Lecture Content Overview - Michaelmas Term

This has been updated with all the 96 lectures in Michaelmas term.

Contents

1 Numbers and Sets

- **Lecture 1.** Examples of proofs and non-proofs; basic proof techniques.
- **Lecture 2.** More proofs and non-proofs. Construction of natural numbers with Peano axioms; definition of addition by induction. Overview of arithmetic properties satisfied by addition/multiplication.
- **Lecture 3.** Strong induction and equivalent forms. Construction of integers and rationals. Definition of primes; proof of existence of prime factorisation for all naturals.
- **Lecture 4.** Proof that there are infinitely many primes. Highest common factor definition. Euclid's algorithm for finding HCF, with proof that it does give the HCF.
- **Lecture 5.** Proof that there's a solution to $ax + by = \text{hcf}(a, b)$; Bezout's theorem as a corollary. Applications to solving integer linear equations. Proof that $p|ab \implies p|a$ or $p|b$. Proof of Fundamental Theorem of Arithmetic.
- **Lecture 6.** Applications of Fundamental Theorem of Arithmetic; factors, HCF and LCM. Introduction to modular arithmetic; addition, multiplication and inverses.
- **Lecture 7.** Invertible elements mod p and mod n. The Euler totient function. Proof of Fermat's little theorem and Fermat-Euler theorem. Solutions to $x^2 \equiv 1 \bmod p$.
- **Lecture 8.** Proof of Wilson's theorem. Condition for -1 to be a square mod p. Solving linear congruence equations. The Chinese Remainder Theorem.
- **Lecture 9.** RSA encryption. Proofs that $\sqrt{2}$ is irrational. Construction of the reals, including the least upper bound axiom.
- **Lecture 10.** General remarks on axioms for the reals. Examples of sets and least upper bounds; the intervals $[0, 1]$ and $(0, 1)$. Axiom of Archimedes and related corollary.
- **Lecture 11.** Supremum and infimum of sets. Proof that $\sqrt{2}$ exists. Proof that rationals and irrationals are both dense. Definition of what it means for a sequence to "tend to" a limit.
- **Lecture 12.** Convergent and divergent sequences; proofs and examples. Limits and series; combining the limits of sequences.
- **Lecture 13.** Proof that increasing, bounded-above sequences are convergent. Divergence of harmonic series and convergence of sum of reciprocals of squares. Decimal expansions as the limits of sequences. Remarks on recurrence and uniqueness of decimal expansions.

- **Lecture 14.** Definition of e and proof that it converges. Proof that e is irrational. Definition of algebraic and transcendental. Proof that $\sum_{n=1}^{\infty} 1/10^{n!}$ is transcendental.
- **Lecture 15.** Construction of the complex numbers. Sets introduction: sets, subsets, unions and intersections with proof of distributive property.
- **Lecture 16.** Ordered pairs and power sets. Proofs that a set of size n has exactly 2^n subsets. Introduction to binomial coefficients.
- **Lecture 17.** Proof of binomial coefficient formula. Proof of binomial theorem. Inclusion-exclusion principle. Introduction to functions.
- **Lecture 18.** Injective, surjective and bijective functions, including examples for finite and infinite sets. Composition of functions and proof that composition is associative.
- **Lecture 19.** Invertible functions and bijections. Definition of equivalence relation with properties. Equivalence class introduction; equivalence classes as partitions of a set.
- **Lecture 20.** Proof that equivalence classes partition a set; definition of quotient map. Introduction to countability; definition of a countable set (including $\mathbb{N}$ and $\mathbb{Z}$. Proof that a set is countable iff it injects into $\mathbb{N}$.
- **Lecture 21.** Proofs that $\mathbb{N} \times \mathbb{N}$ is countable. Proof that the union of countable sets is countable. Proof that algebraic numbers are countable. Cantor's diagonal argument: proof that $\mathbb{R}$ is uncountable.
- **Lecture 22.** Proof that the power set of $\mathbb{N}$ is uncountable. Countability of pairwise disjoint open intervals.
- **Lecture 23.** Intuition on sizes of sets and properties. Proof of Schröder-Bernstein Theorem; showing whether sets biject. "Repeated" power sets.
- **Lecture 24 (Non-examinable).** Introduction to generators in ${Z_p}^*$. Propositions and proof that ${Z_p}^*$ has a generator. Other results in number theory that relate to generators.

2 Differential Equations

- **Lecture 1.** Informal outline of limits and derivatives. Review of differentiation techniques. Orders of magnitude: definition of "little o" and "big o" with examples. First two terms of Taylor series from definition of derivative. Link to equation of tangent line.
- **Lecture 2.** Taylor series definition. Error term and generalisation of "first two terms" expression to more terms. Statement of Taylor's theorem. Proof of L'Hôpital's Rule.
- **Lecture 3.** Definite integrals as areas. Mean value theorem statement. Derivation of integral definition from Taylor series and MVT. Proof of Fundamental Theorem of Calculus and corollaries. Review of integration techniques.
- **Lecture 4.** Introduction to multivariate functions. Partial derivatives - definition and notation. Definition of differential. Derivation of multivariate chain rule in differential, standard and integral form; interpretation as "integrating over a path".
- **Lecture 5.** Multivariate chain rule applications. Change of variables. Implicit differentiation. Reciprocal rule. Differentiation of integral with respect to its parameters.
- **Lecture 6.** First order linear ODEs; terminology definitions. The exponential function as an eigenfunction of the differential operator. Rules for nth order linear ODEs.

- **Lecture 7.** Inhomogeneous first order ODEs with constant coefficients; constant forcing and eigenfunction forcing, solving with particular integral and complementary function. First order ODEs with non-constant coefficients; integrating factor method.
- **Lecture 8.** Numerical integration and use of recurrence. Series solutions to ODEs. Nonlinear first order ODEs: separable equations and exact equations, including use of multivariate chain rule.
- **Lecture 9.** Sketching DE solution curves. Isoclines and use in sketching curves. Fixed (equilibrium) points; analysing stability by perturbation analysis. Stable and unstable fixed points. Autonomous DE special case.
- **Lecture 10.** 2D and 1D phase portraits. Examples: chemical kinetics, population dynamics and logistic map. Fixed points in discrete equations: perturbation analysis to show if stable/unstable.
- **Lecture 11.** Linear second order ODEs with constant coefficients. Definition of linear independence for functions. Cases where the auxillary equation has 1 or 2 distinct roots (to include degenerate case).
- **Lecture 12.** Homogeneous second order linear ODEs with non-constant coefficients; reduction of order method. Solutions as vectors in solution space; linear independence and Wronskian of fundamental matrix. Reconstruction of original ODE.
- **Lecture 13.** Abel's theorem and Abel's identity; sketch proof. Bessel's equation: finding Wronskian without solving. Abel's identity applied to find a second solution to an ODE. Solving equidimensional equations.
- **Lecture 14.** Solving forced inhomogeneous second order ODEs: finding particular integrals with guesswork or variation of parameters. Oscillating system ODE; unforced case and graphs of displacement against time.
- **Lecture 15.** Damped oscillating system with forced response (non-homogeneous). Resonance condition and resonance in undamped systems. Detuning and taking limit as $\omega \to \omega_0$. Impulses and point forces; force-time graph and displacement-time graph. Impulse as integral of force over time.
- **Lecture 16.** Dirac delta function, Heaviside step function and ramp function. Solving delta function forcing ODEs with example, including jump conditions. Heaviside step function forcing with jump conditions.
- **Lecture 17.** Solving higher order discrete (difference) equations by finding complementary function and particular integral. Finding a closed form for the Fibonacci sequence. Series solutions for higher order ODEs; introduction to Frobenius method and identifying points.
- **Lecture 18.** Method of Frobenius and use of Fuch's theorem with ordinary point and regular singular point cases. Finding linearly independent series solutions.
- **Lecture 19.** Special cases of the indicial equation with general solutions. Two examples of using Frobenius method/Fuch's theorem to find series solutions.
- **Lecture 20.** Introduction to gradient vector; use of multivariate chain rule. Properties of gradient vector. Types of stationary points with properties. Taylor series for multivariate functions in vector and Cartesian form; Hessian matrix introduction.
- **Lecture 21.** The $n-$dimensional Hessian matrix and diagonalisation. Classifying types of stationary points, including signature of Hessian/Sylvester's criterion. Contours of a function near stationary points. Systems of linear ODEs introduction; writing nth order ODE as a system of first order ODEs.
- **Lecture 22.** Solving systems of linear ODEs using matrix methods. Phase portraits revisited - cases for different eigenvalues. Nonlinear systems of ODEs; perturbation analysis.

- **Lecture 23.** Predator-prey model example with phase portrait. Partial differential equations introduction. Examples: first order wave equation using method of characteristics. Unforced and forced wave equation cases.
- **Lecture 24.** Second order wave equation; factorising differential operator. Derivation of diffusion equation using Taylor expansion. Solving the diffusion equation given initial/boundary conditions.

3 Groups

- **Lecture 1.** Introduction to groups as the study of symmetries. Definition of a group with examples and non-examples.
- **Lecture 2.** Basic properties of group elements. Subgroup criteria including shorter "check". Proof that subgroups of $\mathbb{Z}$ are precisely $n\mathbb{Z}$. Basic subgroup relations.
- **Lecture 3.** Definition of $\langle X \rangle$ for a set X and proof of the form it takes. Group homomorphisms; definition, examples and elementary properties. Isomorphisms and examples.
- **Lecture 4.** Image and kernel of a homomorphism; properties and examples. Direct products of groups and construction of a direct product. The direct product theorem. Introduction to cyclic groups.
- **Lecture 5.** Proof that cyclic groups are isomorphic to $\mathbb{Z}$ or C_n. Dihedral groups: introduction, group axioms and generation from a rotation and reflection. Presentation of groups. Introduction to permutation groups.
- **Lecture 6.** Permutation notation, including cycle notation. Examples of symmetric groups. Proof that disjoint cycles commute. Proof of disjoint cycle decomposition theorem.
- **Lecture 7.** Cycle type; order of permutation as LCM of cycle type lengths. Permutation as product of transpositions. Proof that parity of no. of transpositions is fixed. Definition of $\text{sign}(\sigma)$ and surjective homomorphism proof. Alternating group definition. Consequences of parity result.
- **Lecture 8.** Möbius map definition; proof that such functions form a group under composition. Möbius maps as compositions of other maps. Coset definition and examples.
- **Lecture 9.** Proof of Lagrange's theorem; corollaries. Application to number theory; proof of Fermat-Euler theorem and Fermat's Little Theorem. Investigating subgroups of small groups by Lagrange.
- **Lecture 10.** Investigating groups of small order, particularly $|G| = 4$ and $|G| = 6$. Definition of normal subgroup, with examples. Proof that subgroups of abelian groups, and subgroups of index 2 are normal. Proof that kernel of a homomorphism is a normal subgroup; examples.
- **Lecture 11.** Cosets acting as a group; proof that operation is well-defined. Quotient group definition and examples. Relation to property of being a normal subgroup. Proof that normal subgroups are exactly the kernels of homomorphisms.
- **Lecture 12.** First isomorphism theorem and examples. Correspondence theorem and subgroups of $C_4 \times C_2$. Second and third isomorphism theorems.
- **Lecture 13.** More quotient group examples (using isomorphism theorems). Simple group definition. Group actions introduction, axioms and examples. Proof that group actions are bijective; connections to $\text{Sym}(X)$ including homomorphism proposition with more examples.
- **Lecture 14.** Definitions of orbit and stabiliser with examples. Proof that stabiliser is a subgroup of G. Proof that orbits partition the set. Proof of Orbit-Stabiliser Theorem and corollaries.

- **Lecture 15.** Symmetries of the tetrahedron and the cube. Statement and proof of Cauchy's theorem.

- **Lecture 16.** Left regular action; proof of Cayley's theorem. Conjugation actions; examples in D_{2n} and $GL_n\mathbb{R}$. Definitions of centre, conjugacy class and centraliser. Action of a group on itself by conjugation.

- **Lecture 17.** Proof that normal subgroups are unions of conjugacy classes. Conjugation of $k-$cycles in S_n; proof that conjugate elements have the same cycle type. Conjugacy classes of S_4 and use to find normal subgroups and quotients. Remarks on elements conjugate in S_n but not in A_n.

- **Lecture 18.** Conjugacy classes in S_n and A_n and splitting conjugacy classes. Examples: A_4 and A_5 (tables). Proof that A_5 is a simple group. Möbius groups as actions on $\hat{\mathbb{C}}$. Proof that only the identity map fixes 3 distinct points. Proof that two Möbius maps that coincide on 3 distinct points are equal.

- **Lecture 19.** Proof that there is a unique Möbius map sending any 3 distinct points to any 3 distinct points. Conjugates of Möbius maps. Proof that every non-identity map has 1 or 2 fixed points (conjugate to $z \mapsto z+1$ and $z \mapsto az$ respectively). Circles and lines in $\hat{\mathbb{C}}$. Proof that Möbius maps send circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$.

- **Lecture 20.** Cross-ratio definition and proof that it's fixed by double transpositions. Proof that Möbius maps preserve the cross-ratio. Proof that four distinct points in $\hat{\mathbb{C}}$ lie on a circle iff the cross-ratio is real. Matrix groups introduction; determinant as a surjective homomorphism. Möbius maps via matrices. Actions of matrix groups.

- **Lecture 21.** Change of basis matrices. Action of general linear group on $n \times n$ matrices by conjugation. Jordan normal form examples and conjugation. Orbits and stabilisers for action of $GL_2(\mathbb{C})$. Geometrical interpretations for orthogonal groups.

- **Lecture 22.** Proof that SO_n consists of rotations and $O_n \setminus SO_n$ consists of reflections; corollaries of results. Proof that elements of O_2 and O_3 are the composition of at most 2 or 3 reflections respectively. Symmetries of Platonic solids as subgroups of orthogonal groups. Quaternion group introduction.

- **Lecture 23.** Proof that a group with all non-identity elements of order 2 is isomorphic to $C_2 \times ... \times C_2$. Classification of all groups of order 8.

- **Lecture 24 (Non-examinable).** Non-examinable introduction to geometric group theory. Free groups and group presentations with examples. Cayley graphs - examples and properties.

4 Vectors and Matrices

- **Lecture 1.** Complex numbers; arithmetic and properties. Polar and exponential form. Argand diagrams. Composition property and triangle inequality.

- **Lecture 2.** Series definitions of exp, sin and cos for complex numbers. Proof of indices multiplication rule. Properties of e^z and roots of unity. Complex logarithms and powers. Transformations, lines and circles in the complex plane.

- **Lecture 3.** Vectors in 3 dimensions; addition and scalar multiplication with properties. Definition of span. Scalar (dot) and vector (cross) product; definition, interpretations and algebraic properties.

- **Lecture 4.** Orthonormal bases and component form. Scalar and vector triple products with interpretations. Vector equations for lines and planes (parametric and non-parametric). Other vector equations with geometrical interpretations.

- **Lecture 5.** Vectors in component (index) form. Kronecker delta and Levi-Civita epsilon; dot and cross products in this form. Summation convention with examples. Summation convention rules; proof of vector triple product identity. Contracted epsilon identities and derivations.

- **Lecture 6.** Vectors in $\mathbb{R}^n$. Inner product and properties. Cauchy-Schwarz and triangle inequalities. Additional comments on definitions of products using δ and ϵ. Axioms for vector spaces.
- **Lecture 7.** Vector subspaces with examples. Linear dependence and independence with examples. Generalisation of inner product. Proof that nonzero orthogonal vectors are linearly independent. Bases and dimension; proof of the dimension theorem.
- **Lecture 8.** (Non-examinable) proposition relating dimension of vector space to spanning subset and linearly independent subset. Infinite-dimension vector space example. Complex vector spaces: axioms and properties including inner product.
- **Lecture 9.** Linear map definition. Kernel and image of a linear map - examples. (Non-examinable) proof of Rank-Nullity Theorem. Geometrical linear transformations: rotations, reflections, dilations and shears in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- **Lecture 10.** Matrices as linear maps; definition of matrix multiplication; examples. Geometrical examples: rotations, dilations, reflections and shears. Construction of matrices for these transformations by considering action on basis vectors or components. Matrix of a general linear map.
- **Lecture 11.** Matrix algebra: linear combinations and matrix multiplication using index notation. Properties of matrix products and helpful points of view. Matrix inverses; construction of 2×2 inverse and geometrical examples.
- **Lecture 12.** Transpose, Hermitian conjugate and trace of a matrix. Orthogonal and unitary matrices; properties and preservation of inner product. Finding a general 2×2 orthogonal matrix.
- **Lecture 13.** Determinants and inverses: finding the determinant of a matrix explicitly. Generalised "scalar product" construction in n dimensions. Alternating symbol definition and properties.
- **Lecture 14.** Check that generalised "scalar product" is antisymmetric. Proof that $[\mathbf{v}_1, ..., \mathbf{v}_n] \neq 0$ iff vectors are linearly independent. Numerical example of evaluating this expression. Definition of determinant in $\mathbb{R}^n$ or $\mathbb{C}^n$ and examples. Proof that $\det M = \det M^{\mathrm{T}}$. Evaluating determinants by expanding rows or columns.
- **Lecture 15.** Simplifying determinants by row and column operations. Proof of multiplicative property of determinants with consequences. Minors, cofactors and inverses and use to evaluate determinants. Adjugates to evaluate inverses.
- **Lecture 16.** Systems of linear equations; possibilities for number of solutions with conditions. Solving homogeneous problem and finding particular solution with example. Geometrical interpretations in $\mathbb{R}^3$.
- **Lecture 17.** Eigenvalues and eigenvectors definition. Characteristic equation and polynomial with examples. Trace as sum of eigenvalues and determinant as product of eigenvalues. Eigenspace definition and examples; algebraic and geometric multiplicity.
- **Lecture 18.** Proof that if eigenvalues are distinct, then eigenvectors are linearly independent; forming a basis for the eigenspace. Conditions for diagonalisability and similarity with examples.
- **Lecture 19.** Proof that eigenvalues of Hermitian matrices are real and that eigenvalues with distinct values are orthogonal. Forming an orthonormal basis with examples. Statement that Hermitian matrices are diagonalisable (with related conditions - proof not in this lecture).
- **Lecture 20.** Quadratic forms introduction and diagonalisability. Examples in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- **Lecture 21.** (Non-examinable) proof of Cayley-Hamilton theorem. Changes of basis in general; finding change of basis matrices.

- **Lecture 22.** Change of basis; vector components with special cases. Jordan canonical/normal form in 2×2 case. Generalisation to higher dimensions; statement of Jordan normal form matrix for $n \times n$ matrices (without proof).
- **Lecture 23.** Introduction to quadrics. Conics as quadrics in $\mathbb{R}^2$; cases with zero and nonzero determinant. Conic sections in Cartesian and polar coordinates; conics as sections of a cone.
- **Lecture 24.** Symmetries and transformation groups. Orthogonal transformations and rotations in $\mathbb{R}^n$. 2D Minkowski space and Lorentz transformations: definition of Minkowski metric and properties including Lorentz group. Physical interpretation in special relativity. (Non-examinable) overview of length contraction and time dilation.