MAT 137Y: Calculus with proofs
Assignment 2
Due on Thursday, October 15 by 11:59pm via Crowdmark

Instructions:

- You will need to submit your solutions electronically via Crowdmark. See MAT137 Crowdmark help page for instructions. Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.

- You may submit individually or as a team of two students. See the link above for more details.

- You will need to submit your answer to each question separately.

- This problem set is about the the definition and basic properties of limits (Unit 2 up to Video 2.13).

0. Read "Notes on collaboration" on the course website.

   Write out the following sentence and sign below it, to certify that you have read it.

   "I have read and understood the notes on collaboration for this course, as explained in the course website."

1. *Note:* Before you attempt this question, solve Q1 and Q3 on the practice problems for Unit 2. Otherwise, you will find this question too hard.

   Sketch the graph of a function h that satisfies all the following properties at once:

   (a) The domain of h is $\mathbb{R}$.

   (b) $\lim\limits_{x \to 2} h(x) = 0$ and $\lim\limits_{x \to 2} h(h(x)) = \infty$.

   (c) $\lim\limits_{x \to 4} h(x) = 0$ and $\lim\limits_{x \to 4} h(h(x)) = -\infty$.

   (d) $\lim\limits_{x \to 0} h(h(x)) = 3$.

   (e) $\lim\limits_{x \to -3+} h(x) = 0$ and

   $\lim\limits_{x \to -3+} h(h(x))$ does not exist, is not ∞, and is not $-\infty$.

   To clarify, we want one single function h that satisfies all the conditions in all the parts, all at once. Make your graph tidy and unambiguous. If necessary, add words to explain what is happening in the graph.

   the picture of the function h is on the next page.
   Here is the definition of function h that satisfies conditions a-e all at once.

$$y = \left\{ x \le -3 : \frac{1}{x + \frac{8}{3}} + 3 \right\}$$

$$y = \{ -3 \le x \le -2 : 0 \}$$

$$y = \left\{ -2 \le x < 0 : \frac{1}{x} \right\}$$

$$y = \{ x = 0 : 0 \}$$

$$y = \left\{ 0 < x \le 2 : \frac{1}{x} - \frac{1}{2} \right\}$$

$$y = \{ 2 \le x < 3 : x - 2 \}$$

$$y = \{ 3 < x \le 5 : -(x - 4)^2 \}$$

$$y = \left\{ 5 < x : \frac{1}{-x + \frac{19}{4}} + 3 \right\}$$

Here is what the function looks like.
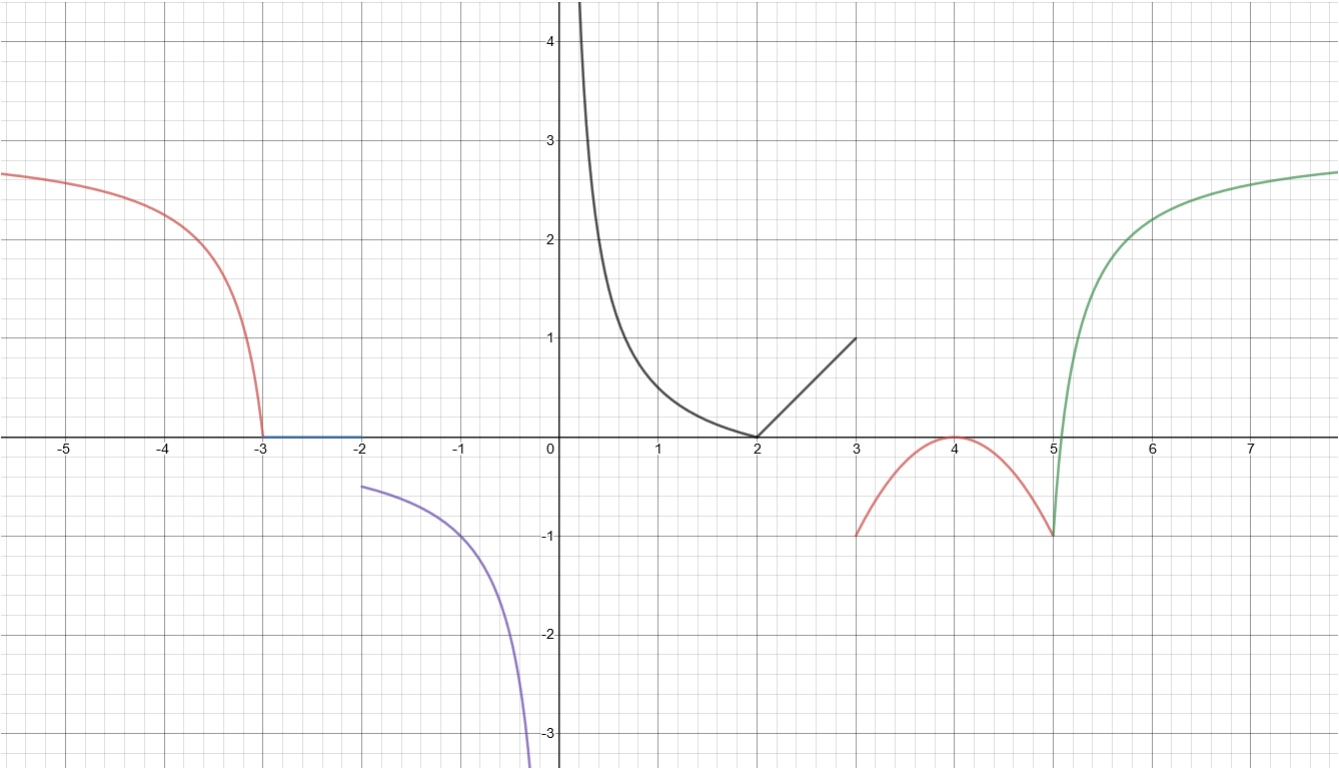
2. Let $a \in \mathbb{R}$. Let f and g be two functions that are defined, at least, on an interval centered at a, except maybe at a. Assume that $\lim_{x \to a} f(x)$ does not exist, and that $\lim_{x \to a} g(x)$ does not exist. Based only on this information, can you conclude whether $\lim_{x \to a} [f(x) + g(x)]$ exists or does not exist? Prove it.

claim: We can't conclude whether $\lim_{x \to a} [f(x) + g(x)]$ exists or does not exist only based on the given information.
given information.
Let $h(x) = f(x) + g(x)$

proof:

- WTS that based on the given conditions of f and g, h COULD exist.

We can define function f as:

$$\begin{cases} x = -3, & \text{if } x < a \\ x = 3, & \text{if } x > a \end{cases}$$

f is defined, at least, on an interval centered at a, except maybe at a.

WTS that $\lim_{x \to a} f(x)$ does not exist. This can be written as: $\forall L \in \mathbb{R}, \exists \epsilon > 0$, s.t $\forall \delta > 0, \exists x \in \mathbb{R}, 0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$

Let $L \in \mathbb{R}$. Take $\epsilon = 1$. Let $\delta \in \mathbb{R}$
At least one of the following must be true:

- Case A: $3 \notin (L - \epsilon, L + \epsilon)$.
  I can choose any $x \in (a, a + \delta)$, satisfying $0 < |x - a| < \delta$. From definition we know that $f(x) = 3$

- Case B: $-3 \notin (L - \epsilon, L + \epsilon)$.
  I can choose any $x \in (a - \delta, a)$, satisfying $0 < |x - a| < \delta$. From definition we know that $f(x) = -3$

Either way, it satisfies $x \in (a - \delta, a + \delta)$ and $|f(x) - L| \geq \epsilon$.

We can define function g as:

$$\begin{cases} x = 3, & \text{if } x < a \\ x = -3, & \text{if } x > a \end{cases}$$

g is defined, at least, on an interval centered at a, except maybe at a.

WTS that $\lim\limits_{x \to a} g(x)$ does not exist. This can be written as: $\forall L \in \mathbb{R}, \exists \epsilon > 0$, s.t $\forall \delta > 0, \exists x \in \mathbb{R}, 0 < |x - a| < \delta$ and $|g(x) - L| \geq \epsilon$

Let $L \in \mathbb{R}$. Take $\epsilon = 1$. Let $\delta \in \mathbb{R}$.
At least one of the following must be true:

- Case A: $-3 \notin (L - \epsilon, L + \epsilon)$.
  I can choose any $x \in (a, a + \delta)$, satisfying $0 < |x - a| < \delta$. From definition we know that $g(x) = -3$

- Case B: $3 \notin (L - \epsilon, L + \epsilon)$.
  I can choose any $x \in (a - \delta, a)$, satisfying $0 < |x - a| < \delta$. From definition we know that $g(x) = 3$

Either way, it satisfies $x \in (a - \delta, a + \delta)$ and $|g(x) - L| \geq \epsilon$.

From definition we know that $h(x) = f(x) + g(x)$

We can write function $h(x)$ as:

$$\begin{cases} x = 0, & \text{if } x < a \\ x = 0, & \text{if } x > a \end{cases}$$

WTS that $\lim\limits_{x \to a} h(x) = 0$. This can be written as: $\forall \epsilon > 0, \exists \delta > 0$, s.t $0 < |x - a| < \delta \implies |h(x)| < \epsilon$

Let $\epsilon > 0$. I take $\delta = \epsilon$. Let $X \in \mathbb{R}$. Assume that $0 < |x - a| < \delta$.
Then, $|h(x)| = 0 < \delta = \epsilon \implies h(x) < \epsilon$ I have proven that $h(x) < \epsilon$, as needed.

- WTS that based on the given conditions of f and g, h COULD BE NOT exist.

We can define function f, g as:

$$\begin{cases} x = -3, & \text{if } x < a \\ x = 3, & \text{if } x > a \end{cases}$$

f, g is defined, at least, on an interval centered at a, except maybe at a.

We have proven that f, g does not have limits.

We can write function $h(x)$ as:

$$\begin{cases} x = -6, & \text{if } x < a \\ x = 6, & \text{if } x > a \end{cases}$$

WTS that $\lim\limits_{x \to a} h(x)$ does not exist. This can be written as: $\forall L \in \mathbb{R}, \exists \epsilon > 0$, s.t $\forall \delta > 0, \exists x \in \mathbb{R}, 0 < |x - a| < \delta$ and $|h(x) - L| \geq \epsilon$

Let $L \in \mathbb{R}$. Take $\epsilon = 1$. Let $\delta \in \mathbb{R}$.
At least one of the following must be true:

- Case A: $6 \notin (L - \epsilon, L + \epsilon)$.
  I can choose any $x \in (a, a + \delta)$, satisfying $0 < |x - a| < \delta$. From definition we know that $h(x) = 6$

- Case B: $-6 \notin (L - \epsilon, L + \epsilon)$.
  I can choose any $x \in (a - \delta, a)$, satisfying $0 < |x - a| < \delta$. From definition we know that $h(x) = -6$

Either way, it satisfies $x \in (a - \delta, a + \delta)$ and $|h(x) - L| \geq \epsilon$.

Since $h(x)$ could exists or not exist, we can't conclude whether $\lim\limits_{x \to a} [f(x) + g(x)]$ exists or does not exist only based on the given information. ∎

3. Prove that $\lim\limits_{x \to 2} x^3 = 8$. Write a proof directly from the definition of limit, without using any of the limit laws or other theorems.

Claim: $\lim\limits_{x \to 2} x^3 = 8$. This can be written as: $\forall \epsilon > 0, \delta > 0$ s.t. $0 < |x - 2| < \delta \implies |x^3 - 8| < \epsilon$

Proof:

Let $\epsilon > 0$.

I take $\delta = min\{1, \frac{\epsilon}{19}\}$.   Therefore $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{19}$

Let $x \in \mathbb{R}$.   Assume $0 < |x - 2| < \delta$.   This implies

$$|x - 2| < \frac{\epsilon}{19} \quad and \quad |x - 2| < 1$$

Hence

$$1 < x < 3, \quad 2 < x + 1 < 4,$$

So

$$2^2 = 4 < (x + 1)^2 = x^2 + 2x + 1 < 4^2 = 16, \quad 7 < x^2 + 2x + 4 < 19.$$

Thus

$$|x^3 - 8| = |x - 2||x^2 + 2x + 4| < \frac{\epsilon}{19}19 = \epsilon$$

We have proven: $|x^3 - 8| < \epsilon$, as needed.   ■

4. Let f and g be two functions with domain $\mathbb{R}$. Let $h = f + g$. Prove that

$$\text{IF} \qquad \lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} g(x) \text{ exists,}$$
$$\text{THEN} \quad \lim_{x \to \infty} h(x) = \infty.$$

Write a proof directly from the definition of limit, without using any of the limit laws or other theorems.

Definition:

$\lim_{x \to \infty} f(x) = \infty$. This can be written as: $\forall M \in \mathbb{R}, \exists K \in \mathbb{R}$, s.t. $x > K \implies f(x) > M$

$\lim_{x \to \infty} g(x)$ exists. This can be written as: $\exists L \in \mathbb{R}$, s.t. $\forall \epsilon > 0, \exists M \in \mathbb{R}$, s.t. $x > M \implies |g(x) - L| < \epsilon$

Since $h = f + g$, $\lim_{x \to \infty} h(x) = \infty$. This can be written as: $\forall M \in \mathbb{R}, \exists K \in \mathbb{R}$, s.t. $x > K \implies h(x) > M$

Proof:

Let $M \in \mathbb{R}$. Assume $\lim_{x \to \infty} g(x) = L \in \mathbb{R}$. Let $\epsilon > 0$.

I use $M - L + \epsilon$ in the definition of $\lim_{x \to \infty} f(x) = \infty$. Since $M - L + \epsilon \in \mathbb{R}$, we can get:

$$\exists K_1 \in \mathbb{R}, \text{ s.t. } x > K_1 \implies f(x) > M - L + \epsilon \implies f(x) - M + L - \epsilon > 0$$

I use the same ϵ in the definition of $\lim_{x \to \infty} g(x)$ exists, we have:

$$\exists L \in \mathbb{R}, \text{ s.t. } \forall \epsilon > 0, \exists M_1 \in \mathbb{R}, \text{ s.t. } x > M_1 \implies |g(x) - L| < \epsilon \implies L - \epsilon < g(x) \implies g(x) - L + \epsilon > 0$$

Take $K = \max\{K_1, M_1\}$.

Let $x \in \mathbb{R}$. Assume $x > K$. This implies:

$x > K_1$. Thus $f(x) - M + L - \epsilon > 0$

$x > M_1$. Thus $g(x) - L + \epsilon > 0$

Then $h(x) - M = [f(x) - M + L - \epsilon] + [g(x) - L + \epsilon] > 0$. So $h(x) > M$.

We have proven: $h(x) > M$ as needed. ∎