Differential Equations Methods

Abigail Tan

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This quick-reference document summarises methods only without derivations, and excludes most content found in the A level course.

Contents

1 Partial Derivatives

1.1 Multivariate Chain Rule

Given $f(x(t), y(t))$ we have the standard form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

which generalises to a larger number of variables. We also have the differential form

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y.$$

Application. Given $f(x, y, z(x, y)) = c$, using the multivariate chain rule we can obtain the following.

$$\left.\frac{\partial z}{\partial x}\right|_y = \frac{-\left.\frac{\partial f}{\partial x}\right|_{yz}}{\left.\frac{\partial f}{\partial z}\right|_{xy}}$$

1.2 Classifying Stationary Points

Let $f(x_1, x_2, ..., x_n)$ be a function of n variables. Then the following matrix defines the Hessian matrix.

$$H = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}$$

We write H_k for the $k \times k$ matrix obtained by taking the top left $k \times k$ grid from H, so for example

$$H_1 = f_{x_1x_1},\ H_2 = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{pmatrix}, \ldots, H_n = H.$$

We write the *signature* of H as $|H_1|, |H_2|, \ldots, |H_{n-1}|$. The following signatures correspond to the following stationary points.

- $+, +, +, \ldots$ minimum
- $-, +, -, \ldots$ maximum
- Otherwise it is a saddle point.

2 Solving Ordinary Differential Equations

2.1 Exact Equations (nonlinear 1st order)

For a nonlinear first order ODE

$$Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x, y) = 0,$$

this is an exact equation iff

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

If this holds, then the solution takes the form $f(x, y) = c$ for some function f satisfying

$$\frac{\partial f}{\partial x} = P(x, y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = Q(x, y).$$

Solve this by integrating.

2.2 Reduction of Order (linear homogeneous 2nd order)

Suppose we have one solution $y_1(x)$ for the following homogeneous 2nd order linear ODE with non-constant coefficients:

$$y'' + p(x)y' + q(x)y = 0.$$

To find a second linearly independent solution, we try a solution of the form $y_2(x) = v(x)y_1(x)$. We obtain $v(x)$ using the following method. Set $u(x) = v'(x)$, then u is the solution of the first order ODE

$$u'y_1 + u(2y_1' + py_1) = 0.$$

Then integrate to find $v(x)$ and hence obtain $y_2(x)$.

2.3 Abel's Identity (linear homogeneous 2nd order)

(A special case of) Abel's identity states that for a homogeneous 2nd order linear ODE with non-constant coefficients

$$y'' + p(x)y' + q(x)y = 0,$$

we have

$$W(x) = W(x_0)\, e^{-\int_{x_0}^{x} p(u)\,\mathrm{d}u}$$

where $W(x)$ is the Wronskian. Let y_1 and y_2 be linearly independent solutions to the ODE. Since $W(x) = y_1y_2' - y_2y_1'$, if we know one solution y_1 then

$$y_1y_2' - y_2y_1' = W_0\, e^{-\int_{x_0}^{x} p(u)\,\mathrm{d}u}$$

gives a first order ODE that we can solve to obtain the second solution y_2.

2.4 Equidimensional Equations (2nd order)

An ODE is equidimensional if the differential operator is unaffected by a multiplicative scaling. The general form for second order is

$$ax^2y'' + bxy' + cy = f(x).$$

To solve, try $y = x^k$ which gives a quadratic in k:

$$ak(k-1) + bk + c = 0.$$

Then solve this to find the two roots k_1 and k_2. If $k_1 \neq k_2$ then we have complementary function

$$y_c = Ax^{k_1} + Bx^{k_2}$$

and if $k_1 = k_2 = k$ then

$$y_c = Ax^k + Bx^k \ln x.$$

2.5 Dirac Delta Function Forcing (2nd order)

For this it may be easier to refer to a worked example, but the basic information is summarised here.

Consider the second order ODE

$$y'' + p(x)y + q(x)y = \delta(x).$$

First solve the homogeneous equation for both $x < 0$ and $x > 0$ (replace x by $x - x_0$ if that's the form the equation is given in). This should give 4 unknown constants. To find these constants, use the following jump conditions.

1. $y(x)$ is continuous at $x = 0$:

$$\lim_{\epsilon \to 0} \left[y\right]_{x=-\epsilon}^{x=\epsilon} = 0$$

2. $y'(x)$ has a jump of 1 at $x = 0$:

$$\lim_{\epsilon \to 0} \left[y'\right]_{-\epsilon}^{\epsilon} = 1$$

2.6 Heaviside Step Function Forcing (2nd order)

Consider the ODE

$$y'' + p(x)y' + q(x)y = H(x - x_0).$$

First solve the equations

$$y'' + py' + qy = 0 \qquad (x < x_0)$$

and

$$y'' + py' + qy = 1 \qquad (x > x_0).$$

Then to find the unknown constants, use the following jump conditions, which come from y and y' both being continuous at x_0.

$$\lim_{\epsilon \to 0} \left[y\right]_{x_0-\epsilon}^{x_0+\epsilon} = 0$$

$$\lim_{\epsilon \to 0} \left[y'\right]_{x_0-\epsilon}^{x_0+\epsilon} = 0$$

2.7 Series Solutions - Method of Frobenius (linear homogeneous 2nd order)

We might want to find a power series solution (expanded about $x = x_0$) for the equation

$$p(x)y'' + q(x)y' + r(x)y = 0.$$

First, classify the point $x = x_0$. This is:

- an ordinary point if the Taylor series for q/p and r/p converge around x_0
- a regular singular point if q/p has a pole/singularity up to order 1 at x_0 and r/p has a pole up to order 2 at x_0
- an irregular singular point otherwise.

Ordinary points. There are 2 linearly independent solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Write

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,\ y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},\ y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

and substitute back into the ODE (multiplying up by a power of x if necessary). Then equate the coefficients of $(x - x_0)^n$ for $n \geq 2$ (setting them equal to 0) to obtain a recurrence for a_n, which can be solved to find the coefficients in the series.

Regular singular points. There is at least one solution of the following form, where $a_0 \neq 0$ and $\sigma \in \mathbb{R}$:

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma}.$$

Again, write y, y' and y'' in series form, substitute back into the ODE, and equate coefficients of $(x - x_0)^{n+\sigma}$ to find a recurrence for a_n. Then to find σ, equate the coefficients of the lowest power of $x - x_0$ (usually $(x - x_0)^\sigma$) to obtain the indicial equation for σ. We can then use these values of σ to solve the recurrences for a_n.

There are special cases of the indicial equation.

Case 1. $\sigma_1 - \sigma_2$ not an integer: 2 linearly independent solutions

$$y = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n + (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

Case 2. $\sigma_1 - \sigma_2$ is an integer: linearly independent solutions take the following forms (c may be 0).

$$y_1 = (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$y_2 = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} b_n (x - x_0)^n + c y_1 \ln(x - x_0).$$

Case 3. $\sigma_1 = \sigma_2$: same as above but with $\sigma = \sigma_1 = \sigma_2$ and $c \neq 0$.

2.8 Matrix Methods (systems of linear ODEs)

We can write

$$\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

to represent n solutions to an nth order ODE in vector form. Then the equation

$$\dot{\mathbf{Y}} = M\mathbf{Y} + \mathbf{F}$$

represents a system of n linear ODEs (where M is a matrix and $\mathbf{F}$ is a forcing term).
First solve the homogeneous equation $\dot{\mathbf{Y}} = M\mathbf{Y}$ to find the complementary function. We try the form $\mathbf{Y}_c = \mathbf{v}e^{\lambda t}$ where $\mathbf{v}$ and λ are the eigenvectors and corresponding eigenvalues for M. We can then write $\mathbf{Y}_c$ as a linear combination of the solutions we obtain.
We then find the particular integral using the form of $\mathbf{F}$ (same ideas as in section 4 apply). For example if

$$\mathbf{F} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^t$$

then we may try

$$\mathbf{Y}_p = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} e^t$$

and plug into the equation to solve for u_1 and u_2. The general solution is then $\mathbf{Y} = \mathbf{Y}_c + \mathbf{Y}_p$.

3 Perturbation Analysis

3.1 Stability of Fixed Points

A fixed point of

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t)$$

is a value $y = a$ such that $f(a, t) = 0$ for all t.
To check whether it's a stable or unstable fixed point, let $\epsilon(t)$ be the solution to the first order ODE

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \epsilon \frac{\partial f}{\partial y}(a, t).$$

If $\lim_{t\to\infty} \epsilon = 0$ then it's a stable fixed point.
If $\lim_{t\to\infty} \epsilon = \pm\infty$ then it's an unstable fixed point.

3.2 Autonomous Differential Equations - Stability

This is a special case of subsection 3.1 where

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$$

i.e. f is a function independent of t. In this case, if $f'(a) < 0$ then it is a stable fixed point, and if $f'(a) > 0$ then it is an unstable fixed point.

3.3 Fixed Points in Discrete Equations - Stability

For a first order discrete equation of the form

$$x_{n+1} = f(x_n)$$

a fixed point x_f is the value of x_n where $x_{n+1} = x_n$. Then

$$\left|\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_f}\right| < 1 \implies x_f \text{ is stable and}$$

$$\left|\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_f}\right| > 1 \implies x_f \text{ is unstable.}$$

3.4 Fixed Points in Systems of ODEs - Stability

Consider an autonomous system of two nonlinear first order ODEs. (Note f, g are nonlinear functions which are independent of t).

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

Given a fixed point (x_0, y_0) with $\dot{x}$ and $\dot{y}$ both zero at this point, let

$$M = \left.\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\right|_{x_0, y_0}$$

and then the eigenvalues λ_1, λ_2 of M determine the type of fixed point as follows.

- λ_1, λ_2 real and $\lambda_1\lambda_2 < 0$: saddle node
- λ_1, λ_2 real and $\lambda_1\lambda_2 > 0$:
  - $\lambda_1, \lambda_2 < 0$: stable node
  - $\lambda_1, \lambda_2 > 0$: unstable node
- λ_1, λ_2 a complex conjugate pair:
  - $\mathrm{Re}(\lambda_1, \lambda_2) < 0$: stable spiral
  - $\mathrm{Re}(\lambda_1, \lambda_2) > 0$: unstable spiral
  - $\mathrm{Re}(\lambda_1, \lambda_2) = 0$: centre

4 Finding Particular Integrals

4.1 Guesswork

Given a forcing function $f(x)$ on one side of a second order inhomogeneous ODE, there are some forms that we can guess for the particular integral.

Form of $f(x)$	Form of $y_p(x)$
e^{mx}	Ae^{mx}
$\sin kx, \cos kx$	$A\sin kx + B\cos kx$
$x^n, P_n(x)$ (polynomial)	$a_n x^n + a_{n-1}x^{n-1} + ... + a_0$

4.2 Variation of Parameters

We use this to find the particular integral y_p given that we know the complementary function. For 2nd order, suppose $y_1(x)$ and $y_2(x)$ are two linearly independent complementary functions, with $W(x)$ the determinant of their fundamental matrix. Then (with $f(x)$ the forcing function) we have

$$y_p = y_2 \int^x \frac{y_2(t)f(t)}{W(t)}\,\mathrm{d}t - y_1 \int^x \frac{y_2(t)f(t)}{W(t)}\,\mathrm{d}t.$$

5 Solving Difference Equations

5.1 Inhomogeneous Equations (second order)

Consider the difference equation

$$a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = f_n.$$

To find the complementary function $y_c(n)$, we try $y_n = k^n$ for some k, which leads to the quadratic

$$a_2 k^2 + a_1 k + a_0 = 0.$$

This can be solved for roots k_1 and k_2 which leads to

$$y_c(n) = \begin{cases} A{k_1}^n + B{k_2}^n & \text{if } k_1 \neq k_2 \\ Ak^n + Bnk^n & \text{if } k_1 = k_2 = k. \end{cases}$$

The following table gives the common forms for particular integrals.

Form of f_n	Form of $y_n(p)$
k^n	Ak^n for $k \neq k_1, k_2$
${k_1}^n, {k_2}^n$	$Ank_1{}^n + Bnk_2{}^n$
n^p	$An^p + Bn^{p-1} + ... + Cn + D$

6 Solving Partial Differential Equations

6.1 Unforced First Order Wave Equation

Given $y(x,t)$ where

$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0,$$

subject to the initial condition $y(x,0) = f(x)$ we have that the general solution is $y = f(x+ct)$.

6.2 Forced First Order Wave Equation

Here we have

$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = f(t)$$

subject to $y(x, 0) = g(x)$. Using the multivariate chain rule, we can see that along paths where $\mathrm{d}x/\mathrm{d}t = -c$ (giving $x = x_0 - ct$) we have $\mathrm{d}y/\mathrm{d}t = f(t)$.
Then integrate $f(t)$ to find an expression for y. To find the "constant", use the initial conditions $x = x_0$ and $y(x, 0) = g(x)$, taking care to write $x_0 = x + ct$ in the final expression to consider all possible such paths.

6.3 Second Order Wave Equation

This is

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0.$$

Here the solution takes the general form

$$y = f(x + ct) + g(x - ct)$$

for some functions f and g, which can be found using the initial conditions.

6.4 Diffusion Equation

This equation takes the form

$$\frac{\partial y}{\partial t} = \kappa\frac{\partial^2 y}{\partial x^2}$$

subject to initial/boundary conditions.
We first define the similarity variable

$$\beta = \frac{x^2}{4\kappa t}$$

and then look for solutions of the form

$$y = t^{-\alpha} f(\beta).$$

We can plug this into the original PDE, which leads to the following ODE for $f(\beta)$:

$$\alpha f + \beta f' + \beta f'' + \frac{f'}{2} = 0$$

or equivalently

$$\beta\frac{\mathrm{d}}{\mathrm{d}\beta}(f + f') + \frac{1}{2}(f' + 2\alpha f) = 0$$

and since α is arbitrary, we can choose $\alpha = 1/2$ and set $F = f + f'$ to give

$$\beta\frac{\mathrm{d}F}{\mathrm{d}\beta} + \frac{F}{2} = 0.$$

We can then just solve this to find $F = f + f'$ and then use the initial/boundary conditions to find the "constant" that arises and hence the general solution for y.