

# Noncommutative Spaces

Lecture Course, Summer Semester 2025

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# 1 Introduction

TODO: Motivation

## 2 $C^*$ -Algebras

### 2.1 Banach algebras

**Definition 1.** A *Banach algebra* is a (not necessarily unital or commutative)  $\mathbb{C}$ -algebra  $A$  together with a norm  $\|\cdot\| : A \rightarrow \mathbb{R}$  such that:

- $\|\cdot\|$  is *submultiplicative*:  $\|ab\| \leq \|a\| \|b\|$  for all  $a, b \in A$ .
- $(A, \|\cdot\|)$  is a *Banach space*:  $A$  complete normed vector space.

**Remark 1.1.** The multiplication on a Banach algebra  $A$  is continuous: As for all  $a, b \in A$  we have  $\|ab\| \leq \|a\| \|b\|$ , the linear map  $a \cdot (-) : A \rightarrow A$  is a bounded operator, hence continuous.

**Remark 1.2.** We can usually assume  $A$  to be *unital* (i.e. there is some  $1 \in A$  with  $1 \cdot a = a \cdot 1 = a$  for all  $a \in A$ ), otherwise replacing it by the *unitization*  $\tilde{A}$  of  $A$ , given by:

$$\tilde{A} := A \oplus \mathbb{C}$$

with the multiplication

$$(a, \lambda) \cdot (b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$$

and the norm

$$\|(a, \lambda)\| := \|a\| + |\lambda|.$$

This is in fact a unital Banach algebra: The unit is given by  $(0, 1)$ , as witnessed by

$$(0, 1) \cdot (a, \lambda) = (a, \lambda) = (a, \lambda) \cdot (0, 1)$$

for  $(a, \lambda) \in \tilde{A}$ .  $\tilde{A}$  is a Banach space as  $\mathbb{C}$  is one and the sum of Banach spaces is

again a Banach space. Submultiplicativity follows from

$$\begin{aligned}\|(a, \lambda) \cdot (b, \mu)\| &= \|ab + \lambda b + \mu a\| + |\lambda\mu| \\ &\leq \|ab\| + \|\lambda b\| + \|\mu a\| + |\lambda\mu| \\ &\leq \|a\| \|b\| + |\lambda| \|b\| + |\mu| \|a\| + |\lambda| |\mu| \\ &= \|(a, \lambda)\| \|(b, \mu)\|.\end{aligned}$$

Confirming the algebra structure is a straightforward check. **Maybe: Remark on adjunction**

**Example 2.** 1. Let  $V$  be a Banach space. Then

$$\mathcal{B}(V) := \{T: V \rightarrow V \mid T \text{ bounded linear}\}$$

with norm  $\|T\| := \sup_{v \in V} \frac{\|Tv\|}{\|v\|}$  and composition as multiplication is a unital Banach algebra.

2. Let  $X$  be a topological space. We can define

$$\mathcal{C}_b(X) := \left\{ f: X \rightarrow \mathbb{C} \mid f \text{ continuous, } \sup_{x \in X} |f(x)| < \infty \right\}$$

and

$$\mathcal{C}_0(X) := \{f \in \mathcal{C}_b(X) \mid \forall \varepsilon > 0 \exists K \subseteq X \text{ compact, } f^{-1}((-\varepsilon, \varepsilon)) \subseteq K\}$$

with pointwise multiplication and  $\|f\| := \sup_{x \in X} |f(x)|$ . Both of these form Banach algebras.  $\mathcal{C}_b$  is always unital with unit  $\text{const}_1$ , whereas  $\mathcal{C}_0$  is unital if and only if  $X$  is compact.

**Definition 3.** A *(twosided) ideal*  $J \subseteq A$  is a subspace  $J \subseteq A$  with  $AJ \subseteq J$  and  $JA \subseteq J$ . This is equivalent to  $J$  being a twosided ideal of  $A$  viewed as an ordinary (non-unital) ring.

**Lemma 4.** If  $J \subseteq A$  is a closed ideal, the quotient ring  $A/J$  equipped with the norm

$$\|a + J\| := \inf_{j \in J} \|a + j\|$$

is again a Banach algebra.

*Proof.* Quotients of algebras under twosided ideals are again algebras, hence so is  $A/J$ . Further, the underlying normed vector space of  $A/J$  agrees with the quotient  $A/J$  of

underlying normed vector spaces, hence is the quotient of a Banach space by a closed subspace and as such again a Banach space. Lastly, as  $J$  is an ideal, for  $a, b \in A$  and  $j, k \in J$  we have  $aj + bk + jk \in J$ , hence

$$\begin{aligned}
\|(a + J)(b + J)\| &= \inf_{j \in J} \|ab + j\| \\
&\leq \inf_{j, k \in J} \|ab + aj + bk + jk\| \\
&= \inf_{j, k \in J} \|(a + j)(b + k)\| \\
&\leq \inf_{j, k \in J} \|a + j\| \|b + k\| \\
&= \left( \inf_{j \in J} \|a + j\| \right) \left( \inf_{k \in J} \|b + k\| \right) \\
&= \|a + J\| \|b + J\|,
\end{aligned}$$

so submultiplicativity holds.  $\square$

**Example 5.** For a Banach algebra  $A$ ,  $A \subseteq \tilde{A}$  is a twosided ideal.

*Proof.* The map  $p : \tilde{A} \rightarrow \mathbb{C}, (a, \lambda) \mapsto \lambda$  is a ring homomorphism, hence the kernel  $\ker p = A$  is a twosided ideal. Further,  $p$  is continuous and  $\{0\} \subseteq \mathbb{C}$  is closed, hence so is  $A$ .  $\square$

**Definition 6.** For a unital Banach algebra  $A$  and an element  $a \in A$ , we define the *spectrum*

$$\text{sp}(a) := \{\lambda \in \mathbb{C} \mid (\lambda \cdot 1 - a) \notin A^\times\},$$

where  $A^\times$  is the group of units of  $A$ . We further define the spectral radius

$$r(a) := \sup_{\lambda \in \text{sp}(a)} |\lambda|.$$

This spectrum for elements of a general Banach algebra behaves just like the familiar one:

**Theorem 7.** Let  $A$  be a unital Banach algebra and  $a \in A$ . Then:

- $\text{sp}(a) \subseteq \mathbb{C}$  is non-empty and compact.

- The following formula describes the spectral radius:

$$r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}.$$

Especially, by submultiplicativity,  $r(a) \leq \|a\|$ .

**Corollary 8** (Gelfand-Mazur). Let  $A$  be a unital Banach algebra with  $A^\times = A \setminus \{0\}$ . Then  $A \cong \mathbb{C}$ .

*Proof sketch.* Let  $a \in A^\times$ . By Theorem 7, there is some  $\lambda \in \mathbb{C}$  with  $\lambda \cdot 1 - a \in A \setminus A^\times = \{0\}$ , so  $a = \lambda \cdot 1$ . This provides an isomorphism  $A \cong \mathbb{C}$ . [Maybe: More detail.](#)  $\square$

## 2.2 Commutative Banach algebras

For this section, fix a commutative Banach algebra  $A$ .

**Definition 9.**

- A *character* on  $A$  is a non-zero  $\mathbb{C}$ -algebra homomorphism  $\chi : A \rightarrow \mathbb{C}$ .
- The *spectrum* of  $A$  is

$$\hat{A} := \{\chi : A \rightarrow \mathbb{C} \mid \chi \text{ character}\}$$

**Remark 9.1.** If  $A$  is unital, for an algebra homomorphism  $\chi : A \rightarrow \mathbb{C}$  to be non-zero is equivalent to being unital, i.e. satisfying  $\chi(1) = 1$ . This is because  $\chi(a) = \chi(1)\chi(a)$  for all  $a \in A$ .

**Example 10.** As we will see in [TODO: reference](#), we have:

- For a locally compact Hausdorff space  $X$ ,

$$X \cong \mathcal{C}_0(\hat{X})$$

- For a unital  $C^*$ -algebra  $A$  and  $a \in A$  with  $aa^* = a^*a$  we have

$$\text{sp}(a) \cong \langle 1, \hat{a} \rangle,$$

where  $\langle 1, \hat{a} \rangle \subseteq A$  is the sub- $C^*$ -algebra of  $A$  generated by 1 and  $a$ .

**Fact 10.1.** For a commutative Banach algebra  $A$ , the following hold:

- An ideal  $m \subseteq A$  is maximal if and only if it has codimension 1.
- Maximal ideals are closed.

**Proposition 11.** If additionally  $A$  is unital, the map

$$\begin{aligned}\hat{A} &\rightarrow \text{mSpec}(A) \\ \chi &\mapsto \ker \chi,\end{aligned}$$

where  $\text{mSpec } A$  denotes the set of maximal ideals of  $A$ , is a bijection.

*Proof.* As  $\ker \chi$  has codimension 1 and is closed, it is maximal by Fact 10.1, hence the map is well defined.

Let  $m \in \text{mSpec}(A)$ .  $m$  is closed by Fact 10.1. Thus,  $A/m$  is a Banach algebra by Lemma 4, and a field by maximality of  $m$ , hence we have  $A/m \cong \mathbb{C}$  by Corollary 8. But then, the quotient map  $A \rightarrow A/m \cong \mathbb{C}$  is a character with kernel  $m$ , hence surjectivity.

For injectivity, let  $\chi, \chi'$  be characters with  $\ker \chi = \ker \chi' =: m$ . Note that  $A/m \cong \mathbb{C}$  by the argument above and fix such an isomorphism. Especially, both  $\chi$  and  $\chi'$  factor as a composition  $A \rightarrow A/m \cong \mathbb{C} \rightarrow \mathbb{C}$ , where the first map is the quotient map and the second map is the fixed isomorphism. But now, as any non-zero homomorphism of  $\mathbb{C}$ -algebras from  $\mathbb{C}$  to  $\mathbb{C}$  is the identity<sup>1</sup>, the composition above is unique and  $\chi = \chi'$ .  $\square$

Maybe: Something about the topology on the spectrum

**Lemma 12.** For  $\chi \in \hat{A}$ , we have  $\|\chi\| \leq 1$ . In particular,  $\hat{A} \subseteq A^* := \mathcal{B}(A, \mathbb{C})$ , where  $\mathcal{B}(V, W)$  denotes the Banach space of bounded linear maps  $V \rightarrow W$ , equipped with the operator norm.

*Proof.* Suppose  $A$  is unital and let  $\chi \in \hat{A}$ . By the proof of Proposition 11,  $\chi$  is already isomorphic to the quotient map  $p : A \rightarrow A/(\ker \chi)$ . But that quotient map satisfies  $\|p(a)\| = \|a + \ker \chi\| \leq \|a\|$  for all  $a \in A$ , hence so does  $\chi$ .

Now, consider the case where  $A$  is not unital. We can define

$$\begin{aligned}\tilde{\chi} : \tilde{A} &\rightarrow \mathbb{C} \\ (a, \lambda) &\mapsto \chi(a) + \lambda\end{aligned}$$

It is easy to see that this is again an algebra homomorphism. But then, by the unital case:

$$|\chi(a)| = |\tilde{\chi}((a, 0))| \leq \|(a, 0)\| = \|a\|.$$

Hence,  $\|\chi\| \leq 1$  and especially  $\chi$  is continuous.  $\square$

<sup>1</sup>To see this, first note that any such homomorphism  $\varphi$  is already unital. Then,  $\varphi(\lambda) = \lambda\varphi(1) = \lambda$  for all  $\lambda \in \mathbb{C}$ .

**Definition 13.** We equip  $A^* = \mathcal{B}(A, \mathbb{C})$  with the *weak-\** topology: The coarsest topology such that for all  $a \in A$

$$\begin{aligned} \text{ev}_a : A^* &\rightarrow \mathbb{C} \\ \varphi &\mapsto \varphi(a) \end{aligned}$$

is continuous. In other words,  $\varphi_n \rightarrow \varphi$  in  $A^*$  if and only if  $\varphi_n(a) \rightarrow \varphi(a)$  for all  $a \in A$ . We further equip  $\hat{A} \subseteq A^*$  with the subspace topology.

**Lemma 14.**  $A^*$  is Hausdorff. In particular, so is  $\hat{A}$ .

*Proof.* Let  $\varphi, \varphi' \in A^*$ . Then there is some  $a \in A$  with  $\varphi(a) \neq \varphi'(a)$ . Now, picking disjoint opens  $U, V \subseteq \mathbb{C}$  with  $\varphi(a) \in U$ ,  $\varphi'(a) \in V$ , their preimages form disjoint opens  $\text{ev}_a^{-1}(U), \text{ev}_a^{-1}(V) \subseteq A^*$  with  $\varphi \in \text{ev}_a^{-1}(U)$  and  $\varphi' \in \text{ev}_a^{-1}(V)$ .  $\square$

**Theorem 15** (Banach-Alaoglu). The closed unit ball  $B^* := \{\varphi \in A^* \mid \|\varphi\| \leq 1\}$  is compact.

*Proof.* Omitted.  $\square$

**Proposition 16.**  $\hat{A}$  is locally compact. If  $A$  is unital,  $\hat{A}$  is compact.

*Proof.* By Lemma 12,  $\hat{A}$  is a subspace of the closed unit ball  $B^* := \{\varphi \in A^* \mid \|\varphi\| \leq 1\}$ , which is compact by Theorem 15 and Hausdorff by Lemma 14, hence for the first part of the statement it suffices to show that  $\hat{A} \cup \{0\} \subseteq B^*$  is closed. Pick a sequence  $(\varphi_n)_{n \in \mathbb{N}} \in (\hat{A} \cup \{0\})^{\mathbb{N}}$  converging to  $\varphi \in B^*$ . Then

$$\varphi(ab) = \lim_{n \rightarrow \infty} \varphi_n(ab) = \lim_{n \rightarrow \infty} \varphi_n(a) \lim_{n \rightarrow \infty} \varphi_n(b) = \varphi(a)\varphi(b)$$

and analogous arguments for  $\mathbb{C}$ -linearity work, hence  $\varphi$  is again an algebra homomorphism and hence  $\varphi \in \hat{A}$ . If  $A$  is unital and all  $\varphi_n$  are non-zero, then  $\varphi_n(1) = 1$  for all  $n \in \mathbb{N}$  by Remark 9.1 and especially  $\varphi(1) = 1$ , so  $\varphi$  is non-zero as well - hence in that case,  $\hat{A}$  is already compact.  $\square$

## 2.3 The Gelfand transform

**Definition 17.** We define the *Gelfand transform* to be the map

$$\begin{aligned} \Gamma : A &\rightarrow \mathcal{C}_0(\hat{A}) \\ a &\mapsto (\chi \mapsto \chi(a)). \end{aligned}$$



**Remark 17.1.** This is well defined:  $\Gamma(a)$  is bounded as

$$|\Gamma(a)(\chi)| = |\chi(a)| \leq \|\chi\| \|a\| \leq \|a\|.$$

In other words,  $\Gamma(a) \in \mathcal{C}_b(\hat{A})$ . Now, let  $\varepsilon > 0$  and pick a sequence  $(\chi_n)_{n \in \mathbb{N}}$  of elements in  $\hat{A} \cup \{0\}$  with  $|\chi_n(a)| \leq \varepsilon$  for all  $n \in \mathbb{N}$  that converges to an element  $\chi \in \hat{A} \cup \{0\}$ . Then  $|\chi(a)| = \lim_{n \rightarrow \infty} |\chi_n(a)| \leq \varepsilon$ , hence the set

$$K_{a,\varepsilon} := \{\chi \in \hat{A} \cup \{0\} \mid |\chi(a)| \leq \varepsilon\}$$

is a closed subspace of the compact space  $\hat{A} \cup \{0\}$  and thus compact. But further

$$\Gamma(a)^{-1}((-\varepsilon, \varepsilon)) \subseteq K_{a,\varepsilon},$$

so  $\Gamma(a) \in \mathcal{C}_0(\hat{A})$ .

**Theorem 18.**  $\Gamma$  is a norm-decreasing algebra homomorphism and  $\Gamma(A)$  separates points in  $\hat{A}$ .

*Proof.* We have

$$\|\Gamma(a)\| = \sup_{\chi \in \hat{A}} |\chi(a)| \leq \sup_{\chi \in \hat{A}} \|\chi\| \|a\| \leq \|a\|,$$

so  $\Gamma$  is indeed norm-decreasing. Furthermore,

$$\Gamma(ab) = (\chi \mapsto \chi(ab)) = (\chi \mapsto \chi(a))(\chi \mapsto \chi(b)) = \Gamma(a)\Gamma(b)$$

and an analogous argument shows  $\mathbb{C}$ -linearity, so  $\Gamma$  is an algebra homomorphism. Lastly, for  $\varphi \neq \varphi' \in \hat{A}$ , there is some  $a \in A$  with

$$\Gamma(a)(\chi) = \chi(a) \neq \chi'(a) = \Gamma(a)(\chi'),$$

hence  $\Gamma(A)$  separates points in  $\hat{A}$ . □

**Example 19.** We will see in [TODO: reference](#) that for  $A = \mathcal{C}_0(X)$  for a locally compact topological space  $X$ , the Gelfand transform  $\Gamma: A \rightarrow \mathcal{C}_0(\hat{A})$  is an isomorphism.

**Proposition 20.** If  $A$  is unital and  $a \in A$ , we have  $\text{sp}(a) = \text{Im}(\Gamma(a))$ . If  $A$  is not unital,  $\tilde{\text{sp}}(a) = \text{Im}(\Gamma(a)) \cup \{0\}$ , where  $\tilde{\text{sp}}(a) := \text{sp}((a, 0))$  in  $\tilde{A}$ .

*Proof.* [TODO: Think about this](#) □

**Corollary 21.** For  $A$  unital and  $a \in A$  we have  $\|\Gamma(a)\| = r(a)$ . In particular,  $\ker \Gamma = \{a \in A \mid r(a) = 0\}$ .

## 2.4 $C^*$ -algebras

**Definition 22.** An *involution* on a Banach algebra  $A$  is a map  $(-)^* : A \rightarrow A$  such that:

- For all  $\lambda \in \mathbb{C}, a, b \in A$  we have  $(\lambda a + b)^* = \bar{\lambda}a^* + b^*$ .
- For  $a, b \in A$  we have  $(ab)^* = b^*a^*$ .
- For all  $a \in A$  we have  $a^{**} = a$ .

**Remark 22.1.** One can, more generally, define an involution on a (non-unital) ring  $R$  as a (non-unital) ring automorphism  $(-)^* : R \rightarrow R^{\text{op}}$  with  $((-)^*)^{-1} = ((-)^*)^{\text{op}}$ , where  $R^{\text{op}}$  is the ring with reversed multiplication. For such a ring with involution  $R$ , an involution on an  $R$ -algebra  $A$  is an involution  $(-)^*$  on  $A$  making

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow (-)^* & & \downarrow (-)^* \\ R & \longrightarrow & A \end{array}$$

commute. An involution on a Banach algebra is then just an involution of the  $\mathbb{C}$ -algebra, where the involution on  $\mathbb{C}$  is  $\overline{(-)}$ . This approach works in any category with a  $C_2$ -action.

**Definition 23.** A  $C^*$ -algebra is a Banach algebra  $(A, \|\cdot\|)$  with involution  $(-)^*$  such that  $\|a^*a\| = \|a\|^2$  for all  $a \in A$ .

**Lemma 24.** Let  $A$  be a  $C^*$ -algebra and  $a \in A$ . Then  $\|a^*\| = \|a\|$ .

*Proof.* For  $a = 0$  we have  $a^* = \bar{0} = 0$ , hence  $\|a\| = \|a^*\|$  is clear. Otherwise, we have  $\|a\| > 0$  and

$$\|a\|^2 = \|a^*a\| \leq \|a^*\| \|a\|,$$

thus dividing by  $\|a\|$  yields  $\|a\| \leq \|a^*\|$ . Applying the same result to  $a^*$ , we get

$$\|a\| \leq \|a^*\| \leq \|a^{**}\| = \|a\|,$$

so  $\|a\| = \|a^*\|$ . □

**Example 25.** The following Banach algebras with involution are  $C^*$ -algebras:

- $\mathcal{C}_b(X)$  and  $\mathcal{C}_0(X)$  with (pointwise) complex conjugation.
- For any Hilbert space  $\mathcal{H}$ , the algebra  $\mathcal{B}(\mathcal{H})$  of bounded operators on  $\mathcal{H}$  with involution induced by  $\langle T^*v|w \rangle = \langle v|Tw \rangle$ .

*Proof.* We have already seen that  $\mathcal{C}_b(X)$  and  $\mathcal{C}_0(X)$  are commutative Banach algebras. Pointwise conjugation being an involution is a simple check. Lastly,

$$\|a^*a\| = \sup_{x \in X} |a(x)\overline{a(x)}| = \sup_{x \in X} |a(x)|^2 = \|a\|^2,$$

so the given algebra with involution is indeed a  $C^*$ -algebra.

We have also already seen that  $\mathcal{B}(\mathcal{H})$  is a Banach algebra. The axioms for an involution are again a straightforward check. Further, we have

$$\begin{aligned} \|T^*T\| &= \sup_{\substack{v \in V \\ \|v\|=1}} \|T^*Tv\| \\ &= \sup_{\substack{v \in V \\ \|v\|=1}} |\langle T^*Tv|T^*Tv \rangle|^{\frac{1}{2}} \\ &= \sup_{\substack{v \in V \\ \|v\|=1}} |\langle TTv|TTv \rangle|^{\frac{1}{2}} \\ &= \sup_{\substack{v \in V \\ \|v\|=1}} \|TTv\| \\ &= \|TT\| \end{aligned}$$

hence  $\mathcal{B}(\mathcal{H})$  is a  $C^*$ -algebra. □