Noncommutative Spaces

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2.1 Banach algebras

Definition 1. A *Banach algebra* is a (not necessarily unital or commutative) $\mathbb{C}$-algebra A together with a norm $\|.\| : A \to \mathbb{R}$ such that:

- $\|.\|$ is *submultiplicative*: $\|ab\| \leqslant \|a\| \|b\|$ for all $a, b \in A$.
- $(A, \|.\|)$ is a *Banach space*: A complete normed vector space.

Remark 1.1. The multiplication on a Banach algebra A is continuous: As for all $a, b \in A$ we have $\|ab\| \leqslant \|a\| \|b\|$, the linear map $a \cdot (-): A \to A$ is a bounded operator, hence continuous.

Remark 1.2. We can usually assume A to be *unital* (i.e. there is some $1 \in A$ with $1 \cdot a = a \cdot 1 = a$ for all $a \in A$), otherwise replacing it by the *unitization* $\tilde{A}$ of A, given by:

$$\tilde{A} := A \oplus \mathbb{C}$$

with the multiplication

$$(a, \lambda) \cdot (b, \mu) := (ab + \lambda B + \mu A, \lambda\mu)$$

and the norm

$$\|(a, \lambda)\| := \|a\| + |\lambda|.$$

This is in fact a unital Banach algebra: The unit is given by $(0, 1)$, as witnessed by

$$(0, 1) \cdot (a, \lambda) = (a, \lambda) = (a, \lambda) \cdot (0, 1)$$

for $(a, \lambda) \in \tilde{A}$. $\tilde{A}$ is a Banach space as $\mathbb{C}$ is one and the sum of Banach spaces is again a Banach space. Submultiplicativity follows from

$$\begin{aligned}\|(a,\lambda)\cdot(b,\mu)\| &= \|ab+\lambda b+\mu a\| + |\lambda\mu| \\ &\leqslant \|ab\| + \|\lambda b\| + \|\mu a\| + |\lambda\mu| \\ &\leqslant \|a\|\,\|b\| + |\lambda|\,\|b\| + |\mu|\,\|a\| + |\lambda||\mu| \\ &= \|(a,\lambda)\|\,\|(b,\mu)\|\,.\end{aligned}$$

Confirming the algebra structure is a straightforward check. Maybe: Remark on adjunction

Example 2. 1. Let V be a Banach space. Then

$$\mathcal{B}(V) := \{T\colon V \to V \mid T \text{ bounded linear}\}$$

with norm $\|T\| := \sup_{v\in V} \frac{\|Tv\|}{\|v\|}$ and composition as multiplication is a unital Banach algebra.

2. Let X be a topological space. We can define

$$\mathcal{C}_b(X) := \left\{ f : X \to \mathbb{C} \mid f \text{ continuous, } \sup_{x\in X}|f(x)| < \infty \right\}$$

and

$$\mathcal{C}_0(X) := \left\{ f \in \mathcal{C}_b(X) \mid \forall\varepsilon > 0 \exists K \subseteq X \text{ compact, } f^{-1}((-\varepsilon,\varepsilon)) \subseteq K \right\}$$

with pointwise multiplication and $\|f\| := \sup_{x\in X}|f(x)|$. Both of these form Banach algebras. $\mathcal{C}_b$ is always unital with unit const_1, whereas $\mathcal{C}_0$ is unital if and only if X is compact.

Definition 3. A *(twosided) ideal* $J \subseteq A$ is a subspace $J \subseteq A$ with $AJ \subseteq J$ and $JA \subseteq J$. This is equivalent to J being a twosided ideal of A viewed as an ordinary (non-unital) ring.

Lemma 4. If $J \subseteq A$ is a closed ideal, the quotient ring A/J equipped with the norm

$$\|a + J\| := \inf_{j\in J} \|a + j\|$$

is again a Banach algebra.

Proof. Quotients of algebras under twosided ideals are again algebras, hence so is A/J. Further, the underlying normed vector space of A/J agrees with the quotient A/J of underlying normed vector spaces, hence is the quotient of a Banach space by a closed subspace and as such again a Banach space. Lastly, as J is an ideal, for $a, b \in A$ and $j, k \in J$ we have $aj + bk + jk \in J$, hence

$$\begin{aligned} \|(a+J)(b+J)\| &= \inf_{j\in J} \|ab+j\| \\ &\leqslant \inf_{j,k\in J} \|ab+aj+bk+jk\| \\ &= \inf_{j,k\in J} \|(a+j)(b+k)\| \\ &\leqslant \inf_{j,k\in J} \|a+j\| \, \|b+k\| \\ &= \left(\inf_{j\in J} \|a+j\|\right)\left(\inf_{k\in J} \|b+k\|\right) \\ &= \|a+J\| \, \|b+J\|, \end{aligned}$$

so submultiplicativity holds. □

Example 5. For a Banach algebra A, $A \subseteq \tilde{A}$ is a twosided ideal.

Proof. The map $p : \tilde{A} \to \mathbb{C}, (a, \lambda) \mapsto \lambda$ is a ring homomorphism, hence the kernel $\ker p = A$ is a twosided ideal. Further, p is continuous and $\{0\} \subseteq \mathbb{C}$ is closed, hence so is A. □

Definition 6. For a unital Banach algebra A and an element $a \in A$, we define the *spectrum*

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C} \mid (\lambda \cdot 1 - a) \notin A^\times\},$$

where $A^\times$ is the group of units of A. We further define the spectral radius

$$r(a) := \sup_{\lambda \in \mathrm{sp}(a)} |\lambda|.$$

This spectrum for elements of a general Banach algebra behaves just like the familiar one:

Theorem 7. Let A be a unital Banach algebra and $a \in A$. Then:

- $\mathrm{sp}(a) \subseteq \mathbb{C}$ is non-empty and compact.

- The following formula describes the spectral radius:

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

Especially, by submultiplicativity, $r(a) \leqslant \|a\|$.

Corollary 8 (Gelfand-Mazur)**.** Let A be a unital Banach algebra with $A^\times = A\backslash\{0\}$. Then $A \cong \mathbb{C}$.

Proof sketch. Let $a \in A^\times$. By Theorem 7, there is some $\lambda \in \mathbb{C}$ with $\lambda\cdot 1 - a \in A\backslash A^\times = \{0\}$, so $a = \lambda \cdot 1$. This provides an isomorphism $A \cong \mathbb{C}$. Maybe: More detail. ☐

2.2 Commutative Banach algebras

For this section, fix a commutative Banach algebra A.

Definition 9.

- A *character* on A is a (TODO: non-zero?) algebra homomorphism $\chi : A \to \mathbb{C}$.
- The *spectrum* of A is

$$\hat{A} := \{\chi : A \to \mathbb{C} \mid \chi \text{ character}\}$$

Example 10. As we will see in TODO: reference, we have:

- For a locally compact Hausdorff space X,

$$X \cong \mathcal{C}_0\hat{(X)}$$

- For a unital C^*-algebra A and $a \in A$ with $aa^* = a^*a$ we have

$$\mathrm{sp}(a) \cong \langle 1,\hat{} a\rangle,$$

where $\langle 1, a\rangle \subseteq A$ is the sub-C^*-algebra of A generated by 1 and a.

Proposition 11. The map

$$\begin{aligned} \hat{A} &\to \mathrm{mSpec}(A) \\ \chi &\mapsto \ker \chi, \end{aligned}$$

where mSpec A denotes the set of maximal ideals of A, is a bijection.