Noncommutative Spaces

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2.1 Banach algebras

Definition 1. A *Banach algebra* is a (not necessarily unital or commutative) $\mathbb{C}$-algebra A together with a norm $\|.\| : A \to \mathbb{R}$ such that:

- $\|.\|$ is *submultiplicative*: $\|ab\| \leqslant \|a\| \|b\|$ for all $a, b \in A$.
- $(A, \|.\|)$ is a *Banach space*: A complete normed vector space.

Remark 1.1. The multiplication on a Banach algebra A is continuous: As for all $a, b \in A$ we have $\|ab\| \leqslant \|a\| \|b\|$, the linear map $a \cdot (-) \colon A \to A$ is a bounded operator, hence continuous.

Remark 1.2. We can usually assume A to be *unital* (i.e. there is some $1 \in A$ with $1 \cdot a = a \cdot 1 = a$ for all $a \in A$), otherwise replacing it by the *unitization* $\tilde{A}$ of A, given by:

$$\tilde{A} := A \oplus \mathbb{C}$$

with the multiplication

$$(a, \lambda) \cdot (b, \mu) := (ab + \lambda B + \mu A, \lambda\mu)$$

and the norm

$$\|(a, \lambda)\| := \|a\| + |\lambda|.$$

This is in fact a unital Banach algebra: The unit is given by $(0, 1)$, as witnessed by

$$(0, 1) \cdot (a, \lambda) = (a, \lambda) = (a, \lambda) \cdot (0, 1)$$

for $(a, \lambda) \in \tilde{A}$. $\tilde{A}$ is a Banach space as $\mathbb{C}$ is one and the sum of Banach spaces is again a Banach space. Submultiplicativity follows from

$$\begin{aligned}\|(a,\lambda)\cdot(b,\mu)\| &= \|ab+\lambda b+\mu a\| + |\lambda\mu| \\ &\leqslant \|ab\| + \|\lambda b\| + \|\mu a\| + |\lambda\mu| \\ &\leqslant \|a\|\,\|b\| + |\lambda|\,\|b\| + |\mu|\,\|a\| + |\lambda||\mu| \\ &= \|(a,\lambda)\|\,\|(b,\mu)\|\,.\end{aligned}$$

Confirming the algebra structure is a straightforward check. Maybe: Remark on adjunction

Example 2. 1. Let V be a Banach space. Then

$$\mathcal{B}(V) := \{T\colon V\to V \mid T \text{ bounded linear}\}$$

with norm $\|T\| := \sup_{v\in V}\frac{\|Tv\|}{\|v\|}$ and composition as multiplication is a unital Banach algebra.

2. Let X be a topological space. We can define

$$\mathcal{C}_b(X) := \left\{f : X\to\mathbb{C} \mid f \text{ continuous, } \sup_{x\in X}|f(x)| < \infty\right\}$$

and

$$\mathcal{C}_0(X) := \left\{f\in\mathcal{C}_b(X) \mid \forall\varepsilon>0\exists K\subseteq X \text{ compact, } f^{-1}((-\varepsilon,\varepsilon))\subseteq K\right\}$$

with pointwise multiplication and $\|f\| := \sup_{x\in X}|f(x)|$. Both of these form Banach algebras. $\mathcal{C}_b$ is always unital with unit const_1, whereas $\mathcal{C}_0$ is unital if and only if X is compact.

Definition 3. A *(twosided) ideal* $J\subseteq A$ is a subspace $J\subseteq A$ with $AJ\subseteq J$ and $JA\subseteq J$. This is equivalent to J being a twosided ideal of A viewed as an ordinary (non-unital) ring.

Lemma 4. If $J\subseteq A$ is a closed ideal, the quotient ring A/J equipped with the norm

$$\|a+J\| := \inf_{j\in J}\|a+j\|$$

is again a Banach algebra.

Proof. Quotients of algebras under twosided ideals are again algebras, hence so is A/J. Further, the underlying normed vector space of A/J agrees with the quotient A/J of underlying normed vector spaces, hence is the quotient of a Banach space by a closed subspace and as such again a Banach space. Lastly, as J is an ideal, for $a, b \in A$ and $j, k \in J$ we have $aj + bk + jk \in J$, hence

$$\begin{aligned}
\|(a+J)(b+J)\| &= \inf_{j \in J} \|ab + j\| \\
&\leqslant \inf_{j,k \in J} \|ab + aj + bk + jk\| \\
&= \inf_{j,k \in J} \|(a+j)(b+k)\| \\
&\leqslant \inf_{j,k \in J} \|a+j\| \, \|b+k\| \\
&= \left(\inf_{j \in J} \|a+j\|\right) \left(\inf_{k \in J} \|b+k\|\right) \\
&= \|a+J\| \, \|b+J\| \,,
\end{aligned}$$

so submultiplicativity holds. ☐

Example 5. For a Banach algebra A, $A \subseteq \tilde{A}$ is a twosided ideal.

Proof. The map $p : \tilde{A} \to \mathbb{C}, (a, \lambda) \mapsto \lambda$ is a ring homomorphism, hence the kernel $\ker p = A$ is a twosided ideal. Further, p is continuous and $\{0\} \subseteq \mathbb{C}$ is closed, hence so is A. ☐

Definition 6. For a unital Banach algebra A and an element $a \in A$, we define the *spectrum*

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C} \mid (\lambda \cdot 1 - a) \notin A^{\times}\},$$

where $A^{\times}$ is the group of units of A. We further define the spectral radius

$$r(a) := \sup_{\lambda \in \mathrm{sp}(a)} |\lambda|.$$

This spectrum for elements of a general Banach algebra behaves just like the familiar one:

Theorem 7. Let A be a unital Banach algebra and $a \in A$. Then:

- $\mathrm{sp}(a) \subseteq \mathbb{C}$ is non-empty and compact.

- The following formula describes the spectral radius:

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

Especially, by submultiplicativity, $r(a) \leqslant \|a\|$.

Corollary 8 (Gelfand-Mazur)**.** Let A be a unital Banach algebra with $A^\times = A\backslash\{0\}$. Then $A \cong \mathbb{C}$.

Proof sketch. Let $a \in A^\times$. By Theorem 7, there is some $\lambda \in \mathbb{C}$ with $\lambda\cdot 1 - a \in A\backslash A^\times = \{0\}$, so $a = \lambda \cdot 1$. This provides an isomorphism $A \cong \mathbb{C}$. Maybe: More detail. □

2.2 Commutative Banach algebras

For this section, fix a commutative Banach algebra A.

Definition 9.

- A *character* on A is a non-zero $\mathbb{C}$-algebra homomorphism $\chi : A \to \mathbb{C}$.
- The *spectrum* of A is

$$\hat{A} := \{\chi : A \to \mathbb{C} \mid \chi \text{ character}\}$$

Remark 9.1. If A is unital, for an algebra homomorphism $\chi : A \to \mathbb{C}$ to be non-zero is equivalent to being unital, i.e. satisfying $\chi(1) = 1$. This is because $\chi(a) = \chi(1)\chi(a)$ for all $a \in A$.

Example 10. As we will see in TODO: reference, we have:

- For a locally compact Hausdorff space X,

$$X \cong \widehat{\mathcal{C}_0(X)}$$

- For a unital C^*-algebra A and $a \in A$ with $aa^* = a^*a$ we have

$$\mathrm{sp}(a) \cong \widehat{\langle 1, a\rangle},$$

where $\langle 1, a\rangle \subseteq A$ is the sub-C^*-algebra of A generated by 1 and a.

Fact 10.1. For a commutative Banach algebra A, the following hold:

- An ideal $m \subseteq A$ is maximal if and only if it has codimension 1.
- Maximal ideals are closed.

Proposition 11. If additionally A is unital, the map

$$\begin{aligned} \hat{A} &\to \mathrm{mSpec}(A) \\ \chi &\mapsto \ker \chi, \end{aligned}$$

where $\mathrm{mSpec}\, A$ denotes the set of maximal ideals of A, is a bijection.

Proof. As $\ker \chi$ has codimension 1 and is closed, it is maximal by Fact 10.1, hence the map is well defined.

Let $m \in \mathrm{mSpec}(A)$. m is closed by Fact 10.1. Thus, A/m is a Banach algebra by Lemma 4, and a field by maximality of m, hence we have $A/m \cong \mathbb{C}$ by Corollary 8. But then, the quotient map $A \to A/m \cong \mathbb{C}$ is a character with kernel m, hence surjectivity.

For injectivity, let χ, χ' be characters with $\ker \chi = \ker \chi' =: m$. Note that $A/m \cong \mathbb{C}$ by the argument above and fix such an isomorphism. Especially, both χ and χ' factor as a composition $A \to A/m \cong \mathbb{C} \to \mathbb{C}$, where the first map is the quotient map and the second map is the fixed isomorphism. But now, as any non-zero homomorphism of $\mathbb{C}$-algebras from $\mathbb{C}$ to $\mathbb{C}$ is the identity[1], the composition above is unique and $\chi = \chi'$. $\square$

Maybe: Something about the topology on the spectrum

Lemma 12. For $\chi \in \hat{A}$, we have $\|\chi\| \leqslant 1$. In particular, $\hat{A} \subseteq A^* := \mathcal{B}(A, \mathbb{C})$, where $\mathcal{B}(V, W)$ denotes the Banach space of bounded linear maps $V \to W$, equipped with the operator norm.

Proof. Suppose A is unital and let $\chi \in \hat{A}$. By the proof of Proposition 11, χ is already isomorphic to the quotient map $p : A \to A/(\ker \chi)$. But that quotient map satisfies $\|p(a)\| = \|a + \ker \chi\| \leqslant \|a\|$ for all $a \in A$, hence so does χ.
Now, consider the case where A is not unital. We can define

$$\begin{aligned} \tilde{\chi} : \tilde{A} &\to \mathbb{C} \\ (a, \lambda) &\mapsto \chi(a) + \lambda \end{aligned}$$

It is easy to see that this is again an algebra homomorphism. But then, by the unital case:

$$|\chi(a)| = |\tilde{\chi}((a, 0))| \leqslant \|(a, 0)\| = \|a\|.$$

Hence, $\|\chi\| \leqslant 1$ and especially χ is continuous. $\square$

[1] To see this, first note that any such homomorphism φ is already unital. Then, $\varphi(\lambda) = \lambda\varphi(1) = \lambda$ for all $\lambda \in \mathbb{C}$.

Definition 13. We equip $A^* = \mathcal{B}(A, \mathbb{C})$ with the *weak-$*$* topology: The coarsest topology such that for all $a \in A$

$$\begin{aligned}\mathrm{ev}_a : A^* &\to \mathbb{C}\\ \varphi &\mapsto \varphi(a)\end{aligned}$$

is continuous. In other words, $\varphi_n \to \varphi$ in A^* if and only if $\varphi_n(a) \to \varphi(a)$ for all $a \in A$. We further equip $\hat{A} \subseteq A^*$ with the subspace topology.

Lemma 14. A^* is Hausdorff. In particular, so is $\hat{A}$.

Proof. Let $\varphi, \varphi' \in A^*$. Then there is some $a \in A$ with $\varphi(a) \neq \varphi'(a)$. Now, picking disjoint opens $U, V \subseteq \mathbb{C}$ with $\varphi(a) \in U$, $\varphi'(a) \in V$, their preimages form disjoint opens $\mathrm{ev}_a^{-1}(U), \mathrm{ev}_a^{-1}(V) \subseteq A^*$ with $\varphi \in \mathrm{ev}_a^{-1}(U)$ and $\varphi' \in \mathrm{ev}_a^{-1}(V)$. ☐

Theorem 15 (Banach-Alaoglu)**.** The closed unit ball $B^* := \{\varphi \in A^* \mid \|\varphi\| \leqslant 1\}$ is compact.

Proof. Omitted. ☐

Proposition 16. $\hat{A}$ is locally compact. If A is unital, $\hat{A}$ is compact.

Proof. By Lemma 12, $\hat{A}$ is a subspace of the closed unit ball $B^* := \{\varphi \in A^* \mid \|\varphi\| \leqslant 1\}$, which is compact by Theorem 15 and Hausdorff by Lemma 14, hence for the first part of the statement it suffices to show that $\hat{A} \cup \{0\} \subseteq B^*$ is closed. Pick a sequence $(\varphi_n)_{n\in\mathbb{N}} \in (\hat{A} \cup \{0\})^{\mathbb{N}}$ converging to $\varphi \in B^*$. Then

$$\varphi(ab) = \lim_{n\to\infty} \varphi_n(ab) = \lim_{n\to\infty} \varphi_n(a) \lim_{n\to\infty} \varphi_n(b) = \varphi(a)\varphi(b)$$

and analogous arguments for $\mathbb{C}$-linearity work, hence φ is again an algebra homomorphism and hence $\varphi \in \hat{A}$. If A is unital and all φ_n are non-zero, then $\varphi_n(1) = 1$ for all $n \in \mathbb{N}$ by Remark 9.1 and especially $\varphi(1) = 1$, so φ is non-zero as well - hence in that case, $\hat{A}$ is already compact. ☐

2.3 The Gelfand transform

Definition 17. We define the *Gelfand transform* to be the map

$$\begin{aligned}\Gamma\colon A &\to \mathcal{C}_0(\hat{A})\\ a &\mapsto (\chi \mapsto \chi(a)).\end{aligned}$$

Remark 17.1. This is well defined: $\Gamma(a)$ is bounded as

$$|\Gamma(a)(\chi)| = |\chi(a)| \leqslant \|\chi\| \, \|a\| \leqslant \|a\| \, .$$

In other words, $\Gamma(a) \in \mathcal{C}_b(\hat{A})$. Now, let $\varepsilon > 0$ and pick a sequence $(\chi_n)_{n\in\mathbb{N}}$ of elements in $\hat{A} \cup \{0\}$ with $|\chi_n(a)| \leqslant \varepsilon$ for all $n \in \mathbb{N}$ that converges to an element $\chi \in \hat{A} \cup \{0\}$. Then $|\chi(a)| = \lim_{n\to\infty} |\chi_n(a)| \leqslant \varepsilon$, hence the set

$$K_{a,\varepsilon} := \{\chi \in \hat{A} \cup \{0\} \mid |\chi(a)| \leqslant \varepsilon\}$$

is a closed subspace of the compact space $\hat{A} \cup \{0\}$ and thus compact. But further

$$\Gamma(a)^{-1}((-\varepsilon, \varepsilon)) \subseteq K_{a,\varepsilon},$$

so $\Gamma(a) \in \mathcal{C}_0(\hat{A})$.

Theorem 18. Γ is a norm-decreasing algebra homomorphism and $\Gamma(A)$ separates points in $\hat{A}$.

Proof. We have

$$\|\Gamma(a)\| = \sup_{\chi\in\hat{A}} |\chi(a)| \leqslant \sup_{\chi\in\hat{A}} \|\chi\| \, \|a\| \leqslant \|a\| \, ,$$

so Γ is indeed norm-decreasing. Furthermore,

$$\Gamma(ab) = (\chi \mapsto \chi(ab)) = (\chi \mapsto \chi(a))(\chi \mapsto \chi(b)) = \Gamma(a)\Gamma(b)$$

and an analogous argument shows $\mathbb{C}$-linearity, so Γ is an algebra homomorphism. Lastly, for $\varphi \neq \varphi' \in \hat{A}$, there is some $a \in A$ with

$$\Gamma(a)(\chi) = \chi(a) \neq \chi'(a) = \Gamma(a)(\chi'),$$

hence $\Gamma(A)$ separates points in $\hat{A}$. □

Example 19. We will see in TODO: reference that for $A = \mathcal{C}_0(X)$ for a locally compact topological space X, the Gelfand transform $\Gamma\colon A \to \mathcal{C}_0(\hat{A})$ is an isomorphism.

Proposition 20. If A is unital and $a \in A$, we have $\mathrm{sp}(a) = \mathrm{Im}(\Gamma(a))$. If A is not unital, $\widetilde{\mathrm{sp}}(a) = \mathrm{Im}(\Gamma(a)) \cup \{0\}$, where $\widetilde{\mathrm{sp}}(a) := \mathrm{sp}\left((a, 0)\right)$ in $\tilde{A}$.

Proof. TODO: Think about this □

Corollary 21. For A unital and $a \in A$ we have $\|\Gamma(a)\| = r(a)$. In particular, $\ker \Gamma = \{a \in A | r(a) = 0\}$.

2.4 C^*-algebras

Definition 22. An *involution* on a Banach algebra A is a map $(-)^* : A \to A$ such that:

- For all $\lambda \in \mathbb{C}, a, b \in A$ we have $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$.
- For $a, b \in A$ we have $(ab)^* = b^* a^*$.
- For all $a \in A$ we have $a^{**} = a$.

Remark 22.1. One can, more generally, define an involution on a (non-unital) ring R as a (non-unital) ring automorphism $(-)^* : R \to R^{\mathrm{op}}$ with $((-)^*)^{-1} = ((-)^*)^{\mathrm{op}}$, where R^{op} is the ring with reversed multiplication. For such a ring with involution R, an involution on an R-algebra A is an involution $(-)^*$ on A making

$$\begin{tikzcd} R \arrow[r] \arrow[d, "(-)^*"] & A \arrow[d, "(-)^*"] \\ R \arrow[r] & A \end{tikzcd}$$

commute. An involution on a Banach algebra is then just an involution of the $\mathbb{C}$-algebra, where the involution on $\mathbb{C}$ is $\overline{(-)}$. This approach works in any category with a C_2-action.

Definition 23. A C^*-algebra is a Banach algebra $(A, \|.\|)$ with involution $(-)^*$ such that $\|a^* a\| = \|a\|^2$ for all $a \in A$.

Lemma 24. Let A be a C^*-algebra and $a \in A$. Then $\|a^*\| = \|a\|$.

Proof. For $a = 0$ we have $a^* = \overline{0} = 0$, hence $\|a\| = \|a^*\|$ is clear. Otherwise, we have $\|a\| > 0$ and

$$\|a\|^2 = \|a^* a\| \leqslant \|a^*\| \|a\| ,$$

thus dividing by $\|a\|$ yields $\|a\| \leqslant \|a^*\|$. Applying the same result to a^*, we get

$$\|a\| \leqslant \|a^*\| \leqslant \|a^{**}\| = \|a\| ,$$

so $\|a\| = \|a^*\|$. ∎

Example 25. The following Banach algebras with involution are C^*-algebras:

- $\mathcal{C}_b(X)$ and $\mathcal{C}_0(X)$ with (pointwise) complex conjugation.
- For any Hilbert space $\mathcal{H}$, the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ with involution induced by $\langle T^*v|w\rangle = \langle v|Tw\rangle$.

Proof. We have already seen that $\mathcal{C}_b(X)$ and $\mathcal{C}_0(X)$ are commutative Banach algebras. Pointwise conjugation being an involution is a simple check. Lastly,

$$\|a^*a\| = \sup_{x\in X}\left|a(x)\overline{a(x)}\right| = \sup_{x\in X}|a(x)|^2 = \|a\|^2,$$

so the given algebra with involution is indeed a C^*-algebra.

We have also already seen that $\mathcal{B}(\mathcal{H})$ is a Banach algebra. The axioms for an involution are again a straightforward check. Further, we have

$$\begin{aligned}\|T^*T\| &= \sup_{\substack{v\in V\\ \|v\|=1}} \|T^*Tv\| \\ &= \sup_{\substack{v\in V\\ \|v\|=1}} |\langle T^*Tv|T^*Tv\rangle|^{\frac{1}{2}} \\ &= \sup_{\substack{v\in V\\ \|v\|=1}} |\langle TTv|TTv\rangle|^{\frac{1}{2}} \\ &= \sup_{\substack{v\in V\\ \|v\|=1}} \|TTv\| \\ &= \|TT\|\end{aligned}$$

hence $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. □

Remark 25.1. Let A be a non-unital C^*-algebra. We can consider the unitization $\tilde{A}$ of the underlying Banach algebra: This is a unital Banach algebra with norm $\|(a,\lambda)\| = \|a\|, |\lambda|$, but with that norm it does not neccessarily admit the structure of a C^*-algebra compatible with that of A (Maybe: Example for this phenomenon). Hence we have to provide a different norm on $\tilde{A}$ if we aim to construct a unitization for C^*-algebras.

Definition 26. Let A be a C^*-algebra and consider the embedding

$$\begin{aligned}L\colon \tilde{A} &\to \mathcal{B}(A)\\ (a,\lambda) &\mapsto (b \mapsto ab + \lambda b).\end{aligned}$$

We will see in Lemma 27 that L is an injective algebra homomorphism, hence we define the norm and $(-)^*$ on $\tilde{A} \cong L(\tilde{A}) \subseteq \mathcal{B}(A)$ to be the ones inherited from $\mathcal{B}(A)$. We refer to this C^*-algebra as the *unitization* of A.

Lemma 27. Let A be a C^*-algebra. The map $L\colon \tilde{A} \to \mathcal{B}(A)$ from Definition 26 is an injective algebra homomorphism. Further, $\tilde{A} \cong L(\tilde{A})$ with the norm and $(-)^*$ inherited from $\mathcal{B}(A)$ forms a unital C^*-algebra and $A \subseteq \tilde{A}$ is a C^*-subalgebra.

Maybe: Prove at least parts of this

Note. *The norm on $\tilde{A}$ is $\|(a,\lambda)\| = \sup_{b\in A} \|ab + \lambda b\|$, the involution is $(a,\lambda)^* = (a^*, \overline{\lambda})$.*

Maybe: Note on the adjunction

Definition 28. Let A be a C^*-algebra and $a \in A$. We say that a is

- *self-adjoint* if $a^* = a$,
- *unitary* if $a^*a = aa^* = 1$,
- *normal* if $a^*a = aa^*$,
- *positive* if $a = b^*b$ for some $b \in A$.

Definition 29. For a (not neccessarily unital) C^*-algebra A and $a \in A$, we define the spectral radius of a to be $r(a) := r((a,0))$, the spectral radius of $(a,0) \in \tilde{A}$.

Proposition 30. For a C^*-algebra A, $a \in A$ normal and $n \in \mathbb{N}$, we have $\|a^n\| = \|a\|^n$.

Proof. Using normality and $(aa)^* = a^*a^*$ for the second equation, we have

$$\begin{aligned}
\left\|a^2\right\| &= \left\|a^2 \left(a^2\right)^*\right\|^{\frac{1}{2}} \\
&= \|(a^*a)(a^*a)^*\|^{\frac{1}{2}} \\
&= \|a^*a\| \\
&= \|a\|^2 .
\end{aligned}$$

Inductively, $\left\|a^{2^n}\right\| = \|a\|^{2^n}$. Further, by submultiplicativity, for any $n \in \mathbb{N}$ we have

$\|a^n\| \leqslant \|a\|^n$. But then, letting $k \in \mathbb{N}$ with $2^k > n$, we get

$$\begin{aligned} \|a\|^n \left\| a^{2^k - n} \right\| &\leqslant \|a\|^n \|a\|^{2^k - n} \\ &= \|a\|^{2^k} \\ &= \left\| a^{2^k} \right\| \\ &\leqslant \|a^n\| \left\| a^{2^k - n} \right\| . \end{aligned}$$

Now, if $a = 0$, the statement to be proven is trivial, otherwise we get

$$\|a\|^n \leqslant \|a^n\| \leqslant \|a\|^n ,$$

which yields the desired equality. □

Corollary 31. For a C^*-algebra A and $a \in A$ normal, we have $r(a) = \|a\|$.

Proof. Let $\alpha = (a, 0) \in \tilde{A}$. As a was normal, so is α. By Theorem 7 and Lemma 30, we have

$$r(a) = r(\alpha) = \lim_{n \to \infty} \|\alpha^n\|^{\frac{1}{n}} = \lim_{n \to \infty} (\|\alpha\|^n)^{\frac{1}{n}} = \|\alpha\| .$$

□

Corollary 32. For a $C*$-algebra A and $a \in A$, we have $\|a\| = r(a^*a)^{\frac{1}{2}}$. In particular, the norm of a C^*-algebra is already uniquely determined by the $\mathbb{C}$-algebra structure.

Proof. a^*a is self-adjoint, hence normal, so we get

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = r(a^*a)^{\frac{1}{2}} .$$

□