Noncommutative Spaces

Lecture Course, Summer Semester 2025

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July 8, 2025

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TODO: Motivation

2 C^*-Algebras

Definition 1. A *Banach algebra* is a (not necessarily unital or commutative) $\mathbb{C}$-algebra A together with a norm $\|.\| : A \to \mathbb{R}$ such that:

- $\|.\|$ is *submultiplicative*: $\|ab\| \leqslant \|a\| \, \|b\|$ for all $a, b \in A$.
- $(A, \|.\|)$ is a *Banach space*: A complete normed vector space.

Remark 1.1. The multiplication on a Banach algebra A is continuous: As for all $a, b \in A$ we have $\|ab\| \leqslant \|a\| \, \|b\|$, the linear map $a \cdot (-) \colon A \to A$ is a bounded operator, hence continuous.

Remark 1.2. We can usually assume A to be *unital* (i.e. there is some $1 \in A$ with $1 \cdot a = a \cdot 1 = a$ for all $a \in A$), otherwise replacing it by the *unitization* $\tilde{A}$ of A, given by:

$$\tilde{A} \coloneqq A \oplus \mathbb{C}$$

with the multiplication

$$(a, \lambda) \cdot (b, \mu) \coloneqq (ab + \lambda B + \mu A, \lambda\mu)$$

and the norm

$$\|(a, \lambda)\| \coloneqq \|a\| + |\lambda|.$$

This is in fact a unital Banach algebra: The unit is given by $(0, 1)$, as witnessed by

$$(0, 1) \cdot (a, \lambda) = (a, \lambda) = (a, \lambda) \cdot (0, 1)$$

for $(a, \lambda) \in \tilde{A}$. $\tilde{A}$ is a Banach space as $\mathbb{C}$ is one and the sum of Banach spaces is again a Banach space. Submultiplicativity follows from

$$\begin{aligned}
\|(a, \lambda) \cdot (b, \mu)\| &= \|ab + \lambda b + \mu a\| + |\lambda\mu| \\
&\leqslant \|ab\| + \|\lambda b\| + \|\mu a\| + |\lambda\mu| \\
&\leqslant \|a\| \, \|b\| + |\lambda| \, \|b\| + |\mu| \, \|a\| + |\lambda||\mu| \\
&= \|(a, \lambda)\| \, \|(b, \mu)\| .
\end{aligned}$$

Confirming the algebra structure is a straightforward check. Maybe: Remark on adjunction

Example 2.

1. Let V be a Banach space. Then

$$\mathcal{B}(V) := \{T\colon V \to V \mid T \text{ bounded linear}\}$$

with norm $\|T\| := \sup_{v\in V} \frac{\|Tv\|}{\|v\|}$ and composition as multiplication is a unital Banach algebra.

2. Let X be a topological space. We can define

$$\mathcal{C}_b(X) := \left\{f : X \to \mathbb{C} \mid f \text{ continuous, } \sup_{x\in X}|f(x)| < \infty\right\}$$

and

$$\mathcal{C}_0(X) := \left\{f \in \mathcal{C}_b(X) \mid \forall\varepsilon > 0 \exists K \subseteq X \text{ compact, } f^{-1}((-\varepsilon, \varepsilon)) \subseteq K\right\}$$

with pointwise multiplication and $\|f\| := \sup_{x\in X}|f(x)|$. Both of these form Banach algebras. $\mathcal{C}_b$ is always unital with unit const_1, whereas $\mathcal{C}_0$ is unital if and only if X is compact.

Definition 3. A *(twosided) ideal* $J \subseteq A$ is a subspace $J \subseteq A$ with $AJ \subseteq J$ and $JA \subseteq J$. This is equivalent to J being a twosided ideal of A viewed as an ordinary (non-unital) ring.

Lemma 4. If $J \subseteq A$ is a closed ideal, the quotient ring A/J equipped with the norm

$$\|a + J\| := \inf_{j\in J} \|a + j\|$$

is again a Banach algebra.

Proof. Quotients of algebras under twosided ideals are again algebras, hence so is A/J. Further, the underlying normed vector space of A/J agrees with the quotient A/J of underlying normed vector spaces, hence is the quotient of a Banach space by a closed subspace and as such again a Banach space. □

Example 5. For a Banach algebra A, $A \subseteq \tilde{A}$ is a twosided ideal.

Proof. The map $p : \tilde{A} \to \mathbb{C}, (a, \lambda) \mapsto \lambda$ is a ring homomorphism, hence the kernel $\ker p = A$ is a twosided ideal. Further, p is continuous and $\{0\} \subseteq \mathbb{C}$ is closed, hence so is A. $\square$

Definition 6. For a unital Banach algebra A and an element $a \in A$, we define the *spectrum*

$$(a) := \{\lambda \in \mathbb{C} \mid (\lambda - a) \notin A^\times\},$$

where $A^\times$ is the group of units of A. We further define the spectral radius

$$r(a) := \sup_{\lambda \in (a)} |\lambda|.$$