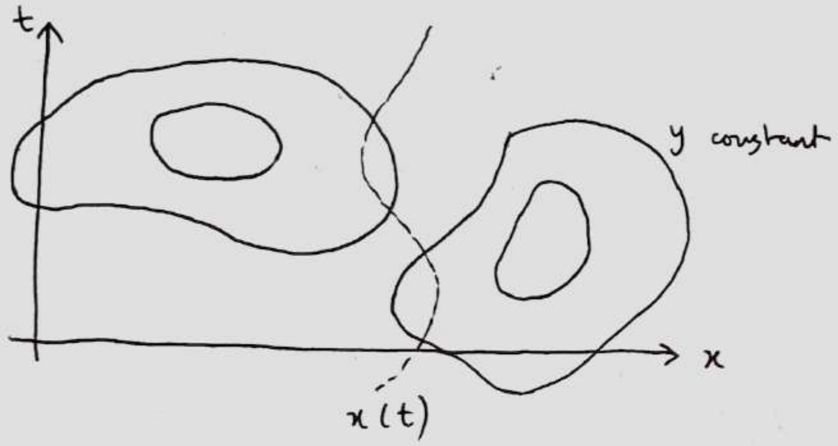
Here, consider 3 examples.

1. First order wave equation

Consider $y(x,t)$ where

$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0 \qquad (c \text{ constant}) \qquad (23.1)$$

Use method of characteristics



Sample $y(x,t)$ along "path" given by $x(t)$.

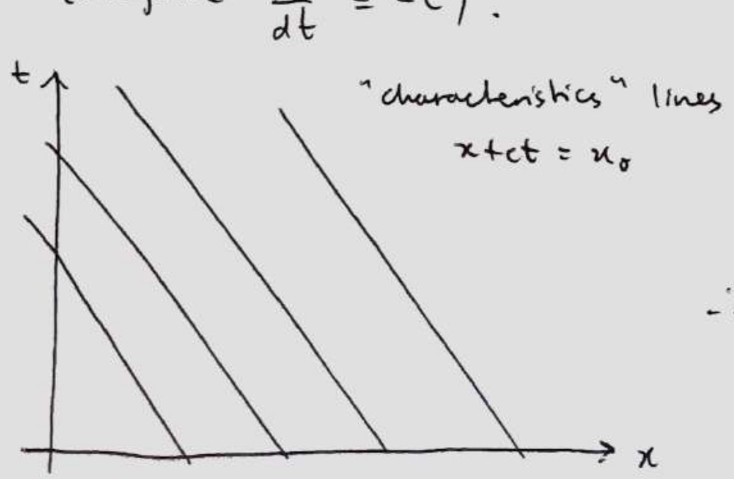
Along path: $y(x(t), t)$

Multivariate chain rule:

$$\frac{dy}{dt} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x}\frac{dx}{dt}$$

Compare with (23.1): note that if $\frac{dx}{dt} = -c$ then $\frac{dy}{dt} = 0$

So y is constant along paths with $x = x_0 - ct$ (integrate $\frac{dx}{dt} = -c$).



If $y(x, t=0) = f(x)$, we have $y = f(x_0)$ along characteristics.

$\therefore$ general solution is

$y = f(x+ct)$.

Example 1 unforced wave equation

$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0 \quad \text{with} \quad y(x,0) = x^2 - 3$$

Then $\underline{y(x,t) = (x+ct)^2 - 3}$

Example 2 Forced wave equation

$$\frac{\partial y}{\partial t} + 5\frac{\partial y}{\partial x} = e^{-t} \quad \text{with} \quad y(x,0) = e^{-x^2}$$

$\frac{dy}{dt} = e^{-t}$ along paths with $\frac{dx}{dt} = 5$ or $x = x_0 + 5t$

$\downarrow$ $y = A - e^{-t}$ along paths

Use ICs: at $t=0$, $x = x_0$ and $y(x,0) = A - 1$
$= e^{-x_0^2}$

$\Rightarrow$ $A = 1 + e^{-x_0^2}$ so we have

$$\underline{y = 1 + e^{-(x-5t)^2} - e^{-t}} \leftarrow \text{general solution}$$

(considers $x_0 = x - 5t$ - all possible paths).

Differential Equations - Lecture 24

2. Second order wave equation

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0 \qquad (23.2)$$

Factor differential operator:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) y = 0$$

Operators commute: so either $\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0$

or $\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = 0.$

Compare with (23.1) from previous lecture, so

$y = f(x+ct)$ or $y = g(x-ct)$. Linear in y (23.2)

so general solution is $y = f(x+ct) + g(x-ct)$.

<u>Example</u> Solve $y_{tt} - c^2 y_{xx} = 0$ subject to $y = \frac{1}{1+x^2}$ and $y_t = 0$ at $x = 0$, and $y \to 0$ as $x \to \pm\infty$.

Solution form $y = f(x+ct) + g(x-ct)$

Use ICs: $t = 0$: $f(x) + g(x) = \frac{1}{1+x^2}$

$$cf'(x) - cg'(x) = 0 \Rightarrow f' = g' \Rightarrow f = g + A$$

so $2g(x) + A = \frac{1}{1+x^2} \Rightarrow g(x) = \frac{1}{2}\left(\frac{1}{1+x^2}\right) - \frac{A}{2}$

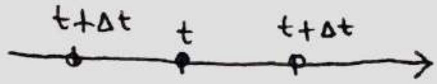
and $f(x) = \frac{1}{2}\left(\frac{1}{1+x^2}\right) + \frac{A}{2}$

Note: A doesn't show up since we take $f(x) + g(x)$.

So $y(x,t) = \frac{1}{2}\left[\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right]$

For $c > 0$ moves left moves right

Diffusion equation



Let $c(x,t)$ be # of particles at x, t

After Δt, let $p =$ prob. of moving right one step

$p =$ prob. of moving left one step

$1-2p =$ prob. of staying there

$$c(x, t+\Delta t) = (1-2p)\,c(x,t) + p\left(c(x+\Delta x, t) + c(x-\Delta x, t)\right) \qquad (24.2)$$

Expand in Taylor series for small Δx, Δt

$$c(x, t+\Delta t) = c(x,t) + \Delta t \frac{\partial c}{\partial t}(x,t) + O(\Delta t^2)$$

$$c(x+\Delta x, t) = c(x,t) + \Delta x \frac{\partial c}{\partial x}(x,t) + \frac{\Delta x^2}{2}\frac{\partial^2 c}{\partial x^2}(x,t) + O(\Delta x^3)$$

(24.2): $$c + \Delta t \frac{\partial c}{\partial t} + O(\Delta t^2) = (1-2p)c + p\left(2c + \Delta x^2 \frac{\partial^2 c}{\partial x^2} + O(\Delta x^2)\right)$$

so $$\frac{\partial c}{\partial t} + O(\Delta t) = p \frac{\Delta x^2}{\Delta t}\frac{\partial^2 c}{\partial x^2} + O\left(\frac{\Delta x^3}{\Delta t}\right)$$

Let $\Delta x, \Delta t \to 0$ such that $\dfrac{\Delta x^2}{\Delta t}$ is constant.

Then $$\underline{\frac{\partial c}{\partial t} = k \frac{\partial^2 c}{\partial x^2}} \quad \text{with } k = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} p \frac{\Delta x^2}{\Delta t} \qquad (24.3)$$

diffusion equation

k = diffusion coefficient

Example $$\frac{\partial y}{\partial t} = k \frac{\partial^2 y}{\partial x^2} \quad \text{with } y(x,0) = \delta(x)$$

$$y \to 0 \text{ as } x \to \pm\infty$$

Define $\eta = \dfrac{x^2}{4kt}$, similarity variable (think dimensional analysis)

Seek solutions of form $\underline{y = t^{-\alpha} f(\eta)}$

Then $y_t = -\alpha t^{-\alpha-1} f(\eta) + t^{-\alpha} f_\eta \eta_t$

$y_x = t^{-\alpha} f_\eta \eta_x$, $y_{xx} = t^{-\alpha} f_{\eta\eta} (\eta_x)^2 + t^{-\alpha} f_\eta \eta_{xx}$

so (24.4) $\Rightarrow$ $-\frac{\alpha}{t} f + f' \eta_t = \kappa f'' (\eta_x)^2 + \kappa f' \eta_{xx}$ (24.5)

and notice $\eta_t = \frac{-x^2}{4\kappa t^2} = \frac{-\eta}{t}$

$\eta_x = \frac{2x}{4\kappa t}$, $(\eta_x)^2 = \frac{4x^2}{16\kappa^2 t^2} = \frac{\eta}{\kappa t}$

$\eta_{xx} = \frac{2}{4\kappa t}$ so (24.5) gives an ODE

$$\Rightarrow \quad \alpha f + f'\eta + f''\eta + \frac{f'}{2} = 0$$

or $\eta \frac{d}{d\eta}(f + f') + \frac{1}{2}(f' + 2\alpha f) = 0$ an ODE for $f(\eta)$

(note α arbitrary)

Let $\alpha = \frac{1}{2}$ (24.6) $\Rightarrow$ $\eta \frac{dF}{d\eta} + \frac{F}{2} = 0$ where $F = f + f'$

One solution: $F = 0 \;\; \forall \eta \Rightarrow f + f' = 0 \Rightarrow f = Ae^{-\eta}$

recall $y = t^{-\alpha} f(\eta)$ hence $\underline{y = A t^{-1/2} e^{-x^2/4\kappa t}}$

Set A from ICs.

Recall $\delta(x) = \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon\sqrt{\pi}} e^{-x^2/\varepsilon^2} \right]$ let $\varepsilon^2 = 4\kappa t$ as $t \to 0$, $\varepsilon \to 0$.

Then $\frac{1}{\varepsilon\sqrt{\pi}} = \frac{1}{\sqrt{4\pi\kappa}} t^{-1/2}$ hence $A = \frac{1}{\sqrt{4\pi\kappa}}$

and $$\underline{y(x,t) = \frac{1}{\sqrt{4\pi\kappa}} t^{-1/2} e^{-x^2/4\kappa t}}$$