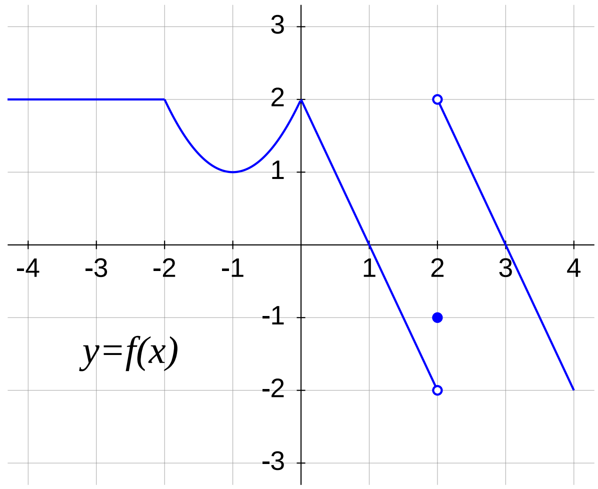
MAT 137Y – Practice problems
Unit 2 : Limits and continuity

1. Below is the graph of the function f:



Compute the following limits

(a) $\lim_{x \to 2} f(x)$

(b) $\lim_{x \to 0} f(f(x))$

(c) $\lim_{x \to -3} f(f(x))$

(d) $\lim_{x \to 0} f(2 \sec x)$

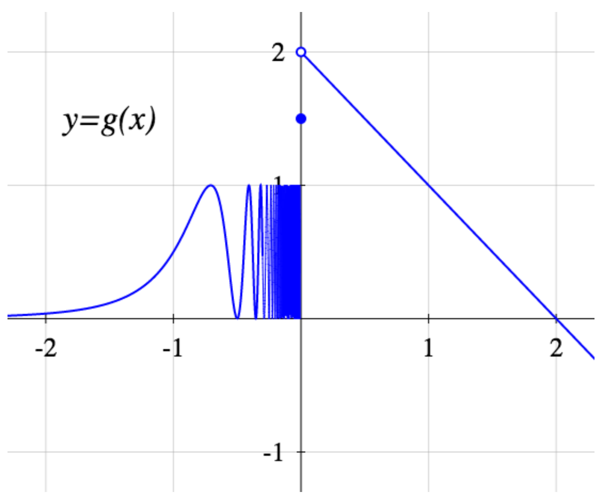
(e) $\lim_{x \to 2} (f(x))^2$

2. Given a real number x, we defined the *floor of* x, denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to x. For example, $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$, and $\lfloor -0.5 \rfloor = -1$.

   (a) Sketch the graph of this function. At which points is the function $f(x) = \lfloor x \rfloor$ continuous? Which discontinuities are removable and which ones are non-removable?

   (b) Consider the function $h(x) = \lfloor \sin x \rfloor$. Show that h has exactly one removable and one non-removable discontinuity inside the interval $(0, 2\pi)$.

3. Below is the graph of the function g:



For clarification, when $-1 < x < 0$, $g(x)$ "oscillates" between 0 and 1; as x approaches 0 from the left, these oscillations become faster and faster. The behaviour is similar to that of the function $f(x) = \sin(\pi/2x)$, which you can see on Video 2.2. Find the following limits:

(a) $\lim_{x\to 0^+} g(x)$

(b) $\lim_{x\to 0^+} \lfloor g(x) \rfloor$

(c) $\lim_{x\to 0^+} g(\lfloor x \rfloor)$

(d) $\lim_{x\to 0^-} g(x)$

(e) $\lim_{x\to 0^-} \lfloor g(x) \rfloor$

(f) $\lim_{x\to 0^-} \lfloor \frac{g(x)}{2} \rfloor$

(g) $\lim_{x\to 0^-} g(\lfloor x \rfloor)$

4. Compute the following limits

(a) $\displaystyle\lim_{x\to 1} \frac{x+1}{x+2}$

(b) $\displaystyle\lim_{x\to 2} \frac{x^2+3x-10}{x^2-4}$

(c) $\displaystyle\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$

(d) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)}$

(e) $\displaystyle\lim_{x\to \infty} \frac{x^3+2x^2+1}{5x^3+6x-1}$

(f) $\displaystyle\lim_{x\to -\infty} \frac{x^5+2x^2+1}{5x^3+6x-1}$

(g) $\displaystyle\lim_{x\to \infty} \frac{\sqrt{x^4+2x+1}+3x^2+1}{x^2}$

(h) $\displaystyle\lim_{x\to 1} \frac{x^3-1}{x^2-2x+1}$

(i) $\displaystyle\lim_{x\to 0} \frac{\sin^{10}(2\sin^{10}(3x))}{x^{100}}$

5. Write the formal definition of the following concepts:

(a) $\lim_{x\to a} f(x) = L$

(b) $\lim_{x\to a} f(x)$ exists

(c) $\lim_{x\to a} f(x) \neq L$

(d) $\lim_{x\to a} f(x)$ doesn't exist

(e) $\lim_{x\to a^+} f(x) = L$

(f) $\lim_{x\to a} f(x) = \infty$

(g) $\lim_{x\to a^-} f(x) = -\infty$

(h) $\lim_{x\to \infty} f(x) = L$

(i) $\lim_{x\to -\infty} f(x) = \infty$

6. Prove the following claims directly from the formal definitions.

(a) $\lim_{x\to 2}(4x+1) = 9$

(b) $\displaystyle\lim_{x\to \infty} \frac{1}{x^2} = 0$

(c) $\lim_{x\to 1} x^3 = 1$

(d) $\displaystyle\lim_{x\to 1} \frac{1}{x^2+1} = \frac{1}{2}$

(e) $\displaystyle\lim_{x\to 0} \frac{x}{|x|}$ does not exist

(f) $\displaystyle\lim_{x\to 1^+} \frac{1}{1-x} = -\infty$

7. Let $a, L, M \in \mathbb{R}$. Let f be a function defined, at least, on an interval centered at a, except maybe at a. Prove that

$$\text{IF } \lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M \qquad \text{THEN } \lim_{x\to a} [f(x) - g(x)] = L - M.$$

Write a proof directly from the formal definitions, without using any of the limit laws.

8. Let $a \in \mathbb{R}$. Let f be a function defined at least on an interval centered at a, except possibly at a. Prove that

$$\text{IF } \lim_{x\to a} f(x) = \infty \qquad \text{THEN } \lim_{x\to a} \frac{1}{f(x)} = 0.$$

Write a proof directly from the formal definitions, without using any of the limit laws.

9. Construct a function f with domain $\mathbb{R}$ such that $\lim_{x \to 0} f(x) = 0$ but $\lim_{x \to 0} f(f(x)) \neq 0$.

10. Prove Theorem 3 on Video 2.16. More specifically:

    Let $a, L \in \mathbb{R}$. Let f be a function defined, at least, on an interval centered at a, except maybe at a. Let g be a function defined at least on an interval centered at L. Prove that

    $$\text{IF } \lim_{x \to a} f(x) = L \text{ and } g \text{ is continuous at } L \qquad \text{THEN } \lim_{x \to a} g(f(x)) = g(L).$$

    Write a proof directly from the formal definitions, without using any of the limit laws.

11. Use the Intermediate Value Theorem to prove that the equation

    $$\sin x = 2\cos^2 x + 0.5$$

    has at least one solution.

12. Use the Squeeze Theorem to explain why $\lim_{x \to 0} x \cos \frac{1}{x}$ exists, even though $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist. Explain why the same argument does not work for $\lim_{x \to 0} x e^{1/x^2}$.

Bonus question:
Do you *really* understand the definition of limit?

13. Let f be a function. Let $a, L \in \mathbb{R}$. Assume that f is defined on some open interval around a, except maybe at a. Below is a list of nine statements.

    a. $\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.
    b. $\forall \varepsilon > 0, \exists \delta > 0$ such that $|x-a| < \delta \implies |f(x) - L| < \varepsilon$.
    c. $\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x-a| < \delta \implies \mathbf{0} < |f(x) - L| < \varepsilon$.
    d. $\forall \varepsilon \geq \mathbf{0}, \exists \delta > 0$ such that $0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.
    e. $\forall \varepsilon > 0, \exists \delta \geq \mathbf{0}$ such that $0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.
    f. $\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x-a| < \delta \implies |f(x) - L| \leq \varepsilon$.
    g. $\forall \delta > \mathbf{0}, \exists \varepsilon > \mathbf{0}$ such that $0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.
    h. $\forall \delta > \mathbf{0}, \exists \varepsilon > \mathbf{0}$ such that $0 < |x-a| < \varepsilon \implies |f(x) - L| < \delta$.
    i. $\exists \delta > \mathbf{0}$ such that $\forall \varepsilon > \mathbf{0}$, $0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.

    Match each of the statements above to one of the following (there may be repeats):

    A. Every function satisfies this statement.
    B. There isn't any function which satisfies this statement.
    C. This statement is (equivalent to) the definition of $\lim_{x \to a} f(x) = L$.
    D. This statement is (equivalent to) the definition of "f is continuous at a".
    E. This statement means that $\lim_{x \to a} f(x) = L$ and that, in addition, f does not take the value L anywhere on some interval centered at a, except maybe at a.
    F. This statement is equivalent to saying that f must be constantly equal to L on an interval centered at a, except maybe at a.
    G. This statement means that f is bounded on every interval centered at a.

Some answers and hints

1. (a) DNE (b) -2 (c) -1 (d) 2 (e) 4

2. (a) f is discontinuous at a when $a \in \mathbb{Z}$. f is continuous everywhere else. All the discontinuities are non-removable.
   (b) g has a removable discontinuity at $\frac{\pi}{2}$ and a non-removable discontinuity at π.

3. (a) 2 (b) 1 (c) 1.5 (d) DNE (e) DNE (f) 0 (g) 0.5

4. (a) $2/3$ (b) $7/4$ (c) $1/4$ (d) $3/2$ (e) $1/5$ (f) ∞ (g) 4 (h) DNE (i) $2^{10}3^{100}$

5. There are various equivalent ways to write each definition. The parts in blue (and only the parts in blue) are often omitted and are considered implicit.
   (a) $\forall \varepsilon > 0, \exists \delta > 0$ such that $(\forall x \in \mathbb{R},)\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$
   (b) $\exists L \in \mathbb{R}$ such that $\forall \varepsilon > 0, \exists \delta > 0$ such that $(\forall x \in \mathbb{R},)\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$
   (c) $\exists \varepsilon > 0$ such that $\forall \delta > 0, \exists x \in \mathbb{R}$ such that $\quad [0 < |x-a| < \delta \text{ and } |f(x) - L| \geq \varepsilon]$
   (d) $\forall L \in \mathbb{R}, \exists \varepsilon > 0$ such that $\forall \delta > 0, \exists x \in \mathbb{R}$ such that $\quad [0 < |x-a| < \delta \text{ and } |f(x) - L| \geq \varepsilon]$
   (e) $\forall \varepsilon > 0, \exists \delta > 0$ such that $(\forall x \in \mathbb{R},)\quad a < x < a + \delta \implies |f(x) - L| < \varepsilon$
   (f) $\forall M \in \mathbb{R}, \exists \delta > 0$ such that $(\forall x \in \mathbb{R},)\quad 0 < |x-a| < \delta \implies f(x) > M$
   (g) $\forall M \in \mathbb{R}, \exists \delta > 0$ such that $(\forall x \in \mathbb{R},)\quad a - \delta < x < a \implies f(x) < M$
   (h) $\forall \varepsilon > 0, \exists K \in \mathbb{R}$ such that $(\forall x \in \mathbb{R},)\quad x > K \implies |f(x) - L| < \varepsilon$
   (i) $\forall M \in \mathbb{R}, \exists K \in \mathbb{R}$ such that $(\forall x \in \mathbb{R},)\quad x < K \implies f(x) > M$

6. (a) This is similar to the proof in Video 2.7.
   (b) WTS: $\forall \varepsilon > 0, \exists K \in \mathbb{R}$ such that $\forall x \in \mathbb{R},\quad x > K \implies \left| \frac{1}{x^2} - 0 \right| < \varepsilon$
   - Fix $\varepsilon > 0$
   - Take $K = \frac{1}{\sqrt{\varepsilon}}$.
   - Fix $x \in \mathbb{R}$. Assume $x > K$. I need to verify that $\frac{1}{x^2} < \varepsilon$.

$$\frac{1}{x^2} < \frac{1}{K^2} = \varepsilon.$$

   (c) This is similar to the proof in Video 2.8
   (d) WTS: $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in \mathbb{R},\quad 0 < |x-1| < \delta \implies \left| \frac{1}{x^2+1} - \frac{1}{2} \right| < \varepsilon$
   - Fix $\varepsilon > 0$

- Take $\delta = \min\{1, 2\varepsilon/3\}$. Thus $\delta \leq 1$ and $\delta \leq 2\varepsilon/3$.
- Fix $x \in \mathbb{R}$. Assume $0 < |x-1| < \delta$. I need to verify that $\left|\dfrac{1}{x^2+1} - \dfrac{1}{2}\right| < \varepsilon$.
  By assumption, $0 \leq 1-\delta < x < 1+\delta \leq 2$. Thus $|1+x| < 3$.
  In addition $\dfrac{1}{x^2+1} \leq 1$.

$$\left|\frac{1}{x^2+1} - \frac{1}{2}\right| = \frac{|x+1||x-1|}{2(x^2+1)} < \frac{3\delta}{2\cdot 1} \leq \varepsilon.$$

(e) This is somewhat similar to the proof in Video 2.9.

(f) WTS $\forall M \in \mathbb{R}, \exists \delta > 0$ such that $\forall x \in \mathbb{R}, \quad 1 < x < 1+\delta \implies \dfrac{1}{1-x} < M$

- Fix $M \in \mathbb{R}$
- Next we need to choose δ. It is probably easiest to break this into two cases.
  - If $M > 0$, take $\delta = 1$ for example.
  - If $M \leq 0$ take $\delta = \dfrac{1}{|M|}$
- Fix $x \in \mathbb{R}$. Assume $1 < x < 1+\delta$. I need to verify that $\dfrac{1}{1-x} < M$.

$$\ldots$$

  (Pay careful attention to the signs. Sometimes you will be working with negative numbers.)

7. This proof is very similar to the one in Video 2.11.

8. WTS $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in \mathbb{R}, \quad 0 < |x-a| < \delta \implies \left|\dfrac{1}{f(x)}\right| < \varepsilon$

- Fix an arbitrary $\varepsilon > 0$.
- Using $\dfrac{1}{\varepsilon}$ as the bound in the definition of $\lim\limits_{x\to a} f(x) = \infty$, we can conclude that

$$\exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, \quad 0 < |x-a| < \delta \implies f(x) > \frac{1}{\varepsilon}$$

  This is the value of δ I take.
- Let $x \in \mathbb{R}$. Assume $0 < |x-a| < \delta$. I need to verify that $\left|\dfrac{1}{f(x)}\right| < \varepsilon$.

  This follows immediately from knowing that $f(x) > \dfrac{1}{\varepsilon} > 0$.

9. This is definitely possible. You will need a function that is not continuous at 0, although being discontinuous at 0 is not enough.

10. I want to prove that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, \quad 0 < |x-a| < \delta \implies |g(f(x)) - g(L)| < \varepsilon.$$

- Fix an arbitrary $\varepsilon > 0$.
- First I use this value of ε in the definition of "g is continuous at L" to conclude that

  $$\exists \delta_0 > 0 \text{ such that } \forall y \in \mathbb{R}, \quad |y - L| < \delta_0 \implies |g(y) - g(L)| < \varepsilon.$$

  Second I use this value of δ_0 "as the epsilon" in the definition of "$\lim\limits_{x \to a} f(x) = L$" to conclude that

  $$\exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, \quad 0 < |x - a| < \delta \implies |f(x) - L| < \delta_0.$$

  This is the value of δ I take.
- Fix $x \in \mathbb{R}$. Assume $0 < |x - a| < \delta$. I need to verify that $|g(f(x)) - g(L)| < \varepsilon$.
  - Since $0 < |x - a| < \delta$, we conclude that $|f(x) - L| < \delta_0$.
  - Since $|f(x) - L| < \delta_0$, we conclude that $|g(f(x)) - g(L)| < \varepsilon$.

11. Consider the function f defined by $f(x) = \sin x - 2\cos^2 x$. f has domain $\mathbb{R}$ and is continuous everywhere.

    $$f(0) = -2 < 0.5, \qquad f(\pi/2) = 1 > 0.5.$$

    Therefore, by the Intermediate Value Theorem, $\exists x \in (0, \pi/2)$ such that $f(x) = 0.5$.

12. This is similar to the argument in Video 2.12.

13. A. e   C. a, f, h   E. c   G. g
    B. d   D. b   F. i