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# Universal spaces for asymptotic dimension via factorization

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The main goal of this paper is to construct universal spaces for asymptotic dimension by generalizing to the coarse context an approach to constructing universal spaces for covering dimension using a factorization result due to Mardesic.

## 1 Introduction

In classical dimension theory, a universal space for the class of separable metric spaces of covering dimension at most $n$ is a separable metric space of covering dimension $n$ into which every separable metric space of dimension at most $n$ embeds. Such spaces provide a common structure with which the embedded metric spaces can be studied. One way such universal spaces can be directly constructed was described by Menger in [10] where one imitates the construction of the Cantor set (a universal space for dimension 0) by taking a cube in $2n+1$-dimensional Euclidean space and removing successively smaller cubes, much in the same way one removes intervals in the construction of the Cantor set. An accessible explanation of the construction can be found in [5]. The resulting space is a compact subset of $\mathbb{R}^{2n+1}$ that is of covering dimension $n$ and has the desired universal space property.

This construction of a universal space for covering dimension is very geometric, but can hardly be adjusted to the more general settings of Cohomological Dimension, Extension Theory and Coarse Geometry. In this regard a more efficient means of showing the existence of universal spaces for covering dimension is via a factorization result due to Mardesic in [9]. Specifically, the following:

***Theorem* 1.1 (*Mardesic Factorization Theorem*)** *Let $f : X \to Y$ be a map from a compact Hausdorff space $X$ to a compact metric space $Y$. Then there is a compact metric space $Z$ with $dim(Z) \leq dim(X)$ and maps $g : X \to Z$, $h : Z \to Y$ such that $f$ factors through $g$ and $h$.*

How this result is used to construct universal spaces is fairly easy to state. One takes their space $X$ to be the Stone-Cech compactification of the disjoint union of all separable

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metric spaces of covering dimension at most $n$. Then $Y$ is taken to be the Hilbert cube into which every separable metric space embeds. Using this embedding property we get a map from $X$ to $Y$ that is an embedding on each separable metric space as a subset of $X$. Then the Mardesic factorization theorem yields a compact metric space $Z$ with dimension equal to the dimension of $X$, $n$ in this case. As the map from $X$ to $Y$ is an embedding on each separable metric space that makes up $X$ we have that its factor map into $Z$ has the same property, making $Z$ the desired universal space. A striking feature of this approach that it makes use of the Stone-Cech compactification, a space very far from being separable and metric, to get a result that belongs to separable metric topology.

In this paper we will focus on generalizing this approach to universal spaces to the field of coarse geometry and asymptotic dimension.

The asymptotic dimension of metric spaces was introduced by Gromov in [6] in order to study geometric properties of infinite discrete groups. As the small scale structure of such spaces is trivial the asymptotic dimension is a property of the "large scale" or "coarse" structure of the space. Basic results surrounding asymptotic dimension are well summarized in [4] and [12]. In order to generalize the aforementioned approach to universal spaces to the context of asymptotic dimension we have to replace the constituent parts used in the application of the Mardesic factorization theorem with analogs more suitable to coarse geometry. In the place of the Stone-Cech compactification of the disjoint union of separable metric spaces we use a "wedge" construction of all separable metric spaces of a given asymptotic dimension as described at the beginning of section 3. In place of the Hilbert cube we use the Urysohn universal metric space. Of course, using these pieces to construct a universal space for asymptotic dimension is predicated on us first proving a large scale analog of Mardesic's factorization result. Specifically, we need to prove the following:

***Theorem* 1.2** *Let $f : X \to Y$ be a coarsely continuous map from a metric space $X$ to a separable metric space $Y$. Then $f$ factors through coarsely continuous maps $g : X \to Z$ and $h : Z \to Y$ with* $\operatorname{asdim} Z \le \operatorname{asdim} X$ *and $Z$ is separable metric.*

The existence of the desired universal space (Theorem 3.2) follows quickly from the above result which the majority of this paper is devoted to proving. Similarly to the above remark regarding the Stone-Cech compactification, this "coarse Mardesic" factorization theorem is an example of a result that uses nonseparable metric spaces to prove a result for separable metric spaces. Said differently, we apply this result to a nonseparable metric space to get universal spaces for separable metric spaces of a given asymptotic dimension.

Our construction of universal spaces for asymptotic dimension is not the first such construction, but appears to be the first to be done via factorization. In [3] Dranishnikov and Zarichnyi constructed universal spaces for proper metric spaces with bounded geometry (for an explanation of these terms see [12]). Then in [2] universal spaces for separable metric spaces were constructed. In [8] universal spaces for general proper metric spaces of asymptotic dimension zero were constructed. Finally in [14] it was shown that universal spaces for general proper metric spaces of asymptotic dimension greater than or equal to one do not exist.

The outline of this paper is as follows. In the next section we cover some basic preliminary notions in coarse geometry and the theory surrounding asymptotic dimension. The existence of universal spaces for separable metric spaces of a given asymptotic dimension is proved in section 4 and Theorem 1.2 is proved in Section 5. As the constructions we make in section 5 seem like they may be applicable to other coarse properties we provide some open questions in section 6 about the possible existence of universal spaces for asymptotic property C, finite decomposition complexity, and Yu's coarse property A.

## 2 Preliminaries

We say that a map $f : X \to Y$ between metric spaces is **coarsely continuous** if $f$ sends uniformly bounded covers in $X$ to uniformly bounded families in $Y$. Maps $f, g : X \to Y$ are **coarsely close** if there is $r > 0$ such that $f(x)$ and $g(x)$ are such that the distance between $f(x)$ and $g(x)$ is less than $r$ for all $x \in X$. A coarsely continuous map $f : X \to Y$ is a **coarse equivalence** if there is a coarsely continuous map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are coarsely close to the identity maps of $X$ and $Y$ respectively. A map $f : X \to Y$ is a **coarse embedding** if $f$ is a coarse equivalence between $X$ and $f(X)$. Note that a coarsely continuous map $f : X \to Y$ is a coarse embedding if and only if the preimages of uniformly bounded covers of $Y$ are uniformly bounded in $X$. We recall that the **asymptotic dimension** of a metric space $X$ is bounded by $n$, written $\mathrm{asdim} X \leq n$, if for every $r > 0$ there is a uniformly bounded cover of $X$ that splits into $n + 1$ families of $r$-disjoint sets. All the spaces are assumed to be metric.

Let $\mathcal{F}$ be a family of subsets of $X$. By the **star** $\mathrm{st}(A, \mathcal{F})$ of a subset $A$ of $X$ we denote the union of $A$ with all the sets $F \in \mathcal{F}$ meeting $A$, and $\mathrm{st}\mathcal{F}$ stands for the cover of $X$ consisting of $\mathrm{st}(x, \mathcal{F})$ for all $x \in X$. We say that $\mathcal{F}$ **separates** subsets $A$ and $B$ of $X$ if $\mathrm{st}(A, \mathcal{F})$ does not meet $B$.

For a metric space $(X, d)$ and a subset $A \subset X$ we denote by $\mathbb{B}(A, r)$ the set of points at distance at most $r$ from $A$. We say that a subset $B$ of $X$ is $r$-**close** to $A$ if in $B \subset \mathbb{B}(A, r)$.

*Proposition* 2.1 *Let $X$ be a metric space with $\mathrm{asdim} X \leq n$. Then for every $r > 0$ there is a uniformly bounded cover of $X$ which has a Lebesgue number $r$ and which splits into $n + 1$ families of $r$-disjoint sets. Moreover if $b$ is a base point in $X$ then we can assume that $\mathbb{B}(b, r)$ is contained in an element of each family of this splitting.*

**Proof.** Let $\mathcal{F}$ be a uniformly bounded cover of $X$ that splits into families $\mathcal{F}_j$, $1 \leq j \leq n + 1$, of $(3r)$-disjoint sets. Then the families $\mathcal{F}'_j = \{\mathbb{B}(F, r) : F \in \mathcal{F}_j\}$ are $r$-disjoint and form the cover $\mathcal{F}' = \mathcal{F}'_1 \cup \cdots \cup \mathcal{F}'_{n+1}$ whose Lebesgue number is $r$.

Now suppose $b$ is a base point of $X$. Define $\mathcal{F}''_j$ to be the family consisting of elements of $\mathcal{F}'_j$ not intersecting $B(b, 2r)$ and one extra element being the the union of $B(b, r)$ with the elements of $\mathcal{F}'_j$ intersecting $B(b, 2r)$. Then the cover $\mathcal{F}'' = \mathcal{F}''_1 \cup \cdots \cup \mathcal{F}''_{n+1}$ satisfies the second conclusion of the proposition. ■

*Definition* 2.1 Let $X$ be a metric space of asdim $\leq n$. A sequence $\mathcal{F}$ of covers $\mathcal{F}_i$, $i \in \mathbb{N}$, of $X$ is said to **witness** asdim $\leq n$ if $\mathcal{F}_i$ is $R_i$-bounded, $r_i$ is a Lebesgue number of $\mathcal{F}_i$


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and $\mathcal{F}_i$ splits into the union of $n+1$ families $\mathcal{F}_{ij}$, $1 \le j \le n+1$, of $r_i$-disjoint sets with $r_{i+1} > (100i+1)R_i$ and $R_i > r_i$. We will say that $\mathcal{F}$ is **guided** by the pairs $(R_i, r_i)$. For a pointed space $X$ with a base point $b$ the sequence $\mathcal{F}$ is said to **respect the base point** if $\mathbb{B}(b, r_i)$ is contained in an element of $\mathcal{F}_{ij}$ for every $j$.

**Remark.** For the results presented in this section and the following one it is sufficient to assume $r_{i+1} > 2R_i$. The inequality $r_{i+1} > (100i+1)R_i$ will be used in the last section for proving the factorization theorem.

*Definition* 2.2 Let $X$ be a set. A sequence $\mathcal{F}$ of covers $\mathcal{F}_i$, $i \in \mathbb{N}$, of $X$ is said to **define** asdim $\le n$ if $\mathrm{st}\mathcal{F}_i$ refines $\mathcal{F}_{i+1}$, $\mathcal{F}_i$ splits into the union of $n+1$ families $\mathcal{F}_{ij}$, $1 \le j \le n+1$, of disjoint sets, the sets of $\mathcal{F}_{i+1,j}$ are separated by $\mathcal{F}_i$ and for every $x, y \in X$ there is $i$ such that $y \in \mathrm{st}(x, \mathcal{F}_i)$. We say that $\mathcal{F}$ **separates the points** of a subset $Z$ of $X$ if $\mathcal{F}_1$ separates the points of $Z$.

*Proposition* 2.2

*(i) A metric space of* asdim $\le n$ *admits a sequence of covers witnessing* asdim $\le n$*. A pointed metric space of* asdim $\le n$ *admits a sequence of covers witnessing* asdim $\le n$ *and respecting the base point.*

*(ii) A sequence of covers witnessing* asdim $\le n$ *is also a sequence defining* asdim $\le n$*.*

**Proof.**

(i) follows from Proposition 2.1.

(ii) follows from the definitions of families witnessing asdim $\le n$ and defining asdim $\le n$. ■

*Proposition* 2.3 *Let $\mathcal{F}$ be a sequence of covers $\mathcal{F}_i$ of a set $X$ with the splittings $\mathcal{F}_{ij}$ such that $\mathcal{F}$ defines* asdim $\le n$*. For $x, y \in X$ set $d_{\mathcal{F}}(x, y)$ to be the maximal $i$ such that $\mathcal{F}_i$ separates $x$ and $y$ if such $i$ exists and $d_{\mathcal{F}}(x, y) = 0$ otherwise. Then for a subset $Z \subset X$ separated by $\mathcal{F}$ we have:*

*(i) $d_{\mathcal{F}}$ is a metric on $Z$;*

*(ii) $\mathcal{F}_{i+1}$ restricted to $Z$ is $i$-bounded and has a Lebesgue number $i-1$, $\mathcal{F}_{i+1,j}$ restricted to $Z$ is $(i-1)$-disjoint and, as a result,* asdim$Z \le n$ *(everything here with respect to $d_{\mathcal{F}}$);*

*(iii) if $\mathcal{F}$ is a sequence witnessing* asdim $\le n$ *for a metric space $(X, d)$ then $d$ and $d_{\mathcal{F}}$ are coarsely equivalent on $Z$*

**Proof.**

(i) The only thing needed to be checked is the triangle inequality. Take $x, y, z \in Z$. Clearly we may assume that $i = d_{\mathcal{F}}(x, z) \ge d_{\mathcal{F}}(z, y) \ge 1$. Then $x, y \in \mathrm{st}(z, \mathcal{F}_{i+1})$ and hence $x$ and $y$ are contained in an element of $\mathcal{F}_{i+2}$ and therefore $d_{\mathcal{F}}(x, y) \le i+1$. Thus $d_{\mathcal{F}}(x, y) \le d_{\mathcal{F}}(x, z) + d_{\mathcal{F}}(z, y)$.

(ii) follows from the definition of $d_{\mathcal{F}}$ and the definition of a family defining asdim $\le n$.

(iii) Take a uniformly bounded cover $\mathcal{B}$ of $Z$ with respect to $d_{\mathcal{F}}$. By (ii), $\mathcal{B}$ refines $\mathcal{F}_i$ for some $i$ and therefore $\mathcal{B}$ is uniformly bounded with respect to $d$. A similar argument also works in the other direction. ■


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## 3 Universal space

Let $(X^\alpha, d^\alpha)$ be a collection of separable metric spaces with asdim $\leq n$ representing, up to coarse equivalence, all the separable metric spaces with asdim $\leq n$. In each $X^\alpha$ we fix a base point $b^\alpha$ and, by (i) of Proposition 2.2, take a sequence $\mathcal{F}^\alpha$ of covers $\mathcal{F}_i^\alpha$ of $X^\alpha$ with the splittings $\mathcal{F}_{ij}^\alpha$ that witnesses asdim $\leq n$, respects the base point $b^\alpha$ and guided by the pairs $(R_i^\alpha, r_i^\alpha)$. Replacing $X^\alpha$ by a coarsely equivalent subset we may assume that $\mathcal{F}^\alpha$ separates the points of $X^\alpha$.

Denote by $X = \vee X^\alpha$ the wedge sum of the spaces $X^\alpha$ with the base point $b \in X$ obtained by identifying the base points of all $X^\alpha$. We consider each $X^\alpha$ as a subset of $X$ and denote by $\mathcal{F}$ the sequence of covers $\mathcal{F}_i$ of $X$ with the splittings $\mathcal{F}_{ij}$ defined as follows: $\mathcal{F}_{ij}$ is the union of $\mathcal{F}_{ij}^\alpha$ for all $\alpha$ where the sets containing the base point $b$ being replaced by their union.

***Proposition* 3.1** *In the above setting the following holds:*

*(i) $\mathcal{F}$ is a sequence of covers of $X$ defining* asdim $\leq n$ *and separating the points of $X$, and hence, by (i) and (ii) of Proposition 2.3, we have* $\operatorname{asdim}(X, d_{\mathcal{F}}) \leq n$*;*

*(ii) each $(X^\alpha, d^\alpha)$ is coarsely embedded into $(X, d_{\mathcal{F}})$;*

*(iii) if a function $f : X \longrightarrow Y$ to a metric space $Y$ isometrically embeds each $X^\alpha$ into $Y$ with respect to $d_{\mathcal{F}}$ then $f$ is coarsely continuous.*

**Proof**

(i) Since $\mathcal{F}^\alpha$ separates the points of $X^\alpha$ we get that $\mathcal{F}_1^\alpha$ consists of singletons. Then $\mathcal{F}_1$ consists of singletons as well and therefore $\mathcal{F}$ separates the points of $X$.

Let $x^\alpha \in X^\alpha$ and $x^\beta \in X^\beta$, and let $i$ be such that $x^\alpha \in \mathbb{B}(b^\alpha, r_i^\alpha)$ and $x^\beta \in \mathbb{B}(b^\beta, r_i^\beta)$. Then $x^\alpha$ and $x^\beta$ are contained in an element of $\mathcal{F}_i$ because $\mathcal{F}^\alpha$ and $\mathcal{F}^\beta$ respect the base points.

Thus to show that $\mathcal{F}$ defines asdim $\leq n$ we only need to show that $\mathcal{F}_i$ refines $\mathcal{F}_{i+1}$ and separates $\mathcal{F}_{i+1,j}$. Take a point $x \in X$ and consider the following cases.

Case 1: $b \notin \operatorname{st}(x, \mathcal{F}_i)$. Then for $X^\alpha$ such that $x \in X^\alpha$ we have that the sets of $\mathcal{F}_i$ containing $x$ are exactly the sets of $\mathcal{F}_i^\alpha$ containing $x$. By (ii) of Proposition 2.2, $\mathcal{F}_i^\alpha$ is also defining asdim $\leq n$ and hence $\operatorname{st}(x, \mathcal{F}_i^\alpha)$ is contained in an element of $\mathcal{F}_{i+1}^\alpha$ and no element of $\mathcal{F}_i^\alpha$ containing $x$ meets disjoint elements of $\mathcal{F}_{i+1,j}^\alpha$ for every $j$. Since $\mathcal{F}_i$ restricted to $X^\alpha$ coincides with $\mathcal{F}_i^\alpha$ we get that $\operatorname{st}(x, \mathcal{F}_i)$ is contained in an element of $\mathcal{F}_{i+1}$ and no element of $\mathcal{F}_i$ containing $x$ meets disjoint elements of $\mathcal{F}_{i+1,j}$ for every $j$.

Case 2: $b \in \operatorname{st}(x, \mathcal{F}_i)$. Recall that $\mathcal{F}^\alpha$ witnesses asdim $\leq n$. Then, since $r_{i+1}^\alpha > 2R_i^\alpha$, we have that $\operatorname{st}(x, \mathcal{F}_i)$ is contained in the union of the balls $\mathbb{B}(b^\alpha, r_{i+1}^\alpha)$ for all $\alpha$ and this union in its turn is contained in an element of $\mathcal{F}_{i+1,j}$ for every $j$ because each $\mathcal{F}^\alpha$ respects the base point. Thus we get that $\operatorname{st}(x, \mathcal{F}_i)$ is contained in an element of $\mathcal{F}_{i+1}$ and no element of $\mathcal{F}_i$ containing $x$ meets disjoint elements of $\mathcal{F}_{i+1,j}$ for every $j$.

(ii) follows from (iii) of Proposition 2.3 and the fact that $\mathcal{F}_i$ restricted to $X^\alpha$ coincides with $\mathcal{F}_i^\alpha$.

(iii) Consider $F \in \mathcal{F}_i$. By (ii) of Proposition 2.3, $\mathcal{F}_i$ is $(i-1)$-bounded with respect to $d_{\mathcal{F}}$. If $F$ does not contain $b$ then $F$ is contained in some $X^\alpha$ and therefore


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$\operatorname{diam} f(F) = \operatorname{diam} F \le i-1$. If $F$ does contain $b$ then $F$ is contained in the union of the elements of $\mathcal{F}_i^\alpha$ containing $b$ for all $\alpha$ and therefore $\operatorname{diam} f(F) \le 2(i-1)$. Thus $f(\mathcal{F}_i)$ is uniformly bounded. Moreover, by (ii) of Proposition 2.3, any uniformly bounded cover of $(X, d_{\mathcal{F}})$ refines $\mathcal{F}_i$ for some $i$ and, hence, $f$ is coarsely continuous. ■

***Theorem* 3.2** *(Bell-Nagorko [2])*
*For every $n$ there is a separable metric space of $asdim = n$ which is universal for separable metric spaces of $asdim \le n$ (i.e. it contains a coarsely equivalent copy of every separable metric space of $asdim \le n$).*

**Proof.** Consider the space $X$ from Proposition 3.1 and assume that $X$ is equipped with the metric $d_{\mathcal{F}}$. Let $\mathbb{U}$ be the Urysohn space [11]. Recall that each separable metric space isometrically embeds into $\mathbb{U}$ and $\mathbb{U}$ is homogeneous by isometries. Then one can define isometric embeddings $f^\alpha : X^\alpha \to \mathbb{U}$ (with respect to $d_{\mathcal{F}}$ restricted to $X^\alpha$) sending all the base points $b^\alpha$ to the same point in $\mathbb{U}$ and this way to define a function $f : X \to \mathbb{U}$ that isometrically embeds each $X^\alpha \subset X$ into $\mathbb{U}$ (with respect to $d_{\mathcal{F}}$). By (iii) of Proposition 3.1, $f$ is coarsely continuous. Apply Theorem 1.2 to factorize $f$ through coarsely continuous maps $g : X \to Z$ and $h : Z \to \mathbb{U}$ with $Z$ being separable metric with $\operatorname{asdim} Z \le \operatorname{asdim} X$. Recall that, by (i) of Proposition 3.1, $\operatorname{asdim} X \le n$ and hence $\operatorname{asdim} Z \le n$.

Since $f$ coarsely (even isometrically) embeds $X^\alpha$ into $\mathbb{U}$ we get that $g$ coarsely embeds $X^\alpha$ into $Z$. Indeed, if $\mathcal{B}$ is a uniformly bounded cover of $Z$ then $h(\mathcal{B})$ is uniformly bounded in $\mathbb{U}$ and, hence, the cover $g^{-1}(\mathcal{B})$ is a uniformly bounded on $X^\alpha$ since $g^{-1}(\mathcal{B})$ and $f^{-1}(h(\mathcal{B}))$ coincide on $X^\alpha$ and $f^{-1}(h(\mathcal{B}))$ is uniformly bounded on $X^\alpha$ with respect to $d_{\mathcal{F}}$. Finally note that, by (ii) of Proposition 3.1, the metrics $d^\alpha$ and $d_{\mathcal{F}}$ are coarsely equivalent on $X^\alpha$ and this shows that $Z$ is a universal space for separable metric spaces of $\operatorname{asdim} \le n$. ■

## 4 Factorization Theorem

We actually prove the following more general result obviously implying Theorem 1.2:

***Theorem* 4.1** *Let $f : X \to Y$ be a coarsely continuous map of metric spaces and let $\mathrm{w}X$ denote the topological weight of the space $X$. Then $f$ factors through coarsely continuous maps $g : X \to Z$ and $h : Z \to Y$ with $\operatorname{asdim} Z \le \operatorname{asdim} X$ and $\mathrm{w}Z \le \mathrm{w}Y$.*

Let us first make the following observation:

***Proposition* 4.2** *Let $f : X \to Y$, $g : X \to Z$ and $h : Z \to Y$ be coarsely continuous maps of metric spaces such that $f$ and $h \circ g$ are coarsely close and $\mathrm{w}Y$ is infinite. Then there is a metric space $Z'$ and coarsely continuous maps $g' : X \to Z'$ and $h' : Z' \to Y$ such that $\operatorname{asdim} Z' \le \operatorname{asdim} Z$, $\mathrm{w}Z' \le \max\{\mathrm{w}Z, \mathrm{w}Y\}$ and $f = h' \circ g'$.*


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**Proof.** Set $Z' = \{(f(x), g(x)) : x \in X\} \subset Y \times Z$ and consider $Z'$ with the metric inherited from $Y \times Z$ and defined as the maximum of the distances between the coordinates. Define $g' : X \to Z'$ by $g'(x) = (f(x), g(x)), x \in X$ and let $h' : Z' \to Y$ and $\pi : Z' \to Z$ be the projections. Clearly $g'$, $h'$ and $\pi$ are coarsely continuous, $f = h' \circ g'$ and $\mathrm{w}Z' \leq \max\{\mathrm{w}Z, \mathrm{w}Y\}$.

Let us show that $\pi$ is a coarse embedding. Take a uniformly bounded cover $\mathcal{B}$ of $Z$. Then, since $h(\mathcal{B})$ is uniformly bounded and $f$ and $h \circ g$ are coarsely close, we get that $f(g^{-1}(\mathcal{B}))$ is uniformly bounded as well. Note that $\pi^{-1}(B) \subset f(g^{-1}(B)) \times B$ for every $B \in \mathcal{B}$ and hence $\pi^{-1}(\mathcal{B})$ is uniformly bounded. This implies that $\pi$ is a coarse embedding and hence $\mathrm{asdim} Z' \leq \mathrm{asdim} Z$. ■

**Proof of Theorem 4.1.** The theorem is trivial if $\mathrm{w}Y$ is finite, so we may assume that $\mathrm{w}Y$ is infinite. Let $\mathrm{asdim} X \leq n$. Fix a base point $b$ in $X$. Take a sequence $\mathcal{F}^X$ of covers $\mathcal{F}_i^X$ of $X$ with splittings $\mathcal{F}_{ij}^X$ guided by the pairs $(R_i, r_i)$ and witnessing asdim $\leq n$. For each $i$ take a uniformly bounded cover $\mathcal{F}_i^Y$ of $Y$ that is refined by $f(\mathcal{F}_i^X)$ with the cardinality of $\mathcal{F}_i^Y$ bounded by $\mathrm{w}Y$ and define $\mathcal{F}_{ij}^0$ as a collection of disjoint subsets of $X$ such that each element of $\mathcal{F}_{ij}^0$ is a union of elements of $\mathcal{F}_{ij}^X$ contained in an element of $f^{-1}(\mathcal{F}_i^Y)$, each set of $\mathcal{F}_{ij}^X$ is contained in some element of $\mathcal{F}_{ij}^0$ and no element of $\mathcal{F}_{ij}^X$ is contained in different elements of $\mathcal{F}_{ij}^0$. Note that $f(\mathcal{F}_{ij}^0)$ refines $\mathcal{F}_i^Y$.

We will construct by induction collections $\mathcal{F}_{ij}^p$, $p \in \mathbb{N}$, of subsets of $X$ satisfying the following conditions (the families $\mathcal{F}^p$ and $\mathcal{F}_i^p$ below are determined in the standard way by $\mathcal{F}_{ij}^p$, namely $\mathcal{F}_i^p$ is the union of the families $\mathcal{F}_{ij}^p$ for $1 \leq j \leq n+1$ and $\mathcal{F}^p$ is the sequence of the covers $\mathcal{F}_i^p$ of $X$):

(†1) the cardinality of $\mathcal{F}_{ij}^p$ is bounded by $\mathrm{w}Y$;

(†2) the elements of $\mathcal{F}_{ij}^p$ are disjoint, each element of $\mathcal{F}_{ij}^p$ is a union of elements of $\mathcal{F}_{ij}^X$ and each element of $\mathcal{F}_{ij}^X$ is contained in some element of $\mathcal{F}_{ij}^p$;

(†3) $\mathcal{F}_{ij}^{p+1}$ and $\mathcal{F}_{ij}^p$ restricted to $\mathbb{B}(b, ir_{i+p})$ coincide and $\mathcal{F}_{ij}^{p+1}$ refines $\mathcal{F}_{ij}^p$;

(†4) $\mathrm{st}\mathcal{F}_i^{p+1}$ refines $\mathcal{F}_{i+1}^p$;

(†5) the elements of $\mathcal{F}_{i+1,j}^p$ are separated by $\mathcal{F}_i^{p+1}$.

Clearly the relevant properties hold for $\mathcal{F}^0$. Assume that the construction is completed for $\mathcal{F}_i^p$ with $p + i = m$ and proceed to $m+1$ in the following order $\mathcal{F}_{m+1}^0, \mathcal{F}_m^1, \mathcal{F}_{m-1}^2, \ldots, \mathcal{F}_1^m$. Recall that $\mathcal{F}_{m+1}^0$ is already defined and assume that $\mathcal{F}_{m+1}^0, \ldots, \mathcal{F}_{i+1}^{m-i}$ are already constructed. We will construct $\mathcal{F}_i^{m-i+1}$ as follows. Denote $p = m - i$.

Let $1 \leq j, t \leq n+1$ and let a set $A$ be the union of some elements of $\mathcal{F}_{ij}^X$. By splitting $A$ by $\mathcal{F}_{i+1,t}^p$ we mean replacing $A$ by the family of disjoint subsets of $X$ which is the union of the following collections:

collection 1: the union of the elements of $\mathcal{F}_{ij}^X$ contained in $A$ and intersecting $\mathbb{B}(b, ir_m)$ (collection 1 consists of only one set);


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collection 2: for each element $F$ of $\mathcal{F}^p_{i+1,t}$ take the union of the elements of $\mathcal{F}^X_{ij}$ which are contained in both $A$ and $F$ and which were not used in constructing collection 1 (by (†1) the cardinality of collection 2 is bounded by w$Y$);

collection 3: for each element $F$ of $\mathcal{F}^p_{i+1,t}$ take the union of the elements of $\mathcal{F}^X_{ij}$ which are contained in $A$ and $(r_{i+1}/10)$-close to $F$ and which were not used in constructing collections 1 and 2 (by (†1) the cardinality of collection 3 is bounded by w$Y$, and the elements of collection 3 are disjoint because (†2) implies that $\mathcal{F}^p_{i+1,t}$ is $r_{i+1}$-disjoint);

collection 4: the union of the elements of $\mathcal{F}^X_{ij}$ contained in $A$ and that were not used in constructing collections 1, 2 and 3 (collection 4 consists of only one set).

Now split the elements of $\mathcal{F}^p_{ij}$ by $\mathcal{F}^p_{i+1,1}$, the elements of the resulting family split by $\mathcal{F}^p_{i+1,2}$ and proceed by induction to the last splitting by $\mathcal{F}^p_{i+1,n+1}$ to finally get the family which we denote by $\mathcal{F}^{p+1}_{ij}$. The construction is completed.

**Claim** All the †-conditions hold. ■

We will prove Claim 4.3 later. Let us show now how to derive from Claim 4.3 the proof of the factorization theorem. Condition (†3) implies that there is a unique collection $\mathcal{F}_{ij}$ of subsets of $X$ such that $\mathcal{F}_{ij}$ coincides with $\mathcal{F}^p_{ij}$ on the ball $\mathbb{B}(b, ir_{i+p})$ for every $p$. Condition (†2) implies that $\mathcal{F}_{ij}$ is a family of disjoint sets such that $\mathcal{F}_i = \mathcal{F}_{i,1} \cup \cdots \cup \mathcal{F}_{i,n+1}$ covers $X$. Let $\mathcal{F}$ be the sequence of the covers $\mathcal{F}_i$ with the splittings $\mathcal{F}_{ij}$.

**Claim**
(i) $\mathcal{F}^X_i$ refines $\mathcal{F}_i$ and $\mathcal{F}_i$ refines $\mathcal{F}^p_i$ for every $p$;
(ii) $\mathcal{F}$ defines asdim $\leq n$.

■

**Proof:**
(i) follows from (†2) and (†3).
(ii) Since $\mathcal{F}^X$ is a sequence witnessing asdim $\leq n$, any pair of points is contained in an element of $\mathcal{F}^X_i$ for some $i$. Then, by (†1) this pair is contained in an element of $\mathcal{F}^p_i$ for every $p$ and hence, by (†3), in an element of $\mathcal{F}_i$. Thus in order to show that $\mathcal{F}$ defines asdim $\leq n$ we only need to show that st$\mathcal{F}_i$ refines $\mathcal{F}_{i+1}$ and $\mathcal{F}_i$ separates $\mathcal{F}_{i+1,j}$.

Take a point $x \in X$. By (†3) and (†4), for every $p$ there is $F^p \in \mathcal{F}_{i+1}$ such that $\mathrm{st}(x, \mathcal{F}_i) \cap \mathbb{B}(b, ir_{i+p}) \subset F^p \cap \mathbb{B}(b, ir_{i+p})$. Note that $x$ belongs to at most $n+1$ elements of $\mathcal{F}_{i+1}$ and hence there is $F$ in $\mathcal{F}_{i+1}$ such that $F = F^p$ for infinitely many $p$ and then $\mathrm{st}(x, \mathcal{F}_i) \subset F$. Thus st$\mathcal{F}_i$ refines $\mathcal{F}_{i+1}$.

The property that $\mathcal{F}_i$ separates $\mathcal{F}_{i+1,j}$ follows from (†3) and (†5). ■

Let $Z$ be a maximal subset of $X$ separated by $\mathcal{F}_1$. Define a function $g : X \to Z$ by sending $x \in X$ to a point $z \in Z$ such that $x \in \mathrm{st}(z, \mathcal{F}_1)$. Consider $Z$ with the metric $d_{\mathcal{F}}$ determined by $\mathcal{F}$ as described in Proposition 2.3.


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**Claim** We have that $\mathrm{w}Z \leq \mathrm{w}Y$, $\mathrm{asdim}Z \leq n$, the functions $g : X \to Z$ and $h = f|Z : Z \to Y$ are coarsely continuous (everything here with respect to $d_{\mathcal{F}}$), and $f$ and $h \circ g$ are coarsely close.

■

**Proof.** Since $\mathcal{F}_1$ separates the points of $Z$, we get that $\mathcal{F}_1$ restricted to $Z$ consists of singletons and therefore the cardinality of $Z$ is bounded by the cardinality of $\mathcal{F}_1$. Then, since $\mathcal{F}_1$ restricted to $\mathbb{B}(b, r_{1+p})$ coincides with $\mathcal{F}_1^p$ for every $p$, we get by (†1) that the cardinality of $\mathcal{F}_1$ is bounded by $\mathrm{w}Y$ and hence $\mathrm{w}Z \leq \mathrm{w}Y$. The property asdim $\leq n$ follows from (ii) of Proposition 2.3.

Let us show that $g$ is coarsely continuous. Take $F \in \mathcal{F}_i^X$ and $x_1, x_2 \in F$ and let $z_1 = g(x_1)$ and $z_2 = g(x_2)$. Then $x_1 \in \mathrm{st}(z_1, \mathcal{F}_1)$ and $x_2 \in \mathrm{st}(z_2, \mathcal{F}_1)$. By (ii) of Claim 4.4, $x_1 \in \mathrm{st}(z_1, \mathcal{F}_i)$ and $x_2 \in \mathrm{st}(z_2, \mathcal{F}_i)$ and then $F \subset \mathrm{st}(z_1, \mathcal{F}_{i+1})$ and $F \subset \mathrm{st}(z_2, \mathcal{F}_{i+1})$. Therefore, by (ii) of Proposition 2.3, $z_1$ and $z_2$ are $(i+1)$-close with respect to $d_{\mathcal{F}}$. Thus, $g(\mathcal{F}_i^X)$ is uniformly bounded with respect to $d_{\mathcal{F}}$ and hence $g$ is coarsely continuous.

Let us show that $h$ is coarsely continuous. Recall that $f(\mathcal{F}_i^0)$ refines $\mathcal{F}_i^Y$. Then, by (i) of Claim 4.4, $\mathcal{F}_i$ refines $\mathcal{F}_i^0$ and we get that $f(\mathcal{F}_i)$ is uniformly bounded and hence, again by (ii) of Proposition 2.3, $h$ is coarsely continuous.

Let us finally show that $f$ and $h \circ g$ are coarsely close. Take a point $x \in X$ and let $z = g(x)$. Since $x \in \mathrm{st}(z, \mathcal{F}_1)$, $\mathcal{F}_1$ refines $\mathcal{F}_1^0$ and $f(\mathcal{F}_1^0)$ refines $\mathcal{F}_1^Y$ we get that $f(x)$ and $h(g(x)) = f(z)$ are contained in an element of $\mathcal{F}_1^Y$ and hence $f$ and $h \circ g$ are coarsely close. ■

Thus Theorem 1.2 follows from Claim 4.5 and Proposition 4.2. The only thing left is:

**Proof of Claim 4.3**. Conditions (†1), (†2) and (†3) obviously follow from the construction. So the only conditions we need to verify are (†4) and (†5) for $\mathcal{F}_i^{p+1}$ assuming that all the †-conditions hold for all the covers constructed before $\mathcal{F}_i^{p+1}$. Recall that $m = p + i$. Note that, by (†2),

(*) $r_{i+1}$ is a Lebesgue number of $\mathcal{F}_{i+1}^p$ and
(**) $\mathcal{F}_{i+1,t}^p$ is $r_{i+1}$-disjoint for every $t$.

Also recall that
(***) $\mathcal{F}_i^X$ is $R_i$-bounded with $(100i + 1)R_i < r_{i+1}$ (in particular $R_i < r_{1+1}/10$).

Fix a point $x \in X$. We will say that $x$ satisfies (†4) if $\mathrm{st}(x, \mathcal{F}_i^{p+1})$ is contained in an element of $\mathcal{F}_{i+1}^p$ and we will say that $x$ satisfies (†5) if for every $j$ no element of $\mathcal{F}_i^{p+1}$ containing $x$ meets disjoint elements of $\mathcal{F}_{i+1,j}^p$ one of which contains $x$. Note that if every point of $X$ satisfies (†4) and (†5) then $\mathcal{F}_i^{p+1}$ satisfies (†4) and (†5) as well.

Let $E$ be an element of $\mathcal{F}_i^{p+1}$. Recall that $E$ is constructed from an element of $\mathcal{F}_i^p$ by a sequence of splittings by the families $\mathcal{F}_{i,1}^p, \ldots \mathcal{F}_{i,n+1}^p$. On each step of this construction we create 4 collections (collections 1, 2, 3 and 4). Only one of these collections has an element containing $E$ and we will refer to this collection as the collection refined by $E$. We will refer to the step of splitting by $\mathcal{F}_{i,t}^p$ in the construction of $E$ as step $t$. For given

$x$ and $t$ we assume throughout the proof that

($\diamond$) $E$ is an element of $\mathcal{F}_i^{p+1}$ that contains $x$, $\mathcal{E}$ is the collection refined by $E$ created on step $t$ of the construction of $E$, and $E_{\mathcal{E}}$ is the unique element of $\mathcal{E}$ that contains $E$.

Consider the following cases.

**Case 1:** $\mathrm{st}(x, \mathcal{F}_i^X)$ does not meet $\mathbb{B}(b, ir_m)$.

We will show that $x$ satisfies (†4). By (*) and (***), take an element $F \in \mathcal{F}_{i+1,t}^p$ containing $\mathrm{st}(x, \mathcal{F}_i^X)$. Having $x$ and $t$ chosen we let $E$, $\mathcal{E}$, and $E_{\mathcal{E}}$ be as in ($\diamond$). By the assumption of Case 1, $\mathcal{E}$ cannot be collection 1. Then $\mathcal{E}$ must be collection 2 with $E_{\mathcal{E}} \subset F$. Thus $E \subset F$ and hence $x$ satisfies (†4).

Now we will show that $x$ satisfies (†5). Fix any $t$ and suppose $F$ is an element of $\mathcal{F}_{i+1,t}^p$ containing $x$. Having $x$ and $t$ chosen we let $E$, $\mathcal{E}$, and $E_{\mathcal{E}}$ be as in ($\diamond$). By the assumption of Case 1, $\mathcal{E}$ is not collection 1. Then, by (**) and (***), $\mathcal{E}$ must be either collection 2 or collection 3 and then $E_{\mathcal{E}}$ is either contained in $F$ or $(r_{i+1}/10)$-close to $F$. Hence, by (**), $E_{\mathcal{E}}$ meets no element of $\mathcal{F}_{i+1,t}^p$ different from $F$. Thus $x$ satisfies (†5).

**Case 2:** $\mathrm{st}(x, \mathcal{F}_i^X)$ does meet $\mathbb{B}(b, ir_m)$ and $p = 0$. Note that in this case $m = i$ and hence $2(ir_m + 2R_i) < r_{i+1}/10$.

We will show that $x$ satisfies (†4). By (*) take an element $F \in \mathcal{F}_{i+1,t}^p$ containing $\mathbb{B}(b, ir_m + 2R_i)$. Then, by (***), $\mathrm{st}(x, \mathcal{F}_i^X) \subset F$. Having $x$ and $t$ chosen we let $E$, $\mathcal{E}$, and $E_{\mathcal{E}}$ be as in ($\diamond$). If $\mathcal{E}$ is collection 1 then, again by (***), $E_{\mathcal{E}} \subset \mathbb{B}(b, ir_m + 2R_i)$ and hence $E_{\mathcal{E}} \subset F$. If $\mathcal{E}$ is not collection 1 then, since $\mathrm{st}(x, \mathcal{F}_i^X) \subset F$, the only option left is that $\mathcal{E}$ is collection 2 and hence $E_{\mathcal{E}} \subset F$. Thus $x$ satisfies (†4).

Now let us show that $x$ satisfies (†5). Fix any $t$ and suppose $F$ is an element of $\mathcal{F}_{i+1,t}^p$ containing $x$. Having $x$ and $t$ chosen we let $E$, $\mathcal{E}$, and $E_{\mathcal{E}}$ be as in ($\diamond$). If $\mathcal{E}$ is collection 1 then $E_{\mathcal{E}}$ is contained in $\mathbb{B}(b, ir_m + 2R_i)$ and hence $E_{\mathcal{E}}$ is $(r_{i+1}/10)$-close to $F$. If $\mathcal{E}$ is either collection 2 or collection 3 and then in both cases $E_{\mathcal{E}}$ is $(r_{i+1}/10)$-close to $F$. By (**) and (***), $\mathcal{E}$ cannot be collection 4. Thus, by (**), $E_{\mathcal{E}}$ cannot meet an element of $\mathcal{F}_{i+1,t}^p$ different from $F$ and therefore $x$ satisfies (†5).

**Case 3:** $\mathrm{st}(x, \mathcal{F}_i^X)$ does meet $\mathbb{B}(b, ir_m)$ and $p > 0$. Note that then, by (***), $\mathrm{st}(x, \mathcal{F}_i^X) \subset \mathbb{B}(b, (i+1)r_m)$.

Let us show that $x$ satisfies (†4). By the inductive assumption for (†4), $\mathrm{st}\mathcal{F}_i^p$ refines $\mathcal{F}_{i+1}^{p-1}$. Then, by (†3), there is $F \in \mathcal{F}_{i+1}^p$ such that

($\bullet$) $\operatorname{st}(x, \mathcal{F}_i^p) \cap \mathbb{B}(b, (i+1)r_m) \subset F \cap \mathbb{B}(b, (i+1)r_m)$.

Note that then, by (***) and (†2), we also have $\operatorname{st}(x, \mathcal{F}_i^X) \subset F$. Suppose $F \in \mathcal{F}_{i+1,t}^p$. Having $x$ and $t$ chosen we let $E$, $\mathcal{E}$, and $E_{\mathcal{E}}$ be as in ($\diamond$). If $\mathcal{E}$ is collection 1 then $E_{\mathcal{E}} \subset \mathbb{B}(b, (i+1)r_m)$ and, then by (†3), $E$ is contained in an element of $\mathcal{F}_i^p$, and hence, by ($\bullet$), $E \subset F$. If $\mathcal{E}$ is not collection 1 then, since $\operatorname{st}(x, \mathcal{F}_i^X) \subset F$, the only option left is that $\mathcal{E}$ is collection 2 and hence $E_{\mathcal{E}} \subset F$. Thus $x$ satisfies (†4).

Now we will show that $x$ satisfies (†5). Fix any $t$ and suppose $F$ is an element of $\mathcal{F}_{i+1,t}^p$ containing $x$. Having $x$ and $t$ chosen we let $E$, $\mathcal{E}$, and $E_{\mathcal{E}}$ be as in ($\diamond$). If $\mathcal{E}$ is collection 1 then $E_{\mathcal{E}} \subset \mathbb{B}(b, (i+1)r_m)$ and, hence by (†3), $E$ is contained in an element of $\mathcal{F}_i^p$. Thus, by the inductive assumption for (†5), $E$ cannot meet disjoint elements of $\mathcal{F}_{i+1,t}^{p-1}$ and hence, by (†3), $E$ cannot meet disjoint elements of $\mathcal{F}_{i+1,t}^p$ as well. If $\mathcal{E}$ is collection 2 or collection 3 then $E_{\mathcal{E}}$ is $(r_{i+1}/10)$-close to $F$ and, by (**), $E_{\mathcal{E}}$ cannot meet disjoint elements of $\mathcal{F}_{i+1,t}^p$. By (**) and (***), $\mathcal{E}$ cannot be collection 4. Thus $F$ is the only element $\mathcal{F}_{i+1,t}^p$ that $E$ meets and therefore $x$ satisfies (†5).

Thus every point of $X$ satisfies (†4) and (†5) and the claim is proved. ∎

## 5 Coarse Property C and Finite Decomposition Complexity

Yu's result in [13] that proper metric spaces that admit uniform embeddings into Hilbert space satisfy the Novikov conjecture inspired many to seek out sufficient conditions for this same property. Yu showed that spaces with finite asymptotic dimension have this property, but there are also known examples of spaces with infinite asymptotic dimension that also have this property. Thus, properties weaker than finite asymptotic dimension that still imply the existence of uniform embeddings into Hilbert spaces have been sought out and explored. One such property is asymptotic property C, described by Dranishnikov in [4]. The property is related to the property of having finite asymptotic dimension. Indeed, metric spaces with finite asymptotic dimension have asymptotic property C. It's similarity with having finite asymptotic dimension leads us to the main goal of this section, which is to ask if the methods for constructing universal spaces for asymptotic dimension can be extended to coarse property C.

Below is the general definition for coarse property C (the general version of asymptotic property C for coarse spaces) as given by Bell, Moran, and Nagorko in [1].

***Definition* 5.1** A metric space $X$ has **coarse property C** if for any sequence $\{L_i\}_{i\in\mathbb{N}}$ of uniformly bounded covers such that $L_i$ refines $L_{i+1}$ there is an $n$ and a finite sequence $\mathcal{U}^1, \mathcal{U}^2, \ldots, \mathcal{U}^n$ of families of subsets of $X$ such that:

(1) $\mathcal{U} = \bigcup_{i=1}^n \mathcal{U}^i$ covers $X$;
(2) Each $\mathcal{U}^i$ is uniformly bounded; and
(3) each $\mathcal{U}^i$ is $L_i$-disjoint.

We then ask the following.


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**Question**

(i) Is there a (not necessarily separable) metric space $X$ that has coarse property C and contains a coarsely equivalent copy of each separable metric spaces with coarse property C?

(ii) Let $f : X \to Y$ be a coarsely continuous map from a metric space $X$ with coarse property C to a separable metric space $Y$. Does $f$ factor through a separable metric space with coarse property C?

■

We end by noting that there are other properties one might also want to try constructing universal spaces for using the methods of this paper. These include finite decomposition complexity ([7]) and coarse property A ([13]).

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