Vector Calculus - Lecture 1

22/1/21

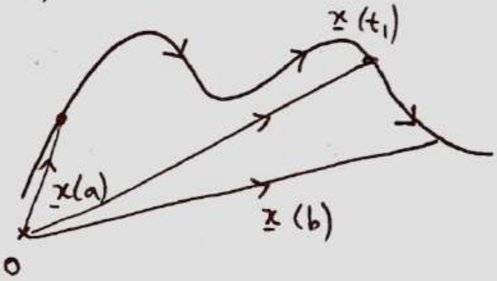
Throughout this course, a column vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is to be interpreted as $\underline{x} = a\underline{e}_x + b\underline{e}_y + c\underline{e}_z$ $\quad \{\underline{e}_x, \underline{e}_y, \underline{e}_z\}$ basis vectors aligned with the fixed Cartesian x-, y, z- axes. (in $\mathbb{R}^3$).

S1: Differential Geometry of Curves

1.1 Parametrised Curves and Arc Length

A parametrised curve C in $\mathbb{R}^3$ is just the image of a continuous map $\underline{x} : [a,b] \to \mathbb{R}^3$ in which $t \mapsto \underline{x}(t)$

Cartesian: $\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$



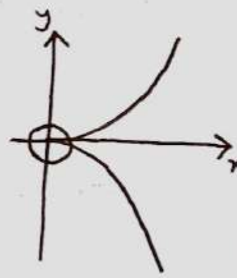
We say C is a <u>differentiable</u> curve if each of the components $\{x_i(t)\}_{i=1}^{3}$ are differentiable functions.

C is <u>regular</u> if $|\underline{x}'(t)| \neq 0$. If C is differentiable and regular, we say C is <u>smooth</u>.

Why "regular" condition? Consider $\underline{x}(t) = (t^2, t^3)$ clearly differentiable but $\underline{x}(t)$ has a cusp at $t = 0$

Note $|\underline{x}'(0)| = 0$.

Regularity prevents this.



Recall: $x_i(t)$ is differentiable at t if $x_i(t+h) = x_i(t) + x_i'(t)h + o(h)$

$\left(\frac{o(h)}{h} \to 0 \text{ as } h \to 0\right)$

Vector form: $\underline{x}(t+h) = \underline{x}(t) + \underline{x}'(t)h + o(h)$

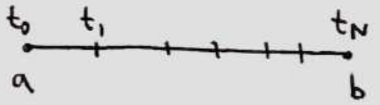
<u>Length of C</u>

Approximate C by straight lines

$C : [a,b] \ni t \mapsto \underline{x}(t)$

Introduce a partition P of $[a,b]$ with $t_0 = a$, $t_N = b$

$t_0 < t_1 < t_2 < \dots < t_N$



Set $\Delta t_i = t_{i+1} - t_i$ and $\Delta t = \max\limits_i \Delta t_i$

Define length of C relative to P by $l(C,P) = \sum\limits_{i=0}^{N-1} |\underline{x}(t_{i+1}) - \underline{x}(t_i)|$

As $\Delta t \to 0$, expect $l(C,P)$ to give a better approximation to length $l(C)$. Define length of C by

$$l(C) = \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} |\underline{x}(t_{i+1}) - \underline{x}(t_i)| = \lim_{\Delta t \to 0} l(C,P)$$

If limit doesn't exist, then curve is <u>non-rectifiable</u>.

Suppose C is differentiable. Then $\underline{x}(t_{i+1}) = \underline{x}(t_i + t_{i+1} - t_i)$

$= \underline{x}(t_i + \Delta t_i) = \underline{x}(t_i) + \underline{x}'(t_i)\Delta t_i) + o(\Delta t_i)$

It follows that $|\underline{x}(t_{i+1}) - \underline{x}(t_i)| = |\underline{x}'(t_i)|\Delta t_i + o(\Delta t_i)$

So $l(C,P) = \sum\limits_{i=0}^{N-1} \left(|\underline{x}'(t_i)| \Delta t_i + o(\Delta t_i) \right)$

Recall that $o(\Delta t_i)$ represents a function s.t. $\frac{o(\Delta t_i)}{\Delta t_i} \to 0$ as $\Delta t_i \to 0$.

So for any $\varepsilon > 0$, if $\Delta t = \max\limits_i \Delta t_i$ is sufficiently small,

have $|o(\Delta t_i)| < \varepsilon$ or $|o(\Delta t_i)| < \left(\frac{\varepsilon}{b-a}\right) \Delta t_i \quad \forall i$

so $$\left| l(C,P) - \sum_{i=0}^{N-1} |\underline{x}'(t_i)| \Delta t_i \right| = \left| \sum_{i=0}^{N-1} o(\Delta t_i) \right|$$

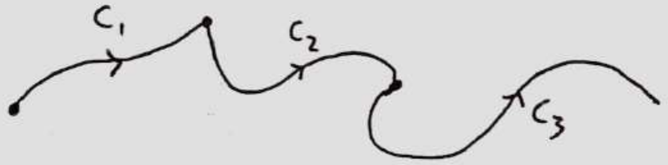
$$< \frac{\varepsilon}{b-a} \underbrace{\sum_{i=0}^{N-1} \Delta t_i}_{b-a} = \varepsilon$$ so LHS $\to 0$ as $\Delta t \to 0$

$$l(C) = \lim_{t \to 0} l(C,P) = \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} |\underline{x}'(t_i)| \Delta t_i = \boxed{\int_a^b |x'(t)|\, dt}$$

Write as $\int_C ds$ ← arc-length element "$ds = |\underline{x}'(t)|\, dt$"

Similarly define $\int_C f(\underline{x})\, ds = \int_a^b f(\underline{x}(t))\, |\underline{x}'(t)|\, dt$. (to integrate function along a curve)

If C is made up of M smooth curves $C_1, C_2, \ldots, C_M$



(piecewise smooth)

write $C = C_1 + C_2 + \ldots + C_M$

$$\int_C f(\underline{x})\, ds = \sum_{i=1}^{M} \int_{C_i} f(\underline{x})\, ds$$

Note <u>informally</u> $ds = |\underline{x}'(t)|\, dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

would give $ds^2 = dx^2 + dy^2 + dz^2$ (intuitively Pythagoras)

<u>Example</u> Let C be circle of radius $r > 0$ in $\mathbb{R}^3$

$$\underline{x}(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ 0 \end{pmatrix} \qquad t \in [0, 2\pi]$$

$$\underline{x}'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \\ 0 \end{pmatrix}$$

$$\int_C ds = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt = \int_0^{2\pi} r\, dt = \underline{2\pi r}$$

Also consider

$$\int_C x^2 y\, ds = \int_0^{2\pi} (r\cos t)^2 (r\sin t) \overbrace{r\, dt}^{|\underline{x}'(t)|\, dt} = 0$$

Does $L(C)$ depend on parametrisation?

$$\underline{x}(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ 0 \end{pmatrix} \quad t \in [0, 2\pi]$$

$$\tilde{\underline{x}}(t) = \begin{pmatrix} r\cos 2t \\ r\sin 2t \\ 0 \end{pmatrix} \quad t \in [0, \pi]$$

Both give different parametrisation of circle of radius r

Suppose C has 2 different parametrisations

$\underline{x} = \underline{x}_1(t) \quad a \leq t \leq b$

$\underline{x} = \underline{x}_2(\tau) \quad \alpha \leq \tau \leq \beta$

Must have $\underline{x}_2(\tau) = \underline{x}_1(t(\tau))$ for some $t(\tau)$ function.

Assume $\frac{dt}{d\tau} \neq 0$ so map between t and τ is invertible and differentiable (inverse function theorem, IB Analysis & Topology)

Note $\underline{x}_2'(\tau) = \frac{d}{d\tau}\underline{x}_2(\tau) = \frac{d}{d\tau}\underline{x}_1(t(\tau)) = \frac{dt}{d\tau}\underline{x}_1'(t(\tau))$
(chain rule)

From definitions:

$$\int_C f(\underline{x})\, ds = \int_a^b f(\underline{x}_1(t))\, |\underline{x}_1'(t)|\, dt$$

Substitution $t = t(\tau)$, assume $\frac{dt}{d\tau} > 0$, latter integral becomes

$$\int_\alpha^\beta f(\underline{x}_2(\tau))\, \underbrace{|\underline{x}_1'(t(\tau))| \frac{dt}{d\tau}\, d\tau}_{= |\underline{x}_2'(\tau)|\, d\tau \text{ (see above)}}$$

which is <u>precisely</u> the same as $\int_C f(\underline{x})\, ds$ but with $\underline{x}_2(\tau)$ instead of $\underline{x}_1(\tau)$ parametrisation. (same holds for $\frac{dt}{d\tau} < 0$).

So integral does <u>not</u> depend on choice of parametrisation of C.

Vector Calculus - Lecture 2

25/1/21

Can parametrise curves in many different ways. Define arc-length function for curve $[a,b] \ni t \mapsto \underline{x}(t)$ by

$$s(t) = \int_a^t |\underline{x}'(\tau)| \, d\tau \qquad \text{so } s(a) = 0 \text{ and } s(b) = l(C).$$

By FTC: $\frac{ds}{dt} = |\underline{x}'(t)| \geq 0$. For regular curves $\frac{ds}{dt} > 0$ so can invert relationship between s and t. Can write $t(s)$.

So we can parametrise regular curves with respect to arc length. If we write $\underline{r}(s) = \underline{x}(t(s))$ where $0 \leq s \leq l(C)$ then by chain rule:

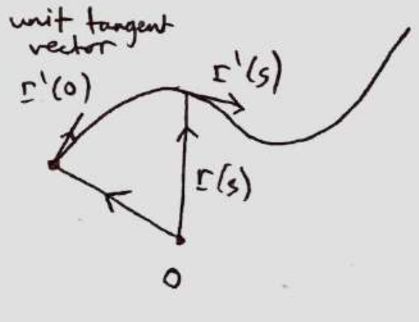
$$\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|\underline{x}'(t(s))|} \qquad \text{so } \underline{r}'(s) = \frac{d}{ds}\underline{x}(t(s)) = \frac{dt}{ds}\underline{x}'(t(s))$$

$$\Rightarrow \underline{r}'(s) = \frac{\underline{x}'(t(s))}{|\underline{x}'(t(s))|}.$$

i.e. $|\underline{r}'(s)| = 1$.

This (consistently) gives

$$l(C) = \int_0^{l(C)} |\underline{r}'(s)| \, ds = \int_0^{l(C)} ds$$



$\underline{r}'(s)$ represents a unit tangent vector to the curve.

1.2 Curvature and Torsion

Throughout this section, talk about generic regular curve C parametrised by arc length, write $s \mapsto \underline{r}(s)$.

Define tangent vector

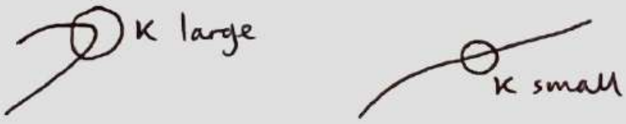
$\underline{t}(s) = \underline{r}'(s)$. This is a unit vector: $|\underline{t}(s)| = 1$.

Since $|\underline{t}(s)|$ doesn't change, the second derivative $\underline{r}''(s) = \underline{t}'(s)$ only measures change in direction.

Intuitively: if $|\underline{r}''(s)|$ is large then curve rapidly changes direction and vice versa. $|\underline{r}''(s)|$ small - curve approximately flat

Definition (Curvature)

Define $\quad \kappa(s) = |\underline{r}''(s)| = |\underline{t}'(s)|$.



Since $\underline{t} = \underline{r}'(s)$ is a unit vector, differentiating $\underline{t}\cdot\underline{t} = 1$ gives $\underline{t}\cdot\underline{t}' = 0$. Define principal normal by

$\underline{t}' = \kappa \underline{n}$ ← principal normal (note unit vector) $\qquad$ note $\underline{n}$ is everywhere normal to C since $\underline{t}\cdot\underline{n} = 0$.

Can extend $\{\underline{t}, \underline{n}\}$ to ~~orthonormal~~ orthonormal basis by defining the binormal $\quad \underline{b} = \underline{t} \times \underline{n}$.

$|\underline{t}| = |\underline{n}| = 1 \Rightarrow |\underline{b}| = 1$ so $\underline{b}'\cdot\underline{b} = 0$. Also since $\underline{t}\cdot\underline{b} = 0$ and $\underline{n}\cdot\underline{b} = 0$, have

$$0 = (\underline{t}\cdot\underline{b})' = \underline{t}'\cdot\underline{b} + \underline{t}\cdot\underline{b}' = \kappa\underbrace{\underline{n}\cdot\underline{b}}_{0} + \underline{t}\cdot\underline{b}'$$

$$\Rightarrow \underline{t}\cdot\underline{b}' = 0.$$

So $\underline{b}'$ is orthogonal to both $\underline{t}$ and $\underline{b}$, i.e. it's parallel to $\underline{n}$.

Define torsion of a curve by $\quad \underline{b}' = -\tau \underline{n}$ ← torsion.

Have ~~to~~ two equations:

$$\boxed{\underline{t}' = \kappa \underline{n}, \quad \underline{b}' = -\tau \underline{n}}$$

Proposition $\quad$ The curvature $\kappa(s)$ and torsion $\tau(s)$ define a curve up to translation / orientation.

Proof $\quad$ Since $\underline{n} = \underline{b} \times \underline{t}$, have $\underline{t}' = \kappa(\underline{b}\times\underline{t})$ and $\underline{b}' = -\tau(\underline{b}\times\underline{t})$ so have 6 eqns for 6 unknowns.

Given $\kappa(s)$, $\tau(s)$, $\underline{t}(0)$, $\underline{b}(0)$, can construct $\underline{t}(s)$, $\underline{b}(s)$ and hence $\underline{n} = \underline{b} \times \underline{t}$. Hence result holds. $\square$

determines curve

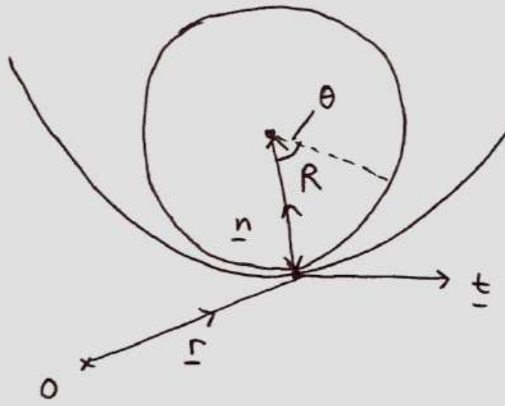
"Fundamental theorem of differential geometry of curves"

1.3 Radius of Curvature

Taylor expand generic curve about $s = 0$. Write $\underline{t} = \underline{t}(0)$, $\underline{n} = \underline{n}(0)$ etc.

$$\begin{aligned} \underline{r}(s) &= \underline{r}(0) + s\underline{r}'(0) + \tfrac{1}{2}s^2 \underline{r}''(0) + o(s^2) \\ &= \underline{r} + s\underline{t} + \tfrac{1}{2}s^2 \kappa \underline{n} + o(s^2) \qquad (1) \end{aligned}$$

Suppose WLOG that $\underline{t}$ is horizontal. What circle, which goes through the curve tangentially at point $\underline{r} = \underline{r}(0)$ is best fit?



Eqn of circle

$$\underline{x}(\theta) = \underline{r} + R(1-\cos\theta)\underline{n} + R\sin\theta\, \underline{t}$$

Expand for small $|\theta|$:

$$\underline{x}(\theta) \simeq \underline{r} + R\theta\, \underline{t} + \tfrac{1}{2}R\theta^2 \underline{n} + o(\theta^2)$$

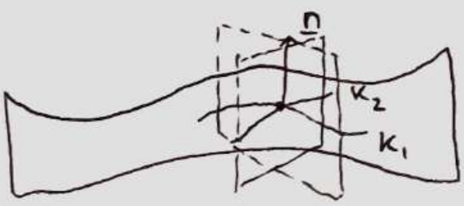
compare

Arc length on circle is $\underline{s = R\theta}$. So $\underline{x}(\theta) = \underline{r} + s\underline{t} + \frac{1}{2}\frac{1}{R}s^2\underline{n} + o(s^2)$ (2)

To match eqns (1) and (2) for curve up to 2nd order, require $\underline{R = \frac{1}{\kappa}}$

Define $R(s) = \dfrac{1}{\kappa(s)}$ as <u>radius of curvature</u>.

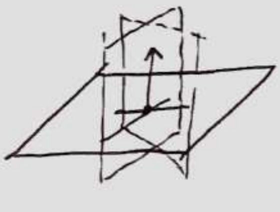
Section 1.4 Gaussian Curvature (non-examinable) for surfaces



Draw normal, take a plane containing normal and consider curve formed by intersection of plane and surface.

$\underline{K_G = \kappa_{min} \kappa_{max}}$ - define Gaussian curvature

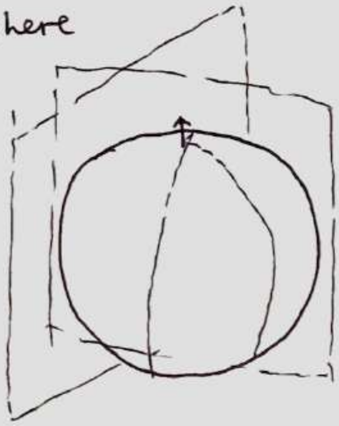
e.g.



for flat sheet:

$K_G = 0$ as $\kappa_{min} = \kappa_{max} = 0$.

Sphere



$\kappa_{min} = \frac{1}{R}$, $\kappa_{max} = \frac{1}{R}$ (great circle)

$\Rightarrow K_G = \frac{1}{R^2}$

<u>Theorem</u> (Remarkable Theorem) Gaussian curvature of a surface S is invariant if you bend the surface without stretching it.

Vector Calculus - Lecture 3

27/1/21

2 Coordinates, Differentials and Gradients

2.1 : Differentials and 1st order changes

Recall : $f = f(u_1, \dots, u_n)$ define differential of f by

$$df = \frac{\partial f}{\partial u_i} du_i \qquad (\Sigma \text{conv})$$

$\{du_i\}$ differential forms, lin. indep if $\{u_1, \dots, u_n\}$ indep.
i.e. if $\underline{\alpha_i du_i = 0}$ then $\underline{\alpha_i = 0}$ for $i = 1, \dots, n$.

Similarly : $\underline{x} = \underline{x}(u_1, \dots, u_n)$ - define $d\underline{x} = \dfrac{\partial \underline{x}}{\partial u_i} du_i$

<u>Example</u> $f(u, v, w) = u^2 + w \sin v$

$$df = 2u\, du + w \cos v\, dv + \sin v\, dw$$

If $\underline{x}(u, v, w) = \begin{pmatrix} u^2 - v^2 \\ w \\ e^v \end{pmatrix}$ then $d\underline{x} = \begin{pmatrix} 2u \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2v \\ 0 \\ e^v \end{pmatrix} dv + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dw$

Differentials encode info about how functions / vector field changes when we change the coordinate values.

$$f(u_1 + \delta u_1, \dots, u_n + \delta u_n) - f(u_1, \dots, u_n) = \frac{\partial f}{\partial u_i} \delta u_i + o(\delta \underline{u})$$

So if δf denotes change in $f(u_1, \dots, u_n)$ under perturbation of coords $(u_1, \dots, u_n) \mapsto (u_1 + \delta u_1, \dots, u_n + \delta u_n)$ then to first order $\underline{\delta f \approx \dfrac{\partial f}{\partial u_i} \delta u_i}$ (similarly $\delta \underline{x} \approx \dfrac{\partial \underline{x}}{\partial u_i} \delta u_i$)

2.2 : Coordinates and Line Elements

A general set of coords (u, v) on $\mathbb{R}^2$ can be specified by their relationship to Cartesian coords (x, y) i.e. specify smooth functions
$x = x(u, v) \quad y = y(u, v)$
(see: diffeomorphism)
which can be inverted to give smooth functions
$u = u(x, y), \quad v = v(x, y)$

Similarly (u, v, w) for $\mathbb{R}^3$: specify

$x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w)$

Standard Cartesian coords : $\underline{x}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix} = x\underline{e}_x + y\underline{e}_y$

$\{\underline{e}_x, \underline{e}_y\}$ orthonormal vectors - $\underline{e}_x$ points in direction of change in x with y fixed and vice versa

$$\underline{e}_x = \frac{\frac{\partial}{\partial x}\underline{x}(x,y)}{\left|\frac{\partial}{\partial x}\underline{x}(x,y)\right|} \qquad \underline{e}_y = \frac{\frac{\partial}{\partial y}\underline{x}(x,y)}{\left|\frac{\partial}{\partial y}\underline{x}(x,y)\right|}$$

Feature of Cartesian coords: $d\underline{x} = \frac{\partial \underline{x}}{\partial x}dx + \frac{\partial \underline{x}}{\partial y}dy = dx\,\underline{e}_x + dy\,\underline{e}_y$

i.e. change $x \to x + \delta x \Rightarrow$ vector changes (to 1st order) by $\underline{x} \to \underline{x} + \delta x\,\underline{e}_x$. Call $d\underline{x}$ the <u>line element</u> - tells us how small changes in coords produce changes in position vectors.

For polar coords : $\underline{x}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = r\underline{e}_r$

using basis vectors $\underline{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, $\underline{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ orthonormal

but <u>not</u> the same for each (r,θ).

As before : $$\underline{e}_r = \frac{\frac{\partial}{\partial r}\underline{x}(r,\theta)}{\left|\frac{\partial}{\partial r}\underline{x}(r,\theta)\right|} \qquad \underline{e}_\theta = \frac{\frac{\partial}{\partial \theta}\underline{x}(r,\theta)}{\left|\frac{\partial}{\partial \theta}\underline{x}(r,\theta)\right|}$$

Since $\{\underline{e}_r, \underline{e}_\theta)$ are orthogonal, it makes sense to call (r,θ) <u>orthogonal curvilinear coordinates</u>.

(check)

Also have <u>line element</u> $\underline{d\underline{x} = \frac{\partial \underline{x}}{\partial r}dr + \frac{\partial \underline{x}}{\partial \theta}d\theta} = \underline{e}_r\,dr + r\,d\theta\,\underline{e}_\theta$

$\theta \to \delta\theta$: up to 1st order, have $\underline{x} \mapsto \underline{x} + r\,\delta\theta\,\underline{e}_\theta$

(not $\underline{x} \mapsto \underline{x} + \delta\theta\,\underline{e}_\theta$)

i.e. change in θ depends also on r, if r is large then small change in $\theta \to$ large change in $\underline{x}$.

2.2.1 Orthogonal Curvilinear Coordinates

Say that (u, v, w) are a set of orthogonal curvilinear coords

if $\underline{e}_u = \dfrac{\partial \underline{x}/\partial u}{|\partial \underline{x}/\partial u|}$, $\underline{e}_v = \dfrac{\partial \underline{x}/\partial v}{|\partial \underline{x}/\partial v|}$, $\underline{e}_w = \dfrac{\partial \underline{x}/\partial w}{|\partial \underline{x}/\partial w|}$

form a right-handed orthonormal basis, for each u, v, w

$\underline{e}_u \times \underline{e}_v = \underline{e}_w$ (not always the same).

$\{\underline{e}_u, \underline{e}_v, \underline{e}_w\}$ form orthonormal basis for R^3 at each choice of (u, v, w) - not necessarily the same basis at each pt (like polar).

Standard: $h_u = \left|\dfrac{\partial \underline{x}}{\partial u}\right|$, $h_v = \left|\dfrac{\partial \underline{x}}{\partial v}\right|$, $h_w = \left|\dfrac{\partial \underline{x}}{\partial w}\right|$ scale factors

tell us how small coord. changes scale up to change in $\underline{x}$

Note: line element is

$$d\underline{x} = \frac{\partial \underline{x}}{\partial u} du + \frac{\partial \underline{x}}{\partial v} dv + \frac{\partial \underline{x}}{\partial w} dw$$

$$= h_u \underline{e}_u \, du + h_v \underline{e}_v \, dv + h_w \underline{e}_w \, dw$$

2.2.2 Cylindrical Polar Coordinates

Define (ρ, ϕ, z) by $\underline{x}(\rho, \phi, z) = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix}$ $\begin{matrix} 0 \le \rho < \infty \\ 0 \le \phi < 2\pi \\ -\infty < z < \infty \end{matrix}$

$\underline{e}_\rho = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$, $\underline{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$, $\underline{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$

$$d\underline{x} = d\rho \, \underline{e}_\rho + \rho \, d\phi \, \underline{e}_\phi + dz \, \underline{e}_z$$

i.e. small change in ρ corresponds to change $d\rho$ in $\underline{e}_\rho$ direction but small change in ϕ corresponds to change by $\rho \, d\phi$ in $\underline{e}_\phi$ direction

Note: $\underline{x} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix} = \rho \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

2.2.3 Spherical Polar Coordinates

Define (r, θ, ϕ) by

$$\underline{x}(r, \theta, \phi) = \begin{pmatrix} r\cos\phi \sin\theta \\ r\sin\phi \sin\theta \\ r\cos\theta \end{pmatrix} \qquad \begin{array}{l} 0 \leq r < \infty \\ 0 \leq \theta \leq \pi \\ 0 \leq \phi < 2\pi \end{array}$$

Check:

$$\underline{e}_r = \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix} \qquad \underline{e}_\theta = \begin{pmatrix} \cos\phi \cos\theta \\ \sin\phi \cos\theta \\ -\sin\theta \end{pmatrix} \qquad \underline{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta$

i.e. $d\underline{x} = dr\,\underline{e}_r + r\,d\theta\,\underline{e}_\theta + r\sin\theta\,d\phi\,\underline{e}_\phi$

and $$\underline{x} = r\begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix} = r\,\underline{e}_r(\theta, \phi)$$

Vector Calculus - Lecture 4

29/1/21

2.3 Gradient Operator

For $f: \mathbb{R}^3 \to \mathbb{R}$, define <u>gradient</u> of f $\underline{\nabla} f$ by

$$f(\underline{x}+\underline{h}) = f(\underline{x}) + \nabla f(\underline{x})\cdot \underline{h} + o(\underline{h}) \qquad (\underline{h}\to 0) \qquad (*)$$

<u>Directional derivative</u> of f in direction $\underline{v}$, denoted $D_{\underline{v}} f$ or $\frac{\partial f}{\partial \underline{v}}$ is defined by

$$D_v f(\underline{x}) = \lim_{t\to 0} \frac{f(\underline{x}+t\underline{v}) - f(\underline{x})}{t}$$

i.e. $f(\underline{x}+t\underline{v}) = f(\underline{x}) + t D_v f(\underline{x}) + o(t) \qquad t\to 0 \qquad (\dagger)$

Setting $\underline{h} = t\underline{v}$ in $(*)$:

$$f(\underline{x}+t\underline{v}) = f(\underline{x}) + t\nabla f(\underline{x})\,\underline{v} + o(t) \qquad t\to 0$$

Comparing to $(\dagger)$, see $\underline{D_v f = \underline{v}\cdot\nabla f}$

by Cauchy-Schwarz, $\underline{a}\cdot\underline{b}$ is maximised when a, b in same direction

so ∇f points in direction of <u>greatest increase</u> of f.

Similarly $-\nabla f$ points in direction of greatest decrease of f.

<u>Example</u> $\quad f(x) = \frac{1}{2}|\underline{x}|^2$

then $f(\underline{x}+\underline{h}) = \frac{1}{2}(\underline{x}+\underline{h})\cdot(\underline{x}+\underline{h}) = \frac{1}{2}|\underline{x}|^2 + \frac{1}{2}(2\underline{x}\cdot\underline{h}) + \frac{1}{2}|\underline{h}|^2$

$\qquad = f(\underline{x}) + \underline{x}\cdot\underline{h} + o(\underline{h}) \implies \underline{\nabla f(\underline{x}) = \underline{x}}$

Consider curve $t \mapsto \underline{x}(t)$. How does f change along this curve?

Write $F(t) = f(\underline{x}(t))$, $\delta\underline{x} = \underline{x}(t+\delta t) - \underline{x}(t)$

$F(t+\delta t) = f(\underline{x}(t+\delta t)) = f(\underline{x}(t)+\delta x) = f(\underline{x}(t)) + \nabla f(\underline{x}(t))\cdot\delta x + o(\delta\underline{x})$

Since $\delta\underline{x} = \underline{x}'(t)\,\delta t + o(\delta t)$

$\implies F(t+\delta t) = F(t) + \underline{x}'(t)\cdot\nabla f(\underline{x}(t))\,\delta t + o(\delta t)$

i.e. $$\boxed{\frac{dF}{dt} = \frac{d}{dt} f(\underline{x}(t)) = \frac{d\underline{x}}{dt}\cdot\nabla f(\underline{x}(t))}$$

Suppose surface S is defined implicitly:

$$S = \{ \underline{x} \in \mathbb{R}^3 : f(\underline{x}) = 0 \} \qquad \text{(2 deg. of freedom)}$$

If $t \mapsto \underline{x}(t)$ is any curve in S, then $f(\underline{x}(t)) = 0$ identically

so $\quad 0 = \dfrac{d}{dt} f(\underline{x}(t)) = \nabla f(\underline{x}(t)) \cdot \dfrac{d\underline{x}}{dt}$

$\Rightarrow$ ∇f is <u>orthogonal</u> to tangent vector of <u>any</u> curve in S

i.e. $\nabla f(\underline{x})$ is <u>normal</u> to surface at $\underline{x}$.

2.4 Computing the Gradient

If working with O.C.C (u, v, w) not clear how to find ∇f.
How to change (u, v, w) so that $\underline{x}(u, v, w) \to \underline{x} + \underline{h}$?

Cartesian: $\quad x \mapsto x + h_1, \quad y \mapsto y + h_2, \quad z \mapsto z + h_3$

$$\Rightarrow f(\underline{x} + \underline{h}) = f(\underline{x}) + \frac{\partial f}{\partial x} h_1 + \frac{\partial f}{\partial y} h_2 + \frac{\partial f}{\partial z} h_3 + o(\underline{h})$$

$$= f(\underline{x}) + \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} \cdot \underline{h} + o(\underline{h})$$

i.e. $\quad \underline{\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}} \qquad$ or $\quad \nabla f = \underline{e}_i \dfrac{\partial f}{\partial x_i}, \quad (\nabla f)_i = \dfrac{\partial f}{\partial x_i}$

~~∇f is~~ ∇ is a vector differential operator.

e.g. $\quad f = \frac{1}{2}(x^2 + y^2 + z^2) = \frac{1}{2}|\underline{x}|^2 \quad$ (as earlier)

$$[\nabla f]_i = \frac{\partial}{\partial x_i}\left[\tfrac{1}{2} x_j x_j\right] = \tfrac{1}{2}\left[\delta_{ij} x_j + x_j \delta_{ij}\right] = \underline{x_i} \text{ as expected.}$$

Recall in Cartesian: $\quad d\underline{x} = dx\, \underline{e}_x + dy\, \underline{e}_y + dz\, \underline{e}_z = dx_i\, \underline{e}_i$
(line element)

and $f(x, y, z)$ has differential $\quad df = \dfrac{\partial f}{\partial x_i} dx_i$

then $\quad \nabla f \cdot d\underline{x} = \left(\underline{e}_i \dfrac{\partial f}{\partial x_i}\right) \cdot \left(\underline{e}_j\, dx_j\right) = \dfrac{\partial f}{\partial x_i} \underbrace{(\underline{e}_i \cdot \underline{e}_j)}_{\delta_{ij}} dx_j = \dfrac{\partial f}{\partial x_i} dx_i$

$$\boxed{\nabla f \cdot d\underline{x} = df} \quad \text{- coordinate independent} \qquad = \underline{df}$$

(slight abuse of notation: should write $F(x, y, z) = f(\underline{x}(x, y, z))$ or
$F(u, v, w) = f(\underline{x}(u, v, w))$ rather than $f(\underline{x})$ just as $f(x, y, z)$)

Proposition If (u,v,w) are OCC, $f=f(u,v,w)$: $\boxed{\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\underline{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\underline{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\underline{e}_w}$

Proof If $f=f(u,v,w)$ and $\underline{x} = \underline{x}(u,v,w)$:

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw, \quad d\underline{x} = h_u\,du\,\underline{e}_u + h_v\,dv\,\underline{e}_v + h_w\,dw\,\underline{e}_w$$

by $df = \nabla f \cdot d\underline{x}$ $\quad (\nabla f = (\nabla f)_u\underline{e}_u + (\nabla f)_v\underline{e}_v + (\nabla f)_w\underline{e}_w)$

we find:

$$\frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw = h_u(\nabla f)_u\,du + h_v(\nabla f)_v\,dv + h_w(\nabla f)_w\,dw$$

Since $\{du, dv, dw\}$ are lin. indep, compare components/coeffs:

$(\nabla f)_u = \frac{1}{h_u}\frac{\partial f}{\partial u}$, $\quad (\nabla f)_v = \frac{1}{h_v}\frac{\partial f}{\partial v}$, $\quad (\nabla f)_w = \frac{1}{h_w}\frac{\partial f}{\partial w}$ as required. $\square$

Cylindrical: (ρ, ϕ, z), $h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$ we get

$$\nabla f = \frac{\partial f}{\partial \rho}\underline{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\underline{e}_\phi + \frac{\partial f}{\partial z}\underline{e}_z$$

Spherical: (r,θ,ϕ), $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$:

$$\nabla f = \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{e}_\phi$$

Example $f(\underline{x}) = \frac{1}{2}|\underline{x}|^2$

then
$$f = \begin{cases} \frac{1}{2}(x^2+y^2+z^2) & \text{cartesian} \\ \frac{1}{2}(\rho^2+z^2) & \text{cylindrical} \\ \frac{1}{2}r^2 & \text{spherical} \end{cases}$$

$$\Rightarrow \quad \nabla f = \begin{cases} x\underline{e}_x + y\underline{e}_y + z\underline{e}_z & \text{cartesian} \\ \rho\underline{e}_\rho + z\underline{e}_z & \text{cylindrical} \\ r & \text{spherical} \end{cases} = \underline{x}$$

same in each coordinate system.

Vector Calculus - Lecture 5 1/2/21

3 Integration over lines / surfaces / volumes

3.1 Line Integrals

For vector field $\underline{F} = \underline{F}(\underline{x})$ and piecewise smooth parametrised curve C $[a,b] \ni t \mapsto \underline{x}(t)$, define line integral

$$\int_C \underline{F} \cdot d\underline{x} = \int_a^b \underline{F}(\underline{x}(t)) \cdot \frac{d\underline{x}}{dt}\, dt$$

To integrate in opposite direction, usually write $\int_{-C} \underline{F} \cdot d\underline{x}$

(interpret: work done by particle moving along C in presence of force $\underline{F}$)

Example Consider $\underline{F} = \begin{pmatrix} x^2 y \\ yz \\ 2zx \end{pmatrix}$ field

Consider 2 curves connecting origin to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$:

$$C_1 : [0,1] \ni t \mapsto \begin{pmatrix} t \\ t \\ t \end{pmatrix}, \qquad C_2 : [0,1] \ni t \mapsto \begin{pmatrix} t \\ t \\ t^2 \end{pmatrix}$$

so $$\int_{C_1} \underline{F} \cdot d\underline{x} = \int_0^1 \begin{pmatrix} t^3 \\ t^2 \\ 2t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dt = \frac{5}{4}$$

depends on path

$$\int_{C_2} \underline{F} \cdot d\underline{x} = \int_0^1 \begin{pmatrix} t^3 \\ t^3 \\ 2t^3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2t \end{pmatrix} dt = \frac{13}{10}$$

Example 2 A particle at $\underline{x}$ experiences force in cylindrical polars

$\underline{F}(\underline{x}) = z\rho\, \underline{e}_\phi$. Calculate work done by travelling along

$$C : [0, 2\pi] \ni t \mapsto \begin{pmatrix} a\cos t \\ a \sin t \\ t \end{pmatrix} \quad (a > 0)$$

Recall line element in cylindrical polars

$$d\underline{x} = d\rho\, \underline{e}_\rho + \rho\, d\phi\, \underline{e}_\phi + dz\, \underline{e}_z$$

so $\underline{F} \cdot d\underline{x} = z\rho^2\, d\phi$

and as $\underline{x}(\rho, \phi, z) = \begin{pmatrix} \rho\cos\phi \\ \rho \sin\phi \\ z \end{pmatrix}$

have $(\rho, \phi, z) = (a, t, t)$

$\Rightarrow (d\rho, d\phi, dz) = (0, dt, dt)$

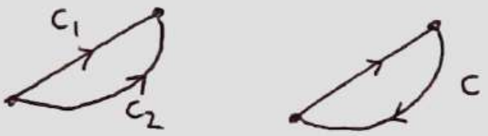
so $\underline{F} \cdot d\underline{x} = a^2 t\, dt$

$$\Rightarrow \int_C \underline{F} \cdot d\underline{x} = a^2 \int_0^{2\pi} t\, dt = \underline{2\pi^2 a^2}$$

A curve $[a,b] \ni t \mapsto \underline{x}(t)$ may be such that $\underline{x}(a) = \underline{x}(b)$ (closed)

Then we write $\oint \underline{F} \cdot d\underline{x}$ - sometimes called circulation of $\underline{F}$ about C.

Example $C = C_1 - C_2$ from previous example 1:



$$\oint_C \underline{F} \cdot d\underline{x} = \int_{C_1} \underline{F} \cdot d\underline{x} - \int_{C_2} \underline{F} \cdot d\underline{x} = \underline{-\frac{2}{15}}$$

3.2 Conservative forces and exact differentials

We've seen how to interpret things like $\underline{F} \cdot d\underline{x}$ inside an integral (another example of differential form) i.e. in coords (u, v, w)

$$\underline{F} \cdot d\underline{x} = (\,)\,du + (\,)\,dv + (\,)\,dw$$

We say $\underline{F} \cdot d\underline{x}$ is exact if $\underline{F} \cdot d\underline{x} = df$ for some scalar f.

Recall: $df = \nabla f \cdot d\underline{x}$ so $\underline{F} \cdot d\underline{x}$ is exact iff $\underline{F} = \nabla f$ for some scalar f. Call such fields conservative. (see D&R L5)

So $\underline{F} \cdot d\underline{x}$ is exact $\iff$ $\underline{F}$ is conservative.

Using properties $d(\alpha f + \beta g) = \alpha\, df + \beta\, dg$ (α, β) constant, $d(fg) = g\,df + f\,dg$, usually easy to see if a differential form is exact.

Proposition If θ is an exact differential form, then $\oint_C \theta = 0$ for any closed curve C.

Proof If θ is exact then $\theta = \nabla f \cdot d\underline{x}$ for some scalar f.
If $C = [a,b] \ni t \mapsto \underline{x}(t)$:

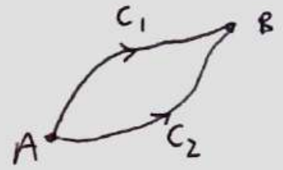
$$\oint_C \theta = \oint_C \nabla f \cdot d\underline{x} = \int_a^b \nabla f(\underline{x}(t)) \cdot \frac{d\underline{x}}{dt}\, dt$$

$$= \int_a^b \frac{d}{dt}\Big(f(\underline{x}(t))\Big)\, dt \quad \text{(by VC L4)}$$

$$= \underset{\text{(FTC)}}{f(\underline{x}(a)) - f(\underline{x}(b))} = \underline{0} \quad \text{if } \underline{x}(a) = \underline{x}(b). \qquad \square$$

Equivalently, if $\underline{F}$ is conservative, then circulation of $\underline{F}$ around any closed loop curve C vanishes: $\oint_C \underline{F} \cdot d\underline{x} = 0$.

If $\underline{F}$ conservative ($\underline{F} \cdot d\underline{x}$ exact) then line integral between $A = \underline{x}(a)$ and $B = \underline{x}(b)$ is <u>independent</u> of path.



$$C = C_1 - C_2 \qquad \int_C \underline{F} \cdot d\underline{x} = 0$$

$$\iff \int_{C_1} \underline{F} \cdot d\underline{x} = \int_{C_2} \underline{F} \cdot d\underline{x}$$

Let $(u_1, u_2, u_3) = (u, v, w)$ be arbitrary OCCs. Let

$$\underline{F} \cdot d\underline{x} = \theta = \underbrace{A(u,v,w)\, du}_{\theta_1\, du} + \underbrace{B(u,v,w)\, dv}_{\theta_2\, dv} + \underbrace{C(u,v,w)\, dw}_{\theta_3\, dw}$$

$$= \theta_i\, du_i$$

A <u>necessary condition</u> for θ to be exact is $\dfrac{\partial \theta_i}{\partial u_j} = \dfrac{\partial \theta_j}{\partial u_i}$ for each i, j $\quad (+)$

Indeed: if θ exact then $\theta = df$ so $\theta = \dfrac{\partial f}{\partial u_i} du_i \iff \theta_i = \dfrac{\partial f}{\partial u_i}$

and so $\dfrac{\partial \theta_i}{\partial u_j} = \dfrac{\partial^2 f}{\partial u_j \partial u_i} = \dfrac{\partial^2 f}{\partial u_i \partial u_j} = \dfrac{\partial \theta_j}{\partial u_i}$

Call differential forms $\theta = \theta_i\, du_i$ that obey $(+)$ <u>closed</u>.

θ exact $\Rightarrow$ θ closed.

The reverse implication is true if the domain $\Omega \subset \mathbb{R}^3$ on which θ is defined is simply connected: all closed loops in Ω can be "continuously shrunk" to any point inside Ω without leaving it)

<u>Example</u> (a) $\theta = y\, dx - x\, dy$. Is it exact?

Check necessity: is it closed?

$\dfrac{\partial}{\partial y}(y) \overset{?}{=} \dfrac{\partial}{\partial x}(-x)$ No: $1 \neq -1$ so it's not exact

(b) Compute line integral $\oint_C 3x^2 y\, dx + x^3\, dy$ $\qquad C : [\alpha_1, \alpha_{100}] \ni t \mapsto \begin{bmatrix} \cos[\mathrm{Im}(\zeta(\frac{1}{2}+it))] \\ \sin[\mathrm{Re}(\zeta(\frac{1}{2}+it))] \\ 0 \end{bmatrix}$

where α_1 and α_{100} are the 1st and 100th zero of $\zeta(\frac{1}{2}+it)$

but $d(x^3 y) = 3x^2 y\, dx + x^3\, dy$: it's 0. Check if it's closed!

Example

$$\text{Work done} = \int_C \underline{F} \cdot d\underline{x} \qquad C : [a,b] \ni t \mapsto \underline{x}(t)$$

$$= m \int_a^b \ddot{\underline{x}} \cdot \dot{\underline{x}} \, dt = \frac{1}{2} m |\dot{\underline{x}}|^2 \Big|_a^b$$

= change in KE

If $\underline{F} = -\nabla V$ ($\underline{F}$ conservative) then

$$\oint_C \underline{F} \cdot d\underline{x} = -\int_C \nabla V \cdot d\underline{x} = V(\underline{x}(a)) - V(\underline{x}(b))$$

$$\Rightarrow \left(\frac{1}{2} m |\dot{\underline{x}}|^2 + V(\underline{x}(t)) \right) \Big|_{t=a} = \left(\frac{1}{2} m |\dot{\underline{x}}|^2 + V(\underline{x}(t)) \right) \Big|_{t=b}$$

energy conserved.

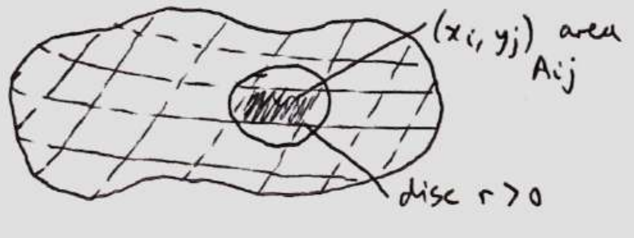
Vector Calculus - Lecture 5

4/2/21

3.3 Integration in $\mathbb{R}^2$

Want to integrate over bounded region $D \subset \mathbb{R}^2$

To do this: cover D with small disjoint sets A_{ij} each with area δA_{ij}, each set contained in a disc of radius $\varepsilon > 0$. Let (x_i, y_j) be points contained in each A_{ij}

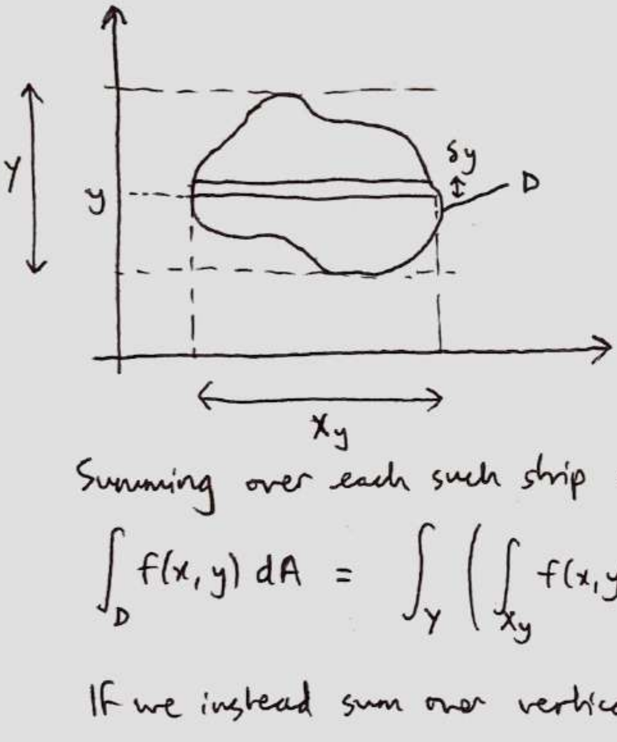


Now define

$$\int_D f(\underline{x})\, dA = \lim_{\varepsilon \to 0} \sum_{i,j} f(x_i, y_j)\, \delta A_{ij}$$

Say integral exists if it is independent of choice of A_{ij} and choice of (x_i, y_j).

Obvious choice of partition - use rectangles with $\delta A_{ij} = \delta x\, \delta y$



can sum over horizontal strips of width δy, then take limit as $\delta x \to 0$:

$$\delta y \int_{X_y} f(x, y)\, dx$$

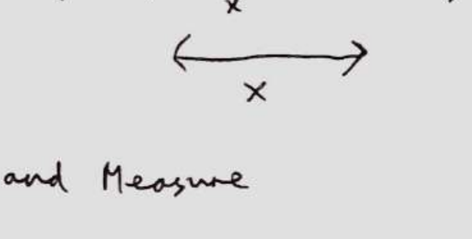
where X_y is $\{x : (x, y) \in D\}$

Summing over each such strip and letting $\delta y \to 0$:

$$\int_D f(x, y)\, dA = \int_Y \left(\int_{X_y} f(x, y)\, dx \right) dy$$

If we instead sum over vertical strips:

$$\int_D f(x, y)\, dA = \int_X \left(\int_{Y_x} f(x, y)\, dy \right) dx$$



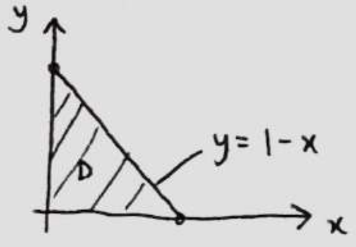
More concisely: $dA = dx\, dy = dy\, dx$

See Fubini's theorem in Part II Probability and Measure

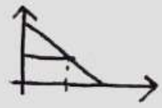
(This states $\int_D |f(x,y)|\, dA < \infty$ then

$$\int \left(\int f(x,y)\, dx \right) dy = \int \left(\int f(x,y)\, dy \right) dx = \iint_D f(x,y)\, dA \Bigg)$$

<u>Example</u> Let D be a triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$



If $f(x,y) = xy^2$: if we integrate over horizontal strips



$$\int_D f\, dA = \int_0^1 \left(\int_0^{1-y} xy^2\, dx \right) dy$$

$$= \int_0^1 y^2 \left[\tfrac{1}{2} x^2 \right]_0^{1-y} dy = \int_0^1 \tfrac{1}{2} y^2 (1-y)^2\, dy = \underline{\tfrac{1}{60}}$$

Vertical:

$$\int_D f\, dA = \int_0^1 \left(\int_0^{1-x} xy^2\, dy \right) dx$$

$$= \int_0^1 x \left[\tfrac{1}{3} y^3 \right]_0^{1-x} dx = \underline{\tfrac{1}{60}}$$

If $f(x,y) = g(x)\, h(y)$ and D is a rectangle

$D = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$ then

$$\int_A f(x,y)\, dA = \left(\int_a^b g(x)\, dx \right) \left(\int_c^d h(y)\, dy \right)$$

Change of variables is often useful to find $\int_a^b f(x)\, dx$ in 1D

If we introduce $x = x(u)$ with $\cancel{x(a) = \alpha}$, $\cancel{x(b) = \beta}$ then
$x(\alpha) = a$, $x(\beta) = b$

$$\int_a^b f(x)\, dx = \begin{cases} + \int_\alpha^\beta f(x(u)) \frac{dx}{du}\, du & \beta > \alpha \quad \frac{dx}{du} > 0 \\ - \int_\beta^\alpha f(x(u)) \frac{dx}{du}\, du & \alpha > \beta \quad \frac{dx}{du} < 0 \end{cases}$$

If $I = [a,b]$ and $I' = x(I)$:

$$\int_I f(x)\, dx = \int_{I'} f(x(u)) \left| \frac{dx}{du} \right| du$$

Similar Formula in 2D

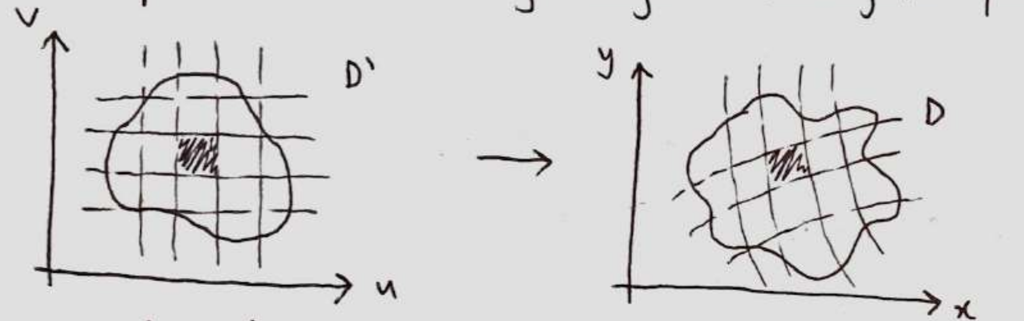
Proposition Let $x = x(u,v)$ and $y = y(u,v)$ be a smooth invertible transformation with smooth inverse that maps the region D' in (u,v) plane to D in (x,y)-plane. Write $\underline{x} = \underline{x}(u,v)$ then

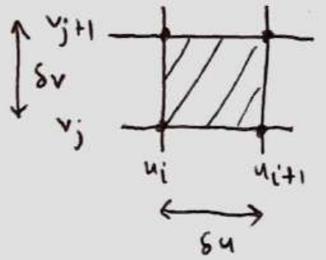
$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

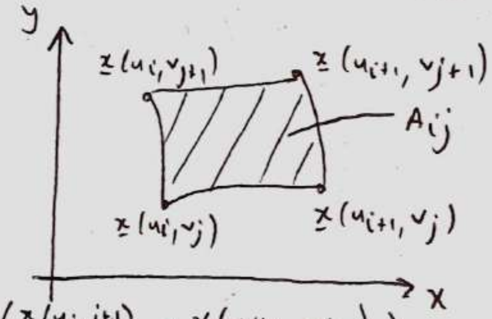
where $\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = \det\left(\dfrac{\partial \underline{x}}{\partial u} \,\Big|\, \dfrac{\partial \underline{x}}{\partial v}\right)$

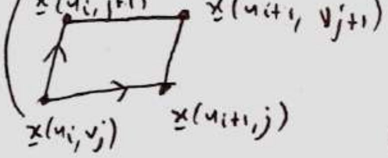
is the Jacobian (J). Short version: $dx\,dy = |J|\,du\,dv$

Form partition of D using image of rectangular partition of D'







Note $\delta A_{ij} \approx$ Area $\Bigg($  $\Bigg)$

$u_i + \delta u$

$$\underline{x}(u_{i+1}, v_j) - \underline{x}(u_i, v_j) \approx \frac{\partial \underline{x}}{\partial u}(u_i, v_j)\,\delta u$$

$$\underline{x}(u_i, v_{j+1}) - \underline{x}(u_i, v_j) \approx \frac{\partial \underline{x}}{\partial v}(u_i, v_j)\,\delta v$$

$$= \left|\det\left(\frac{\partial \underline{x}}{\partial u}(u_i, v_j)\,\delta u \,\Big|\, \frac{\partial \underline{x}}{\partial v}(u_i, v_j)\,\delta v\right)\right| = \underbrace{J(u_i, v_j)\,\delta u\,\delta v}_{\delta A_{ij}}$$

$$\int_D f\,dA = \lim_{\varepsilon \to 0} \sum_{i,j} f(x(u_i, v_j), y(u_i, v_j))\, J(u_i, v_j)\,\delta u\,\delta v$$

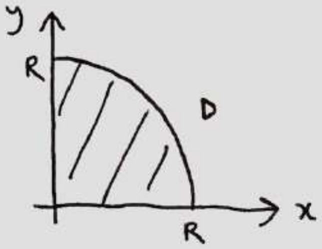
$$\iint_{D'} f(x(u,v),\ y(u,v))\ |J(u,v)|\ du\ dv$$

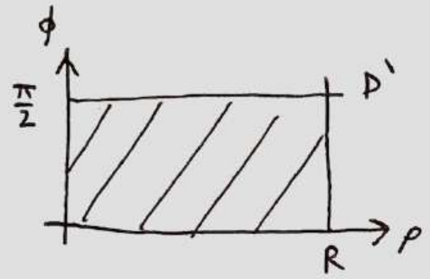
$$= \iint_D f(x,y)\ dx\ dy \qquad \underline{dx\ dy = |J|\ du\ dv}$$

<u>Example</u> Use polar coords (ρ, ϕ) $\quad x(\rho,\phi) = \rho\cos\phi,\quad y(\rho,\phi) = \rho\sin\phi$

$$|J| = \left| \det\begin{pmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{pmatrix} \right| = |\rho| = \underline{\rho}$$

If $D = \{(x,y) : x > 0,\ y > 0,\ x^2 + y^2 < R^2\}$





$$D' = \{(\rho,\phi) : 0 < \rho < R,\ 0 < \phi < \tfrac{\pi}{2})$$

$$\iint_D f(x,y)\ dx\ dy = \iint_{D'} f(\rho\cos\phi,\ \rho\sin\phi)\ \rho\ d\rho\ d\phi$$

i.e. $dx\ dy = \rho\ d\rho\ d\phi$

take $R \to \infty$:

$$\int_{x=0}^{\infty}\int_{y=0}^{\infty} f(x,y)\ dy = \int_{\phi=0}^{\pi/2}\int_{\rho=0}^{\infty} f(\rho\cos\phi,\ \rho\sin\phi)\rho\ d\rho\ d\phi$$

Consider $I = \int_0^\infty e^{-x^2}\ dx$

$$I^2 = \int_0^\infty e^{-x^2}\ dx \int_0^\infty e^{-y^2}\ dy = \int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-x^2-y^2}\ dx\ dy$$

$$= \int_{\phi=0}^{\pi/2}\left(\int_{\rho=0}^{\infty} e^{-\rho^2}\rho\ d\rho\right) d\phi = \frac{\pi}{2}\int_0^\infty \frac{d}{d\rho}\left(-\frac{1}{2}e^{-\rho^2}\right) d\rho = \frac{\pi}{4}$$

$$\Rightarrow \underline{I = \frac{\sqrt{\pi}}{2}}$$

3.4 Integration in $\mathbb{R}^3$

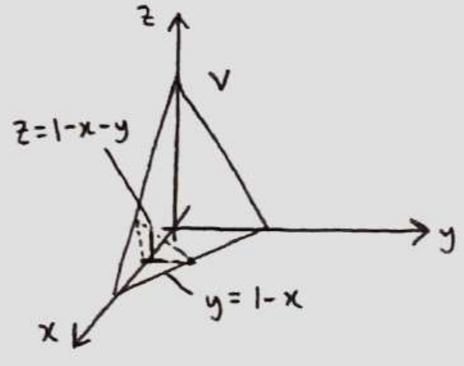
To integrate over regions V in $\mathbb{R}^3$, use similar ideas to in $\mathbb{R}^2$

$$\int_V f(\underline{x})\, dV = \lim_{\varepsilon \to 0} \sum_{i,j,k} f(x_i, y_j, z_k)\, \delta V_{ijk}$$

In this case, the <u>volume element</u> satisfies $dV = dx\, dy\, dz$

Can do the integrals in any order.

<u>Example</u>



V bound by plane $x+y+z=1$ and 3 planes $x=0,\ y=0,\ z=0$

$$\int_V dV = \int_{x=0}^{1}\left(\int_{y=0}^{1-x}\left(\int_{z=0}^{1-x-y} dz\right) dy\right) dx$$

$$= \int_{x=0}^{1} dx \int_{y=0}^{1-x} (1-x-y)\, dy = \frac{1}{6}$$

Could compute centre of mass of V (constant density $\rho = 1$)

centre of mass $\underline{X} = \frac{1}{M}\int_V \rho\, \underline{x}\, dV = \frac{1}{4}\begin{pmatrix}1\\1\\1\end{pmatrix}$ (check)

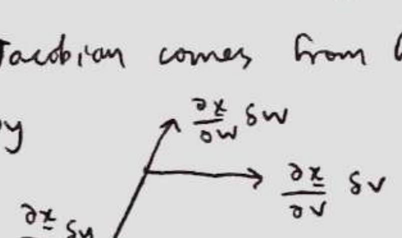
<u>Proposition</u> Let $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ be a continuously differentiable bijection with continuously differentiable inverse mapping volume V' to volume V. Then

$$\iiint_V f(x,y,z)\, dx\, dy\, dz = \iiint_{V'} f(x(u,v,w), y(u,v,w), z(u,v,w))\, |J|\, du\, dv\, dw$$

where $J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\left(\frac{\partial \underline{x}}{\partial u} \middle| \frac{\partial \underline{x}}{\partial v} \middle| \frac{\partial \underline{x}}{\partial w}\right)$, $\underline{x} = \begin{pmatrix} x(u,v,w) \\ \vdots \\ z(u,v,w)\end{pmatrix}$

in short: $dx\, dy\, dz = |J|\, du\, dv\, dw$

Jacobian comes from fact that volume of a parallelepiped generated by



is $\det\left(\frac{\partial \underline{x}}{\partial u} \middle| \frac{\partial \underline{x}}{\partial v} \middle| \frac{\partial \underline{x}}{\partial w}\right) \delta u\, \delta v\, \delta w$

The rest is similar to 2D case.

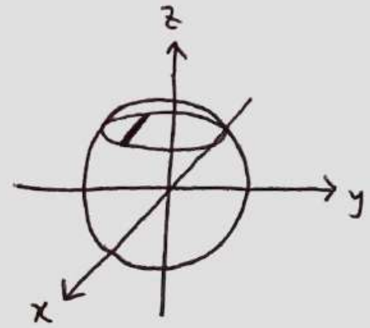
Example

Find in cylindrical polars $(u, v, w) = (\rho, \phi, z)$ $\quad dV = \rho \, d\rho \, d\phi \, dz$

In spherical polars $(u, v, w) = (r, \theta, \phi)$ $\quad dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$|J| = \rho$ (cylindrical) $\qquad |J| = r^2 \sin\theta$ (spherical)

Calculate volume of ball of radius R:



$$V = \{ (x, y, z) : x^2 + y^2 + z^2 \leq R^2 \}$$

$$\int_V dV = \int_{z=-R}^{z=R} \left(\int_{y=-\sqrt{R^2-z^2}}^{y=\sqrt{R^2-z^2}} \left(\int_{x=-\sqrt{R^2-z^2-y^2}}^{\sqrt{R^2-z^2-y^2}} dx \right) dy \right) dz$$

$$= \int_{z=-R}^{z=R} \left(\int_{y=-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} 2\sqrt{R^2-z^2-y^2} \, dy \right) dz$$

$$= \int_{z=-R}^{R} \left[y\sqrt{R^2-z^2-y^2} + (R^2-z^2) \tan^{-1}\left[\frac{y}{\sqrt{R^2-z^2-y^2}} \right]_{y=-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \right]$$

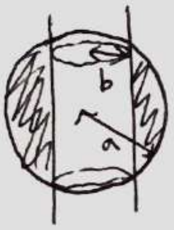
$$= \int_{-R}^{R} \pi (R^2 - z^2) \, dz = \underline{\frac{4\pi R^3}{3}}$$

Alternatively use spherical polars

$$V' = \{ (r, \theta, \phi) : 0 \leq r \leq R, \ 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi \}$$

$$\text{Volume} = \int_{\phi=0}^{2\pi} \left(\int_{\theta=0}^{\pi} \left(\int_{r=0}^{R} r^2 \sin\theta \, dr \right) d\theta \right) d\phi = \int_{\theta=0}^{\pi} \frac{2\pi R^3}{3} \sin\theta \, d\theta = \underline{\frac{4\pi R^3}{3}}$$

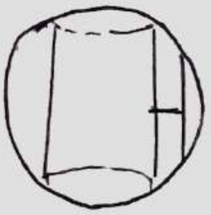
Example ball radius a with cylinder of radius $b < a$ removed



Symmetry about z-axis: use cylindrical polars

$$V = \{ (x, y, z) : x^2 + y^2 + z^2 \leq a^2, \ x^2 + y^2 \geq b^2 \}$$

Cylindrical: $V' = \{ (\rho, \phi, z) : b \leq \rho \leq a, \ 0 \leq z^2 + \rho^2 \leq a^2,$
$0 \leq \phi \leq 2\pi \}$



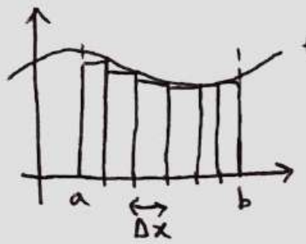
$$\int_V dV = \int_{\rho=b}^{a} \left(\int_{\phi=0}^{2\pi} \left(\int_{z=-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}} dz \right) d\phi \right) \rho \, d\rho \qquad |J| = \rho$$

$$= 2\pi \int_b^a 2\rho \sqrt{a^2-\rho^2} \, d\rho$$

$$= \underline{\frac{4\pi}{3} \left(a^2 - b^2\right)^{3/2}}$$

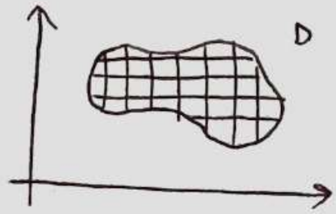
(note as $b \to 0$, approaches $\frac{4\pi a^3}{3}$)

<u>(Additional - not for course)</u>



Riemannian ← integration

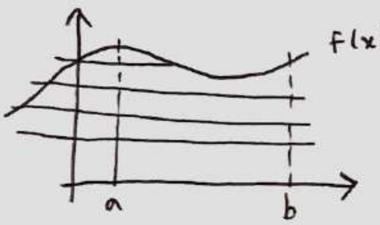
$\Delta x \to 0$



$\int_D f \, dA$

consider $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ not Riemannian integrable

<u>Lebesgue</u> :



split up range of f

consider $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

countable

uncountable

then using Lebesgue integration

$$\int_0^1 f(x) \, dx = 0$$

(as you might expect from countability)

(see Part II Prob & Measure)

Vector Calculus - Lecture 8

3.5 Integration Over Surfaces

A 2D surface in $\mathbb{R}^3$ can be defined implicitly using function $f: \mathbb{R}^3 \to \mathbb{R}$: $S = \{ \underline{x} \in \mathbb{R}^3 : f(\underline{x}) = 0 \}$ 2 DoF

Normal to S at $\underline{x}$ is parallel to $\nabla f(\underline{x})$. Call surface <u>regular</u> if $\nabla f(x) \neq 0 \quad \forall \underline{x} \in S$.

<u>Example</u> $S = \{ (x, y, z) : x^2 + y^2 + z^2 - 1 = 0 \}$ so

$$\nabla f(\underline{x}) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 2\underline{x} \quad \text{normal to } S \text{ at point } \underline{x}$$

Some surfaces have a boundary e.g.

$$S = \{ (x, y, z) : x^2 + y^2 + z^2 = 1, \ z \geq 0 \}$$

Label boundary by ∂S : $\partial S = \{ (x, y, z) : x^2 + y^2 = 1, z = 0 \}$

In this course S either has no boundary or has boundary made of piecewise smooth curves.

Surface S has no boundary: say S is a <u>closed</u> surface

It is often useful to parametrise a surface using some coordinates (u, v)

$S = \{ \underline{x} : \underline{x}(u, v), (u, v) \in D \}$ D : region in (u, v) plane

<u>Example</u>

For hemisphere use spherical polars

$$S = \left\{ \underline{x} = \underline{x}(\theta, \phi) = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix}, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq 2\pi \right\}$$

Call parametrisation of S regular if $\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \neq 0$ on S

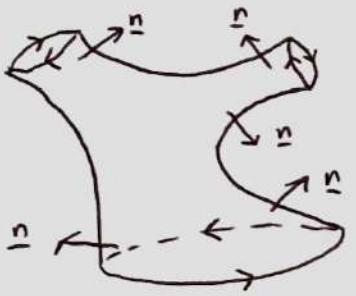
In this case we can define normal

$$\underline{n} = \frac{\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v}}{\left| \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right|}$$

This $\underline{n}$ varies smoothly wrt (u, v).

Normal choice gives us a way of orientating the boundary.

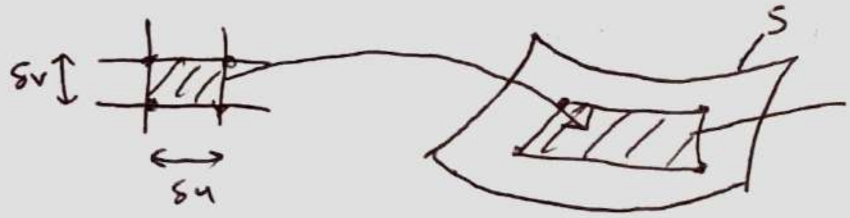
Boundary ∂S: convention that normal vectors in your immediate vicinity should be on your left as you traverse ∂S (if normals are chosen consistently)



How to find area of a surface

$$S = \{ \underline{x} = \underline{x}(u,v),\ (u,v) \in D \} \ ?$$

Might think it's $\iint_D du\, dv$ – $\boxed{\text{wrong}}$



not necessarily $\delta u \delta v$

small change $u \mapsto u + \delta u$ produces

$$\underline{x}(u+\delta u, v) - \underline{x}(u,v) \approx \frac{\partial \underline{x}}{\partial u}\delta u \quad \text{to 1st order}$$

$$\underline{x}(u, v+\delta v) - \underline{x}(u,v) \approx \frac{\partial \underline{x}}{\partial v}\delta v \quad \text{for } v \mapsto v + \delta v$$

so patch of area $\delta u \delta v$ in D corresponds (to 1st order) to a parallelogram of area

$$\left(\frac{\partial \underline{x}}{\partial v}\delta v,\ \frac{\partial \underline{x}}{\partial u}\delta u \right) = \left| \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right| \delta u \, \delta v$$

This leads us to define scalar area element and vector area element.

$$dS = \left| \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right| du\, dv \qquad d\underline{S} = \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial u}\, du\, dv = \underline{n}\, dS$$

Gives area of S: $\text{area}(S) = \int_S dS = \iint_D \left| \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right| du\, dv$

---

and $$\int_S f\, dS = \iint_D f(\underline{x}(u,v)) \left| \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right| du\, dv$$

Example hemisphere, radius R

$$S = \left\{ \underline{x}(\theta, \phi) = \begin{pmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{pmatrix} = R\underline{e}_r,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi < 2\pi \right\}$$

So $\quad \dfrac{\partial \underline{x}}{\partial \theta} = \begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix} = R\underline{e}_\theta \qquad \dfrac{\partial \underline{x}}{\partial \phi} = \begin{pmatrix} -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi \\ 0 \end{pmatrix} = R\sin\theta\,\underline{e}_\phi$

$$\Rightarrow dS = R^2\sin\theta\,|\underline{e}_\theta \times \underline{e}_\phi|\,d\theta\,d\phi = R^2\sin\theta\,d\theta\,d\phi$$

$$\text{Area}(S) = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} R^2\sin\theta\,d\phi\,d\theta = \underline{2\pi R^2}$$

<u>Example</u> Fluid velocity $\underline{u} = \underline{u}(\underline{x})$

Given S, how to calculate how much fluid passes through per unit time?

On small patch δS on S, fluid passing through would be

$(\underline{u}\cdot\delta\underline{S})\,\delta t$ in time δt so amount of fluid passing over S in δt

is $\delta t \underbrace{\int_S \underline{u}\cdot d\underline{S}}$ rate at which fluid passes through surface S

Called flux integrals

Are these surface integrals dependent on parametrisation?

$\underline{x} = \underline{x}(u,v) \qquad \underline{x} = \tilde{\underline{x}}(\tilde{\underline{u}}, \tilde{\underline{v}}) \qquad (u,v)\in D, \ (\tilde{u},\tilde{v})\in\tilde{D}$

Must have relationship $\quad \underline{x}(u,v) = \tilde{\underline{x}}(\tilde{u}(u,v), \tilde{v}(u,v))$

For appropriate $\tilde{u}(u,v)$, $\tilde{v}(u,v)$. Chain rule:

$$\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} = \left(\frac{\partial \tilde{\underline{x}}}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial u} + \frac{\partial \tilde{\underline{x}}}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial u}\right) \times \left(\frac{\partial \tilde{\underline{x}}}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial v} + \frac{\partial \tilde{\underline{x}}}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial v}\right)$$

$$= \left(\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial v} - \frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{v}}{\partial u}\right)\frac{\partial \tilde{\underline{x}}}{\partial \tilde{u}} \times \frac{\partial \tilde{\underline{x}}}{\partial \tilde{v}} = \underline{\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}\frac{\partial \tilde{\underline{x}}}{\partial \tilde{u}} \times \frac{\partial \tilde{\underline{x}}}{\partial \tilde{v}}}$$

$$\int_S f\,dS = \iint_{\tilde{D}} f(\tilde{\underline{x}}(\tilde{u},\tilde{v}))\left|\frac{\partial \tilde{\underline{x}}}{\partial \tilde{u}} \times \frac{\partial \tilde{\underline{x}}}{\partial \tilde{v}}\right| d\tilde{u}\,d\tilde{v}$$

change of vars $\quad \tilde{u} = \tilde{u}(u,v), \qquad \tilde{v} = \tilde{v}(u,v)$

$$= \iint_D f(\underline{x}(u,v))\left|\frac{\partial \tilde{\underline{x}}}{\partial \tilde{u}} \times \frac{\partial \tilde{\underline{x}}}{\partial \tilde{v}}\right|\left|\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}\right| du\,dv$$

$$= \iint_D f(\underline{x}(u,v))\left|\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v}\right| du\,dv$$

: <u>independent</u> of parametrisation.

Vector Calculus - Lecture 9

10/2/21

4 Divergence, Curl and Laplacians

4.1 Definitions

Seen gradient operator ∇, acts on scalar functions $f : \mathbb{R}^3 \to \mathbb{R}$

In Cartesian $\nabla = \underline{e}_i \frac{\partial}{\partial x_i}$

For vector field $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$, define divergence of $\underline{F}$ by

$\text{div}(\underline{F}) = \nabla \cdot \underline{F}$ - in Cartesian: $\nabla \cdot \underline{F} = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right) \cdot (F_j \underline{e}_j)$

$$= \underline{e}_i \cdot \left[\frac{\partial}{\partial x_i}(F_j \underline{e}_j)\right] = (\underline{e}_i \cdot \underline{e}_j) \frac{\partial F_j}{\partial x_i}$$

$$= \delta_{ij} \frac{\partial F_j}{\partial x_i} = \frac{\partial F_i}{\partial x_i}$$

so in Cartesian $\boxed{\nabla \cdot \underline{F} = \frac{\partial F_i}{\partial x_i}}$

Note divergence of vector field is a scalar field

For vector field $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$, define curl of $\underline{F}$ by

$\text{curl}(\underline{F}) = \nabla \times \underline{F}$

in Cartesian $\nabla \times \underline{F} = \left(\underline{e}_j \frac{\partial}{\partial x_j}\right) \times (F_k \underline{e}_k) = \underline{e}_j \times \left[\frac{\partial}{\partial x_j}(F_k \underline{e}_k)\right]$

$$= (\underline{e}_j \times \underline{e}_k) \frac{\partial F_k}{\partial x_j} = \varepsilon_{ijk} \underline{e}_i \frac{\partial F_k}{\partial x_j} = \left(\varepsilon_{ijk} \frac{\partial F_k}{\partial x_j}\right) \underline{e}_i$$

in Cartesian $\boxed{[\nabla \times F]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k}$ (vector field).

In terms of a "formal" determinant: $\nabla \times \underline{F} = \det \begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ F_1 & F_2 & F_3 \end{pmatrix}$

Finally, for scalar field $f : \mathbb{R}^3 \to \mathbb{R}$, define the Laplacian of f

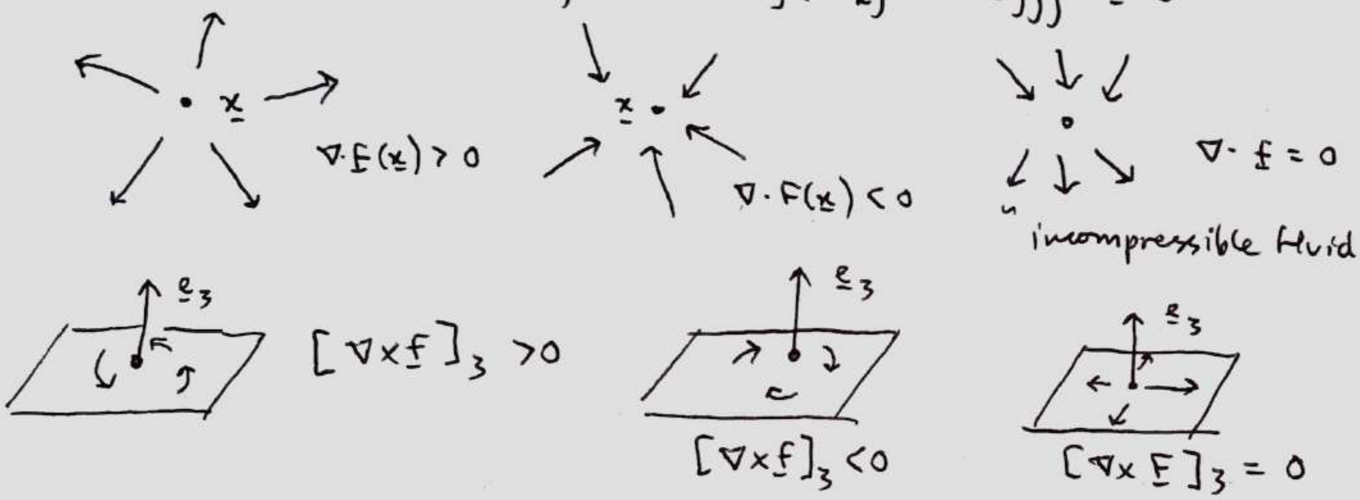
$\nabla^2 f = \nabla \cdot \nabla f$ $(= \text{div}(\text{grad } f))$

In Cartesian $[\nabla f]_i = \frac{\partial f}{\partial x_i}$ so $\boxed{\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i}}$

Example

$\underline{F}(\underline{x}) = \underline{x}$ then in Cartesian $\nabla \cdot \underline{F} = \frac{\partial}{\partial x_i} x_i = \delta_{ii} = 3$

$$[\nabla \times F]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} x_k = \varepsilon_{ijk}\delta_{kj} = \varepsilon_{ijj} = 0$$



Proposition For f, g scalar fields and $\underline{F}, \underline{G}$ vector fields:

See lecture notes (official)

$$\nabla \times (\underline{F} \times \underline{G}) = \underline{F}(\nabla \cdot \underline{G}) - \underline{G}(\underbrace{\nabla \cdot \underline{F}}_{\text{scalar function}}) + (\underbrace{\underline{G} \cdot \nabla}_{\text{differential operator}})\underline{F} - (\underline{F} \cdot \nabla)\underline{G}$$

Proof Note

$$[(\underline{F} \cdot \nabla)\underline{G}]_i = \left(F_j \frac{\partial}{\partial x_j}\right) G_i = F_j \frac{\partial G_i}{\partial x_j}$$

by defns LHS is $[\nabla \times (\underline{F} \times \underline{G})]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\underline{F} \times \underline{G})_k$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_j}\left(\varepsilon_{klm} F_l G_m\right) = \varepsilon_{ijk}\varepsilon_{klm}\left[F_l \frac{\partial G_m}{\partial x_j} + G_m \frac{\partial F_l}{\partial x_j}\right]$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\left[F_l \frac{\partial G_m}{\partial x_j} + G_m \frac{\partial F_l}{\partial x_j}\right]$$

$$= F_i \frac{\partial G_j}{\partial x_j} - F_j \frac{\partial G_i}{\partial x_j} + G_j \frac{\partial F_i}{\partial x_j} - G_i \frac{\partial F_j}{\partial x_j}$$

$$= [\underline{F}(\nabla \cdot \underline{G})]_i - [(\underline{F} \cdot \nabla)\underline{G}]_i + [(\underline{G} \cdot \nabla)\underline{F}]_i - [(\nabla \cdot \underline{F})\underline{G}]_i$$

as required. $\square$

Other similar propositions with f, g scalar, $\underline{F}, \underline{G}$ vector fields:

$$\nabla(fg) = (\nabla f)g + (\nabla g)f$$

$$\nabla \cdot (f\underline{F}) = (\nabla f) \cdot \underline{F} + f(\nabla \cdot \underline{F})$$

$$\nabla \times (f\underline{F}) = (\nabla f) \times \underline{F} + f(\nabla \times \underline{F})$$

$$\nabla(\underline{F}\cdot\underline{G}) = \underline{F}\times(\nabla\times\underline{G}) + \underline{G}\times(\nabla\times\underline{F}) + (\underline{F}\cdot\nabla)\underline{G} + (\underline{G}\cdot\nabla)\underline{F}$$

$$\nabla\cdot(\underline{F}\times\underline{G}) = (\nabla\times\underline{F})\cdot\underline{G} - \underline{F}\cdot(\nabla\times\underline{G})$$

These are proved similarly - left as exercises, use suffix notation

These identities hold in any OCC system: easier to do via Cartesian

For general OCC, divergence is defined by the same formula $\nabla\cdot\underline{F}$,

i.e. $\left(\underline{e}_u \frac{1}{h_u}\frac{\partial}{\partial u} + \underline{e}_v \frac{1}{h_v}\frac{\partial}{\partial v} + \underline{e}_w \frac{1}{h_w}\frac{\partial}{\partial w}\right)\cdot\left(F_u\underline{e}_u + F_v\underline{e}_v + F_w\underline{e}_w\right)$

giving terms like $\left(\underline{e}_u \frac{1}{h_u}\frac{\partial}{\partial u}\right)\cdot(F_v\underline{e}_v) = \frac{1}{h_u}\underline{e}_u\cdot\left(\frac{\partial}{\partial u}(F_v\underline{e}_v)\right)$

$$= \frac{1}{h_u}\underline{e}_u\cdot\left[\cancel{\frac{\partial F_v}{\partial u}\underline{e}_v} + F_v\frac{\partial \underline{e}_v}{\partial u}\right] = \frac{F_v}{h_u}\left(\underline{e}_u\cdot\frac{\partial\underline{e}_v}{\partial u}\right)$$

Gets quite messy as basis vectors depend on u, v, w. State results:

$$\nabla\cdot\underline{F} = \frac{1}{h_uh_vh_w}\left[\frac{\partial}{\partial u}(h_vh_wF_u) + \frac{\partial}{\partial v}(h_uh_wF_v) + \frac{\partial}{\partial w}(h_uh_vF_w)\right]$$

$$\nabla\times\underline{F} = \frac{1}{h_vh_w}\left[\frac{\partial}{\partial v}(h_wF_w) - \frac{\partial}{\partial w}(h_vF_v)\right]\underline{e}_u + \text{cyclic perms}$$

$$= \frac{1}{h_uh_vh_w}\det\begin{pmatrix} h_u\underline{e}_u & h_v\underline{e}_v & h_w\underline{e}_w \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ h_uF_u & h_vF_v & h_wF_w \end{pmatrix}$$

and $$\nabla^2 f = \frac{1}{h_uh_vh_w}\left[\frac{\partial}{\partial u}\left(\frac{h_vh_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_uh_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_uh_v}{h_w}\frac{\partial f}{\partial w}\right)\right]$$

since $[\nabla f]_u = \frac{1}{h_u}\frac{\partial f}{\partial u}$ etc.

cylindrical polars:

$$\nabla^2 f = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial f}{\partial\phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial f}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$$

For Laplacian of vector field, might guess $\nabla\cdot(\nabla\underline{F})$ but haven't defined grad of vector field yet. Would make sense in Cartesian to do $\nabla^2\underline{F} = \nabla^2(F_i\underline{e}_i) = (\nabla^2 F_i)\underline{e}_i$

Using suffix notation, can show we need $\nabla^2 \underline{F} = \nabla(\nabla\cdot\underline{F}) - \nabla\times(\nabla\times\underline{F})$

i.e. $\left[\nabla(\nabla\cdot\underline{F}) - \nabla\times(\nabla\times\underline{F})\right]_i = \frac{\partial^2 F_i}{\partial x_j \partial x_j} = \nabla^2 F_i \qquad (\dagger)$

RHS of $(\dagger)$ is well defined in all OCCs so define it this way.

$$\therefore \quad \underline{\nabla^2 \underline{F} = \nabla(\nabla\cdot\underline{F}) - \nabla\times(\nabla\times\underline{F})}$$

Vector Calculus - Lecture 10

12/2/21

$A_{jk} S_{jk} = 0$

4.2 Relations between div, grad and curl

Proposition For scalar field f and vector field $\underline{F}$ we have:

$\nabla \times \nabla f = 0$ and $\nabla \cdot (\nabla \times \underline{F}) = 0$

Proof $[\nabla \times \nabla f]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right) = \varepsilon_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k} = 0$

~~anti~~ antisymmetric in j, k symmetric in j, k

$\nabla \cdot (\nabla \times \underline{F}) = \frac{\partial}{\partial x_i} \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k = \varepsilon_{ijk} \frac{\partial^2 F_k}{\partial x_i \partial x_j} = 0$ $\square$

Recall $\underline{F}$ was conservative if $\underline{F} = \nabla f$

Say $\underline{F}$ is irrotational if $\nabla \times \underline{F} = 0$.

So from proposition $\underline{F}$ conservative $\Rightarrow$ $\underline{F}$ irrotational.

Reverse implication is true if domain of $\underline{F}$ is simply (1)- connected

Similarly, if there exists a vector potential for $\underline{F}$ i.e. $\underline{F} = \nabla \times \underline{A}$ then $\nabla \cdot \underline{F} = 0$. Here $\underline{A}$ is called vector potential for $\underline{F}$.

When $\nabla \cdot \underline{F} = 0$ we say $\underline{F}$ is solenoidal.

Hence existence of vector potential for $\underline{F}$ $\Rightarrow$ $\underline{F}$ solenoidal.

Reverse implication true if domain of $\underline{F}$ is 2-connected. Say $\Omega \subset \mathbb{R}^3$ is 2-connected if it is 1-connected and every sphere in Ω can be continuously shrunk to any point in Ω.

$\mathbb{R}^3$ is 2-connected, $\mathbb{R}^3 \setminus \{0\}$ is 1- but not 2-connected

4.3 Topology via calculus (non-examinable)

For $\Omega \subset \mathbb{R}^3$, consider $\underline{F} : \Omega \to \mathbb{R}^3$. We know

$\underline{F} : \Omega \to \mathbb{R}^3$ irrotational and Ω 1-connected $\Rightarrow$ $\underline{F}$ conservative

Example Is domain $\Omega = \mathbb{R}^3 \setminus \{z\text{-axis}\}$ simply connected?

Consider $\underline{F} = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ $\underline{F} : \Omega \to \mathbb{R}^3$

well defined on Ω and can check $\nabla \times \underline{F} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$: irrotational VF

If Ω is 1-connected then from $(*)$: $\underline{F} = \nabla f$ then for any closed loop C in Ω $\oint_C \underline{F} \cdot d\underline{x} = \oint_C \nabla f \cdot d\underline{x} = 0$

Take $C : [0, 2\pi] \ni t \mapsto \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$

then $\oint_C \underline{F} \cdot d\underline{x} = \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt = 2\pi \neq 0$ #

Ω not simply connected

5 Integral theorems

5.1 Green's Theorem

Theorem If $P = P(x,y)$, $Q = Q(x,y)$ continuously differentiable functions on $A \cup \partial A$ (boundary) and ∂A is piecewise smooth, then

$$\oint_{\partial A} P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \qquad (\dagger)$$

(orientation of ∂A is s.t. A lies to your left as you traverse it)

Note that it is easy to establish this for a rectangle

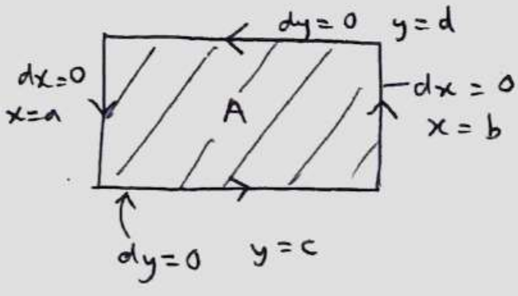
$A = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$

In this case : RHS of $(\dagger)$ is

$$\int_c^d \left(\int_a^b \frac{\partial Q}{\partial x}\,dx \right) dy - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y}\,dy \right) dx$$

$$= \int_c^d \left[Q(b,y) - Q(a,y) \right] dy + \int_a^b \left[P(x,c) - P(x,d) \right] dx$$

$$= \oint_{\partial A} P\,dx + Q\,dy$$



<u>Example</u> Let $P = -\frac{1}{2} y$, $Q = \frac{1}{2} x$

then $\text{area}(A) = \iint_A dx\,dy = \iint_A \Big(\underbrace{\tfrac{1}{2}}_{\frac{\partial Q}{\partial x}} + \underbrace{\tfrac{1}{2}}_{-\frac{\partial P}{\partial y}} \Big)\, dx\,dy$

$$= \frac{1}{2} \oint_{\partial A} x\,dy - y\,dx$$

If A is ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ then ∂A

$[0, 2\pi] \ni t \mapsto \begin{pmatrix} a\cos t \\ b \sin t \end{pmatrix}$ gives area(A) to be

$$\text{area}(A) = \frac{1}{2} \int_0^{2\pi} ab\cos^2 t + ab\sin^2 t \; dt = \underline{\pi ab}$$

5.2 Stokes' Theorem

Proposition If $\underline{F} = \underline{F}(\underline{x})$ is a continuously differentiable vector field and S is an orientable piecewise regular surface with piecewise smooth boundary ∂S, then

$$\int_S (\nabla \times \underline{F}) \cdot d\underline{S} = \oint_{\partial S} \underline{F} \cdot d\underline{x}$$

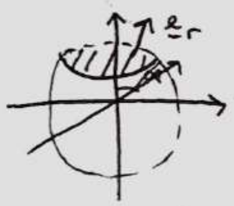
FTC: $\int_I f' dx = f|_{\partial I} = \int_{\partial I} f$ similar

"evaluate at boundary"

(orientable: consistent choice of normal at each point – "two sided")

Example Let S be a cap of a sphere

$$S = \left\{ \underline{x}(\theta, \phi) = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix} \equiv \underline{e}_r, \quad 0 \leq \theta \leq \alpha, \quad 0 \leq \phi \leq 2\pi \right\}$$



$$\underline{F} = \begin{pmatrix} -x^2 y \\ 0 \\ 0 \end{pmatrix} \quad \nabla \times \underline{F} = \begin{pmatrix} 0 \\ 0 \\ x^2 \end{pmatrix}$$

On S: $d\underline{S} = \frac{\partial \underline{x}}{\partial \theta} \times \frac{\partial \underline{x}}{\partial \phi} \, d\theta \, d\phi = \underline{e}_\theta \times (\sin\theta \, \underline{e}_\phi) \, d\theta \, d\phi$

(see OCC stuff)

$= \underline{e}_r \sin\theta \, d\theta \, d\phi$

Note that since $(x^2 \underline{e}_z \cdot \underline{e}_r) = (\sin\theta \cos\phi)^2 \cos\theta$

Then $\int_S \nabla \times \underline{F} \cdot d\underline{S} = \int_{\phi=0}^{2\pi} \left(\int_{\theta=0}^{\alpha} \cos^2\phi \sin^3\theta \cos\theta \, d\theta \right) d\phi$

$= \underline{\frac{\pi}{4} \sin^4 \alpha}$ surface integral

∂S is described by $[0, 2\pi] \ni t \mapsto \begin{pmatrix} \sin\alpha \cos t \\ \sin\alpha \sin t \\ \cos\alpha \end{pmatrix}$

$\Rightarrow d\underline{x} = \frac{d\underline{x}}{dt} dt = \sin\alpha \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt$

Then $\oint_{\partial S} \underline{F} \cdot d\underline{x} = \sin^4\alpha \int_0^{2\pi} (-\cos^2 t \sin t)(-\sin t) \, dt$

$= \underline{\frac{\pi}{4} \sin^4 \alpha}$ as expected.

<u>Example</u> If S is an orientable <u>closed</u> surface, $\underline{F}$ continuously differentiable, then

$$\int_S \nabla \times \underline{F} \cdot d\underline{S} = 0. \qquad \text{(boundary is the empty set)}$$

<u>Proposition</u> If $\underline{F}$ is continuously differentiable and for any loop C

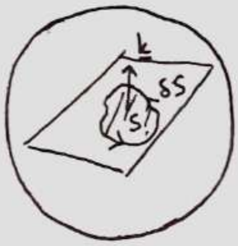
$$\oint_C \underline{F} \cdot d\underline{x} = 0, \quad \text{then } \nabla \times \underline{F} = 0$$

($\Rightarrow$ $\underline{F}$ irrotational iff $\underline{F}$ has zero circulation around any closed loop).

<u>Proof</u> ($\Rightarrow$: $\Leftarrow$ follows quickly):

Suppose false. Then $\exists$ unit vector $\underline{k}$ with $\overbrace{\underline{k} \cdot \nabla \times \underline{F}(\underline{x}_0)}^{=\varepsilon} > 0$ for some $\underline{x}_0$. By continuity for $\delta > 0$ sufficiently small so that by continuity $\underline{k} \cdot \nabla \times \underline{F}(\underline{x}) > \frac{1}{2}\varepsilon$ for $|\underline{x} - \underline{x}_0| < \delta$

Take loop in ball $\{\underline{x} : |\underline{x} - x_0| < \delta\}$ that lies entirely in a plane with normal $\underline{k}$

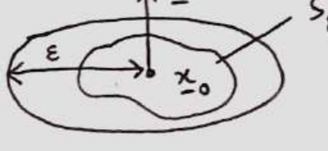


$$0 = \oint_{\partial S} \underline{F} \cdot d\underline{x} = \int_S \nabla \times \underline{F} \cdot \underline{k} \; dS \qquad \text{(by Stokes' thm)}$$

$$> \frac{1}{2}\varepsilon \int dS > 0 \qquad \#$$

$\square$

<u>Example</u> Let S_ε denote a region contained inside a disc with $r = \varepsilon$, centred at $\underline{x}_0$ with normal $\underline{k}$



add and subtract term

$$\int_{S_\varepsilon} \nabla \times \underline{F} \cdot d\underline{S} = \int_{S_\varepsilon} \left(\nabla \times \underline{F}(\underline{x}) - \nabla \times \underline{F}(\underline{x}_0) \right) dS + \underbrace{\int_{S_\varepsilon} \underbrace{\nabla \times \underline{F}(\underline{x}_0) \cdot \underline{k}}_{\text{constant}} \, dS}_{\text{area } S_\varepsilon \; \underline{k} \cdot \nabla \times \underline{F}(\underline{x}_0)}$$

$$= \text{area}(S_\varepsilon) \, \underline{k} \cdot \nabla \times \underline{F}(\underline{x}_0) + \underbrace{\int_{S_\varepsilon} \left(\nabla \times \underline{F}(\underline{x}_0) - \nabla \times \underline{F}(\underline{x}) \right) \cdot d\underline{S}}_{o(\text{area}(S_\varepsilon))}$$

Aside:

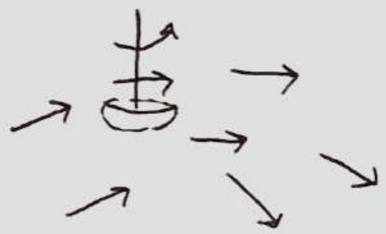
Cauchy-Schwarz

$$\left| \int_{S_\varepsilon} \left(\nabla \times \underline{F}(\underline{x}) - \nabla \times \underline{F}(x_0) \right) \cdot \underline{k} \, dS \right| \leq \int_{S_\varepsilon} \left| \nabla \times F(\underline{x}) - \nabla \times \underline{F}(\underline{x}_0) \right| dS$$

$$\leq \underbrace{\left(\max_{|\underline{x} - \underline{x}_0| < \varepsilon} \left| \nabla \times \underline{F}(\underline{x}) - \nabla \times F(\underline{x}_0) \right| \right)}_{o(1)} \text{ area } S_\varepsilon = o(\text{area}(S_\varepsilon))$$

$$\int_{S_\varepsilon} \nabla \times \underline{F} \cdot d\underline{S} = \underline{k} \cdot \nabla \times \underline{F}(\underline{x}_0) \text{ area}(S_\varepsilon) + o(\text{area}(S_\varepsilon))$$

$$\Rightarrow \underline{k} \cdot \nabla \times \underline{F}(\underline{x}_0) = \lim_{\varepsilon \downarrow 0} \frac{1}{\text{area}(S_\varepsilon)} \oint_{\partial S_\varepsilon} \underline{F} \cdot d\underline{x}$$

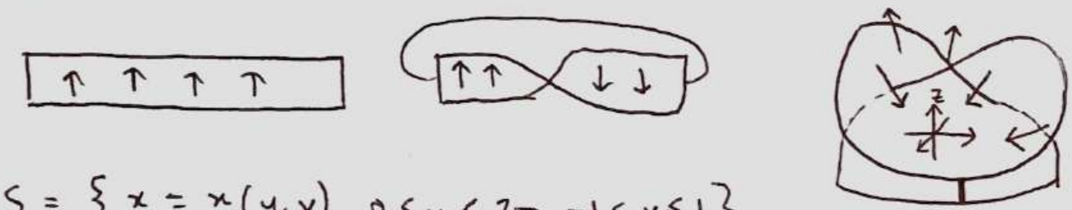


"infinitesimal circulation around k-axis per unit area"

is what $\nabla \times \underline{F}$ is measuring

Non-examinable

5.3 Möbius strips and Stokes' theorem



not orientable

Stokes' theorem won't work

$$S = \{ \underline{x} = \underline{x}(u,v) \quad 0 \leq u < 2\pi \quad -1 \leq v \leq 1 \}$$

$$\underline{x}(u,v) = \begin{pmatrix} \left(1 + \frac{v}{2} \cos\left(\frac{u}{2}\right)\right) \cos u \\ \left(1 + \frac{v}{2} \cos\left(\frac{u}{2}\right)\right) \sin u \\ \frac{v}{2} \sin \frac{u}{2} \end{pmatrix} \qquad \underline{F} = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

$\partial \times \underline{F} = 0 \qquad \oint_{\partial S} \underline{F} \cdot d\underline{x} = \underline{4\pi \neq 0}$ (check!)

$v = 1$

$0 \leq u < 4\pi$

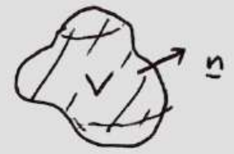
Vector Calculus – Lecture 12

17/2/21

5.4 Divergence Theorem (Gauss's theorem)

<u>Proposition</u> If $\underline{F} = \underline{F}(\underline{x})$ is a cont. diff. vector field and V is a volume with piecewise regular boundary ∂V then

$$\int_V \nabla \cdot \underline{F} \, dV = \int_{\partial V} \underline{F} \cdot d\underline{S}$$



where normal to ∂V points <u>out</u> of V.

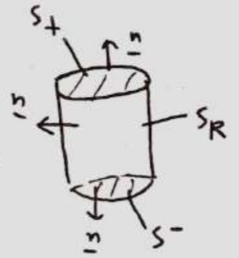
(2D version):

If $\underline{F} = \underline{F}(\underline{x})$ is cont. diff and $D \subset \mathbb{R}^2$ is a planar region with piecewise smooth boundary ∂D then

$$\int_D \nabla \cdot \underline{F} \, dA = \oint_{\partial D} \underline{F} \cdot \underline{n} \; \underset{\uparrow \text{ arc length}}{ds}$$

again $\underline{n}$ points out of D

<u>Example</u> V cylinder in cylindrical polars (ρ, ϕ, z)



$$V = \{ (\rho, \phi, z) : 0 \leq \rho \leq R, \ -h \leq z \leq h, \ 0 \leq \phi < 2\pi \}$$

Consider $\underline{F} = \underline{x}$. So $\nabla \cdot \underline{F} = 3$

$$\int_V \nabla \cdot \underline{F} \, dV = 3 \int_V dV = 6\pi R^2 h$$

Alternatively use divergence theorem. ∂V is made from

$$S_R = \{ (\rho, \phi, z) : \rho = R, \ -h \leq z \leq h, \ 0 \leq \phi < 2\pi \}$$

$$S_\pm = \{ (\rho, \phi, z) : 0 \leq \rho \leq R, \ z = \pm h, \ 0 \leq \phi < 2\pi \}$$

On S_R $d\underline{S} = \underline{e}_\rho R \, d\phi \, dz$ and $\overset{\rho \underline{e}_\rho + z \underline{e}_z}{\underline{x}} \cdot \underline{e}_\rho = R$ so

$$\int_{S_R} \underline{F} \cdot d\underline{S} = \int_{z=-h}^{h} \int_{\phi=0}^{2\pi} R^2 \, d\phi \, dz = 4\pi R^2 h$$

On $S_\pm$ find $d\underline{S} = \pm \underline{e}_z \, \rho \, d\rho \, d\phi$ and $\underline{x} \cdot \underline{e}_z = \pm h$ so

$$\int_{S_\pm} \underline{F}\cdot d\underline{S} = \int_{\phi=0}^{2\pi}\left(\int_{\rho=0}^{R} h\rho\, d\rho\right) d\phi = \pi R^2 h$$

so $$\int_{\partial V} \underline{F}\cdot d\underline{S} = \left(\int_{S_R} + \int_{S_+} + \int_{S_-}\right)\underline{F}\cdot d\underline{S} = 4\pi R^2 h + \pi R^2 h + \pi R^2 h$$
$$= \underline{6\pi R^2 h} \quad \text{as expected}$$

Proposition

If $\underline{F} = \underline{F}(\underline{x})$ is C.D and for every closed surface S

$\int_S \underline{F}\cdot d\underline{S} = 0$, then $\nabla\cdot\underline{F} = 0$.

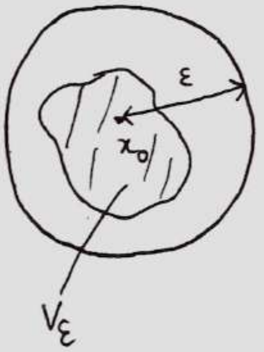
Proof suppose false. So $\nabla\cdot\underline{F}(\underline{x}_0) = \varepsilon > 0$. By continuity, for $\delta > 0$ sufficiently small, have $\nabla\cdot\underline{F}(\underline{x}) > \frac{1}{2}\varepsilon \quad |\underline{x}-\underline{x}_0| < \delta$

Choose volume V inside ball $|\underline{x}-\underline{x}_0| < \delta$

Then by assumption $$0 = \int_{\partial V}\underline{F}\cdot d\underline{S} = \int_V \nabla\cdot\underline{F}\, dV > \frac{1}{2}\varepsilon\int_V dV > 0$$ ※

Conclude that if vector field has zero net flux through any closed surface, then it is solenoidal.

Example Let V_ε be a volume in $\mathbb{R}^3$ contained inside a ball of radius ε centred at $\underline{x}_0$



$$\int_{V_\varepsilon} \nabla\cdot\underline{F}\, dV = \mathrm{vol}(V_\varepsilon)\,\nabla\cdot\underline{F}(\underline{x}_0) + \underbrace{\int_{V_\varepsilon}\left[\nabla\cdot\underline{F}(\underline{x}) - \nabla\cdot\underline{F}(\underline{x}_0)\right] dV}_{o(\mathrm{vol}(V_\varepsilon))}$$

similar reason to last lecture

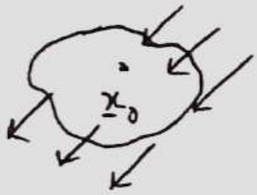
Dividing by $\mathrm{vol}(V_\varepsilon)$ taking $\varepsilon \downarrow 0$, by div thm:

$$\nabla\cdot\underline{F}(\underline{x}_0) = \lim_{\varepsilon\downarrow 0}\frac{1}{\mathrm{vol}(V_\varepsilon)}\int_{\partial V_\varepsilon}\underline{F}\cdot d\underline{S}$$

so $\nabla\cdot\underline{F}$ measures "infinitesimal flux per unit volume"

$\nabla \cdot \underline{F}(\underline{x}_0) > 0$

$\nabla \cdot \underline{F}(\underline{x}_0) < 0$



$\nabla \cdot \underline{F}(\underline{x}_0) = 0$

e.g. incompressible fluid

<u>Example</u> Many eqns in mathematical physics can be written in form

(†) $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0$ conservation laws

Suppose ρ and $|\underline{J}|$ decrease rapidly as $|\underline{x}| \to \infty$ $(\rho(\underline{x},t), \underline{J}(\underline{x},t))$

Define charge $Q = \int_{\mathbb{R}^3} \rho(\underline{x},t)\, dV$. We have conservation of charge

$$\frac{dQ}{dt} = \int_{\mathbb{R}^3} \frac{\partial \rho}{\partial t}\, dV = -\int_{\mathbb{R}^3} \nabla \cdot \underline{J}\, dV = -\lim_{R\to\infty} \int_{|\underline{x}|\le R} \nabla \cdot \underline{J}\, dV$$

$$= -\lim_{R\to\infty} \int_{|\underline{x}|=R} \underline{J} \cdot d\underline{S} = 0 \quad (|\underline{J}| \to 0 \text{ rapidly as } |\underline{x}| \to \infty)$$

(†) conservation of charge

<u>5.5</u> Noether's Theorem (non-examinable)

"For every symmetry of your laws of physics there is a corresponding conserved quantity"

Translational symmetry: $\underline{x} \mapsto \underline{x} + \underline{a}$ linear momentum conserved

Rotational symmetry: $\underline{x} \mapsto R\underline{x}$ angular momentum conserved

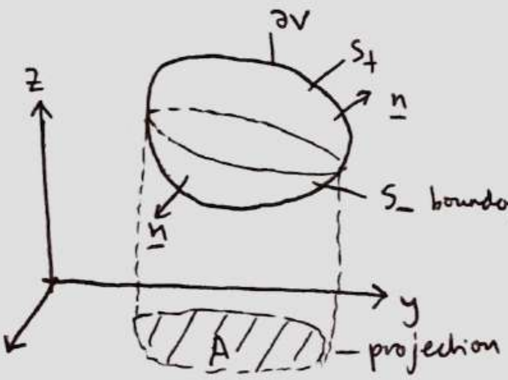
Time symmetry $t \mapsto t + \varepsilon$ energy conserved

5.6 Sketch Proofs

Proposition Divergence theorem is true: $\int_V \nabla \cdot \underline{F} \, dV = \int_{\partial V} \underline{F} \cdot d\underline{S}$

Proof Suppose $\underline{F} = F_z(x, y, z) \, \underline{e}_z$

Then div. thm says
$$\int_V \underbrace{\frac{\partial F_z}{\partial z}}_{\nabla \cdot \underline{F}} \, dV = \int_{\partial V} \underbrace{F_z \underline{e}_z}_{\underline{F}} \cdot d\underline{S} \qquad (\dagger)$$



$\partial V = S_+ \cup S_-$

Write $S_\pm = \left\{ \underline{x}(x, y) = \begin{pmatrix} x \\ y \\ g_\pm(x, y) \end{pmatrix}, (x, y) \in A \right\}$

as function of $(x, y) \in A$, region in x, y plane

Then
$$\int_V \underbrace{\frac{\partial F_z}{\partial z} \, dV}_{dx\,dy\,dz} = \iint \left[\int_{g_-(x,y)}^{g_+(x,y)} \frac{\partial F_z}{\partial z} \, dz \right] dx \, dy$$

by FTC:
$$= \iint_A \left[F_z(x, y, g_+(x, y)) - F_z(x, y, g_-(x, y)) \right] dx \, dy$$

To calculate RHS of $(\dagger)$ over $S_\pm$

$$d\underline{S} = \frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} \, dx \, dy = \begin{pmatrix} -\partial g_\pm / \partial x \\ -\partial g_\pm / \partial y \\ 1 \end{pmatrix} dx \, dy$$

Since we want $\underline{n}$ to point out of V, on $S_\pm$ we have

$$d\underline{S}\Big|_{S_\pm} = \pm \begin{pmatrix} -\partial g_\pm / \partial x \\ -\partial g_\pm / \partial y \\ 1 \end{pmatrix} dx \, dy$$

$$\Rightarrow \int_{\partial V} \underline{F} \cdot d\underline{S} = \left[\int_{S_+} + \int_{S_-} \right] F_z \underline{e}_z \cdot d\underline{S}$$

$$= \iint_A F_z(x, y, g_+(x,y))\, dx\, dy - \iint_A F_z(x, y, g_-(x,y))\, dx\, dy$$

$$= \int_V \frac{\partial F_z}{\partial z}\, dV \qquad \text{so } (\dagger) \text{ holds for this form of } \underline{F}$$

$$\int_V \frac{\partial F_x}{\partial x}\, dV = \int_{\partial V} F_x \underline{e}_x \cdot d\underline{S} \quad \text{and} \quad \int_V \frac{\partial F_y}{\partial y}\, dV = \int_{\partial V} F_y \underline{e}_y \cdot d\underline{S}$$

in exactly the same way.

Adding:

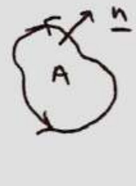
$$\int_V \left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right] dV = \int_{\partial V} \left[F_x \underline{e}_x + F_y \underline{e}_y + F_z \underline{e}_z \right] \cdot d\underline{S}$$

exactly as required. $\square$

(a) Will show div thm $\Rightarrow$ Green's thm

Use 2D div thm with $\underline{F} = \begin{pmatrix} Q(x,y) \\ -P(x,y) \end{pmatrix}$. Then

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_A \nabla \cdot \underline{F}\, dA = \oint_{\partial A} \underline{F} \cdot \underline{n}\, dS$$



If ∂A is parametrised wrt arc length, have unit tangent vector

$$\underline{t} = \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix}$$

Then normal vector must be $\underline{n} = \begin{pmatrix} y'(s) \\ -x'(s) \end{pmatrix}$

Check: if $\underline{t}$ points vertically upwards then A is to our left

And so: $\underline{F} \cdot \underline{n}\, dS = \begin{pmatrix} Q \\ -P \end{pmatrix} \cdot \begin{pmatrix} y'(s) \\ -x'(s) \end{pmatrix} dS$

$$= P \frac{dx}{dS}\, dS = Q \frac{dy}{dS}\, dS = \underline{P\, dx + Q\, dy} \qquad \begin{matrix} x = x(s) \\ y = y(s) \end{matrix}$$

giving Green's theorem:

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \oint_{\partial A} P\, dx + Q\, dy. \qquad \square$$

(b) Green's thm $\Rightarrow$ Stokes' thm

Consider regular surface

$$S = \{ \underline{x} = \underline{x}(u, v) : (u, v) \in A \}$$

Then boundary is $\partial S = \{ \underline{x} = \underline{x}(u, v) : (u, v) \in \partial A \}$

Green's thm gives

$$\oint_{\partial A} P\,du + Q\,dv = \iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) du\,dv$$

Make choices

$$P(u,v) = \underline{F}(\underline{x}(u,v)) \cdot \frac{\partial \underline{x}}{\partial u}, \quad Q(u,v) = \underline{F}(\underline{x}(u,v)) \cdot \frac{\partial \underline{x}}{\partial v}$$

Then $P\,du + Q\,dv = \underline{F}(\underline{x}(u,v)) \cdot \left(\frac{\partial \underline{x}}{\partial u} du + \frac{\partial \underline{x}}{\partial v} dv \right)$

$$= \underline{F}(\underline{x}(u,v)) \cdot d\underline{x}(u,v)$$

And so $\oint_{\partial A} P\,du + Q\,dv = \int_{\partial S} \underline{F} \cdot d\underline{x}$

For other side of Stokes':

$$\frac{\partial Q}{\partial u} = \frac{\partial x_j}{\partial u} \frac{\partial F_i}{\partial x_j} \frac{\partial x_i}{\partial v} + F_i \frac{\partial^2 x_i}{\partial u \partial v}$$

$$\frac{\partial P}{\partial v} = \frac{\partial x_j}{\partial v} \frac{\partial F_i}{\partial x_j} \frac{\partial x_i}{\partial u} + F_i \frac{\partial^2 x_i}{\partial v \partial u}$$

then $\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} = \left(\frac{\partial x_i}{\partial v} \frac{\partial x_j}{\partial u} - \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial v} \right) \frac{\partial F_i}{\partial x_j}$

$$= \left(\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp} \right) \frac{\partial F_i}{\partial x_j} \frac{\partial x_p}{\partial v} \frac{\partial x_q}{\partial u}$$

$$= \varepsilon_{ijk} \varepsilon_{pqk} \frac{\partial F_i}{\partial x_j} \frac{\partial x_p}{\partial v} \frac{\partial x_q}{\partial u}$$

$$= \left[-\nabla \times \underline{F} \right]_k \left(- \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right)_k = (\nabla \times \underline{F}) \cdot \left(\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right)$$

so $\iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) du\,dv = \iint_A (\nabla \times \underline{F}) \cdot \left(\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right) du\,dv = \underline{\int_S \nabla \times \underline{F} \cdot d\underline{S}}$

as required. $\square$

6.1 Electromagnetism

Denote by $\underline{B} = \underline{B}(\underline{x}, t)$ magnetic field, $\underline{E} = \underline{E}(\underline{x}, t)$ electric field

These fields will depend on charge density $\rho = \rho(\underline{x}, t)$ (charge per unit volume) and current density $\underline{J} = \underline{J}(\underline{x}, t)$ (electric current per unit area)

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \quad (1) \qquad \nabla \cdot \underline{B} = 0 \quad (2) \qquad \nabla \times \underline{E} + \frac{\partial B}{\partial t} = 0 \quad (3) \qquad \nabla \times \underline{B} - \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J} \quad (4)$$

Maxwell's equations $\qquad \dfrac{1}{\mu_0 \varepsilon_0} = c^2$

If we take div of (4), using $\nabla \cdot \nabla \times \underline{B} = 0$, we get

$$0 = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \underline{E}) + \mu_0 \nabla \cdot \underline{J}$$

$\left(\text{using } \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial E_i}{\partial t}\right) = \nabla \cdot \left(\dfrac{\partial E}{\partial t}\right) = \dfrac{\partial}{\partial t}\dfrac{\partial E_i}{\partial x_i} = \dfrac{\partial}{\partial t}(\nabla \cdot \underline{E}) \right)$

By (1): we get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0 \qquad \text{a conservation law.}$$

This gives rise to conservation of charge. (Corresponds to "gauge invariance")

6.2 Integral formulations

Integrating (1) over volume V and using divergence theorem:

$$\int_{\partial V} \underline{E} \cdot d\underline{S} = \frac{1}{\varepsilon_0} \int_V \rho \, dV \equiv \frac{Q}{\varepsilon_0} \leftarrow \text{total charge in } V \qquad \text{Gauss's law}$$

For magnetic fields

(2) $\Rightarrow \displaystyle\int_{\partial V} \underline{B} \cdot d\underline{S} = 0$ so there is no net magnetic flux over ~~and~~ any closed surface ∂V. (i.e. there are no magnetic monopoles)

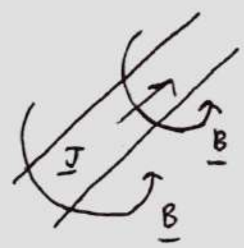
Integrate (3) over surface S, use Stokes' theorem:

$$\oint_{\partial S} \underline{E} \cdot d\underline{x} = -\int_S \frac{\partial \underline{B}}{\partial t} \cdot d\underline{S} = -\frac{d}{dt}\int_S \underline{B} \cdot d\underline{S}$$

Change in magnetic flux through S induces circulation in $\underline{E}$ about ∂S

Integrate (4) over surface S, use Stokes' theorem:

$$\oint_{\partial S} \underline{B} \cdot d\underline{x} = \mu_0 \int_S \underline{J} \cdot d\underline{S} + \mu_0 \varepsilon_0 \frac{d}{dt}\int_S \underline{E} \cdot d\underline{S}$$



6.3 Electromagnetic Waves

In empty space $\rho = 0$, $\underline{J} = 0$, so (1) − (4) give

$$\underset{(1)}{\nabla \cdot \underline{E} = 0}, \quad \underset{(2)}{\nabla \cdot \underline{B} = 0}, \quad \underset{(3)}{\nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0}, \quad \underset{(4)}{\nabla \times \underline{B} - \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t} = 0}$$

Recall Laplacian of vector field $\underline{F}$: $\nabla^2 \underline{F} = \nabla(\nabla \cdot \underline{F}) - \nabla \times (\nabla \times \underline{F})$

(1), (3), (4) give

$$\nabla^2 \underline{E} = \nabla(0) - \nabla \times \left(\frac{-\partial \underline{B}}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla \times \underline{B}) = \frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}\right)$$

$$\mu_0 \varepsilon_0 = \frac{1}{c^2} \Rightarrow \underline{\nabla^2 \underline{E} - \frac{1}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2} = 0}$$

wave eqn for waves travelling at speed c

Similarly by (2), (3), (4)

$$\nabla^2 \underline{B} = \nabla(0) - \nabla \times \left(\mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \underline{E})$$

by (3) $$= \mu_0 \varepsilon_0 \frac{\partial^2 \underline{B}}{\partial t^2} \quad \text{i.e.} \quad \underline{\nabla^2 \underline{B} - \frac{1}{c^2}\frac{\partial^2 \underline{B}}{\partial t^2} = 0}$$

6.4 Electrostatics and Magnetostatics

Suppose all fields and source terms are time independent.

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla \times \underline{E} = 0, \quad \nabla \cdot \underline{B} = 0, \quad \nabla \times \underline{B} = \mu_0 \underline{J}$$

If we work on all of $\mathbb{R}^3$ (2-connected), then $\nabla \times \underline{E} = 0$, $\nabla \cdot \underline{B} = 0$

$$\underline{E} = -\nabla \phi, \quad \underline{B} = \nabla \times \underline{A}$$

Call ϕ electric potential, A magnetic potential, then get

$$-\nabla^2 \phi = \frac{\rho}{\varepsilon_0}, \quad \nabla \times (\nabla \times \underline{A}) = \mu_0 \underline{J}$$

The first is called Poisson's equation (see section 7)

6.5 Gauge Invariance (non-examinable)

The second of Maxwell's eqns is $\nabla \cdot \underline{B} = 0$

Assume working in $\mathbb{R}^3$: $\underline{B} = \nabla \times \underline{A}$. A is not defined uniquely, can always change $\underline{A} \mapsto \underline{A} + \nabla \chi$ and $\underline{B}$ is unchanged

By $\underline{B} = \nabla \times \underline{A}$ in (3): $\nabla \times \left(\underline{E} + \frac{\partial \underline{A}}{\partial t} \right) = 0$

so can write term in brackets in terms of scalar potential, so

$\underline{E} = -\nabla \phi - \frac{\partial \underline{A}}{\partial t}$ and Maxwell's eqns reduce to

$$(1) \Rightarrow \quad -\nabla^2 \phi - \frac{\partial}{\partial t}(\nabla \cdot \underline{A}) = \frac{\rho}{\varepsilon_0}$$

$$(4) \Rightarrow \quad \nabla \times (\nabla \times \underline{A}) + \mu_0 \varepsilon_0 \nabla \left(\frac{\partial \phi}{\partial t} \right) + \mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} = \mu_0 \underline{J}$$

Recall $\nabla \times (\nabla \times \underline{A}) = \nabla(\nabla \cdot \underline{A}) - \nabla^2 \underline{A}$ and $\mu_0 \varepsilon_0 = \frac{1}{c^2}$ so (2) becomes $-\left(\nabla^2 \underline{A} - \frac{1}{c^2} \frac{\partial^2 \underline{A}}{\partial t^2} \right) + \nabla \left(\nabla \cdot \underline{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = \mu_0 \underline{J}$

$\underline{A} \mapsto \underline{A} + \nabla \chi$ in such a way that $\nabla \cdot \underline{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \to 0$

then $(1) \to -\nabla^2 \phi + \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{\rho}{\varepsilon_0}$

$(4) \to -\nabla^2 A + \frac{1}{c^2} \frac{\partial^2 \underline{A}}{\partial t^2} = \mu_0 \underline{J}$

Maxwell's eqns in Lorenz gauge

Solve to get $\underline{B} = \nabla \times \underline{A}$, $\underline{E} = -\nabla \phi - \frac{\partial \underline{A}}{\partial t}$

Vector Calculus - Lecture 15 26/2/21

Poisson's and Laplace's Equations

7.1 Boundary value problem

Many problems can be reduced to Poisson's eqn $\nabla^2 y = F$

or Laplace's eqn $\nabla^2 y = 0$. Solve on $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3, \dots$

with physical boundary conditions

The Dirichlet problem is $\nabla^2 \phi = F$ in Ω with $\phi = f$ on $\partial\Omega$

Neumann problem: $\nabla^2 \phi = F$ in Ω, $\frac{\partial \phi}{\partial \underline{n}} = g$ on $\partial\Omega$

where $\frac{\partial \phi}{\partial \underline{n}} = \underline{n} \cdot \nabla \phi$

Interpret boundary conditions appropriately: assume ϕ or $\frac{\partial \phi}{\partial n}$ approaches boundary continuously as $\underline{x} \to \partial\Omega$ i.e. ϕ and $\nabla\phi$ continuous on $\Omega \cup \partial\Omega$

* Note: if $\nabla^2 \phi = 0$ in Ω then ϕ needs to be well-defined on all of Ω. Don't e.g. assume $\nabla^2 \left(\frac{1}{|\underline{x}|}\right) = 0 \ \forall x \in \mathbb{R}^3$ - only true for $\underline{x} \neq \underline{0}$

Example $r = |\underline{x}|$, $\nabla^2 \phi \overset{(\dagger)}{=} r$ in $r < a$ (a), $\phi = 1$ on $r = a$ (b)

Guess solution of form $\phi = \phi(r)$. Using $\nabla^2 \phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right)$

Sub into $(\dagger)$: $(r^2 \phi')' = r^3$ in $r < a$, $\phi(a) = 1$

General soln is

$$\phi(r) = A + \frac{B}{r} + \frac{1}{12} r^3 \qquad (A, B \text{ constants})$$

Must have $B = 0$ or else ϕ not well defined throughout $\Omega = \{r < a\}$

$(\dagger)$ (b): $1 = \phi(a) = A + \frac{a^3}{12} \Rightarrow A = 1 - \frac{a^3}{12}$

Solution is hence $\phi(r) = 1 + \frac{1}{12}(r^3 - a^3)$.

Wants soln to be unique. Consider generic linear problem

$(\dagger\dagger)$ $L\phi = F$ in Ω, $B\phi = f$ on $\partial\Omega$ $\quad$ L, B linear differential operators

Let ϕ_1, ϕ_2 both solve $(\dagger\dagger)$. Let $\psi = \phi_1 - \phi_2$. By linearity:

$L\psi = 0$ in Ω, $B\psi = 0$ on $\partial\Omega$ $\quad (\Delta)$ homogeneous problem

If we can show that the only solution to (Δ) is $\psi = 0$, then the soln to $(\dagger\dagger)$ is unique.

Proposition The solution to the Dirichlet problem is unique, and the soln to the Neumann problem is unique up to a constant.

Proof Let $\psi = \phi_1 - \phi_2$ be difference of two solutions to Dirichlet/Neumann
$\nabla^2\psi = 0$ in Ω, $B\psi = 0$ on $\partial\Omega$ where $B\psi = \psi$ (Dirichlet) or $B\psi = \frac{\partial\psi}{\partial n}$ (Neumann)

Consider non-negative functional

$$I(\psi) = \int_\Omega |\nabla\psi|^2 \, dV \geq 0. \quad I(\psi) = 0 \Leftrightarrow \nabla\psi = 0 \text{ in } \Omega$$

Note
$$I(\psi) = \int_\Omega \nabla\psi\cdot\nabla\psi \, dV = \int_\Omega \Big(\underbrace{\nabla\cdot(\psi\nabla\psi)}_{\text{use div. thm}} - \psi\underbrace{\cancel{\nabla^2\psi}}_{0 \text{ as } \nabla^2\psi = 0 \text{ in } \Omega}\Big) \, dV$$

$$= \int_{\partial\Omega} (\psi\nabla\psi)\cdot d\underline{S} = \int_{\partial\Omega} \psi\frac{\partial\psi}{\partial \underline{n}} \, dS \quad \left(\text{by } d\underline{S} = \underline{n}\,dS,\ \underline{n}\cdot\nabla\psi = \frac{\partial\psi}{\partial\underline{n}}\right)$$

$= \underline{0}$ since $\psi = 0$ on boundary (Dirichlet) or $\frac{\partial\psi}{\partial\underline{n}} = 0$ on boundary (Neumann)

Conclude $\nabla\psi = 0$ throughout $\Omega \Rightarrow \psi =$ constant throughout Ω

Case a) Dirichlet : $\psi = 0$ on $\partial\Omega$

By continuity of ψ on $\Omega\cup\partial\Omega$ must have $\psi = 0$ everywhere so soln to Dirichlet problem is unique.

Case b) Neumann : $\frac{\partial\psi}{\partial\underline{n}} = 0$ on boundary. So can't say any more, so since ψ is constant, deduce that $\phi_1 = \phi_2 +$ constant so any 2 solutions only differ by a constant. $\square$

Example Charge density $\rho(\underline{x}) = 0$ $(r<a)$, $F(r)$ $\quad r \geq a$

Claim no electric field in $r<a$: indeed know potential ϕ satisfies
$$\nabla^2\phi = -\frac{\rho(\underline{x})}{\varepsilon_0} = 0 \quad r<a.$$
By spherical symmetry $\phi = \phi(r)$.
So $\phi = \phi(a) =$ constant on $r = a$. Unique soln to $\nabla^2\phi = 0$ $(r<a)$ and $\phi =$ const $(r = a)$ is $\phi =$ constant throughout $r \leq a$.
Then $\underline{E} = -\nabla\phi = \underline{0}$ throughout $r<a$.

"Newton's shell theorem"

Vector Calculus - Lecture 16

1/3/21

7.2 Gauss's Flux Method

Suppose source term F is spherically symmetric : $F = F(r)$ $\quad r = |\underline{x}|$

(*) $\nabla \cdot \nabla \phi = F$ $\qquad$ and assume $\Omega = \mathbb{R}^3$

Since RHS only depends on r, same is true of LHS. Assume $\phi = \phi(r)$

then $\nabla \phi = \phi'(r)\, \underline{e}_r$

Integrate (*) over region $|\underline{x}| < R$, use divergence theorem:

$$\int_{|\underline{x}|<R} \nabla \cdot \nabla \phi \, dV = \int_{|\underline{x}|=R} \nabla \phi \cdot d\underline{S} = \int_{|\underline{x}|<R} F(r)\, dV = Q(R)$$

RHS represents amount of e.g. mass inside ball of radius $R > 0$

Then have $\displaystyle\int_{|\underline{x}|=R} \nabla \phi \cdot d\underline{S} = Q(R)$

Recall on sphere of radius R, $\quad d\underline{S} = \underline{e}_r R^2 \sin\theta \, d\theta \, d\phi$

so on $|\underline{x}| = R$: $\quad \nabla \phi \cdot d\underline{S} = \phi'(r)\underline{e}_r \cdot \left(\underline{e}_r R^2 \sin\theta \, d\theta \, d\phi\right)\Big|_{|\underline{x}|=R}$

$$= \phi'(R)\, dS$$

so $\displaystyle Q(R) = \int_{|\underline{x}|=R} \phi'(R)\, dS = \phi'(R) \int_{|\underline{x}|=R} dS = 4\pi R^2 \phi'(R)$

so $\displaystyle \underline{\phi'(R) = \frac{Q(R)}{4\pi R^2}} \quad \Rightarrow \quad \underline{\nabla \phi = \frac{Q(r)}{4\pi r^2}\, \underline{e}_r}$

<u>Example</u> Maxwell's 1st eqn: $\nabla \cdot \underline{E} = \dfrac{\rho}{\varepsilon_0}$. Use electric potential ϕ : $\underline{E} = -\nabla \phi$

then get $-\nabla^2 \phi = \dfrac{\rho}{\varepsilon_0}$. Consider charge density

$$\rho(r) = \begin{cases} \rho_0, & 0 \le r \le a \\ 0, & r > a \end{cases}$$: by previous result $\displaystyle \phi'(r) = -\frac{1}{4\pi\varepsilon_0} \frac{Q(r)}{r^2}$

$$\underline{E}(\underline{x}) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q(r)}{r^2}\, \underline{e}_r & r \le a \\ \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{r^2}\, \underline{e}_r & r > a \end{cases}$$

Q = total charge

Taking $a \to 0$ keeping Q fixed (point charge):

$$\underline{E}(\underline{x}) = \frac{Q}{4\pi\varepsilon_0} \frac{\underline{e}_r}{r^2} = \frac{Q}{4\pi\varepsilon_0} \frac{\underline{x}}{|\underline{x}|^3}$$

corresponding charge density $\rho(\underline{x}) = Q\,\delta(\underline{x})$

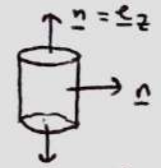
What if problem is symmetric about z-axis?

$\nabla^2 \phi = F(\rho) \qquad \rho^2 = x^2 + y^2 \qquad$ cylindrical symmetry

Integrate $\nabla \cdot \nabla \phi = F(\rho)$ over cylinder radius r, height a

Assuming $\phi = \phi(\rho)$ have $\nabla \phi = \phi'(\rho)\, \underline{e}_\rho$

$\underline{n} \cdot \nabla\phi = 0$
$\underline{n} = \underline{e}_z$



$\underline{n} = -\underline{e}_z$
$\Rightarrow \underline{n} \cdot \nabla \phi = 0$

only curved part contributes

$$\int_{\text{cyl}} \nabla \cdot \nabla \phi \, dV = \int_{\text{cyl}} F(\rho)\, dV = \int_{\partial\,\text{cyl}} \nabla \phi \cdot d\underline{S}$$

$$= \int_{\phi=0}^{2\pi} \int_{z=z_0}^{z_0+a} \phi'(R)\, R \, d\phi \, dz$$

here $d\underline{S} = R \, d\phi \, dz \, \underline{e}_\rho$

$\nabla \phi \cdot d\underline{S} = R \phi'(R) \, d\phi \, dz$

$$= 2\pi a R \phi'(R) = \underbrace{\int_{\text{cyl}} F(\rho)\, dV}_{(\dagger)}$$

$$(\dagger) = \int_{z=z_0}^{z_0+a} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{R} F(\rho)\, \rho \, d\rho \, d\phi \, dz = 2\pi a \int_0^R F(\rho)\, \rho \, d\rho$$

so $\displaystyle \phi'(\rho) = \frac{1}{\rho} \int_0^\rho s F(s)\, ds$

<u>Example</u> How might we describe a <u>line</u> of charge density with constant (λ) charge density per unit length? Could consider cylinder, let $r \to 0$

Let $F(\rho)$ be ~~dens~~ desired charge density. If we integrate over any cylinder of length 1, should have total charge contained to be λ

$$\lambda = \int_{\text{cyl}\,\updownarrow 1} F(\rho)\, dV = \int_{z=z_0}^{z_0+1} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{R} F(\rho)\, \rho \, d\rho \, d\phi \, dz$$

$$= 2\pi \int_0^R \rho F(\rho)\, d\rho$$

so we see that choosing $F(\rho)$ to be

$$F(\rho) = \frac{\lambda \delta(\rho)}{2\pi\rho}$$

so corresponding electric potential would satisfy $\displaystyle \phi'(\rho) = -\frac{1}{\varepsilon_0 \rho} \int_0^\rho \frac{\lambda}{2\pi} \delta(s)\, ds$

$$= -\frac{\lambda}{2\pi\varepsilon_0} \cdot \frac{1}{\rho} \quad \Rightarrow \quad \underline{\underline{E}(\underline{x}) = \frac{1}{2\pi\varepsilon_0} \frac{\underline{e}_\rho}{\rho}}$$

Vector Calculus - Lecture 17 3/3/21

7.3 Superposition Principle

Linear problems are relatively easy since if $L\Psi_n = F_n \quad n = 1,2,3,\dots$

then $L\left(\sum_n \Psi_n\right) = \sum_n F_n$

We can superimpose solutions. Can often break up forcing term $F = \sum_n F_n$, solve each problem $L\Psi_n = F_n$ and to get soln to $L\Psi = F$, write $\Psi = \sum_n \Psi_n$.

Example Electric potential due to pair of point charges Q_a at $\underline{x} = \underline{a}$, Q_b at $\underline{x} = \underline{b}$. Charge density would be

$$\rho(\underline{x}) = Q_a\,\delta(\underline{x}-\underline{a}) + Q_b\,\delta(\underline{x}-\underline{b})$$

For one point charge, electric potential obeys

$$-\nabla^2\phi = \frac{Q_a}{\varepsilon_0}\,\delta(\underline{x}-\underline{a}) \qquad (\text{by } \underline{E} = -\nabla\phi \text{ in H1})$$

solution: $\phi(\underline{x}) = \dfrac{Q_a}{4\pi\varepsilon_0}\dfrac{1}{|\underline{x}-\underline{a}|}$

so by superposition principle, electric potential due to point charges at $\underline{x} = \underline{a}$ and $\underline{x} = \underline{b}$ is $\phi(\underline{x}) = \dfrac{Q_a}{4\pi\varepsilon_0}\dfrac{1}{|\underline{x}-\underline{a}|} + \dfrac{Q_b}{4\pi\varepsilon_0}\dfrac{1}{|\underline{x}-\underline{b}|}$

Example Consider electric potential outside ball of radius $|\underline{x}| < R$ of uniform charge density ρ_0, that has several balls removed from interior $|\underline{x}-\underline{a}_i| < R_i \quad i = 1,\dots,N$ with $|\underline{a}_i| + R_i < R$, $|\underline{a}_i - \underline{a}_j| > R_i + R_j$ for each i, j

Represent each "hole" as a ball of uniform charge density $\underline{-\rho_0}$

Effective potential in $|\underline{x}| > R$ from each hole is

$$\phi(\underline{x}) = -\frac{1}{4\pi\varepsilon_0}\frac{Q_i}{|\underline{x}-\underline{a}_i|} \quad \text{where} \quad \left(\frac{4\pi R_i^3}{3}\right)\rho_0 = Q_i$$

by superposition: $\phi(\underline{x}) = \dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{Q}{|\underline{x}|} - \displaystyle\sum_{i=1}^{N}\dfrac{Q_i}{|\underline{x}-\underline{a}_i|}\right]$

7.4 Integral Solutions

We know electric potential due to a point charge at $\underline{x} = \underline{a}$ is proportional $\frac{1}{|\underline{x} - \underline{a}|}$ or collection of point charges $\sum \frac{Q_i}{|\underline{x} - \underline{a}_i|}$

Leads us to consider superpositions of form $\int_{\mathbb{R}^3} \frac{F(\underline{y})}{|\underline{x} - \underline{y}|} \, dV(\underline{y})$

Proposition Assume $F \to 0$ rapidly as $|\underline{x}| \to \infty$. The unique solution to the Dirichlet problem $\nabla^2 \phi = F$ $x \in \mathbb{R}^3$, $|\phi| \to 0$ as $|\underline{x}| \to \infty$ is given by
$$\phi(\underline{x}) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{F(\underline{y})}{|\underline{x} - \underline{y}|} \, dV(\underline{y})$$

This is another way of saying $\nabla^2 \left(-\frac{1}{4\pi} \cdot \frac{1}{|\underline{x}|} \right) = \delta(\underline{x})$

since by DUTIS $\nabla^2 \left(-\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{F(\underline{y})}{|\underline{x} - \underline{y}|} \, dV(\underline{y}) \right) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} F(\underline{y}) \nabla^2 \left(\frac{1}{|\underline{x} - \underline{y}|} \right) dV(\underline{y})$
$$= \int_{\mathbb{R}^3} F(\underline{y}) \, \delta(\underline{x} - \underline{y}) \, dV(\underline{y}) = F(\underline{x})$$

Note that for $r \neq 0$: $\nabla^2 \left(\frac{1}{r} \right) = \frac{\partial^2}{\partial x_i \partial x_i} \left(\frac{1}{r} \right) = \frac{\partial}{\partial x_i} \left(-\frac{x_i}{r^3} \right)$

$\left(\text{since } \frac{\partial}{\partial x_i} r = \frac{x_i}{r} \right)$ $\quad = -\frac{\delta_{ii}}{r^3} + \frac{3 x_i x_i}{r^5} = -\frac{3}{r^3} + \frac{3}{r^3} = 0$

Certainly have $\nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\underline{x}|} \right) = \delta(\underline{x})$ $\quad \underline{x} \neq 0$

Assuming divergence theorem for δ-functions, on any ball $|\underline{x}| < R$
$$\int_{|\underline{x}| < R} \nabla^2 \left(\frac{1}{|\underline{x}|} \right) dV = \int_{|\underline{x}| = R} \nabla \left(\frac{1}{|\underline{x}|} \right) \cdot d\underline{S}$$
$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(-\frac{\underline{e}_r}{R^2} \right) \cdot \underline{e}_r \, R^2 \sin\theta \, d\phi \, d\theta = -4\pi$$

So for any $R > 0$: $\int_{|\underline{x}| < R} \nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\underline{x}|} \right) dV = 1 = \int_{|\underline{x}| < R} \delta(\underline{x}) \, dV$

So we conclude $\nabla^2 \left(-\frac{1}{4\pi} \cdot \frac{1}{|\underline{x}|} \right) = \delta(\underline{x})$ and result follows. $\square$

3/2/21

Vector Calculus - Lecture 18

7.5 Harmonic Functions

When the forcing term in Poisson's eqn is identically 0, call it Laplace's eqn $\nabla^2 \phi = 0$. Solutions are called harmonic functions.

<u>Proposition</u> If ϕ harmonic on $\Omega \subset \mathbb{R}^3$ then
$$\phi(\underline{a}) = \frac{1}{4\pi r^2} \int_{|\underline{x}-\underline{a}|=r} \phi(\underline{x})\, dS \qquad (\dagger)$$
for $\underline{a} \in \Omega$, r sufficiently small ("mean value property")

<u>Proof</u> Let $F(r)$ denote the RHS of $(\dagger)$

Then $F(r) = \dfrac{1}{4\pi r^2} \displaystyle\int_{|\underline{x}|=r} \phi(\underline{a}+\underline{x})\, dS$

$$= \frac{1}{4\pi \cancel{r^2}} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \phi(\underline{a} + r\underline{e}_r)\, \cancel{r^2} \sin\theta \, d\theta \, d\phi$$

Computing $F'(r)$, using $\dfrac{d}{dr}\phi(\underline{a}+r\underline{e}_r) = \underline{e}_r \cdot \nabla\phi(\underline{a}+r\underline{e}_r)$

$\left(\text{by } \dfrac{d}{dt} f(\underline{x}(t)) = \underline{x}'(t)\ \nabla f(\underline{x}(t))\right)$

have
$$F'(r) = \frac{1}{4\pi r^2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \underline{e}_r \cdot \nabla\phi(\underline{a}+r\underline{e}_r)\, r^2 \sin\theta\, d\theta\, d\phi$$
$$= \frac{1}{4\pi r^2} \int_{|\underline{x}|=r} \underline{e}_r\, \nabla\phi(\underline{a}+r\underline{e}_r)\, dS = \frac{1}{4\pi r^2} \int_{|\underline{x}|=r} \nabla\phi(\underline{a}+\underline{x}) \cdot d\underline{S}$$
$$= \frac{1}{4\pi r^2} \int_{|\underline{x}-\underline{a}|=r} \nabla\phi \cdot d\underline{S} \underset{\text{div thm}}{=} \frac{1}{4\pi r^2} \int_{|\underline{x}-\underline{a}|<r} \nabla^2\phi\, dV = 0$$

So $F(r)$ is constant and we note from $(\dagger)$:

$\lim_{r\to 0} F(r) = \phi(\underline{a})$ so $F(r) = \phi(\underline{a})$ and result follows. $\square$

Can use central idea to ~~eq~~ examine what Laplacian measures.

<u>Proposition</u> For any smooth $\phi: \mathbb{R}^3 \to \mathbb{R}$:
$$\nabla^2 \phi(\underline{a}) = \lim_{r\to 0} \frac{6}{r^2} \left[\frac{1}{4\pi r^2} \int_{|\underline{x}-\underline{a}|=r} \phi(\underline{x})\, dS - \phi(\underline{a}) \right].$$

In particular if ϕ satisfies MVP then it's harmonic.

<u>Proof</u> Consider function $G(r)$ defined by

$$G(r) = \frac{1}{4\pi r^2}\int_{|\underline{x}-\underline{a}|=r} \phi(\underline{x})\, dS - \phi(\underline{a})$$

So G measures extent to which ϕ differs from its average. From previous proof $G'(r) = F'(r) = \frac{1}{4\pi r^2}\int_{|\underline{x}-\underline{a}|<r} \nabla^2\phi \, dV$

(this vanishes <u>if</u> ϕ is harmonic)

Note $\int_{|\underline{x}-\underline{a}|<r} \nabla^2\phi(\underline{x})\, dV = \nabla^2\phi(\underline{a})\int_{|\underline{x}-\underline{a}|<r} dV + \int_{|\underline{x}-\underline{a}|<r}\left(\nabla^2\phi(\underline{x}) - \nabla^2\phi(\underline{a})\right) dV$

$= \frac{4\pi r^3}{3}\nabla^2\phi(\underline{a}) + o(r^3)$ as $r\to 0$ (see lecture notes)

So $G'(r) = \frac{1}{4\pi r^2}\int_{|\underline{x}-\underline{a}|<r} \nabla^2\phi(\underline{x})\, dS = \frac{1}{4\pi r^2}\left[\frac{4\pi r^3}{3}\nabla^2\phi(\underline{a}) + o(r^3)\right]$

$= \frac{r}{3}\nabla^2\phi(\underline{a}) + o(r) \quad (r\to 0)$

Compare with Taylor expansion: $G'(r) = G'(0) + rG''(0) + o(r)$
$(r\to 0)$

We deduce: $G'(0) = 0$, $G''(0) = \frac{1}{3}\nabla^2\phi(a)$

so $G(r) = \cancel{G(0)}^{\,0} + r\,\cancel{G'(0)}_{\,0} + \frac{r^2}{2}G''(0) + o(r^2)$

$F(0) - \phi(a)$
phi

$= \frac{1}{6}\nabla^2\phi(\underline{a})\, r^2 + o(r^2) \quad (r\to 0)$

$\Rightarrow \nabla^2\phi(\underline{a}) = \lim_{r\to 0}\left[\frac{6}{r^2}G(r)\right] \Rightarrow$ result. $\square$

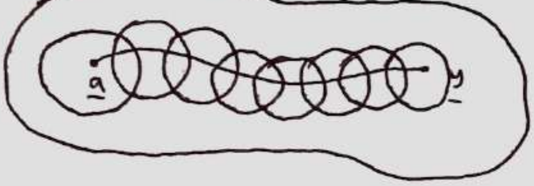
<u>Proposition</u> If ϕ is harmonic on $\Omega \subset \mathbb{R}^3$ then ϕ cannot have a maximum at any interior point of Ω unless ϕ is constant.

<u>Proof</u> Suppose $\underline{a}\in\Omega$ is s.t. $\phi(\underline{a}) \geq \phi(\underline{x})$ $\forall \underline{x}\in\Omega$.

So certainly $\phi(\underline{a}) \geq \phi(\underline{x})$ on $0 < |\underline{x}-\underline{a}| \leq \varepsilon$ for some $\varepsilon > 0$.

By MVT: $\phi(\underline{a}) = \frac{1}{4\pi\varepsilon^2}\int_{|\underline{x}-\underline{a}|=\varepsilon}\phi(\underline{x})\, dS \Rightarrow 0 = \frac{1}{4\pi\varepsilon^2}\int_{|\underline{x}-\underline{a}|=\varepsilon}\overbrace{(\phi(\underline{a})-\phi(\underline{x}))}^{\geq 0}\, dS$

Conclude $\phi(\underline{x}) = \phi(\underline{a})$. Apply same argument to $|\underline{x}-\underline{a}| = \varepsilon' < \varepsilon$, deduce $\phi(\underline{x}) = \phi(\underline{a})$ on $|\underline{x}-\underline{a}| < \varepsilon$.



Introduce overlapping balls s.t. centre of the $(n+1)^{th}$ is contained inside the n^{th}

Everywhere inside 1st ball: $\phi(\underline{x}) = \phi(\underline{a})$, in particular on centre of 2nd have $\phi(\underline{x}) = \phi(\underline{a})$. By previous argument $\phi(\underline{x}) = \phi(\underline{a})$ throughout 2nd. Carry on until you reach $\underline{y}$: $\phi(\underline{y}) = \phi(\underline{a})$ i.e. ϕ constant. $\square$

[(Alternatively): consider $U \subset \Omega$ with $U = \{ \underline{x} \in \Omega : \phi(\underline{x}) = \phi(\underline{a}) \}$
U nonempty, open by MVT, closed by continuity of ϕ, so $U = \Omega$ $\square$]

Immediate corollary: if ϕ harmonic on Ω then

$$\phi(\underline{x}) \leq \max_{y \in \partial\Omega} \phi(\underline{y}) \qquad (\underline{x} \in \Omega) \qquad \text{maximum principle}$$

8 Cartesian Tensors

8.1 A closer look at vectors

Let $\{\underline{e}_i\}$ be a RH, ON basis, wrt fixed Cartesian coordinate axes

Write vector as $\underline{x} = x_i \underline{e}_i$

We shouldn't identify $\underline{x}$ with components $\{x_i\}$ - basis specific

If we used basis $\{\underline{e}_i'\}$ would have $\underline{x} = x_i' \underline{e}_i'$

We must have $x_j \underline{e}_j = x_j' \underline{e}_j'$ (*) and by orthonormality,

$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}, \quad \underline{e}_i' \cdot \underline{e}_j' = \delta_{ij}$

By (*): $x_i' = \delta_{ij} x_j' = (\underline{e}_i' \cdot \underline{e}_j') x_j' = \underline{e}_i' \cdot (\underline{e}_j' x_j')$

$= (\underline{e}_i' \cdot \underline{e}_j) x_j$. Set $R_{ij} = \underline{e}_i' \cdot \underline{e}_j$, then $\underline{x_i' = R_{ij} x_j}$.

Alternatively:

$x_i = \delta_{ij} x_j = (\underline{e}_i \cdot \underline{e}_j) x_j = (\underline{e}_i \cdot \underline{e}_j') x_j' = (\underline{e}_j' \cdot \underline{e}_i) x_j'$

i.e. $x_i = R_{ji} x_j' = \cancel{R_{kj} x_k'}\ R_{ki} x_k'$

$$\underline{x_j = R_{kj} x_k'}$$

$x_i' = R_{ij} x_j = R_{ij} R_{kj} x_k'$ and so we find

$(\delta_{ik} - R_{ij} R_{kj}) x_k' = 0$ for all choices $\{x_k'\}$, get

$\underline{R_{ij} R_{kj} = \delta_{ik}}$. If R is a matrix with entries $\{R_{ij}\}$, this reads

$\underline{R R^T = I}$ so $\{R_{ij}\}$ are components of an orthogonal matrix.

Since $x_j \underline{e}_j = x_i' \underline{e}_i' = R_{ij} x_j \underline{e}_i'$ holds for all $\{x_j\}$, also have

$\underline{\underline{e}_j = R_{ij} \underline{e}_i'}$ and since $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$ are right-handed,

$1 = \underline{e}_1 \cdot (\underline{e}_2 \times \underline{e}_3) = R_{i1} R_{j2} R_{k3}\, \underline{e}_i' \cdot (\underline{e}_j' \times \underline{e}_k')$

$= R_{i1} R_{j2} R_{k3}\, \varepsilon_{ijk} = \underline{\det R}$ so R is orthogonal and $\det R = 1$: is a rotation matrix.

<u>Moral</u>: If we transform from $\{\underline{e}_i\}$ to $\{\underline{e}_i'\}$, then the components of a vector $\underline{v}$ transform according to

$$\underline{v_i' = R_{ij} v_j} \qquad \text{where } R_{ij} = \underline{e}_i' \cdot \underline{e}_j \text{ are components of a rotation matrix.}$$

Call objects whose components transform in this way <u>rank 1 tensors</u> (vectors)

<u>8.2</u> A closer look at scalars

Consider $\sigma = \underline{a} \cdot \underline{b}$. Using RHON $\{\underline{e}_i\}$ with $\underline{a} = a_i \underline{e}_i$ etc:

$$\sigma = a_i b_j (\underline{e}_i \cdot \underline{e}_j) = a_i b_j \delta_{ij} = a_i b_i$$

Instead use $\{\underline{e}_i'\}$, would find $\sigma' = a_i' b_i'$

Using $a_i' = R_{ip} a_p$, $b_i' = R_{iq} b_q$, have

$$\sigma' = R_{ip} R_{iq} a_p b_q = \delta_{pq} a_p b_q = a_p b_p = \sigma$$

We call objects that transform this way <u>scalars</u>

<u>Moral</u>: objects that transform as $\underline{\sigma' = \sigma}$ changing from $\{\underline{e}_i\}$ to $\{\underline{e}_i'\}$ are <u>scalars</u> or <u>rank 0 tensors</u>.

<u>8.3</u> A closer look at linear maps

Let $\underline{n} \in \mathbb{R}^3$ be a fixed unit vector, define linear map $T: \underline{x} \mapsto \underline{y}$

$= T(\underline{x}) = \underline{x} - (\underline{x} \cdot \underline{n})\,\underline{n}$ (orthogonal projection of x into plane with normal n)

Using $\{\underline{e}_i\}$ with $\underline{x} = x_i \underline{e}_i$, $\underline{y} = y_i \underline{e}_i$ etc:

$$y_i \underline{e}_i = T(x_j \underline{e}_j) = x_j T(\underline{e}_j) = x_j (\underline{e}_j - n_i n_j \underline{e}_i)$$
$$= (\delta_{ij} - n_i n_j)\, x_j \underline{e}_i$$

Set $T_{ij} = \delta_{ij} - n_i n_j$. Then $\underline{y_i = (\delta_{ij} - n_i n_j)\, x_j = T_{ij} x_j}$

$\{T_{ij}\}$ are components of linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$ wrt basis $\{\underline{e}_i\}$

If we had instead used $\{\underline{e}_i'\}$, would have found

$y_i' = T_{ij}' x_j'$ where $T_{ij}' = \delta_{ij} - n_i' n_j'$. Using $n_i' = R_{ij} n_j$,

give $T_{ij}' = \delta_{ij} - R_{ip} R_{jq} n_p n_q$

$= R_{ip} R_{jq} (\delta_{pq} - n_p n_q)$ (since $R_{ip} R_{jq} \delta_{pq}$
$= R_{ip} R_{jp} = \delta_{ij}$)

$= R_{ip} R_{jq} T_{pq}$

so components of T transform by $\underline{T_{ij}' = R_{ip} R_{jq} T_{pq}}$

Objects that transform this way are <u>rank 2 tensors</u>

Vector Calculus - Lecture 20 11/3/21

8.4 Cartesian tensors of rank n

$\{\underline{e}_i\} \to \{\underline{e}_i'\}$

Definition An object whose components $\underbrace{T_{ij\ldots k}}_{n \text{ indices}}$ transform according to

$T_{ij\ldots k}' = R_{ip} R_{jq} \ldots R_{kr} T_{pq\ldots r}$ is a (Cartesian) tensor of rank n

Here $R_{ij} = \underline{e}_i' \cdot \underline{e}_j$ are components of a rotation matrix $(R_{ip} R_{jp} = \delta_{ij})$

Example If $u_i, v_j, \ldots, w_k$ are components of n vectors, then

$T_{ij\ldots k} = u_i v_j \ldots w_k$ define the components of a rank n tensor.

Check: $T_{ij\ldots k}' = u_i' v_j' \ldots w_k' = R_{ip} u_p R_{jq} v_q \ldots R_{kr} w_r$

$= R_{ip} R_{jq} \cdots R_{kr} T_{pq\ldots r}$ as required.

Example Kronecker delta defined without reference to any basis:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

So $\delta_{ij}' = \delta_{ij}$ by definition. But note

$R_{ip} R_{jq} \delta_{pq} = R_{ip} R_{jp} = \delta_{ij}$ so we have $\delta_{ij}' = R_{ip} R_{jq} \delta_{pq}$
transforms like a rank 2 tensor

Example Levi-Civita epsilon

$$\varepsilon_{ijk} = \begin{cases} +1 & (ijk) \text{ even perm of } (1\,2\,3) \\ -1 & (ijk) \text{ odd perm of } (1\,2\,3) \\ 0 & \text{otherwise} \end{cases}$$

By definition, $\varepsilon_{ijk}' = \varepsilon_{ijk}$.

But $R_{ip} R_{jq} R_{kr} \varepsilon_{pqr} = \underbrace{\det R}_{1} \varepsilon_{ijk} = \varepsilon_{ijk}$

So we have $\varepsilon_{ijk}' = R_{ip} R_{jq} R_{kr} \varepsilon_{pqr}$
transforms like a rank 3 tensor

Example Experiments suggest linear relationship between current $\underline{J}$ produced in a conductive medium exposed to an electric field $\underline{E}$, so $\underline{J} = \sigma \underline{E}$ or $J_i = \sigma_{ij} E_j$. σ_{ij} is electrical conductivity tensor (rank 2)

Under change of basis, $\sigma_{ij}' E_j' = J_i' = R_{ip} J_p = R_{ip} \sigma_{pq} E_q$

Using $E_j' = R_{jq} E_q \iff E_q = R_{jq} E_j'$, get $\underline{\sigma_{ij}' E_j' = R_{ip} R_{jq} \sigma_{pq} E_j'}$

Holds for $\underline{\text{all}}$ E_j', so $\underline{\sigma_{ij}' = R_{ip} R_{jq} \sigma_{pq}}$ See Quotient Theorem later

Example Not all things are tensors. Given Cartesian RH $\{\underline{e}_i\}$, define

$$A_{ij} = \begin{pmatrix} \pi & 7 & 0 \\ \sqrt{2} & e & -3 \\ \gamma & 1 & 12 \end{pmatrix}$$

and set $A_{ij}' = 0$ in all other bases $\{\underline{e}_i'\}$. This array is not like a rank 2 tensor

If $A_{ij\ldots k}$, $B_{ij\ldots k}$ are n^{th} rank tensors, define

$\underline{(A+B)_{ij\ldots k}} = A_{ij\ldots k} + B_{ij\ldots k}$ is also an n^{th} rank tensor

If α is a scalar, can define $\underline{(\alpha A)_{ij\ldots k}} = \alpha A_{ij\ldots k}$ n^{th} rank tensor

Define $\underline{\text{tensor product}}$ of m^{th} rank tensor $u_{ij\ldots k}$ and n^{th} rank $v_{pq\ldots r}$ by $(u \otimes v)_{ij\ldots k\,pq\ldots r} = \underbrace{u_{ij\ldots k}}_{m} \underbrace{v_{pq\ldots r}}_{n}$ is a rank $\underline{n+m}$ tensor.

$$u_{i\ldots j}' v_{p\ldots q}' = R_{ia} \ldots R_{jb} u_{a\ldots b} R_{pc} \ldots R_{qd} v_{c\ldots d}$$
$$= R_{ia} \ldots R_{jb} R_{pc} \ldots R_{qd} \underbrace{u_{a\ldots b} v_{c\ldots d}}_{(u \otimes v)_{a\ldots b c\ldots d}}$$

Given n^{th} rank tensor $T_{ijk\ldots l}$ $(n \geq 2)$ can define a tensor of rank $\underline{n-2}$ by $\underline{\text{contracting}}$ on a pair of indices

e.g. contracting on i and j is defined by

$\delta_{ij} T_{ijk\ldots l} = T_{iik\ldots l}$. Note that

$$T_{itk\ldots l}' = \underbrace{R_{ip} R_{iq}}_{\delta_{pq}} R_{kr} \ldots R_{ls} T_{pqr\ldots s} = R_{kr} \ldots R_{ls} T_{ppr\ldots s}$$

should be $\underline{iik\ldots l}$

so we get a tensor of rank $n-2$ $\underline{T_{iik\ldots l}}$.

Vector Calculus - Lecture 21

Say $T_{ij\ldots k}$ is <u>symmetric</u> in (i,j) if $T_{ij\ldots k} = T_{ji\ldots k}$

This is well-defined:

$$T'_{ij\ldots k} = R_{ip}R_{jq}\ldots R_{kr}T_{pq\ldots r} = R_{ip}R_{jq}\ldots R_{kr}T_{qp\ldots r}$$

$$= R_{iq}R_{jp}\ldots R_{kr}T_{pq\ldots r} = T'_{ji\ldots k}$$

(interchange dummy indices)

Similarly $A_{ij\ldots k}$ <u>antisymmetric</u> in (i,j) if $A_{ij\ldots k} = -A_{ji\ldots k}$

Say a tensor is <u>totally</u> (anti) symmetric if the property holds for any pair of indices.

<u>Example</u> Tensors δ_{ij} and $a_i a_j a_k$ are both totally symmetric

ε_{ijk} totally antisymmetric

In fact the <u>only</u> totally antisymmetric tensor rank 3 on $\mathbb{R}^3$ is proportional to ε_{ijk} and there are no non-zero higher rank ones.

If $T_{ij\ldots k}$ is T. A. of rank n, then $T_{ij\ldots k} = 0$ <u>if any 2 are the same</u>

By pigeonhole, there will always be ≥ 2 matching if $n > 3$.

If $n = 3$ there are only $3!$ nonzero components

By antisymmetry
$$T_{123} = T_{231} = T_{312} = \lambda \qquad \propto \text{ to } \varepsilon_{ijk}$$
$$T_{213} = T_{321} = T_{132} = -\lambda$$

<u>8.5</u> Tensor Calculus

"vector field" $\rightarrow$ vector $\underline{v}(\underline{x})$ for $\underline{x} \in \mathbb{R}^3$

"scalar field" $\rightarrow$ scalar $\phi(\underline{x})$ for $\underline{x} \in \mathbb{R}^3$

A tensor field of rank n, $T_{ij\ldots k}(\underline{x})$ gives an n^{th} rank tensor at each $\underline{x} \in \mathbb{R}^3$

Recall $x_i' = R_{ij}x_j \iff x_j = R_{ij}x_i'$ (use $R^{-1} = R^T$)

Differentiate wrt x_k'

$$\Rightarrow \quad \frac{\partial x_j}{\partial x_k'} = R_{ij} \frac{\partial x_i'}{\partial x_k'} = R_{ij} \delta_{ik} = R_{kj}$$

By chain rule $\displaystyle \frac{\partial}{\partial x_i'} = \frac{\partial x_j}{\partial x_i'} \frac{\partial}{\partial x_j} = \underline{R_{ij} \frac{\partial}{\partial x_j}}$

Informally " $\frac{\partial}{\partial x_i}$ transforms like a rank 1 tensor "

<u>Proposition</u> If $T_{i \dots j}(\underline{x})$ is a tensor field of rank n then

$$\underbrace{\left(\frac{\partial}{\partial x_p}\right) \cdots \left(\frac{\partial}{\partial x_q}\right)}_{m \text{ terms}} T_{i \dots j}(\underline{x}) \quad \text{is a tensor field of rank } n+m$$

<u>Proof</u> Label LHS by $A_{p \dots q i \dots j}$

$$A'_{p \dots q i \dots j} = \left(\frac{\partial}{\partial x_p'}\right) \cdots \left(\frac{\partial}{\partial x_q'}\right) T'_{i \dots j}(\underline{x})$$

$$= \left(R_{pa} \frac{\partial}{\partial x_a}\right) \cdots \left(R_{qb} \frac{\partial}{\partial x_b}\right) R_{ic} \cdots R_{jd} T_{c \dots d}$$

$$= R_{pa} \dots R_{qb} R_{ic} \dots R_{jd} A_{a \dots b c \dots d}$$ tensor field of rank $n+m$. □

<u>Example</u> If $\phi = \phi(\underline{x})$ scalar field, then

$[\nabla \phi]_i = \frac{\partial \phi}{\partial x_i}$ so $\nabla \phi$ is rank $0+1 = 1$ tensor field i.e. vector field

For vector field $\underline{v}$ have divergence

$\nabla \cdot \underline{v} = \frac{\partial v_i}{\partial x_i}$. Note $\displaystyle \frac{\partial v_i'}{\partial x_i'} = R_{ip} \frac{\partial}{\partial x_p} R_{iq} v_q$

$$~~= R_{ip} R_{iq} v_q~~ \quad = R_{ip} R_{iq} \frac{\partial v_q}{\partial x_p} = \delta_{pq} \frac{\partial v_q}{\partial x_p} = \underline{\frac{\partial v_p}{\partial x_p}}$$

i.e. $\nabla \cdot \underline{v}$ is a scalar field

$\underline{v}$ vector field, $\nabla \times \underline{v}$ curl

$$[\nabla \times \underline{v}]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} v_k \quad \Rightarrow \quad \varepsilon_{ijk}' \frac{\partial v_k'}{\partial x_j'} = R_{ia} R_{jb} R_{kc} \varepsilon_{abc} \cdot R_{jp} \frac{\partial}{\partial x_p} R_{kq} v_q$$

$$= R_{ia} \varepsilon_{abc} \underbrace{R_{jb} R_{jp}}_{\delta_{pb}} \underbrace{R_{kc} R_{kq}}_{\delta_{cq}} \frac{\partial v_q}{\partial x_p} = R_{ia} \varepsilon_{abc} \frac{\partial v_c}{\partial x_b}$$

so $\nabla \times \underline{v}$ is a vector field.

<u>Proposition</u> For tensor field $T_{ij\ldots k\ldots l}(\underline{x})$

$$\int_V \frac{\partial}{\partial x_k} T_{ij\ldots k\ldots l} \, dV = \int_{\partial V} T_{ij\ldots k\ldots l} \, n_k \, dS$$

<u>Proof</u> Apply divergence theorem to $v_k = a_i b_j \ldots c_l T_{ij\ldots k\ldots l}$ only free index k so it's rank 1

where $a_i, b_j, \ldots, c_l$ are components of constant vector fields

Then $$\int_V \frac{\partial v_k}{\partial x_k} \, dV = a_i b_j \ldots c_l \int_V \frac{\partial}{\partial x_k} T_{ij\ldots k\ldots l} \, dV$$

$$= \int_{\partial V} v_k n_k \, dS = a_i b_j \ldots c_l \int_{\partial V} T_{ij\ldots k\ldots l} \, n_k \, dS$$ for every choice of $a_i, b_j, \ldots, c_l$

Result follows since $\underline{a}, \underline{b}, \underline{c}$ arbitrary

To check when all free indices $i, j, \ldots, l = 1$:

$a_i = \delta_{i1}, \quad b_j = \delta_{j1}, \ldots, \quad c_l = \delta_{l1}$

Then LHS $= \int_V \frac{\partial}{\partial x_k} T_{11\ldots k\ldots 1} \, dV$, RHS $= \int_{\partial V} T_{11\ldots k\ldots 1} \, n_k \, dS$

Similar ~~and~~ idea for any choice of free indices. $\square$

8.6 Rank 2 tensors

Observe for rank 2 tensor T_{ij}, $T_{ij} = \overbrace{\frac{1}{2}(T_{ij}+T_{ji})}^{S_{ij}} + \overbrace{\frac{1}{2}(T_{ij}-T_{ji})}^{A_{ij}}$

S_{ij} symmetric, A_{ij} antisymmetric

$$\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \qquad \begin{pmatrix} 0 & * & * \\ & 0 & * \\ & & 0 \end{pmatrix}$$

6 indep. compts 3 indep. compts

$6+3=9$ ← general # indep. compts to specify rank 2 tensor

Seems ~~if~~ info in A_{ij} could be written in vector form

<u>Proposition</u> Every rank 2 tensor can be uniquely written as

$$T_{ij} = S_{ij} + \varepsilon_{ijk}\omega_k \quad \text{where } \omega_i = \frac{1}{2}\varepsilon_{ijk}T_{jk}, \ S_{ij} \text{ symmetric}$$

<u>Proof</u> $S_{ij} = \frac{1}{2}(T_{ij}+T_{ji})$ so remains to show $\frac{1}{2}(T_{ij}-T_{ji}) = \varepsilon_{ijk}\omega_k$

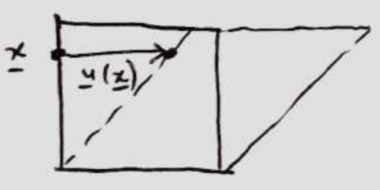
$$\varepsilon_{ijk}\omega_k = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}T_{lm} = \frac{1}{2}(\delta_{il}\delta_{jk} - \delta_{im}\delta_{jl})T_{lm}$$
$$= \frac{1}{2}(T_{ij}-T_{ji})$$

For uniqueness, suppose $\cancel{S_{ij} + \overbrace{\varepsilon_{ijk}\omega_k}^{A}}$ $S_{ij} + A_{ij} = \tilde{S}_{ij} + \tilde{A}_{ij}$

Take symmetric parts: $\frac{1}{2}(T_{ij}+T_{ji}) = \frac{1}{2}(\tilde{T}_{ij}+\tilde{T}_{ji})$ etc.

Then $S_{ij} = \tilde{S}_{ij}$, $A_{ij} = \tilde{A}_{ij}$ so is unique. □

<u>Example</u> Each pt $\underline{x}$ in an elastic body undergoes small displacement $\underline{u}(\underline{x})$



2 nearby pts $\underline{x}+\delta\underline{x}$ and $\underline{x}$ that were initially separated by $\delta\underline{x}$ become separated by

$$(\underline{x}+\delta\underline{x}+\underline{u}(\underline{x}+\delta\underline{x})) - (\underline{x}+\underline{u}(\underline{x}))$$

$$= \delta\underline{x} + [\underline{u}(\underline{x}+\delta\underline{x}) - \underline{u}(\underline{x})]$$ - change in displacement $\underline{u}(\underline{x}+\delta\underline{x}) - \underline{u}(\underline{x})$

This tells us how much deformation happens to the body. Use Taylor's thm:

$$u_i(\underline{x}+\delta\underline{x}) - u_i(\underline{x}) = \frac{\partial u_i}{\partial x_j}\delta x_j + o(\delta\underline{x})$$

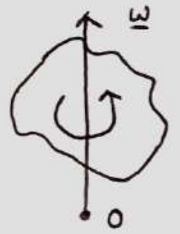
Decompose $\frac{\partial u_i}{\partial x_j} = e_{ij} + \varepsilon_{ijk}\omega_k$ where $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ ← linear strain tensor and $w_i = \frac{1}{2} \varepsilon_{ijk} \frac{\partial u_j}{\partial x_k} = -\frac{1}{2} (\nabla \times \underline{u})_i$

So $u_i(\underline{x} + \delta \underline{x}) - u_i(\underline{x}) = e_{ij} \delta x_j + \underbrace{[\delta \underline{x} \times \underline{w}]_i}_{\text{rotation}} + o(\delta \underline{x})$

$e_{ij} \delta x_j$ measures deformation

A well known symmetric rank 2 tensor is inertia tensor

Suppose body with density $\rho(\underline{x})$ occupies $V \subset \mathbb{R}^3$ - constant $\underline{w}$



velocity of pt $\underline{x} \in V$ is $\underline{v} = \underline{w} \times \underline{x}$

Total A.V about 0 is $\underline{L} = \int_V \rho(\underline{x})(\underline{x} \times \underline{v}) \, dV$

$= \int_V \rho(\underline{x}) [\underline{x} \times (\underline{w} \times \underline{x})] \, dV$

Suffix notation: $L_i = \int_{\mathcal{V}} \rho(\underline{x}) (x_k x_k \underbrace{w_i}_{\delta_{ij} w_j} - x_i x_j w_j) \, dV = I_{ij} w_j$

where I_{ij} is inertia tensor - $I_{ij} = \int_{\mathcal{V}} \rho(\underline{x}) (x_k x_k \delta_{ij} - x_i x_j) \, dV$

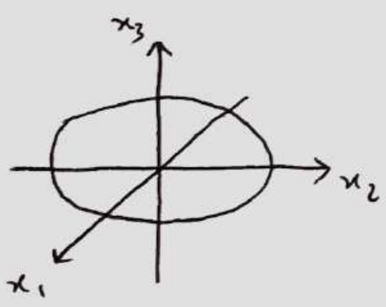
over $\mathcal{V} = \{ x_i : x_i \underline{e}_i \in V \}$

Had we used different frame $\{ \underline{e}_i' \}$, $\underline{x} = x_i' \underline{e}_i'$ etc, would find

$$I_{ij}' = \int_{\mathcal{V}'} \rho(\underline{x}) (x_k' x_k' \delta_{ij} - x_i' x_j') \, dV = R_{ip} R_{jq} \int_{\mathcal{V}} \rho(\underline{x}) (x_k x_k \delta_{pq} - x_p x_q) \, dV$$

$= R_{ip} R_{jq} I_{pq}$ rank 2 tensor, symmetric

<u>Example</u> Ellipsoid



$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} \leq 1$$

with uniform mass density ρ_0 so mass is

$$M = \rho_0 \frac{4\pi}{3} abc$$

To find components of inertia in this frame, use scaled spherical polars to compute integrals:

$x_1 = a r \cos\phi \sin\theta \qquad 0 \leq \phi \leq 2\pi$

$x_2 = b r \sin\phi \sin\theta \qquad 0 \leq \theta \leq \pi$

$x_3 = c r \cos\theta \qquad 0 \leq r \leq 1$

Note if $i \neq j$ then $\int_V \rho_0 x_i x_j \, dV = 0$ by symmetry

Also $I_{11} = \rho_0 \int_V (x_2^2 + x_3^2)\, dV = \rho_0 abc \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{1} r^2(b^2 \sin^2\phi \sin^2\theta + c^2\cos^2\theta)\, r^2 \sin\theta \, dr\, d\theta\, d\phi$

$$= \rho_0 \frac{abc}{5} \int_0^{\pi} (\pi b^2 \sin^2\theta + 2\pi c^2 \cos^2\theta) \sin\theta \, d\theta$$

$$= \frac{3M}{4} \frac{1}{5} \int_0^{\pi} (b^2 \sin^2\theta + (2c^2 - b^2)\cos^2\theta \sin\theta)\, d\theta$$

$$= \frac{3M}{20}\left(2b^2 + \frac{2}{3}(2c^2 - b^2)\right) = \frac{M}{5}(b^2 + c^2)$$

by symmetry $I_{22} = \frac{M}{5}(a^2 + c^2)$, $I_{33} = \frac{M}{5}(a^2 + b^2)$

i.e. $(I_{ij}) = \frac{M}{5}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$ if $a = b = c$: $I_{ij} = \frac{2}{5} M \delta_{ij}$

see P&R sheet 3

<u>Proposition</u> If T_{ij} is symmetric, then $\exists$ choice of $\{\underline{e}_i\}$ for which $(T_{ij}) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ (called principal axes)

<u>Proof</u> Direct consequence of fact that any real symmetric matrix can be diagonalised via orthogonal transformation R for which $\det R = 1$ (WLOG)

$$T' = R^{T} T R$$

So can always choose set of axes so that I_{ij} is diagonal.

8.7 Invariant and isotropic tensors

We say that a tensor is <u>isotropic</u> if it is invariant under changes in Cartesian coordinates, i.e.

$$T_{ij\ldots k}' = R_{ip} R_{jq} \ldots R_{kr} T_{pq\ldots r} = T_{ij\ldots k}$$

for <u>any</u> choice of rotation R.

<u>Example</u>

(i) Every scalar is isotropic

(ii) The Kronecker delta is isotropic : $\delta_{ij}' = R_{ip} R_{jq} \delta_{pq} = R_{ip} R_{jp} = \delta_{ij}$

(iii) The Levi-Civita tensor is isotropic :

$$\varepsilon_{ijk}' = R_{ip} R_{jq} R_{kr} \varepsilon_{pqr} = \det R \; \varepsilon_{ijk} = \varepsilon_{ijk}$$

We can classify all isotropic tensors ~~of~~ on $\mathbb{R}^3$.

<u>Proposition</u> Isotropic tensors on $\mathbb{R}^3$ are classified as follows:

(a) All rank 0 tensors are isotropic

(b) There are no non-zero rank 1 tensors that are isotropic

(c) The most general isotropic tensor of rank 2 is $\underline{\alpha \delta_{ij}}$

(d) The most general isotropic tensor of rank 3 is $\underline{\beta \varepsilon_{ijk}}$

(e) The most general isotropic tensor of rank 4 is
$$\alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$$

(f) Rank > 4 is linear combination of products of δ and ε

<u>Sketch Proof</u>

(a) Follows from definition

(b) If v_i are components of isotropic vector then
$v_i = R_{ij} v_j = v_i' \quad \forall R$ rotation.

Take $(R_{ij}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ π about z-axis

Then $v_1 = \underset{R_{1j}}{R_{ij}} v_j = -v_1$, $v_2 = R_{2j} v_j = -v_2$ i.e. $v_1 = v_2 = 0$

Using $(R_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ π about x-axis

$v_3 = R_{3j} v_j = -v_3 \Rightarrow v_3 = 0$. So $\underline{v_i = 0}$.

(c) If T_{ij} isotropic then $T_{ij} = R_{ip} R_{jq} T_{pq}$ for any R

Take R to be rotation $\frac{\pi}{2}$ about each axis

$(R_{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (z axis, $\frac{\pi}{2}$)

then $T_{13} = R_{1p} R_{3q} T_{pq} = R_{12} R_{33} T_{23} = T_{23}$

$T_{23} = R_{2p} R_{3q} T_{pq} = R_{21} R_{33} T_{13} = -T_{13}$

so $T_{13} = T_{23} = 0$

Also $T_{11} = R_{1p} R_{1q} T_{pq} = R_{12} R_{12} T_{22} = T_{22} \Rightarrow T_{11} = T_{22}$

Now choose rotation by $\frac{\pi}{2}$ about x-axis $(R_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Then $T_{32} = R_{3p} R_{2q} T_{pq} = R_{32} R_{23} T_{23} = -T_{23}$ so $T_{32} = 0$, $T_{23} = 0$

$T_{12} = R_{1p} R_{2q} T_{pq} = R_{11} R_{23} T_{13} = -T_{13} = 0 \Rightarrow T_{12} = 0$

$T_{31} = R_{3p} R_{1q} T_{pq} = R_{32} R_{11} T_{21} = -T_{21}$

$T_{21} = R_{2p} R_{1q} T_{pq} = R_{23} R_{11} T_{31} = T_{31}$

i.e. $T_{31} = T_{21} = 0$

Finally $T_{22} = R_{2p} R_{2q} T_{pq} = R_{23} R_{23} T_{33} = T_{33} \Rightarrow T_{22} = T_{33} = T_{11}$

In conclusion $T_{ij} = 0$ if $i \neq j$, $T_{11} = T_{22} = T_{33}$

so $T_{ij} = \alpha \delta_{ij}$ for some scalar α.

(d) and (e) same idea, more indices. $\square$

Consider integral of form $T_{ij\ldots k} = \int_{|\underline{x}|<R} f(r)\, x_i x_j \ldots x_k \, dV(\underline{x})$

where $x_k x_k = r^2$, $dV(\underline{x}) = dx_1\, dx_2\, dx_3$

Note $f(r)$ and $\{\underline{x} : |\underline{x}| < R\}$ are invariant under rotations.

We have: $T'_{ij\ldots k} = \int_{|\underline{x}|<R} f(r)\, x_i' x_j' \ldots x_k' \underbrace{dV(\underline{x})}_{dx_1' dx_2' dx_3'}$

$= \int_{|\underline{x}|<R} f(r)\, R_{ip} x_p R_{jq} x_q \ldots R_{kr} x_r \, dV(\underline{x})$

Make substitution $y_i = R_{ij} x_j$, $dV = dy_1\, dy_2\, dy_3$

$T'_{ij\ldots k} = \int_{|\underline{x}|<R} f(r)\, y_i y_j \ldots y_k \, dV(\underline{y})$

Since $\{y\}$ is dummy variable: $T'_{ij\ldots k} = \int_{|\underline{x}|<R} f(r)\, x_i x_j \ldots x_k \, dV(\underline{x}) = T_{ij\ldots k}$

so $T_{ij\ldots k}$ is <u>isotropic</u>. Take $R \to \infty$ - corresponds to integrating over $\underline{\mathbb{R}^3}$.

<u>Example</u> $T_{ij} = \int_{\mathbb{R}^3} e^{-r^5} x_i x_j \, dV$. By previous $T_{ij} = \alpha \delta_{ij}$.

Contracting on (i,j): $\alpha \delta_{ii} = 3\alpha = \int_{\mathbb{R}^3} e^{-r^5} r^2 \, dV$

$= 4\pi \int_0^\infty r^2 e^{-r^5} r^2 \, dr = \frac{4\pi}{5} \Rightarrow \alpha = \frac{4\pi}{15}$, $T_{ij} = \frac{4\pi}{15} \delta_{ij}$

<u>Example</u> The inertia tensor of a ball of radius R, constant density ρ_0 (mass $M = \frac{4\pi}{3} R^3 \rho_0$)

$$I_{ij} = \int_{|\underline{x}|<R} \rho_0 \,(x_k x_k \delta_{ij} - x_i x_j)\, dV$$

This is a sum of 2 isotropic tensors, so is isotropic.

Hence $I_{ij} = \alpha \delta_{ij}$ for some α. Contract on i and j:

$3\alpha = \int_{|\underline{x}|<R} \rho_0 \left[3r^2 - r^2\right] dV$ ← $r^2 \sin\theta \, dr\, d\theta\, d\phi$

$= 4\pi \rho_0 \cdot 2 \int_0^R r^4 \, dr = \left(\frac{4\pi}{3} \rho_0 R^3\right) \frac{3}{R^3} \cdot 2 \cdot \frac{R^5}{5}$

$= \frac{6MR^2}{5} \Rightarrow \alpha = \frac{2MR^2}{5}$, $\underline{I_{ij} = \frac{2MR^2}{5} \delta_{ij}}$.

8.8 Tensors as multilinear maps and the quotient rule

For a tensor T_{ij} consider bilinear map $t: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by

$t(\underline{a}, \underline{b}) = T_{ij} a_i b_j$ (RHS scalar - well-defined)

so rank 2 tensor $\rightarrow$ bilinear map.

Conversely, suppose $t: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is bilinear. Then for given basis $\{\underline{e}_i\}$ it defines an array T_{ij} via

$$t(\underline{a}, \underline{b}) = t(a_i \underline{e}_i, b_j \underline{e}_j) = a_i b_j t(\underline{e}_i, \underline{e}_j) := a_i b_j T_{ij}$$

If we use different basis $\{\underline{e}_i'\}$ with $\underline{e}_i' = R_{ip} \underline{e}_p$

then by linearity, $T_{ij}' = t(\underline{e}_i', \underline{e}_j') = t(R_{ip} \underline{e}_p, R_{jq} \underline{e}_q)$

$= R_{ip} R_{jq} t(\underline{e}_p, \underline{e}_q) = R_{ip} R_{jq} T_{pq}$ so T_{ij} is a rank 2 tensor

i.e. bilinear map $\rightarrow$ rank 2 tensor.

Have <u>1 to 1 correspondence</u>.

In particular if the map $(\underline{a}, \underline{b}) \mapsto T_{ij} a_i b_j$ is genuinely bilinear, independent of basis, then T_{ij} are components of rank 2 tensor.

Same idea holds for higher rank: if the map

$(\underline{a}, \underline{b}, \dots, \underline{c}) \mapsto T_{ij\dots k} a_i b_j \dots c_k$ genuinely defines a n-multilinear map, then $T_{ij\dots k}$ are components of rank n tensor.

Recall from earlier that we showed σ_{ij} (conductivity tensor) was tensor from definition $\underline{J_i = \sigma_{ij} E_j}$

Could have used quotient theorem.

<u>Proposition</u> Let $T_{i\dots jp\dots q}$ defined in each Cartesian coord system such that $\overbrace{V_{i\dots j}}^{A} := \underbrace{T_{i\dots jp\dots q}}_{A+B} \overbrace{u_{p\dots q}}^{B}$ indices

is a tensor for $\boxed{\text{each}}$ tensor $u_{p\dots q}$. Then $T_{i\dots jp\dots q}$ is a tensor.

__Proof__ Take special case $u_{p\ldots q} = c_p \ldots d_q$ for vectors $\{\underline{c}, \ldots, \underline{d}\}$.

Then $v_{i\ldots j} = T_{i\ldots j p \ldots q} c_p \ldots d_q$ is a tensor and in particular

$v_{i\ldots j} a_i \ldots b_j = T_{i\ldots j p\ldots q} a_i \ldots b_j c_p \ldots d_q$ is a scalar

for each $\{\underline{a}, \ldots, \underline{b}, \underline{c}, \ldots, \underline{d}\}$ so RHS is scalar (indep. of basis)

and gives rise to well-defined multilinear map via

$$t(\underline{a}, \ldots, \underline{b}, \underline{c}, \ldots, \underline{d}) := T_{i\ldots j p \ldots q} a_i \ldots b_j c_p \ldots d_q$$

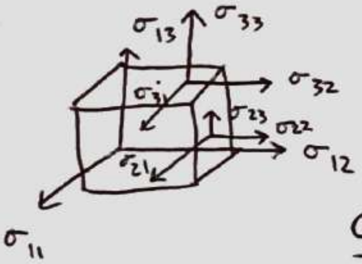
so by previous discussion $T_{i\ldots j p \ldots q}$ is a tensor. $\square$

__Example__ Linear strain tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ where $\underline{u}(\underline{x})$ measures change in displacement at $\underline{x}$.

Experiment suggests that internal forces experienced by a body that has undergone deformation depend linearly on strain at each point.

Stresses are described by a stress tensor σ_{ij}.



So $\exists$ an array of $3^4 = 81$ numbers c_{ijkl} such that $\underline{\sigma_{ij} = c_{ijkl} e_{kl}}$ $(\dagger)$

__CAN'T__ apply quotient theorem: e_{kl} symmetric

If $c_{ijkl} = c_{ijlk}$ then can apply quotient theorem (__see ES4__)

Call this stiffness tensor. ~~Soft~~ Suppose it is a __property__ of the material.

If material is isotropic, expect stiffness tensor to be isotropic

Write:

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{kj}$$

Use in $(\dagger)$: we get

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + \overbrace{\beta e_{ij} + \gamma e_{ji}}^{\text{same - symmetric}} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$$

where $2\mu = \beta + \gamma$.

This is actually a higher dimensional version of Hooke's law ($F = -kx$)

Can invert - contract on (i, j)

$\sigma_{ii} = (3\lambda + 2\mu) e_{ii}$ i.e. $e_{kk} = \dfrac{\sigma_{kk}}{3\lambda + 2\mu}$ $(3\lambda + 2\mu \neq 0)$

Then we get

$$2\mu e_{ij} = \sigma_{ij} - \left(\frac{\lambda}{3\lambda + 2\mu}\right) \sigma_{kk} \delta_{ij}$$

End of course