MAT 137Y: Calculus with proofs
Assignment 1
Due on Thursday, October 1 by 11:59pm via Crowdmark

Instructions:

- You will need to submit your solutions electronically via Crowdmark. See MAT137 Crowdmark help page for instructions. Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.
- You may submit individually or as a team of two students. See the link above for more details.
- You will need to submit your answer to each question separately.
- This problem set is about the introduction to logic, sets, notation, quantifiers, conditionals, definitions, and proofs (Unit 1).

0. Read "Notes on collaboration" on the course website.

   Write out the following sentence and sign below it, to certify that you have read it.

   > "I have read and understood the notes on collaboration for this course, as explained in the course website."

I have read and understood the notes on collabotion for this course, as explained in the course website

— Hanrui Fan

I have read and understood the notes on collaboration for this course, as explained in the course website

- Letian Cheng

1. In this problem, assume all functions have domain $\mathbb{R}$. I will define a new concept. For every pair of functions f and g, we define the set

$$\Omega_f^g = \{x \in \mathbb{R} \ : \ f(x) < g(x)\}$$

We say that the function f *loves* the function g when

$$\forall x \in \Omega_f^g, \ \exists y \in \Omega_g^f \text{ such that } x < y$$

(a) Consider the functions Walt and Tor defined by

$$\text{Walt}(x) = \sin x, \qquad \text{Tor}(x) = -2\sin x.$$

Prove that *Tor* loves *Walt*.

Suggestion: Before doing anything else, find out what the sets $\Omega_{\text{Walt}}^{\text{Tor}}$ and $\Omega_{\text{Tor}}^{\text{Walt}}$ are.

Proof:

$$\Omega_{Walt}^{Tor} = \{x \in \mathbb{R} \ : \ Walt(x) < Tor(x)\}$$
$$\Omega_{Tor}^{Walt} = \{x \in \mathbb{R} \ : \ Tor(x) < Walt(x)\}$$

Let $x, y \in \mathbb{R}$. WTS $\forall x \in \Omega_{Tor}^{Walt}$, $\exists y \in \Omega_{Walt}^{Tor}$ such that $x < y$

By definition,

$$\Omega_{Tor}^{Walt} = \{x \in \mathbb{R} : \ -2\sin(x) < \sin(x)\} = \{x \in \mathbb{R} : \ 2s\pi < x < (2s+1)\pi \ for \ some \ s \in \mathbb{Z}\}$$

$$\Omega_{Walt}^{Tor} = \{y \in \mathbb{R} : \ sin(y) < -2\sin(y)\} = \{y \in \mathbb{R} : \ (2t-1)\pi < y < 2t\pi \ for \ some \ s \in \mathbb{Z}\}$$

We can see that $\Omega_{Walt}^{Tor} = \Omega_{Tor}^{Walt} + \{2\pi\}$
So we can assume $\exists k \in \mathbb{Z}, \ s = k, \ t = k + 2\pi$.

We have

$$x \in \{x \in \mathbb{R} : \ 2k\pi < x < \ (2k+1)\pi \ for \ some \ k \in \mathbb{Z}\}$$
$$y \in \{y \in \mathbb{R} : \ (2k+1)\pi < y < (2k+2)\pi \ for \ some \ k \in \mathbb{Z}\}$$

We can get

$$x < (x+\pi) \in \{x \in \mathbb{R} : \ 2k\pi + \pi < x < (2k+1)\pi + \pi \ for \ some \ k \in \mathbb{Z}\}$$

$$= \{x \in \mathbb{R} : \ (2k+1)\pi < x < (2k+2)\pi \ for \ some \ k \in \mathbb{Z}\}$$

$$\exists y \in \{y \in \mathbb{R} : \ (2k+1)\pi < y < (2k+2)\pi \ for \ some \ k \in \mathbb{Z}\} \ S.T. \ y = x + \pi > x \qquad \blacksquare$$

(b) Let $f(x) = 3$ and let $g(x) = x$. Prove that f doesn't love g.

Proof:
WTS $\exists x \in \Omega_f^g$ and $\forall y \in \Omega_g^f$, we have $x \geq y$
Because $f(x) = 3$ and $g(x) = x$,

we can get $\Omega_f^g = \{x \in \mathbb{R} : \ x > 3\}$, $\Omega_g^f = \{y \in \mathbb{R} : \ y < 3\}$.
Take $x = 5 \in \Omega_f^g = \{x \in \mathbb{R} : \ x > 3\}$, Let $y \in \Omega_g^f = \{y \in \mathbb{R} : \ y < 3\}$,
$x = 5 > 3 > y$ $\blacksquare$

(c) Which functions f satisfy that f loves f?

Proof:

We want to find a function f, S.T. $\forall x \in \Omega_f^f, \exists y \in \Omega_f^f\ S.T.\ x < y$
By definition, $\Omega_f^f = \{x \in \mathbb{R} :\ f(x) < f(x)\} = \emptyset$

Because $\forall x \in \emptyset,\ \exists y \in \emptyset\ S.T.\ x < y$ is always true, all function satisfied f loves f

2. We continue with the assumptions, notation and definitions as in Question 1.

Given a function f and any $t \in \mathbb{R}$, we define a new function, called f_t, via the equation

$$f_t(x) = f(x) + t.$$

Determine whether each of the following claims is true or false. If true, prove it directly. If false, prove it with a counterexample.

(a) Let f, g, and h be functions. IF f loves g and g loves h, THEN f loves h.

Suggestion: It may be helpful to think of functions in terms of graphs instead of in terms of their equations at first.

Proof: We need to prove,

$$f\ loves\ g\ and\ g\ loves\ h \implies f\ loves\ h$$

This is FALSE. To show it, we need to prove its negation is true:

$$\exists f, g, h,\ \text{S.T.}\ f\ loves\ g\ and\ g\ loves\ h\ AND\ f\ doesn't\ love\ h$$

Take $f = \sin x$, $g = \frac{1}{2}$, $h = \sin x + \frac{1}{2}$. By definition to prove $f\ loves\ g$,

$$\Omega_f^g = \{x \in \mathbb{R} :\ \sin(x) < \frac{1}{2}\} = \{x \in \mathbb{R} : (\frac{5}{6} + 2s)\pi < x < (\frac{13}{6} + 2s)\pi\ for\ some\ s \in \mathbb{Z}\}$$

$$\Omega_g^f = \{y \in \mathbb{R} :\ \frac{1}{2} < \sin(x)\} = \{y \in \mathbb{Z} : (\frac{13}{6} + 2t)\pi < y < (\frac{17}{6} + 2t)\pi\ for\ some\ t \in \mathbb{Z}\}$$

$\exists k \in \mathbb{Z}$, so we can assume $s = k,\ t = k$.
We have

$$x \in \{x \in \mathbb{R} : (\frac{5}{6} + 2k)\pi < x < (\frac{13}{6} + 2k)\pi\ for\ some\ k \in \mathbb{Z}\}$$

$$y \in \{y \in \mathbb{R} : (\frac{13}{6} + 2k)\pi < y < (\frac{17}{6} + 2k)\pi\ for\ some\ k \in \mathbb{Z}\}$$

So, $\forall x\ \exists y\ S.T\ x < y\ , f\ loves\ g$
To prove $g\ loves\ h$,

$$\Omega_g^h = \{x \in \mathbb{R} :\ \frac{1}{2} < \sin(x) + \frac{1}{2}\} = \{x \in \mathbb{R} :\ 2s\pi < x < (2s+1)\pi\ for\ some\ s \in \mathbb{Z}\}$$

$$\Omega_h^g = \{y \in \mathbb{R} :\ \sin(y) + \frac{1}{2} < \frac{1}{2}\} = \{y \in \mathbb{R} :\ (2s+1)\pi < y < (2s+2)\pi\ for\ some\ t \in \mathbb{Z}\}$$

$\exists k \in \mathbb{Z}$, so we can assume $s = k,\ t = k$.
We have

$$x \in \{x \in \mathbb{R} :\ 2k\pi < x < (2k+1)\pi\ for\ some\ k \in \mathbb{Z}\}$$

$$y \in \{y \in \mathbb{R} : \ (2k+1)\pi < y < (2k+2)\pi \ for \ some \ k \in \mathbb{Z}\}$$

So, $\forall x \ \exists y \ S.T. \ x < y$, $g \ loves \ h$
To prove $f \ doesn't \ love \ h$,

$$\Omega_f^h = \{x \in \mathbb{R} : \sin(x) < \sin(x) + \frac{1}{2}\} = \{x \in \mathbb{R}\}$$

$$\Omega_h^f = \{y \in \mathbb{R} : \sin(y) + \frac{1}{2} < \sin(y)\} = \emptyset$$

Because $\exists x \in \mathbb{R}, \forall y \in \emptyset \ S.T. \ x \geq y$ is always True, $g \ doesn't \ love \ h$.
The negation of statement is True. The statement is False $\blacksquare$

(b) For every function f there exists a function g such that, for every $t \in \mathbb{R}$, g loves f_t.

Proof: WTS $\forall f, \exists g$ S.T. $\forall t \in \mathbb{R}, g$ loves f_t
Let $g(x) = f(x) + tan(x)$
Using the definition of love, g loves f_t then becomes:

$$\forall x \in \{x \in \mathbb{R} : tan(x) < t\; for\; some\, t \in \mathbb{R}\}, \exists y \in \{y \in \mathbb{R} : t < tan(y)\; for\; some\, t \in \mathbb{R}\}\; S.T.\; x < y$$

We know that if $x \in \mathbb{R}$, the range of $tan(x)$ is from $-\infty$ to $+\infty$
So $\forall x \in \{x \in \mathbb{R} : tan(x) < t\; for\; some\; t \in \mathbb{R}\}$,
we can always find a $y \in \mathbb{R}$ S.T. $tan(y) = t - tan(x) + t > t$ and $y > x$. ■

3. Prove by induction that for every positive integer n, the number $5^{2n} + 11$ is a multiple of 12.

Proof:

1) Base Case ($n = 1$) WTS

$$12 \mid 5^2 + 11$$

We can get,

$$5^2 + 11 = 36$$

$$12 \mid 36$$

2) Induction Step
Let $n \geq 1$.
So we can assume,

$$12 \mid 5^{2n} + 11$$

WTS:

$$12 \mid 5^{2(n+1)} + 11$$

We know that,

$$12 \mid 12 \times 2 \times 5^{2n}$$

We can get,

$$12 \mid 5^{2n} + 11 + 24 \times 5^{2n}$$

$$12 \mid (24 + 1) \times 5^{2n} + 11$$

$$12 \mid 25 \times 5^{2n} + 11$$

$$12 \mid 5^2 \times 5^{2n} + 11$$

$$12 \mid 5^{2n+2} + 11$$

$$12 \mid 5^{2(n+1)} + 11 \quad \blacksquare$$