Lecture Content Overview - Lent Term

This has been updated with all the 96 lectures in Lent term. Here's a clown to brighten up your day.

Contents

1 Analysis I

- **Lecture 1.** Review of limits and convergence from Numbers and Sets. Fundamental axiom of the real numbers/monotone convergence. Proof of basic properties of limits. Proof that $1/n \to 0$ as $n \to \infty$.
- **Lecture 2.** Proof of Bolzano-Weierstrass theorem. Cauchy sequences; proof that sequences converge iff they are Cauchy sequences. Introduction to series.
- **Lecture 3.** Geometric series and proof that it converges iff common ratio x satisfies $|x| < 1$. Proof that if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. Proof that harmonic series diverges. Comparison test and proof that $\sum_{n=1}^{\infty}(1/n^2)$ converges. Root test and ratio test.
- **Lecture 4.** Examples for ratio and root tests. Cauchy's condensation test and application to Riemann zeta function convergence. Alternating series definition; alternating series test.
- **Lecture 5.** Proof that absolutely convergent series are convergent. Conditionally convergent series; rearrangements. Proof that absolutely convergent series have the same sum regardless of rearrangement.
- **Lecture 6.** Equivalent definitions of continuity (in terms of sequences or $\varepsilon - \delta$). Basic properties of continuous functions and examples.
- **Lecture 7.** Definition of limit points and isolated points. Proof of Intermediate Value Theorem. Application to showing existence of nth root of a positive real number.
- **Lecture 8.** Proof that a continuous function on a closed interval is bounded on the interval. Proof that a continuous function on a closed bounded interval is bounded and attains its bounds. Proof that a function continuous and strictly increasing on a closed interval has a continuous and strictly increasing inverse.
- **Lecture 9.** Definition of differentiability; remarks and interpretation of derivative as a linear approximation. Differentiability of $f(x) = |x|$. Differentiation of sums, products and quotients from limit properties.

- **Lecture 10.** Derivative of x^n for $n \in \mathbb{Z}$. Proof of chain rule. Proof of Rolle's theorem and Mean Value Theorem.

- **Lecture 11.** Corollaries of Mean Value Theorem. Proof of inverse function theorem; application to derivative of x^r for rational r. Cauchy's mean value theorem statement.

- **Lecture 12.** Proof of Cauchy's mean value theorem; application to L'Hôpital's rule. Taylor's theorem with Lagrange's remainder and Cauchy's remainder.

- **Lecture 13.** Proof that (Lagrange and Cauchy) remainder terms in "Taylor series" tend to 0 for binomial series expansion. Remarks on complex differentiation: proof that $f(z) = \bar{z}$ is nowhere $\mathbb{C}-$differentiable.

- **Lecture 14.** Proof that every power series has a radius of convergence R with $0 \leq R \leq \infty$. Finding radius of convergence from ratio test or root test. Examples of series and radius of convergence.

- **Lecture 15.** (Non-examinable) proof that a power series function is differentiable at all points within its radius of convergence (i.e. term-by-term differentiation works). Standard functions introduction; properties of $\exp x$ from power series.

- **Lecture 16.** Proof of properties of $\exp(x)$, $\log(x)$ and x^α.

- **Lecture 17.** Remark on "exponentials beat polynomials" limits. Definition of trigonometric functions and proof of standard properties including periodicity.

- **Lecture 18.** Remark on definition of angle. Hyperbolic functions definition. Upper and lower sums and proof of inequalities satisfied; definition of upper and lower integral and Riemann integrability.

- **Lecture 19.** Criterion for Riemann integrability; proof that monotonic functions and continuous functions are integrable.

- **Lecture 20.** Proof that $f(x) = 1/q$ for $x = p/q \in (0, 1]$, $f(x) = 0$ elsewhere is Riemann integrable. Proofs of elementary properties of integrals.

- **Lecture 21.** Proof of "splitting up integrals" property. Proof of Fundamental Theorem of Calculus and corollaries.

- **Lecture 22.** Proof of integration by parts and integration by substitution formulas. Mean value theorem for integrals. Taylor's theorem with integral remainder; obtaining Cauchy's remainder and Lagrange's remainder.

- **Lecture 23.** Improper integrals and examples. Integral test for convergence of series.

- **Lecture 24.** Examples using integral test for series. Piecewise continuous functions. Non-examinable: f is Riemann integrable iff the set of discontinuity points has measure zero.

2 Probability

- **Lecture 1.** Definition of $\sigma-$algebra, probability measure and probability space with examples. Combinatorial analysis examples - partitioning sets and counting increasing/strictly increasing functions.

- **Lecture 2.** Statement of Stirling's formula; proof of weaker statement and non-examinable proof of Stirling's formula. Countable subadditivity property.

- **Lecture 3.** Continuity of probability measures. Inclusion-exclusion formula with Bonferroni inequalities. Counting number of surjections from $\{1, ..., n\}$ to $\{1, ..., m\}$.

- **Lecture 4.** Counting number of derangements. Independence and conditional probability. Law of total probability and Bayes' formula.
- **Lecture 5.** Example applications of Bayes' formula: false positives and probability of having 2 boys given different amounts of information. Aggregating data; Simpson's paradox.
- **Lecture 6.** Discrete probability distributions: Bernoulli, binomial, multinomial, geometric and Poisson distributions. Derivation of Poisson distribution function from binomial distribution. Random variable definition.
- **Lecture 7.** Discrete random variable and probability mass function (pmf) introduction. Independence of discrete random variables. Expectation of non-negative discrete random variables; binomial and Poisson examples. Expectation for general discrete random variables. Properties of expectation.
- **Lecture 8.** Countable additivity for expectation; expectation of random variable that is a function of another. Proof of inclusion-exclusion using expectation. Variance definition and properties; variance of binomial and Poisson distributions.
- **Lecture 9.** Covariance definition and properties. Covariance of independent variables. Proof of Markov's inequality, Chebyshev's inequality and Cauchy-Schwarz inequality.
- **Lecture 10.** Proof of Jensen's inequality; special cases of equality. Proof of AM-GM inequality using Jensen's inequality. Conditional expectation and proof of law of total expectation.
- **Lecture 11.** Joint distribution and conditional distribution. Distribution of sum of independent random variables. Conditional expectation definition, examples and properties.
- **Lecture 12.** More conditional expectation identities. Random walks: hitting and absorption times by solving recurrences.
- **Lecture 13.** Probability generating functions; proof that pgf uniquely determines distribution. Expectation as $\lim_{z\to 1^-} p'(z)$. Obtaining probability distribution from pgf; pgf of independent random variables. Examples of pgf for binomial, geometric and Poisson distributions. Sum of a random number of random variables.
- **Lecture 14.** Branching processes and recurrence; proofs of properties of expectations and generating functions. Probability of extinction; proof that extinction probability is the minimal non-negative solution to $t = G(t)$.
- **Lecture 15.** Continuation of proof from lecture 14; proof that $q < 1$ iff $\mathbb{E}(X_1) > 1$. Proof of properties of continuous random variables; probability density function for discrete and continuous functions.
- **Lecture 16.** Expectation and variance of continuous random variable. Examples for uniform, exponential and normal distribution, including memoryless property of exponential distribution.
- **Lecture 17.** Distribution of a linear function of a normal random variable. Median definition. Multivariate density functions and independence; density of sum of independent random variables.
- **Lecture 18.** Conditional density and law of total probability for continuous random variables. Transformation of a multidimensional RV; statement of theorem with examples. Order statistics for a randm sample; density and examples (including exponential).
- **Lecture 19.** Moment generating functions for continuous random variables. Statement that mgf uniquely determines distribution provided it is defined on some open interval. Examples for gamma, normal and Cauchy distributions. Multivariate mgf and continuity properties. Weak law of large numbers proof.

- **Lecture 20.** Proof that if $X_n \to 0$ almost surely, then $X_n \to 0$ in probability. Non-examinable proof of strong law of large numbers. Proof of central limit theorem. Applications to binomial and Poisson approximations.

- **Lecture 21.** Sampling error via central limit theorem. Buffon's needle and estimating π. Bertrand's paradox. Multidimensional Gaussian random variable definition and example.

- **Lecture 22.** Mean and variance for Gaussian vectors. Moment generating function for multidimensional case. Construction of Gaussian vectors and finding density function.

- **Lecture 23.** Independence for Gaussian vectors. Properties of bivariate Gaussian. Non-examinable overview of multivariate central limit theorem.

- **Lecture 24.** Simulation of random variables. Proof that $F^{-1}(U) = F$ for uniform U. Rejection sampling.

3 Vector Calculus

- **Lecture 1.** Introduction to parametrised curves in $\mathbb{R}^3$. Finding the length of a parametrised curve. Integrating functions along curves. Proof that arc length does not depend on parametrisation.

- **Lecture 2.** Parametrisation of a curve as $\mathbf{r}(s)$ using arc length. Tangent, principal normal and binormal vectors. Definition of curvature and torsion; statement that they define a curve up to translation/orientation. Radius of curvature. Non-examinable introduction to Gaussian curvature.

- **Lecture 3.** Differentials and first order changes. General coordinates; finding orthonormal basis. Line elements and scale factors. Orthogonal curvilinear coordinates; cylindrical and spherical polar coordinates.

- **Lecture 4.** Gradient operator and directional derivative; gradient operator as a vector differential operator. Application of $\nabla f \cdot \mathrm{d}\mathbf{x} = \mathrm{d}f$. Finding ∇f in other orthogonal curvilinear coordinate systems.

- **Lecture 5.** Line integrals and closed curves with examples. Conservative forces and exact differentials. Proof that line integral over a closed curve is zero for an exact differential form. Proof that exact $\implies$ closed for differential forms. Interpretation as work done, with conservation of energy.

- **Lecture 6.** Integration over regions in $\mathbb{R}^2$. Using a change of variables for integral in 2 dimensions; use of Jacobian. Examples using change of variables.

- **Lecture 7.** Integration over regions in $\mathbb{R}^3$. Change of variables in 3 dimensions; Jacobian in cylindrical and spherical polar coordinates. Examples of calculating volumes. Non-examinable introduction to concept of Lebesgue integration.

- **Lecture 8.** Integration over surfaces in $\mathbb{R}^3$. Normals to a surface; boundary orientation convention. Finding areas of surfaces; scalar and vector area element. Proof that surface integral is independent of parametrisation.

- **Lecture 9.** Definitions of divergence, curl and Laplacian. Identities involving these functions. Divergence and curl in a general OCC system. Definition of Laplacian of a vector field.

- **Lecture 10.** Proof that $\nabla \times \nabla f = 0$ and $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. Condition for $\mathbf{F}$ to be irrotational or solenoidal. Non-examinable: showing that $\mathbb{R}^3 \setminus \{(0, 0, z) : z \in \mathbb{R}\}$ is not simply connected. Green's theorem statement; rectangular case. Example for finding area of ellipse.

- **Lecture 11.** Stokes' theorem statement and examples. Interpretation of $\nabla \times \mathbf{F}$ as a measure of infinitesimal circulation per unit area with examples. Non-examinable: Möbius strips and Stokes' theorem (example of non-orientable surface where it doesn't work).
- **Lecture 12.** Divergence theorem statement (in 3D and 2D) with examples. Proof that divergence of a vector field is 0 if integral over every closed surface is 0. Examples; interpretation of divergence as infinitesimal flux per unit volume. Conservation laws.
- **Lecture 13.** Sketch proofs of divergence theorem, Green's theorem and Stokes' theorem.
- **Lecture 14.** Maxwell's equations for electromagnetism: statement and integral formulations. Electric and magnetic fields satisfying the wave equation. Time independent field case. Non-examinable discussion of gauge invariance.
- **Lecture 15.** Boundary value problem: Dirichlet problem and Neumann problem versions. Proof that Dirichlet problem solution is unique and Neumann problem solution is unique up to a constant. Examples including Newton's shell theorem.
- **Lecture 16.** Gauss's flux method for solving Poisson's equation; example with Maxwell's first equation for electric charge. Cylindrical symmetry and examples.
- **Lecture 17.** Superposition principle and examples. Integral solutions for the Dirichlet problem.
- **Lecture 18.** Harmonic functions as solutions to Laplace's equation. Sketch proof for mean value property of harmonic functions. Proof of limit for Laplacian (what exactly do you call this?). Proof that if ϕ is harmonic then ϕ cannot have a maximum at any interior point unless ϕ is constant.
- **Lecture 19.** Cartesian tensors: transforming scalars, vectors and linear maps from one orthonormal right-handed basis to another. Rank 0, rank 1 and rank 2 tensors.
- **Lecture 20.** Definition of rank n Cartesian tensors. Constructing rank n tensors from vectors. Examples of tensors. Constructing tensors from tensor product and from contracting indices.
- **Lecture 21.** Symmetric and antisymmetric tensors with examples. Tensor calculus; transformation of derivative. Constructing higher rank tensors with derivatives. Examples including "general" divergence theorem.
- **Lecture 22.** Symmetric and antisymmetric parts of rank 2 tensors. Deformation of elastic body example. Inertia tensor and example with ellipsoid. Existence of principal axes where tensor is diagonal for symmetric tensors.
- **Lecture 23.** Definition and examples of isotropic tensors. Classification of isotropic tensors; sketch proofs. More examples of isotropic tensors from integrals.
- **Lecture 24.** Equivalence between rank n tensors and n-multilinear maps. Proof of quotient theorem. Example of linear strain tensor.

4 Dynamics and Relativity

- **Lecture 1.** Introduction to Newtonian dynamics; particles, velocity, momentum and acceleration. Product rules for scalar and vector differentiation. Statement of Newton's laws of motion.
- **Lecture 2.** Inertial frames and Galilean boosts/general transformations. Newton's Second Law; gravitational force and electromagnetic force examples.
- **Lecture 3.** Dimensional analysis introduction. Scaling of quantities and balancing dimensions in $n \leq 3$ and $n > 3$ cases with examples.

- **Lecture 4.** Forces, potential energy and kinetic energy. Energy conservation and integral form. Example for motion of particle in 1D in potential, including application of integral form.
- **Lecture 5.** Stable and unstable equilibrium points; Taylor expansion of $V(x)$. Example for pendulum; approximation of period of oscillation. Forces in 3D; work done as a line integral. Conservative force fields and fact that work done is independent of path.
- **Lecture 6.** Gravitational forces and potential; escape velocity. Electromagnetic forces; Lorentz force law, proof of energy conservation and forces due to point charges.
- **Lecture 7.** Static and kinetic friction. Equations and examples for linear and quadratic fluid drag. Range of a projectile and use of dimensional analysis.
- **Lecture 8.** Angular momentum definition. Central forces; conservation of angular momentum. Orbits using polar coordinates; velocity and acceleration in r and θ components. Example of circular motion with constant angular velocity.
- **Lecture 9.** Motion in a central force field; effective potential V_{eff} for radial coordinate with inverse square law example. Conditions for (stable and unstable) circular orbits on V_{eff}. Derivation of the orbit equation.
- **Lecture 10.** Solution of orbit equation for gravitational central force. Solutions for different values of eccentricity (conic sections). Energy of orbit. Kepler's laws of planetary motion.
- **Lecture 11.** Rutherford scattering and finding scattering angle. Rotating frames of reference: rate of change of time-dependent vectors in inertial and rotating frames. Newton's equation of motion in rotating frame; fictitious forces.
- **Lecture 12.** Fictitious forces in rotating frames: Coriolis force and centrifugal force; effective gravity. Example: ball dropped from top of tower. Foucault pendulum and Coriolis force.
- **Lecture 13.** Systems of particles: internal and external forces and use of Newton's 3rd law for internal forces. Centre of mass and total linear/angular momentum (including relative to centre of mass). Condition for conservation of energy.
- **Lecture 14.** Two-body problem formulation; gravitational forces and orbits about centre of mass. Discussion of three-body problem. Rocket equation in one dimension from conservation of momentum.
- **Lecture 15.** Rigid bodies introduction. Angular velocity and moment of inertia for rigid bodies; angular momentum and moment of inertia. Examples calculating moment of inertia in one dimension.
- **Lecture 16.** More examples calculating moment of inertia in 2D and 3D. Perpendicular axes theorem and parallel axes theorem. Motion of rigid bodies; external force and torque in relation to centre of mass. Rigid bodies in uniform gravitational field.
- **Lecture 17.** Motion of uniform rod fixed at pivot point. Sliding and rolling motion conditions; examples for cylinder rolling downhill and sliding to rolling transition (using different methods).
- **Lecture 18.** Postulates of special relativity; derivation of Lorentz transformation and general properties.
- **Lecture 19.** Spacetime diagrams; world lines and x' and t' axes. Composition of velocities. Simultaneity and causality; consideration of past and future light cones. Time dilation.
- **Lecture 20.** Time dilation and twin paradox. Length contraction; example of train alongside platform.

- **Lecture 21.** Invariant interval in spacetime; timelike, spacelike and lightlike separation of events. The Lorentz group and the Minkowski metric. Classes of transformations in the Lorentz group. Rapidity as a way of representing Lorentz transformations.

- **Lecture 22.** Relativistic kinematics; definition of proper time. Definition of 4-velocity; invariance of scalar product. Transformation of 4-velocity between reference frames.

- **Lecture 23.** Definition of 4-momentum; interpretation of P^0 as energy, including $E = mc^2$ derivation. Relation between 3-momentum, mass and energy. $E = |\mathbf{p}|c$ for massless particles. Newton's 2nd law analogue in special relativity.

- **Lecture 24.** Applications of special relativity to particle physics. Examples for particle decay, scattering and particle creation.