# Noncommutative Spaces

**Lecture Course, Summer Semester 2025**

Koen van den Dungen*

July 10, 2025

*Notes by Melvin Weiß

# Contents

# 1 Introduction

TODO: Motivation

# 2 $C^*$-Algebras

## 2.1 Banach algebras

**Definition 1.** A *Banach algebra* is a (not necessarily unital or commutative) $\mathbb{C}$-algebra $A$ together with a norm $\|.\| : A \to \mathbb{R}$ such that:

- $\|.\|$ is *submultiplicative*: $\|ab\| \leqslant \|a\| \, \|b\|$ for all $a, b \in A$.
- $(A, \|.\|)$ is a *Banach space*: A complete normed vector space.

**Remark 1.1.** The multiplication on a Banach algebra $A$ is continuous: As for all $a, b \in A$ we have $\|ab\| \leqslant \|a\| \, \|b\|$, the linear map $a \cdot (-) \colon A \to A$ is a bounded operator, hence continuous.

**Remark 1.2.** We can usually assume $A$ to be *unital* (i.e. there is some $1 \in A$ with $1 \cdot a = a \cdot 1 = a$ for all $a \in A$), otherwise replacing it by the *unitization* $\tilde{A}$ of $A$, given by:

$$\tilde{A} := A \oplus \mathbb{C}$$

with the multiplication

$$(a, \lambda) \cdot (b, \mu) := (ab + \lambda B + \mu A, \lambda\mu)$$

and the norm

$$\|(a, \lambda)\| := \|a\| + |\lambda|.$$

This is in fact a unital Banach algebra: The unit is given by $(0, 1)$, as witnessed by

$$(0, 1) \cdot (a, \lambda) = (a, \lambda) = (a, \lambda) \cdot (0, 1)$$

for $(a, \lambda) \in \tilde{A}$. $\tilde{A}$ is a Banach space as $\mathbb{C}$ is one and the sum of Banach spaces is again a Banach space. Submultiplicativity follows from

$$\begin{aligned}\|(a,\lambda)\cdot(b,\mu)\| &= \|ab+\lambda b+\mu a\| + |\lambda\mu| \\ &\leqslant \|ab\| + \|\lambda b\| + \|\mu a\| + |\lambda\mu| \\ &\leqslant \|a\|\,\|b\| + |\lambda|\,\|b\| + |\mu|\,\|a\| + |\lambda||\mu| \\ &= \|(a,\lambda)\|\,\|(b,\mu)\|\,.\end{aligned}$$

Confirming the algebra structure is a straightforward check. Maybe: Remark on adjunction

**Example 2.** 1. Let $V$ be a Banach space. Then

$$\mathcal{B}(V) := \{T\colon V\to V \mid T \text{ bounded linear}\}$$

with norm $\|T\| := \sup_{v\in V}\frac{\|Tv\|}{\|v\|}$ and composition as multiplication is a unital Banach algebra.

2. Let $X$ be a topological space. We can define

$$\mathcal{C}_b(X) := \left\{f : X\to\mathbb{C} \mid f \text{ continuous, } \sup_{x\in X}|f(x)| < \infty\right\}$$

and

$$\mathcal{C}_0(X) := \left\{f\in\mathcal{C}_b(X) \mid \forall\varepsilon>0\exists K\subseteq X \text{ compact, } f^{-1}((-\varepsilon,\varepsilon))\subseteq K\right\}$$

with pointwise multiplication and $\|f\| := \sup_{x\in X}|f(x)|$. Both of these form Banach algebras. $\mathcal{C}_b$ is always unital with unit $\mathrm{const}_1$, whereas $\mathcal{C}_0$ is unital if and only if $X$ is compact.

**Definition 3.** A *(twosided) ideal* $J\subseteq A$ is a subspace $J\subseteq A$ with $AJ\subseteq J$ and $JA\subseteq J$. This is equivalent to $J$ being a twosided ideal of $A$ viewed as an ordinary (non-unital) ring.

**Lemma 4.** If $J\subseteq A$ is a closed ideal, the quotient ring $A/J$ equipped with the norm

$$\|a+J\| := \inf_{j\in J}\|a+j\|$$

is again a Banach algebra.

*Proof.* Quotients of algebras under twosided ideals are again algebras, hence so is $A/J$. Further, the underlying normed vector space of $A/J$ agrees with the quotient $A/J$ of underlying normed vector spaces, hence is the quotient of a Banach space by a closed subspace and as such again a Banach space. Lastly, as $J$ is an ideal, for $a, b \in A$ and $j, k \in J$ we have $aj + bk + jk \in J$, hence

$$\begin{aligned} \|(a+J)(b+J)\| &= \inf_{j \in J} \|ab + j\| \\ &\leqslant \inf_{j,k \in J} \|ab + aj + bk + jk\| \\ &= \inf_{j,k \in J} \|(a+j)(b+k)\| \\ &\leqslant \inf_{j,k \in J} \|a+j\| \, \|b+k\| \\ &= \left( \inf_{j \in J} \|a+j\| \right) \left( \inf_{k \in J} \|b+k\| \right) \\ &= \|a+J\| \, \|b+J\| \,, \end{aligned}$$

so submultiplicativity holds. □

**Example 5.** For a Banach algebra $A$, $A \subseteq \tilde{A}$ is a twosided ideal.

*Proof.* The map $p : \tilde{A} \to \mathbb{C}, (a, \lambda) \mapsto \lambda$ is a ring homomorphism, hence the kernel $\ker p = A$ is a twosided ideal. Further, $p$ is continuous and $\{0\} \subseteq \mathbb{C}$ is closed, hence so is $A$. □

**Definition 6.** For a unital Banach algebra $A$ and an element $a \in A$, we define the *spectrum*

$$\operatorname{sp}(a) := \{\lambda \in \mathbb{C} \mid (\lambda \cdot 1 - a) \notin A^\times\},$$

where $A^\times$ is the group of units of $A$. We further define the spectral radius

$$r(a) := \sup_{\lambda \in \operatorname{sp}(a)} |\lambda|.$$

This spectrum for elements of a general Banach algebra behaves just like the familiar one:

**Theorem 7.** Let $A$ be a unital Banach algebra and $a \in A$. Then:

- $\operatorname{sp}(a) \subseteq \mathbb{C}$ is non-empty and compact.

- The following formula describes the spectral radius:

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

Especially, by submultiplicativity, $r(a) \leqslant \|a\|$.

**Corollary 8** (Gelfand-Mazur)**.** Let $A$ be a unital Banach algebra with $A^\times = A\backslash\{0\}$. Then $A \cong \mathbb{C}$.

*Proof sketch.* Let $a \in A^\times$. By Theorem 7, there is some $\lambda \in \mathbb{C}$ with $\lambda\cdot 1 - a \in A\backslash A^\times = \{0\}$, so $a = \lambda \cdot 1$. This provides an isomorphism $A \cong \mathbb{C}$. Maybe: More detail. ☐

## 2.2 Commutative Banach algebras

For this section, fix a commutative Banach algebra $A$.

**Definition 9.**

- A *character* on $A$ is a non-zero $\mathbb{C}$-algebra homomorphism $\chi : A \to \mathbb{C}$.
- The *spectrum* of $A$ is

$$\hat{A} := \{\chi : A \to \mathbb{C} \mid \chi \text{ character}\}$$

**Remark 9.1.** If $A$ is unital, for an algebra homomorphism $\chi : A \to \mathbb{C}$ to be non-zero is equivalent to being unital, i.e. satisfying $\chi(1) = 1$. This is because $\chi(a) = \chi(1)\chi(a)$ for all $a \in A$.

**Example 10.** As we will see in TODO: reference, we have:

- For a locally compact Hausdorff space $X$,

$$X \cong \hat{\mathcal{C}_0(X)}$$

- For a unital $C^*$-algebra $A$ and $a \in A$ with $aa^* = a^*a$ we have

$$\mathrm{sp}(a) \cong \hat{\langle 1, a\rangle},$$

where $\langle 1, a\rangle \subseteq A$ is the sub-$C^*$-algebra of $A$ generated by 1 and $a$.

**Fact 10.1.** For a commutative Banach algebra $A$, the following hold:

- An ideal $m \subseteq A$ is maximal if and only if it has codimension 1.
- Maximal ideals are closed.

**Proposition 11.** If additionally $A$ is unital, the map

$$\begin{aligned} \hat{A} &\to \mathrm{mSpec}(A) \\ \chi &\mapsto \ker\chi, \end{aligned}$$

where $\mathrm{mSpec}\, A$ denotes the set of maximal ideals of $A$, is a bijection.

*Proof.* As $\ker\chi$ has codimension 1 and is closed, it is maximal by Fact 10.1, hence the map is well defined.

Let $m \in \mathrm{mSpec}(A)$. $m$ is closed by Fact 10.1. Thus, $A/m$ is a Banach algebra by Lemma 4, and a field by maximality of $m$, hence we have $A/m \cong \mathbb{C}$ by Corollary 8. But then, the quotient map $A \to A/m \cong \mathbb{C}$ is a character with kernel $m$, hence surjectivity.

For injectivity, let $\chi, \chi'$ be characters with $\ker\chi = \ker\chi' =: m$. Note that $A/m \cong \mathbb{C}$ by the argument above and fix such an isomorphism. Especially, both $\chi$ and $\chi'$ factor as a composition $A \to A/m \cong \mathbb{C} \to \mathbb{C}$, where the first map is the quotient map and the second map is the fixed isomorphism. But now, as any non-zero homomorphism of $\mathbb{C}$-algebras from $\mathbb{C}$ to $\mathbb{C}$ is the identity[1], the composition above is unique and $\chi = \chi'$. ☐

Maybe: Something about the topology on the spectrum

**Lemma 12.** For $\chi \in \hat{A}$, we have $\|\chi\| \leqslant 1$. In particular, $\hat{A} \subseteq A^* := \mathcal{B}(A, \mathbb{C})$, where $\mathcal{B}(V, W)$ denotes the Banach space of bounded linear maps $V \to W$, equipped with the operator norm.

*Proof.* Suppose $A$ is unital and let $\chi \in \hat{A}$. By the proof of Proposition 11, $\chi$ is already isomorphic to the quotient map $p : A \to A/(\ker\chi)$. But that quotient map satisfies $\|p(a)\| = \|a + \ker\chi\| \leqslant \|a\|$ for all $a \in A$, hence so does $\chi$.
Now, consider the case where $A$ is not unital. We can define

$$\begin{aligned} \tilde{\chi} : \tilde{A} &\to \mathbb{C} \\ (a, \lambda) &\mapsto \chi(a) + \lambda \end{aligned}$$

It is easy to see that this is again an algebra homomorphism. But then, by the unital case:

$$|\chi(a)| = |\tilde{\chi}((a, 0))| \leqslant \|(a, 0)\| = \|a\|.$$

Hence, $\|\chi\| \leqslant 1$ and especially $\chi$ is continuous. ☐

[1] To see this, first note that any such homomorphism $\varphi$ is already unital. Then, $\varphi(\lambda) = \lambda\varphi(1) = \lambda$ for all $\lambda \in \mathbb{C}$.

**Definition 13.** We equip $A^* = \mathcal{B}(A, \mathbb{C})$ with the *weak-$*$* topology: The coarsest topology such that for all $a \in A$

$$\begin{aligned} \mathrm{ev}_a : A^* &\to \mathbb{C} \\ \varphi &\mapsto \varphi(a) \end{aligned}$$

is continuous. In other words, $\varphi_n \to \varphi$ in $A^*$ if and only if $\varphi_n(a) \to \varphi(a)$ for all $a \in A$. We further equip $\hat{A} \subseteq A^*$ with the subspace topology.

**Lemma 14.** $A^*$ is Hausdorff. In particular, so is $\hat{A}$.

*Proof.* Let $\varphi, \varphi' \in A^*$. Then there is some $a \in A$ with $\varphi(a) \neq \varphi'(a)$. Now, picking disjoint opens $U, V \subseteq \mathbb{C}$ with $\varphi(a) \in U$, $\varphi'(a) \in V$, their preimages form disjoint opens $\mathrm{ev}_a^{-1}(U), \mathrm{ev}_a^{-1}(V) \subseteq A^*$ with $\varphi \in \mathrm{ev}_a^{-1}(U)$ and $\varphi' \in \mathrm{ev}_a^{-1}(V)$. $\square$

**Theorem 15** (Banach-Alaoglu)**.** The closed unit ball $B^* := \{\varphi \in A^* \mid \|\varphi\| \leqslant 1\}$ is compact.

*Proof.* Omitted. $\square$

**Proposition 16.** $\hat{A}$ is locally compact. If $A$ is unital, $\hat{A}$ is compact.

*Proof.* By Lemma 12, $\hat{A}$ is a subspace of the closed unit ball $B^* := \{\varphi \in A^* \mid \|\varphi\| \leqslant 1\}$, which is compact by Theorem 15 and Hausdorff by Lemma 14, hence for the first part of the statement it suffices to show that $\hat{A} \cup \{0\} \subseteq B^*$ is closed. Pick a sequence $(\varphi_n)_{n \in \mathbb{N}} \in (\hat{A} \cup \{0\})^{\mathbb{N}}$ converging to $\varphi \in B^*$. Then

$$\varphi(ab) = \lim_{n \to \infty} \varphi_n(ab) = \lim_{n \to \infty} \varphi_n(a) \lim_{n \to \infty} \varphi_n(b) = \varphi(a)\varphi(b)$$

and analogous arguments for $\mathbb{C}$-linearity work, hence $\varphi$ is again an algebra homomorphism and hence $\varphi \in \hat{A}$. If $A$ is unital and all $\varphi_n$ are non-zero, then $\varphi_n(1) = 1$ for all $n \in \mathbb{N}$ by Remark 9.1 and especially $\varphi(1) = 1$, so $\varphi$ is non-zero as well - hence in that case, $\hat{A}$ is already compact. $\square$

### 2.3 The Gelfand transform

**Definition 17.** We define the *Gelfand transform* to be the map

$$\begin{aligned} \Gamma \colon A &\to \mathcal{C}_0(\hat{A}) \\ a &\mapsto (\chi \mapsto \chi(a)). \end{aligned}$$

**Remark 17.1.** This is well defined: $\Gamma(a)$ is bounded as

$$|\Gamma(a)(\chi)| = |\chi(a)| \leqslant \|\chi\| \, \|a\| \leqslant \|a\| \, .$$

In other words, $\Gamma(a) \in \mathcal{C}_b(\hat{A})$. Now, let $\varepsilon > 0$ and pick a sequence $(\chi_n)_{n\in\mathbb{N}}$ of elements in $\hat{A} \cup \{0\}$ with $|\chi_n(a)| \leqslant \varepsilon$ for all $n \in \mathbb{N}$ that converges to an element $\chi \in \hat{A} \cup \{0\}$. Then $|\chi(a)| = \lim_{n\to\infty} |\chi_n(a)| \leqslant \varepsilon$, hence the set

$$K_{a,\varepsilon} \coloneqq \{\chi \in \hat{A} \cup \{0\} \mid |\chi(a)| \leqslant \varepsilon\}$$

is a closed subspace of the compact space $\hat{A} \cup \{0\}$ and thus compact. But further

$$\Gamma(a)^{-1}((-\varepsilon, \varepsilon)) \subseteq K_{a,\varepsilon},$$

so $\Gamma(a) \in \mathcal{C}_0(\hat{A})$.

**Theorem 18.** $\Gamma$ is a norm-decreasing algebra homomorphism and $\Gamma(A)$ separates points in $\hat{A}$.

*Proof.* We have

$$\|\Gamma(a)\| = \sup_{\chi\in\hat{A}} |\chi(a)| \leqslant \sup_{\chi\in\hat{A}} \|\chi\| \, \|a\| \leqslant \|a\| \, ,$$

so $\Gamma$ is indeed norm-decreasing. Furthermore,

$$\Gamma(ab) = (\chi \mapsto \chi(ab)) = (\chi \mapsto \chi(a))(\chi \mapsto \chi(b)) = \Gamma(a)\Gamma(b)$$

and an analogous argument shows $\mathbb{C}$-linearity, so $\Gamma$ is an algebra homomorphism. Lastly, for $\varphi \neq \varphi' \in \hat{A}$, there is some $a \in A$ with

$$\Gamma(a)(\chi) = \chi(a) \neq \chi'(a) = \Gamma(a)(\chi'),$$

hence $\Gamma(A)$ separates points in $\hat{A}$. □

**Example 19.** We will see in TODO: reference that for $A = \mathcal{C}_0(X)$ for a locally compact topological space $X$, the Gelfand transform $\Gamma\colon A \to \mathcal{C}_0(\hat{A})$ is an isomorphism.

**Proposition 20.** If $A$ is unital and $a \in A$, we have $\mathrm{sp}(a) = \mathrm{Im}(\Gamma(a))$. If $A$ is not unital, $\widetilde{\mathrm{sp}}(a) = \mathrm{Im}(\Gamma(a)) \cup \{0\}$, where $\widetilde{\mathrm{sp}}(a) \coloneqq \mathrm{sp}\left((a, 0)\right)$ in $\tilde{A}$.

*Proof.* Assume first that $A$ is unital. We start by showing $\mathrm{Im}(\Gamma(a)) \subseteq \mathrm{sp}(a)$. Let $\chi \in \hat{A}$. Then

$$\chi(\chi(a) \cdot 1 - a) = \chi(a)\chi(1) - \chi(a) = 0 \notin \mathbb{C}^\times,$$

so $\chi(a) \cdot 1 - a$ is not invertible and hence $\Gamma(a)(\chi) = \chi(a) \in \mathrm{sp}(a)$.

For the opposite inclusion, let $\lambda \in \mathrm{sp}(a)$. We aim to construct a character $\chi_\lambda : A \to \mathbb{C}$ that sends $a$ to $\lambda$. TODO: Think about this □

**Corollary 21.** For $A$ unital and $a \in A$ we have $\|\Gamma(a)\| = r(a)$. In particular, $\ker \Gamma = \{a \in A | r(a) = 0\}$.

### 2.4 $C^*$-algebras

**Definition 22.** An *involution* on a Banach algebra $A$ is a map $(-)^* : A \to A$ such that:

- For all $\lambda \in \mathbb{C}, a, b \in A$ we have $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$.
- For $a, b \in A$ we have $(ab)^* = b^* a^*$.
- For all $a \in A$ we have $a^{**} = a$.

**Remark 22.1.** One can, more generally, define an involution on a (non-unital) ring $R$ as a (non-unital) ring automorphism $(-)^* : R \to R^{\mathrm{op}}$ with $((-)^*)^{-1} = ((-)^*)^{\mathrm{op}}$, where $R^{\mathrm{op}}$ is the ring with reversed multiplication. For such a ring with involution $R$, an involution on an $R$-algebra $A$ is an involution $(-)^*$ on $A$ making

$$\begin{tikzcd} R \arrow[r] \arrow[d, "(-)^*"] & A \arrow[d, "(-)^*"] \\ R \arrow[r] & A \end{tikzcd}$$

commute. An involution on a Banach algebra is then just an involution of the $\mathbb{C}$-algebra, where the involution on $\mathbb{C}$ is $\overline{(-)}$. This approach works in any category with a $C_2$-action.

**Definition 23.** A $C^*$-algebra is a Banach algebra $(A, \|.\|)$ with involution $(-)^*$ such that $\|a^* a\| = \|a\|^2$ for all $a \in A$.

**Lemma 24.** Let $A$ be a $C^*$-algebra and $a \in A$. Then $\|a^*\| = \|a\|$.

*Proof.* For $a = 0$ we have $a^* = \overline{0} = 0$, hence $\|a\| = \|a^*\|$ is clear. Otherwise, we have $\|a\| > 0$ and

$$\|a\|^2 = \|a^* a\| \leqslant \|a^*\| \, \|a\| \, ,$$

thus dividing by $\|a\|$ yields $\|a\| \leqslant \|a^*\|$. Applying the same result to $a^*$, we get

$$\|a\| \leqslant \|a^*\| \leqslant \|a^{**}\| = \|a\| ,$$

so $\|a\| = \|a^*\|$. □

**Example 25.** The following Banach algebras with involution are $C^*$-algebras:

- $\mathcal{C}_b(X)$ and $\mathcal{C}_0(X)$ with (pointwise) complex conjugation.
- For any Hilbert space $\mathcal{H}$, the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ with involution induced by $\langle T^*v|w\rangle = \langle v|Tw\rangle$.

*Proof.* We have already seen that $\mathcal{C}_b(X)$ and $\mathcal{C}_0(X)$ are commutative Banach algebras. Pointwise conjugation being an involution is a simple check. Lastly,

$$\|a^*a\| = \sup_{x\in X}\left|a(x)\overline{a(x)}\right| = \sup_{x\in X}|a(x)|^2 = \|a\|^2 ,$$

so the given algebra with involution is indeed a $C^*$-algebra.

We have also already seen that $\mathcal{B}(\mathcal{H})$ is a Banach algebra. The axioms for an involution are again a straightforward check. Further, we have

$$\begin{aligned}
\|T^*T\| &= \sup_{\substack{v\in V\\ \|v\|=1}} \|T^*Tv\| \\
&= \sup_{\substack{v\in V\\ \|v\|=1}} |\langle T^*Tv|T^*Tv\rangle|^{\frac{1}{2}} \\
&= \sup_{\substack{v\in V\\ \|v\|=1}} |\langle TTv|TTv\rangle|^{\frac{1}{2}} \\
&= \sup_{\substack{v\in V\\ \|v\|=1}} \|TTv\| \\
&= \|TT\|
\end{aligned}$$

hence $\mathcal{B}(\mathcal{H})$ is a $C^*$-algebra. □

**Remark 25.1.** Let $A$ be a non-unital $C^*$-algebra. We can consider the unitization $\tilde{A}$ of the underlying Banach algebra: This is a unital Banach algebra with norm $\|(a,\lambda)\| = \|a\| , |\lambda|$, but with that norm it does not neccessarily admit the structure of a $C^*$-algebra compatible with that of $A$ (Maybe: Example for this phenomenon). Hence we have to provide a different norm on $\tilde{A}$ if we aim to construct a unitization for $C^*$-algebras.

**Definition 26.** Let $A$ be a $C^*$-algebra and consider the embedding

$$\begin{aligned} L\colon \tilde{A} &\to \mathcal{B}(A) \\ (a,\lambda) &\mapsto (b \mapsto ab + \lambda b). \end{aligned}$$

We will see in Lemma 27 that $L$ is an injective algebra homomorphism, hence we define the norm and $(-)^*$ on $\tilde{A} \cong L(\tilde{A}) \subseteq \mathcal{B}(A)$ to be the ones inherited from $\mathcal{B}(A)$. We refer to this $C^*$-algebra as the *unitization* of $A$.

**Lemma 27.** Let $A$ be a $C^*$-algebra. The map $L\colon \tilde{A} \to \mathcal{B}(A)$ from Definition 26 is an injective algebra homomorphism. Further, $\tilde{A} \cong L(\tilde{A})$ with the norm and $(-)^*$ inherited from $\mathcal{B}(A)$ forms a unital $C^*$-algebra and $A \subseteq \tilde{A}$ is a $C^*$-subalgebra.

Maybe: Prove at least parts of this

**Note.** *The norm on $\tilde{A}$ is $\|(a,\lambda)\| = \sup_{b\in A}\|ab+\lambda b\|$, the involution is $(a,\lambda)^* = (a^*,\overline{\lambda})$.*

Maybe: Note on the adjunction

**Definition 28.** Let $A$ be a $C^*$-algebra and $a \in A$. We say that $a$ is

- *self-adjoint* if $a^* = a$,
- *unitary* if $a^*a = aa^* = 1$,
- *normal* if $a^*a = aa^*$,
- *positive* if $a = b^*b$ for some $b \in A$.

**Definition 29.** For a (not neccessarily unital) $C^*$-algebra $A$ and $a \in A$, we define the spectral radius of $a$ to be $r(a) := r((a,0))$, the spectral radius of $(a,0) \in \tilde{A}$.

**Proposition 30.** For a $C^*$-algebra $A$, $a \in A$ normal and $n \in \mathbb{N}$, we have $\|a^n\| = \|a\|^n$.

*Proof.* Using normality and $(aa)^* = a^*a^*$ for the second equation, we have

$$\begin{aligned} \|a^2\| &= \left\|a^2\left(a^2\right)^*\right\|^{\frac{1}{2}} \\ &= \|(a^*a)(a^*a)^*\|^{\frac{1}{2}} \\ &= \|a^*a\| \\ &= \|a\|^2. \end{aligned}$$

Inductively, $\left\|a^{2^n}\right\| = \|a\|^{2^n}$. Further, by submultiplicativity, for any $n \in \mathbb{N}$ we have $\|a^n\| \leqslant \|a\|^n$. But then, letting $k \in \mathbb{N}$ with $2^k > n$, we get

$$\begin{aligned} \|a\|^n \left\|a^{2^k-n}\right\| &\leqslant \|a\|^n \|a\|^{2^k-n} \\ &= \|a\|^{2^k} \\ &= \left\|a^{2^k}\right\| \\ &\leqslant \|a^n\| \left\|a^{2^k-n}\right\| . \end{aligned}$$

Now, if $a = 0$, the statement to be proven is trivial, otherwise we get

$$\|a\|^n \leqslant \|a^n\| \leqslant \|a\|^n ,$$

which yields the desired equality. □

**Corollary 31.** For a $C^*$-algebra $A$ and $a \in A$ normal, we have $r(a) = \|a\|$.

*Proof.* Let $\alpha = (a, 0) \in \tilde{A}$. As $a$ was normal, so is $\alpha$. By Theorem 7 and Lemma 30, we have

$$r(a) = r(\alpha) = \lim_{n\to\infty} \|\alpha^n\|^{\frac{1}{n}} = \lim_{n\to\infty} (\|\alpha\|^n)^{\frac{1}{n}} = \|\alpha\| .$$

□

**Corollary 32.** For a $C^*$-algebra $A$ and $a \in A$, we have $\|a\| = r(a^*a)^{\frac{1}{2}}$. In particular, the norm of a $C^*$-algebra is already uniquely determined by the $\mathbb{C}$-algebra structure.

*Proof.* $a^*a$ is self-adjoint, hence normal, so by Corollary 32 we get

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = r(a^*a)^{\frac{1}{2}}.$$

□

**Corollary 33.** If $A$ is a commutative $C^*$-algebra, the Gelfand transform $\Gamma$ is in isometry.

*Proof.* By Corollary 21 and Corollary 32, we have $\|\Gamma(a)\| = r(a) = \|a\|$ for every normal $a \in A$. But as $A$ is commutative, every element is normal. □

**Proposition 34.** Let $A$ be a unital $C^*$-algebra and $a \in A$. If $a$ is self-adjoint, $\mathrm{sp}(a) \subseteq \mathbb{R}$. If $a$ is unitary, $\mathrm{sp}(a) \subseteq U(1) = S^1$.

TODO: Proof

**Lemma 35.** For a $C^*$-algebra $A$ and $a \in A$, we can find self-adjoint elements $b, c \in A$ with $a = b + ic$.

*Proof.* Let

$$b := \frac{a + a^*}{2}$$

and

$$c := \frac{i(a^* - a)}{2}.$$

Then $b^* = b$, $c^* = c$ and $b + ic = a$. □

We use the following input from functional analysis:

**Theorem 36** (Stone-Weierstrass)**.** If $X$ is a locally compact Hausdorff space and $B \subseteq \mathcal{C}_0(X)$ a nowhere vanishing, self-adjoint algebra which separates points, then $B \subseteq \mathcal{C}_0(X)$ is dense.

Using this, we can show:

**Theorem 37** (Gelfand-Naimark I)**.** Let $A$ be a commutative $C^*$-algebra. Then the Gelfand transform $\Gamma$ is an isomorphism of $C^*$-algebras.

*Proof.* We have already seen that $\Gamma$ is an algebra homomorphism and $\Gamma(a)$ separates points in $\hat{A}$ for all $a \in A$ (Theorem 18). Further, $\Gamma$ is a morphism of $C^*$-algebras: Let $\chi \in \hat{A}$ and $a \in A$. We write $a = b + ic$ with $b, c \in A$ self-adjoint (Lemma 35). Note first that $\chi(b) \in \mathrm{sp}(b) \subseteq \mathbb{R}$ by Proposition 34, and the same holds for $\chi(c)$. But then

$$\begin{aligned}
\chi(a^*) &= \chi(b^* - ic^*) \\
&= \chi(b - ic) \\
&= \chi(b) - i\chi(c) \\
&= \overline{\chi(b) + i\chi(c)} \\
&= \overline{\chi(b + ic)} \\
&= \overline{\chi(a)},
\end{aligned}$$

hence

$$\Gamma(a^*) = (\chi \mapsto \chi(a^*)) = (\chi \mapsto \overline{\chi(a)}) = \Gamma(a)^*.$$

We have already seen that $\Gamma$ is an isometry (Corollary 33), hence it is injective and, as $A$ is complete, the image of $\Gamma$ is closed, so it suffices to show that this image is dense. For that, we apply Theorem 36:

- $\Gamma(A)$ is nowhere vanishing as for each $\chi \in \hat{A}$ we have $\chi \neq 0$, so there is some $a \in A$ with $\Gamma(a)(\chi) = \chi(a) \neq 0$.
- $\Gamma(A)$ is self-adjoint, as $\Gamma(a)^* = \Gamma(a^*)$.
- $\Gamma(A)$ seperates points by Theorem 18.

So the conditions for Stone-Weierstrass are met and $\Gamma(A) \subseteq \mathcal{C}_0(\hat{A})$ is dense, which was left to show. $\square$