Noncommutative Spaces

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2.1 Banach algebras

Definition 1. A *Banach algebra* is a (not necessarily unital or commutative) $\mathbb{C}$-algebra A together with a norm $\|.\| : A \to \mathbb{R}$ such that:

- $\|.\|$ is *submultiplicative*: $\|ab\| \leqslant \|a\| \|b\|$ for all $a, b \in A$.
- $(A, \|.\|)$ is a *Banach space*: A complete normed vector space.

Remark 1.1. The multiplication on a Banach algebra A is continuous: As for all $a, b \in A$ we have $\|ab\| \leqslant \|a\| \|b\|$, the linear map $a \cdot (-) : A \to A$ is a bounded operator, hence continuous.

Remark 1.2. We can usually assume A to be *unital* (i.e. there is some $1 \in A$ with $1 \cdot a = a \cdot 1 = a$ for all $a \in A$), otherwise replacing it by the *unitization* $\tilde{A}$ of A, given by:

$$\tilde{A} := A \oplus \mathbb{C}$$

with the multiplication

$$(a, \lambda) \cdot (b, \mu) := (ab + \lambda B + \mu A, \lambda\mu)$$

and the norm

$$\|(a, \lambda)\| := \|a\| + |\lambda|.$$

This is in fact a unital Banach algebra: The unit is given by $(0, 1)$, as witnessed by

$$(0, 1) \cdot (a, \lambda) = (a, \lambda) = (a, \lambda) \cdot (0, 1)$$

for $(a, \lambda) \in \tilde{A}$. $\tilde{A}$ is a Banach space as $\mathbb{C}$ is one and the sum of Banach spaces is again a Banach space. Submultiplicativity follows from

$$\begin{aligned}\|(a,\lambda)\cdot(b,\mu)\| &= \|ab+\lambda b+\mu a\| + |\lambda\mu| \\ &\leqslant \|ab\| + \|\lambda b\| + \|\mu a\| + |\lambda\mu| \\ &\leqslant \|a\|\,\|b\| + |\lambda|\,\|b\| + |\mu|\,\|a\| + |\lambda||\mu| \\ &= \|(a,\lambda)\|\,\|(b,\mu)\|\,.\end{aligned}$$

Confirming the algebra structure is a straightforward check. Maybe: Remark on adjunction

Example 2. 1. Let V be a Banach space. Then

$$\mathcal{B}(V) := \{T\colon V\to V \mid T \text{ bounded linear}\}$$

with norm $\|T\| := \sup_{v\in V}\frac{\|Tv\|}{\|v\|}$ and composition as multiplication is a unital Banach algebra.

2. Let X be a topological space. We can define

$$\mathcal{C}_b(X) := \left\{ f : X \to \mathbb{C} \mid f \text{ continuous, } \sup_{x\in X}|f(x)| < \infty \right\}$$

and

$$\mathcal{C}_0(X) := \left\{ f\in\mathcal{C}_b(X) \mid \forall\varepsilon>0\,\exists K\subseteq X \text{ compact, } f^{-1}((-\varepsilon,\varepsilon))\subseteq K \right\}$$

with pointwise multiplication and $\|f\| := \sup_{x\in X}|f(x)|$. Both of these form Banach algebras. $\mathcal{C}_b$ is always unital with unit const_1, whereas $\mathcal{C}_0$ is unital if and only if X is compact.

Definition 3. A *(twosided) ideal* $J\subseteq A$ is a subspace $J\subseteq A$ with $AJ\subseteq J$ and $JA\subseteq J$. This is equivalent to J being a twosided ideal of A viewed as an ordinary (non-unital) ring.

Lemma 4. If $J\subseteq A$ is a closed ideal, the quotient ring A/J equipped with the norm

$$\|a+J\| := \inf_{j\in J}\|a+j\|$$

is again a Banach algebra.

Proof. Quotients of algebras under twosided ideals are again algebras, hence so is A/J. Further, the underlying normed vector space of A/J agrees with the quotient A/J of underlying normed vector spaces, hence is the quotient of a Banach space by a closed subspace and as such again a Banach space. Lastly, as J is an ideal, for $a, b \in A$ and $j, k \in J$ we have $aj + bk + jk \in J$, hence

$$\begin{aligned} \|(a+J)(b+J)\| &= \inf_{j \in J} \|ab + j\| \\ &\leqslant \inf_{j,k \in J} \|ab + aj + bk + jk\| \\ &= \inf_{j,k \in J} \|(a+j)(b+k)\| \\ &\leqslant \inf_{j,k \in J} \|a+j\| \, \|b+k\| \\ &= \left(\inf_{j \in J} \|a+j\| \right) \left(\inf_{k \in J} \|b+k\| \right) \\ &= \|a+J\| \, \|b+J\| \,, \end{aligned}$$

so submultiplicativity holds. □

Example 5. For a Banach algebra A, $A \subseteq \tilde{A}$ is a twosided ideal.

Proof. The map $p : \tilde{A} \to \mathbb{C}, (a, \lambda) \mapsto \lambda$ is a ring homomorphism, hence the kernel $\ker p = A$ is a twosided ideal. Further, p is continuous and $\{0\} \subseteq \mathbb{C}$ is closed, hence so is A. □

Definition 6. For a unital Banach algebra A and an element $a \in A$, we define the *spectrum*

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C} \mid (\lambda \cdot 1 - a) \notin A^\times\},$$

where $A^\times$ is the group of units of A. We further define the spectral radius

$$r(a) := \sup_{\lambda \in \mathrm{sp}(a)} |\lambda|.$$

This spectrum for elements of a general Banach algebra behaves just like the familiar one:

Theorem 7. Let A be a unital Banach algebra and $a \in A$. Then:

- $\mathrm{sp}(a) \subseteq \mathbb{C}$ is non-empty and compact.

- The following formula describes the spectral radius:

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

Especially, by submultiplicativity, $r(a) \leqslant \|a\|$.

Corollary 8 (Gelfand-Mazur)**.** Let A be a unital Banach algebra with $A^\times = A\backslash\{0\}$. Then $A \cong \mathbb{C}$.

Proof sketch. Let $a \in A^\times$. By Theorem 7, there is some $\lambda \in \mathbb{C}$ with $\lambda \cdot 1 - a \in A\backslash A^\times = \{0\}$, so $a = \lambda \cdot 1$. This provides an isomorphism $A \cong \mathbb{C}$. Maybe: More detail. ☐

2.2 Commutative Banach algebras

For this section, fix a commutative Banach algebra A.

Definition 9.

- A *character* on A is a non-zero $\mathbb{C}$-algebra homomorphism $\chi : A \to \mathbb{C}$.
- The *spectrum* of A is

$$\hat{A} := \{\chi : A \to \mathbb{C} \mid \chi \text{ character}\}$$

Example 10. As we will see in TODO: reference, we have:

- For a locally compact Hausdorff space X,

$$X \cong \hat{\mathcal{C}_0(X)}$$

- For a unital C^*-algebra A and $a \in A$ with $aa^* = a^*a$ we have

$$\mathrm{sp}(a) \cong \hat{\langle 1, a\rangle},$$

where $\langle 1, a\rangle \subseteq A$ is the sub-C^*-algebra of A generated by 1 and a.

Fact 10.1. For a commutative Banach algebra A, the following hold:

- An ideal $m \subseteq A$ is maximal if and only if it has codimension 1.
- Maximal ideals are closed.

Proposition 11. If additionally A is unital, the map

$$\begin{aligned} \hat{A} &\to \operatorname{mSpec}(A) \\ \chi &\mapsto \ker \chi, \end{aligned}$$

where $\operatorname{mSpec} A$ denotes the set of maximal ideals of A, is a bijection.

Proof. As $\ker \chi$ has codimension 1 and is closed, it is maximal by Fact 10.1, hence the map is well defined.

Let $m \in \operatorname{mSpec}(A)$. m is closed by Fact 10.1. Thus, A/m is a Banach algebra by Lemma 4, and a field by maximality of m, hence we have $A/m \cong \mathbb{C}$ by Corollary 8. But then, the quotient map $A \to A/m \cong \mathbb{C}$ is a character with kernel m, hence surjectivity.

For injectivity, let χ, χ' be characters with $\ker \chi = \ker \chi' =: m$. Note that $A/m \cong \mathbb{C}$ by the argument above and fix such an isomorphism. Especially, both χ and χ' factor as a composition $A \to A/m \cong \mathbb{C} \to \mathbb{C}$, where the first map is the quotient map and the second map is the fixed isomorphism. But now, as any non-zero homomorphism of $\mathbb{C}$-algebras from $\mathbb{C}$ to $\mathbb{C}$ is the identity[1], the composition above is unique and $\chi = \chi'$. $\square$

Maybe: Something about the topology on the spectrum

Lemma 12. For $\chi \in \hat{A}$, we have $\|\chi\| \leqslant 1$. In particular, $\hat{A} \subseteq A^* := \mathcal{B}(A, \mathbb{C})$, where $\mathcal{B}(V, W)$ denotes the Banach space of bounded linear maps $V \to W$, equipped with the operator norm.

Proof. Suppose A is unital and let $\chi \in \hat{A}$. By the proof of Proposition 11, χ is already isomorphic to the quotient map $p : A \to A/(\ker \chi)$. But that quotient map satisfies $\|p(a)\| = \|a + \ker \chi\| \leqslant \|a\|$ for all $a \in A$, hence so does χ.
Now, consider the case where A is not unital. We can define

$$\begin{aligned} \tilde{\chi} : \tilde{A} &\to \mathbb{C} \\ (a, \lambda) &\mapsto \chi(a) + \lambda \end{aligned}$$

It is easy to see that this is again an algebra homomorphism. But then, by the unital case:

$$|\chi(a)| = |\tilde{\chi}((a, 0))| \leqslant \|(a, 0)\| = \|a\|.$$

Hence, $\|\chi\| \leqslant 1$ and especially χ is continuous. $\square$

Definition 13. We equip $A^* = \mathcal{B}(A, \mathbb{C})$ with the *weak-$*$* topology: The coarsest

[1] To see this, first note that any such homomorphism φ is already unital. Then, $\varphi(\lambda) = \lambda\varphi(1) = \lambda$ for all $\lambda \in \mathbb{C}$.

topology such that for all $a \in A$

$$\begin{aligned} \mathrm{ev}_a : A^* &\to \mathbb{C} \\ \varphi &\mapsto \varphi(a) \end{aligned}$$

is continuous. In other words, $\varphi_n \to \varphi$ in A^* if and only if $\varphi_n(a) \to \varphi(a)$ for all $a \in A$. We further equip $\hat{A} \subseteq A^*$ with the subspace topology.

Lemma 14. A^* is Hausdorff. In particular, so is $\hat{A}$.

Proof. Let $\varphi, \varphi' \in A^*$. Then there is some $a \in A$ with $\varphi(a) \neq \varphi'(a)$. Now, picking disjoint opens $U, V \subseteq \mathbb{C}$ with $\varphi(a) \in U$, $\varphi'(a) \in V$, their preimages form disjoint opens $\mathrm{ev}_a^{-1}(U), \mathrm{ev}_a^{-1}(V) \subseteq A^*$ with $\varphi \in \mathrm{ev}_a^{-1}(U)$ and $\varphi' \in \mathrm{ev}_a^{-1}(V)$. □

Theorem 15. [Banach-Alaoglu] The closed unit ball $B^* := \{\varphi \in A^* \mid \|\varphi\| \leqslant 1\}$ is compact.

Proof. Omitted. □

Proposition 16. $\hat{A}$ is locally compact. If A is unital, $\hat{A}$ is compact.

Proof. By Lemma 12, $\hat{A}$ is a subspace of the closed unit ball $B^* := \{\varphi \in A^* \mid \|\varphi\| \leqslant 1\}$, which is compact by Theorem 15 and Hausdorff by Lemma 14, hence for the first part of the statement it suffices to show that $\hat{A} \cup \{0\} \subseteq B^*$ is closed. Pick a sequence $(\varphi_n)_{n \in \mathbb{N}} \in (\hat{A} \cup \{0\})^{\mathbb{N}}$. □