DSP

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March 2, 2025

2.1.1 Euler-Maclaurin Formula: Prove the following version of the Euler-Maclaurin

formula of degree 1 for a differentiable function A(v) over the interval [a, b]:

$$\sum_{n=a}^{b} A(n) = \int_a^b A(v)dv + \int_a^b \psi(v) \left(\frac{\partial}{\partial v} A(v) \right) dv + \frac{1}{2}[A(a) + A(b)]$$

where $\psi(v) = v - \lfloor v \rfloor - \frac{1}{2}$
We will focus on the right side:
First we will open ψ by definition

$$\int_a^b A(v)dv + \int_a^b \left(v - \lfloor v \rfloor - \frac{1}{2} \right) \left(\frac{\partial}{\partial u} A(v) \right) dv + \frac{1}{2}[A(a) + A(b)] =$$

Now we will split $\int_a^b \left(v - \lfloor v \rfloor - \frac{1}{2} \right) \left(\frac{\partial}{\partial v} A(v) \right) dv$ into 3 integrals and solve them individually

$$I_1 = \int_a^b v \left(\frac{\partial}{\partial v} A(v) \right) dv$$

We will do integration by parts:

$$vA(v)|_a^b - \int_a^b A(v)dv = \boxed{bA(b) - aA(a) - \int_a^b A(v)dv = I_1}$$

$$I_3 = \int_a^b -\frac{1}{2} \left(\frac{\partial}{\partial v} A(v) \right) dv = \boxed{\frac{1}{2}A(a) - \frac{1}{2}A(b) = I_3}$$

$$I_2 = \int_a^b \lfloor v \rfloor \left(\frac{\partial}{\partial v} A(v) \right) dv$$

We will split $\lfloor v \rfloor$ into b-a parts:

$$\sum_{n=a}^{b-1} \int_n^{n+1} n \left(\frac{\partial}{\partial v} A(v) \right) dv = \sum_{n=a}^{b-1} n \int_n^{n+1} \left(\frac{\partial}{\partial v} A(v) \right) dv = \sum_{n=a}^{b-1} nA(n+1) - nA(n) =$$

$$\boxed{(b-1)A(b) - \sum_{n=a}^{b-1} A(n) - (a-1)A(a) = I_2}$$

Now we will substitute the integrals into the original equation:

$$\int_a^b A(v)dv + \int_a^b \psi(v)\left(\frac{\partial}{\partial v}A(v)\right)dv + \frac{1}{2}[A(a)+A(b)] =$$

$$\int_a^b A(v)dv + I_1 - I_2 + I_3 + \frac{1}{2}[A(a)+A(b)] =$$

$$\int_a^b A(v)dv + bA(b) - aA(a) - \int_a^b A(v)dv - (b-1)A(b) + \sum_{n=a}^{b-1} A(n) + (a-1)A(a) +$$

$$\frac{1}{2}A(a) - \frac{1}{2}A(b) + \frac{1}{2}[A(a)+A(b)] =$$

$$A(b) - A(a) + \sum_{n=a}^{b-1} A(n) + A(a) = \boxed{\sum_{n=a}^{b} A(n)} \quad \square$$

This version of Euler-Maclaurin allows us to calculate potentially difficult discrete summation with integrals or alternatively, difficult integrals with discrete summation.

2.1.2 Can you find an alternative proof of the above Euler-Maclaurin formula using

δ-functions? Can you make a sense of $\frac{\partial}{\partial v}\psi(v)$?

$$\sum_{n=a}^{b} A(n) = \int_a^b A(v)dv + \int_a^b \psi(v)\left(\frac{\partial}{\partial v}A(v)\right)dv + \frac{1}{2}[A(a)+A(b)]$$

where $\psi(v) = v - \lfloor v \rfloor - \frac{1}{2}$
We will focus on the right side:
First we will find $\psi'(v)$ by definition

$$\psi'(v) = 1 - \sum_{n=-\infty}^{\infty} \delta(v-n)$$

Now we will focus on $\int_a^b \psi(v)\left(\frac{\partial}{\partial v}A(v)\right)dv$:

$$\int_{a-\epsilon}^{b+\epsilon} \psi(v)\left(\frac{\partial}{\partial v}A(v)\right)dv$$

Where $\epsilon \to 0$ Now we shall do integration by parts:

$$\int_{a-\epsilon}^{b+\epsilon} \psi(v)\left(\frac{\partial}{\partial v}A(v)\right)dv = \psi(v)A(v)|_{a-\epsilon}^{b+\epsilon} - \int_{a-\epsilon}^{b+\epsilon} \psi'(v)A(v)dv =$$

$$\psi(v)A(v)|_{a-\epsilon}^{b+\epsilon} - \int_{a-\epsilon}^{b+\epsilon}\left(1 - \sum_{n=-\infty}^{\infty}\delta(v-n)\right)A(v)dv =$$

$$(\psi(b+\epsilon)A(b+\epsilon) - \psi(a-\epsilon)A(a-\epsilon)) - \int_{a-\epsilon}^{b+\epsilon} A(v)dv + \sum_{n=a}^{b} A(n) =$$

We can see $\psi(b+\epsilon) = -\frac{1}{2}$ and $\psi(a-\epsilon) = \frac{1}{2}$

$$-\frac{1}{2}(A(b+\epsilon) + A(a+\epsilon)) - \int_{a-\epsilon}^{b+\epsilon} A(v)dv + \sum_{n=a}^{b} A(n)$$

Now $\epsilon \to 0$:

$$\int_a^b \psi(v)\left(\frac{\partial}{\partial v}A(v)\right)dv = -\frac{1}{2}(A(b)+A(a)) - \int_a^b A(v)dv + \sum_{n=a}^{b} A(n)$$

Now if we rearrange the terms we get:

$$\sum_{n=a}^{b} A(n) = \int_a^b A(v)dv + \int_a^b \psi(v)\left(\frac{\partial}{\partial v}A(v)\right)dv + \frac{1}{2}[A(a)+A(b)] \quad \square$$

And that is exactly Euler-Maclaurin

2.1.3 Compute the trigonometric Fourier series of $\psi(v)$

We will calculate it on $(0,1]$ because it is repeating that period. Also within $(0,1]$ we know $\psi(v) = v - \frac{1}{2}$
We also know $\psi(v)$ is an odd function so we can make a sine series:

$$\psi(v) = \sum_{n=1}^{\infty} a_n sin(\pi n v)$$

$$a_n = 2\int_0^1 \left(v - \frac{1}{2}\right) sin(\pi n v)dv = 2\int_0^1 vsin(\pi n v) - \int_0^1 sin(\pi n v)dv =$$

$$-2\frac{vcos(\pi n v)}{\pi n}|_0^1 + \frac{2}{\pi n}\int_0^1 cos(\pi n v)dv + \frac{cos(\pi n v)}{\pi n}|_0^1 =$$

$$-\frac{2cos(\pi n)}{\pi n} + \frac{1}{\pi^2 n^2} sin(\pi n v)|_0^1 + \frac{cos(\pi n)}{\pi n} - \frac{1}{\pi n} =$$

$$-\frac{2cos(\pi n)}{\pi n} + \frac{cos(\pi n)}{\pi n} - \frac{1}{\pi n} = -\frac{(-1)^n + 1}{\pi n} = \boxed{\frac{(-1)^{n+1} - 1}{\pi n}}$$

So finally we got:

$$\psi(v) = \sum_{n=1}^{\infty} -\frac{(-1)^n + 1}{\pi n} sin(\pi n v)$$

We will take all even components:

$$\psi(v) = \sum_{n=1}^{\infty} -\frac{1}{\pi n} sin(2\pi n v) \quad \square$$

2.1.4 Prove Shannon-Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} x(nT)e^{-2\pi iunT} = \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{x}\left(u - \frac{n}{T}\right),$$

where $x(v) \in L_2(\mathbb{R})$ and its Fourier transform is defined as:

$$\hat{x}(u) = \int_{-\infty}^{\infty} x(v)e^{-2\pi iuv}\, dv.$$

using the Euler-Maclaurin formula. First, we will write down the Euler-Maclaurin formula

$$\sum_{n=a}^{b} A(n) = \int_a^b A(v)dv + \int_a^b \psi(v)\left(\frac{\partial}{\partial v}A(v)\right)dv + \frac{1}{2}[A(a) + A(b)]$$

Where $\psi(v) = v - \lfloor v \rfloor - \frac{1}{2}$ and remember $\psi'(v) = 1 - \sum_{n=-\infty}^{\infty} \delta(v-n)$
Also $lim_{t\to\pm\infty} x(t) = 0$ Now we shall apply the formula to the right side of the equation:

$$\sum_{n\in\mathbb{Z}} x(nT)e^{-2\pi iunT} = \int_{-\infty}^{\infty} x(vT)e^{-2\pi iuvT}dv + \int_{-\infty}^{\infty} \psi(v)\left(\frac{\partial}{\partial v}(x(vT)e^{-2\pi iuvT})\right)dv =$$

Now we will do integration by parts:

$$\frac{1}{T}\hat{x}(\frac{u}{T})dv + \psi(v)x(vT)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi'(v)x(vT)e^{-2\pi iuvT}dv =$$

$\psi(v)x(vT)|_{-\infty}^{\infty} = 0$

$$\frac{1}{T}\hat{x}(\frac{u}{T})dv - \int_{-\infty}^{\infty}\left(1 - \sum_{n=-\infty}^{\infty}\delta(v-n)\right)x(vT)e^{-2\pi iuvT}dv =$$

$$\frac{1}{T}\hat{x}(\frac{u}{T})dv - \int_{-\infty}^{\infty} x(vT)e^{-2\pi iuvT}dv + \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\delta(v-n)\right)x(vT)e^{-2\pi iuvT}dv =$$

$$\int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\delta(v-n)\right)x(vT)e^{-2\pi iuvT}dv =$$

Because $\delta(t) = \frac{1}{a}\delta(\frac{t}{a})$

$$\int_{-\infty}^{\infty} T\left(\sum_{n=-\infty}^{\infty}\delta(vT-nT)\right)x(vT)e^{-2\pi iuvT}dv = T\int_{-\infty}^{\infty}(\text{III}_T(vT)x(vT))e^{-2\pi iuvT}dv =$$

Where $\text{III}_T(v) = \sum_{n=-\infty}^{\infty}\delta(v-nT)$ is dirac comb

$$T\frac{1}{T}\widehat{\text{III}_T \cdot x}(u) = \widehat{\text{III}}_T * \hat{x}(u) = \int_{-\infty}^{\infty}\frac{1}{T}\sum_{n\in\mathbb{Z}}\delta(u - v - \frac{n}{T})\cdot\hat{x}(v)dv = \boxed{\frac{1}{T}\sum_{n\in\mathbb{Z}}\hat{x}\left(u - \frac{n}{T}\right)} \quad \square$$

2.1.5 The special case

$$\sum_{n\in\mathbb{Z}} x(n) = \sum_{n\in\mathbb{Z}} \hat{x}(n)$$

is known as the Poisson summation formula. Prove it by considering the complex Fourier series of the periodic function $F(t) := \sum_{n\in\mathbb{Z}} x(t+n)$.

First, we will write $F(t)$ as its Fourier series on the interval $(0, 1]$:

$$F(t) = \sum_{n=-\infty}^{\infty} x(t+n) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}$$

Now we will find c_n

$$c_n = \int_0^1 F(t)e^{-2\pi int}dt = \int_0^1 \sum_{m=-\infty}^{\infty} x(t+m)e^{-2\pi int}dt =$$

$$\sum_{m=-\infty}^{\infty} \int_0^1 x(t+m)e^{-2\pi int}dt$$

We will substitute $v = t + m$:

$$\sum_{m=-\infty}^{\infty} \int_0^1 x(t+m)e^{-2\pi int}dt = \sum_{m=-\infty}^{\infty} \int_m^{m+1} x(v)e^{-2\pi in(v-m)}dv$$

Now $e^{2\pi inm} = 1$ since $m, n \in \mathbb{Z}$

$$\sum_{m=-\infty}^{\infty} \int_m^{m+1} x(v)e^{-2\pi in(v-m)}dv = \sum_{m=-\infty}^{\infty} \int_m^{m+1} x(v)e^{-2\pi inv}dv =$$

$$\int_{-\infty}^{\infty} x(v)e^{-2\pi inv}dv = \boxed{\widehat{x}(n) = c_n}$$

Now we will plug this in the original equation:

$$F(t) = \sum_{n=-\infty}^{\infty} x(t+n) = \sum_{n=-\infty}^{\infty} \widehat{x}(n)e^{2\pi int}$$

Now if we try $t = 0$

$$F(0) = \sum_{n=-\infty}^{\infty} x(n) = \sum_{n=-\infty}^{\infty} \widehat{x}(n) \quad \square$$

Which is the Poisson summation formula

2.1.6 Derive the full Shannon-Poisson summation formula

$$\sum_{n\in\mathbb{Z}} x(nT)e^{-2\pi iunT} = \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{x}\left(u - \frac{n}{T}\right),$$

from

$$\sum_{n\in\mathbb{Z}} x(n) = \sum_{n\in\mathbb{Z}} \hat{x}(n),$$

by using properties of the Fourier transform. First, let us define a few functions.

$$y(n) = x(n)e^{-2\pi iun}$$

$$z(n) = y(nT)$$

$$\sum_{n\in\mathbb{Z}} x(nT)e^{-2\pi iunT} = \sum_{n\in\mathbb{Z}} z(n) = \sum_{n\in\mathbb{Z}} \hat{z}(n)$$

Now, let us compute $\hat{z}$.
By the change of scale property.

$$\hat{z}(m) = \frac{1}{T}y(\frac{m}{T})$$

Now, we need to compute $\hat{y}$. By the phase shift property.

$$\hat{y}(m) = \hat{x}(m + uT)$$

Thus we get,

$$\hat{z}(m) = \frac{1}{T}\hat{x}(\frac{m + uT}{T}) = \frac{1}{T}\hat{x}(\frac{m}{T} + u)$$

In the summation, we sum all integers positive and negative, so we can change the summation variable to be negative what it is.

$$\sum_{n\in\mathbb{Z}} x(nT)e^{-2\pi iunT} = \sum_{n\in\mathbb{Z}} z(n) = \sum_{m\in\mathbb{Z}} \hat{z}(m) = \frac{1}{T}\sum_{m\in\mathbb{Z}} \hat{x}(\frac{m}{T}+u) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{x}(u-\frac{n}{T})$$

4.1 Haar Wavelet Transform

Haar scaling function ϕ:

$$\phi(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\int_{-\infty}^{\infty} \phi(t)\, dt \ = \int_0^1 1dt = 1$$

this normalization is important because we want $\phi(t)$ to represents averages in signal decomposition if it wasn't 1 then the averages would have scaled incorrectly

Wavelet function ψ :

$$\psi(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} \leq t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$\int_{-\infty}^{\infty} \psi(t)dt = \int_0^{\frac{1}{2}} 1dt + \int_{\frac{1}{2}}^{1} -1dt = \frac{1}{2} - \frac{1}{2} = 0$$

this normalization is important because we want $\psi(t)$ to represents the differences this way, if the function if constant then there won't be any changes and it should be 0,

4.2 Haar Wavelet Transform

Fourier transform of the Haar mother wavelet $\phi(t)$:

$$\Psi(u) = \int_{-\infty}^{\infty} \psi(t)e^{-2\pi iut}dt = \int_{0}^{\frac{1}{2}} e^{-2\pi iut}dt + \int_{\frac{1}{2}}^{1} -e^{-2\pi iut}dt = \frac{e^{-2\pi iut}}{-2\pi iu}\bigg|_{0}^{\frac{1}{2}} - \frac{e^{-2\pi iut}}{-2\pi iu}\bigg|_{\frac{1}{2}}^{1} =$$

$$= \frac{e^{-\pi iu}-1}{-2\pi iu} + \frac{e^{-\pi iu}-e^{-2\pi iu}}{-2\pi iu} == \frac{-e^{-2\pi iu}+2e^{-\pi iu}-1}{-2\pi iu} = \frac{e^{-2\pi iu}-2e^{-\pi iu}+1}{2\pi iu} = \frac{(e^{-\pi iu}-1)^2}{2\pi iu}$$

$\psi(t)$ causes the integral to go only over the interval [0,1)
while getting the difference between the first half of the interval and second half

in addition: (i don;t want to erase this)
we know that:

$$e^{-\pi iu} = \begin{cases} 1, & \text{if } u \text{ is even,} \\ -1, & \text{if } u \text{ is odd.} \end{cases}$$

so for integers number we have:

$$\Psi(u) = 0, \text{if u is even}$$

$$\Psi(u) = \frac{(-2)^2}{2\pi iu} = \frac{2}{\pi iu} = \frac{2}{\pi iu} \cdot \frac{-\pi iu}{-\pi iu} = \frac{-2\pi iu}{(\pi u)^2} = \frac{2}{\pi u}$$

$$\Psi(u) = \begin{cases} 0, & \text{if } u \text{ is even,} \\ \frac{2}{\pi u}, & \text{if } u \text{ is odd.} \end{cases}$$

another cool explanation:

4.1.3 Haar Wavelet Transform

$$\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k), \quad \psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k).$$

explanation:
since the function ψ gives us 1, -1 or 0
then $2^{j/2}$ gives us the value of $\psi_{j,k}$ and $\psi(2^j t - k)$ gives us the range
Also, $j > 0$ compress and $j < 0$ stretches the width of the function,
$\psi(2^{j/2}t - k)$,inside ψ, t is depended by j and because of it
j causes the range of the function to stretch or compress depending on the j
while k can shift the function along the time axes
because t is not depended by k and because of that
k can just shift the time axes

Examples:

$$\psi_{1,0}(t) = \sqrt{2}\psi(2t) = \begin{cases} \sqrt{2}, & 0 \le 2t < 1/2 \\ -\sqrt{2}, & 1/2 \le 2t < 1 \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \sqrt{2}, & 0 \le 2 < 1/4 \\ -\sqrt{2}, & 1/4 \le 2 < 1/2 \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_{2,1}(t) = 2\psi(2^2 t - 1) = 2\psi(4t - 1) = \begin{cases} 2, & 0 \le 4t - 1 < 1/2 \\ -2, & 1/2 \le 4t - 1 < 1 = \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 2, & 1 \le 4t < 3/2 \\ -2, & 3/2 \le 4t < 2 \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 2, & 1/4 \le t < 3/8 \\ -2, & 3/8 \le t < 1/2 \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_{3,0}(t) = 2^{3/2}\psi(2^3 t) = 2^{3/2}\psi(8t) = \begin{cases} 2^{3/2}, & 0 \le 8t < 1/2 \\ -2^{3/2}, & 1/2 \le 8t < 1 \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 2^{3/2}, & 0 \le t < 1/16 \\ -2^{3/2}, & 1/16 \le t < 1/8 \\ 0, & \text{otherwise.} \end{cases}$$

Note:
We have the plots of these graphs at the end of the Jupyter Notebook file.
Python Implementation 4.2(1)(4)
We also include plots of the Fourier transform to visualize their behavior.
Fourier transforms of window functions.

4.1.4 Haar Wavelet Transform

$$c_{j,k} = \int_{-\infty}^{\infty} [x(t)\phi_{j,k}(t)]dt \quad d_{j,k} = \int_{-\infty}^{\infty} [x(t)\psi_{j,k}(t)]dt$$

if we will substitute ϕ:

$$c_{j,k} = \int_{-\infty}^{\infty} [x(t)2^{j/2}\phi(2^j t - k)]dt$$

$$\phi(2^j t - k) = \begin{cases} 1, 0 \leq 2^j t - k < 1 \\ 0, \text{otherwise} \end{cases} = \begin{cases} 1, k \leq 2^j t < 1 + k \\ 0, \text{otherwise} \end{cases} = \begin{cases} 1, \frac{k}{2^j} \leq t < \frac{k+1}{2^j} \\ 0, \text{otherwise} \end{cases}$$

so now our integral will be from $\frac{k}{2^j}$ to $\frac{k+1}{2^j}$:

$$c_{j,k} = \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} [x(t)2^{j/2}]dt = 2^{j/2} \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} x(t)dt$$

lets do the same for $d_{j,k}$
if we will substitute ψ:

$$d_{j,k} = \int_{-\infty}^{\infty} [x(t)2^{j/2}\psi(2^j t - k)]dt$$

$$\psi(2^j t - k) = \begin{cases} 1, 0 \leq 2^j t - k < 1/2 \\ -1, 1/2 \leq 2^j t - k < 1 \\ 0, \text{otherwise} \end{cases} = \begin{cases} 1, k \leq 2^j t < 1/2 + k \\ -1, 1/2 + k \leq 2^j t < 1 + k \\ 0, \text{otherwise} \end{cases} = \begin{cases} 1, \frac{k}{2^j} \leq t < \frac{k+1/2}{2^j} \\ -1, \frac{k+1/2}{2^j} \leq t < \frac{k+1}{2^j} \\ 0, \text{otherwise} \end{cases}$$

so now our integral will be from $\frac{k}{2^j}$ to $\frac{k+1}{2^j}$:

$$d_{j,k} = \int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} [x(t)2^{j/2}]dt - \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} [x(t)2^{j/2}]dt = 2^{j/2} \int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt - 2^{j/2} \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt$$

so now we have the c: average on our interval
and we have the d be the difference in the average of the two intervals
since we know:

The average of a function $x(t)$ over an interval $[a, b]$ is defined as:

$$\frac{1}{b-a} \int_a^b x(t)dt.$$

in our case the average will be:

$$\frac{1}{\frac{k+1}{2^j} - \frac{k}{2^j}} = \frac{1}{\frac{1}{2^j}} = 2^j$$

but because of normalization we have $2^{j/2}$

4.1.5 Haar Wavelet Transform

Consider the piecewise constant signal:

$$x(t) = \begin{cases} 1, & 0 \leq t < 0.5, \\ 2, & 0.5 \leq t < 1. \end{cases}$$

Compute the approximation coefficient $c_{1,0}$:

$$c_{1,0} = \int_{-\infty}^{\infty} [x(t)\phi_{1,0}(t)]dt =$$

$$\phi_{1,0}(t) = 2^{1/2}\phi(2^1 t - 0) = \begin{cases} \sqrt{2}, & 0 \leq 2t < 1, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \sqrt{2}, & 0 \leq t < 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

$$c_{1,0} = \int_0^{0.5} [x(t)\sqrt{2}]dt = \int_0^{0.5} [1 \cdot \sqrt{2}]dt = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Compute the detail coefficient $d_{1,0}$:

$$d_{1,0} = \int_{-\infty}^{\infty} x(t)\psi_{1,0}(t)\, dt.$$

$$\psi_{1,0}(t) = 2^{1/2}\psi(2^1 t - 0) = \begin{cases} \sqrt{2}, & 0 \leq 2t < 1/2 \\ -\sqrt{2}, & 1/2 \leq 2t < 1 \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \sqrt{2}, & 0 \leq t < 1/4 \\ -\sqrt{2}, & 1/4 \leq t < 1/2 \\ 0, & \text{otherwise.} \end{cases}$$

$$d_{1,0} = \int_0^{0.25} [\sqrt{2}x(t)]dt - \int_{0.25}^{0.5} [\sqrt{2}x(t)]dt = \int_0^{0.25} [\sqrt{2} \cdot 1]dt - \int_{0.25}^{0.5} [\sqrt{2} \cdot 1]dt = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} = 0$$

Reconstruct $x(t)$ using:

$$x(t) = c_{1,0}\phi_{1,0}(t) + d_{1,0}\psi_{1,0}(t) = \frac{1}{\sqrt{2}} \cdot \phi_{1,0}(t) + 0 = \begin{cases} 1, & 0 \leq t < 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

we can see that it is not enough to fully recontract, we need more , for example: $c_{1,1}$ and $d_{1,1}$

4.1.6 Haar Wavelet Transform

Prove the general decomposition formula for the Haar wavelet:

$$x(t) = \sum_k c_{0,k}\phi_{0,k}(t) + \sum_{j'=0}^{j}\sum_k d_{j',k}\psi_{j',k}(t).$$

Firstly, lets see the form of $\sum_k c_{j,k}\phi_{j,k}(t)$:
Remember

$$c_{j,k} = 2^{j/2}\int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} x(t)dt$$

and

$$\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$$

$$\sum_k c_{j,k}\phi_{j,k}(t) = \sum_k 2^{j/2}\left(\int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} x(t)dt\right)2^{j/2}\phi(2^j t - k) = \sum_k 2^j\phi(2^j t - k)\int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} x(t)dt$$

Now we will open the integral to 2 parts:

$$(1)\sum_k 2^j\phi(2^j t - k)\int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt + 2^j\phi(2^j t - k)\int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt$$

Now we will split $\phi(2^j t - k) = \phi(2^{j+1}t - 2k) + \phi(2^{j+1}t - 2k + 1)$

$$(2)\sum_k 2^j(\phi(2^{j+1}t - 2k) + \phi(2^{j+1}t - 2k + 1))\int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt +$$

$$2^j(\phi(2^{j+1}t - 2k) + \phi(2^{j+1}t - 2k + 1))\int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt$$

Now lets observe the form of $\sum_k d_{j,k}\psi_{j,k}(t)$
Remember:

$$d_{j,k} = 2^{j/2}\int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt - 2^{j/2}\int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt$$

And:

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k) = 2^{j/2}(\phi(2^{j+1}t - 2k) - \phi(2^{j+1}t - 2k + 1))$$

Now lets plug this in our original equation

$$\sum_k d_{j,k}\psi_{j,k}(t) =$$

$$\sum_k\left(2^{j/2}\int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt - 2^{j/2}\int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt\right)2^{j/2}\psi(2^j t - k) =$$

$$\sum_k 2^j \left(\int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt - \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt\right) \psi(2^j t - k) =$$

$$\sum_k 2^j \left((\phi(2^{j+1}t - 2k) - \phi(2^{j+1}t - 2k + 1)) \int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt\right) +$$

$$2^j \left((-\phi(2^{j+1}t - 2k) + \phi(2^{j+1}t - 2k + 1)) \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt\right) =$$

We can already see this is similar to equation (2)
Now we will add and subtract (I've marked in color what we added and subtracted)

$$\sum_k 2^j \left((2\phi(2^{j+1}t - 2k) - \phi(2^{j+1}t - 2k) - \phi(2^{j+1}t - 2k + 1)) \int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt\right) +$$

$$2^j \left((-\phi(2^{j+1}t - 2k) - \phi(2^{j+1}t - 2k + 1) + 2\phi(2^{j+1}t - 2k + 1)) \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt\right) =$$

We can see that we get equation (2) which I marked in blue

$$\sum_k - \left(2^j (\phi(2^{j+1}t - 2k) + \phi(2^{j+1}t - 2k + 1)) \int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt\right) -$$

$$\left(2^j (\phi(2^{j+1}t - 2k) + \phi(2^{j+1}t - 2k + 1)) \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt\right) +$$

$$2^{j+1}\phi(2^{j+1}t - 2k) \int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt + 2^{j+1}\phi(2^{j+1}t - 2k + 1) \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt =$$

$$2^{j+1}\phi(2^{j+1}t - 2k) \int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt + 2^{j+1}\phi(2^{j+1}t - 2k + 1) \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt - \sum_k c_{j,k}\phi_{j,k}(t) =$$

Now we see equation (1) for j+1 which I marked in purple:

$$2^{j+1}\phi(2^{j+1}t - 2k) \int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt + 2^{j+1}\phi(2^{j+1}t - 2k + 1) \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt - \sum_k c_{j,k}\phi_{j,k}(t) =$$

$$\sum_k c_{j+1,k}\phi_{j+1,k}(t) - \sum_k c_{j,k}\phi_{j,k}(t) =$$

Finally, we see that:

$$(3) \boxed{\sum_k d_{j,k}\psi_{j,k}(t) = \sum_k c_{j+1,k}\phi_{j+1,k}(t) - \sum_k c_{j,k}\phi_{j,k}(t)}$$

So if we plug (3) into our decomposition we get:

$$x(t) = \sum_k c_{0,k}\phi_{0,k}(t) + \sum_{j'=0}^{j}\sum_k d_{j',k}\psi_{j',k}(t) =$$

$$\sum_k c_{0,k}\phi_{0,k}(t) + \left(-\sum_k c_{0,k}\phi_{0,k}(t) + \sum_k c_{1,k}\phi_{1,k}(t) - \sum_k c_{1,k}\phi_{1,k}(t) + \sum_k c_{2,k}\phi_{2,k}(t) - ... + \sum_k c_{j,k}\phi_{j,k}(t) \right)$$

And we get a telescopic series so all terms cancel out except

$$x(t) = \sum_k c_{j,k}\phi_{j,k}(t)$$

Now we need to prove that

$$\lim_{j\to\infty} \sum_k c_{j,k}\phi_{j,k}(t) = x(t)$$

$$\phi_{j,k}(t) \xrightarrow{j\to\infty} 2^{j/2}\delta(t - \frac{k}{2^j})$$

And so

$$c_{j,k} = \int_{-\infty}^{\infty} x(t)\delta\left(t - \frac{k}{2^j}\right) dt = \frac{1}{2^{j/2}} x\left(\frac{k}{2^j}\right)$$

$$\sum_k c_{j,k}\phi_{j,k}(t) \xrightarrow{j\to\infty} \sum_k \frac{1}{2^{j/2}} x\left(\frac{k}{2^j}\right) 2^{j/2}\delta(t - \frac{k}{2^j}) = x(t)$$

So we showed:

$$\lim_{j\to\infty} \sum_k c_{j,k}\phi_{j,k}(t) = x(t)$$

And that means the composition

$$x(t) = \sum_k c_{0,k}\phi_{0,k}(t) + \sum_{j'=0}^{j}\sum_k d_{j',k}\psi_{j',k}(t)$$

Converges to x(t) when $j \to \infty$

4.1.6 Haar Wavelet Transform

now we will verify the formula for $j = 2$:

$$x(t) = \sum_k c_{0,k}\phi_{0,k}(t) + \sum_{j'=0}^{j}\sum_k d_{j',k}\psi_{j',k}(t)$$

$j = 2$, $\quad k = 0$

$$x(t) = c_{0,0}\phi_{0,0}(t) + \sum_{j=2} d_{j,0}\psi_{j,0}(t) =$$

$$= c_{0,0}\phi_{0,0}(t) + d_{0,0}\psi_{0,0}(t) + d_{1,0}\psi_{1,0}(t) + d_{2,0}\psi_{2,0}(t)$$

lets calculate : $\phi_{0,0}, \psi_{0,0}, \psi_{1,0}, \psi_{2,0}$:

$$\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k), \quad \psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k).$$

$$\phi_{0,0} = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise} \end{cases} \quad \psi_{0,0} = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_{1,0} = \begin{cases} \sqrt{2}, & 0 \le t < \frac{1}{4}, \\ -\sqrt{2}, & \frac{1}{4} \le t < \frac{1}{2}, \\ 0, & \text{otherwise} \end{cases} \quad \psi_{1,0} = \begin{cases} 2, & 0 \le t < \frac{1}{8}, \\ -2, & \frac{1}{8} \le t < \frac{1}{4}, \\ 0, & \text{otherwise} \end{cases}$$

now lets calculate $c_{0,0}, d_{0,0}, d_{1,0}, d_{2,0}$
we will use our previous calculations:

$$c_{j,k} = 2^{j/2}\int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} x(t)dt$$

$$d_{j,k} = 2^{j/2}\int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} x(t)dt - 2^{j/2}\int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} x(t)dt$$

$$c_{0,0} = 1.5, \quad d_{0,0} = -\frac{1}{2}, \quad d_{1,0} = 0, \quad d_{2,0} = 0,$$

and now lets substitute what we know:

$$x(t) = c_{0,0}\phi_{0,0}(t) + d_{0,0}\psi_{0,0}(t) + d_{1,0}\psi_{1,0}(t) + d_{2,0}\psi_{2,0}(t) =$$

$$= 1.5 \cdot \phi_{0,0} - \frac{1}{2} \cdot \psi_{0,0} + 0 + 0 = 1.5 \cdot \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise} \end{cases} - \frac{1}{2} \cdot \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, = \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1.5, & 0 \le t < 1, \\ 0, & \text{otherwise} \end{cases} + \begin{cases} -\frac{1}{2}, & 0 \le t < \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} \le t < 1, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ 2, & \frac{1}{2} \le t < 1, \\ 0, & \text{otherwise} \end{cases}$$

looks similar haha

$$x(t) = \begin{cases} 1, & 0 \le t < 0.5, \\ 2, & 0.5 \le t < 1. \end{cases}$$

4.1.7 Haar Wavelet Transform

Explain how the Haar Wavelet Transform can be used for signal compression by:
(a) Describe the role of approximation coefficients $c_{j,k}$ and detail coefficients $d_{j,k}$ in identifying important components of a signal.
Specifically, discuss how the approximation coefficients are considered to capture the overall, low frequency structure of the signal, while the detail coefficients represent the high-frequency variations or fine details.
Can you find a filter interpretation to the approximation and detail coefficients and their properties?

as we found before, $c_{j,k}$ takes the average on the interval $[\frac{k}{2^j}, \frac{k+1}{2^j})$
so the approximation coefficient helps get the structure of the signal,
which is better for low frequencies and more stable signals
because it takes the averages in the whole interval
while $d_{j,k}$ takes the differences in the averages in both halves of the same interval
which is better for spikes and high frequencies or sudden changes
and a combination of $d_{j,k}$ and $c_{j,k}$ multiplied by $\psi_{j,k}$ and $phi_{j,k}$
can give us for each interval for our j,k the value of our signal.
since $\psi_{j,k}$ and $phi_{j,k}$ will go over the whole interval of our signal(depending on our j,k of course)
and $d_{j,k}$ and $c_{j,k}$ will give us the frequencies and changes

$c_{j,k}$ represents the low-pass filter (the average)
for example: $y_n = T(x_n) = \frac{1}{4}(x_{n-1} + x_n + x_{n+1})$
$d_{j,k}$ represents the high-pass filter (difference between averages , Derivative)
for example: $y_n = T(x_n) = x_n - x_{n-1}$

(b) Explain why in well behaved signals x(t) many detail coefficients dj, k are expected to have small magnitudes and how this property can be leveraged for compression.

in well behaved signals where the frequency is very balanced,
the differences between the averages in both halves of the signal is close
which causes the detail coefficient $d_{j,k}$ to get small values and have small magnitudes,
so we would want to compress the detail coefficients in order to prevent that

c).Define a threshold rule for compression. For example:

$$d_{j,k} \begin{cases} d_{j,k}, & \text{if } |d_{j,k}| \geq \text{threshold}, \\ 0, & \text{if } |d_{j,k}| < \text{threshold}. \end{cases}$$

the threshold is important to our reconstruction because:
since $d_{j,k}$ represent the high frequency variations and $c_{j,k}$ the lower frequency
by compressing $d_{j,k}$ it will not affect our signal when there are not changes or when the signal is balanced,
this way we won't have small magnitudes caused by $d_{j,k}$
and when we the signal is balanced or when we have small changes

4.1.8 Haar Wavelet Transform

(8) Suppose the detail coefficients $d_{j,k}$ below a certain threshold are discarded:

(a) Show how this impacts the signal x(t) when reconstructing it using the formula:

$x(t) = \sum_k c_{j,k}\phi_{j,k}(t) + \sum_{j=j}\sum_k d_{j,k}\psi_{j,k}(t)$.

now the detail coefficients below a certain threshold are discarded which means that when there are stable signals, $d_{j,k}$ will have no effect on them, and only when we have big differences in the signal $d_{j,k}$ will have an effect. now that $d_{j,k}$ is composed it will r

(b) Explain the trade-off between the compression ratio and reconstruction accuracy. Include a discussion of mean squared error (MSE) as a metric for reconstruction quality.

The compression ratio depends on the amount of detail coefficients $d_{j,k}$ we discard, The more detail coefficients we discard, the higher the compression ratio The downside of high compression is a loss of accuracy in the reconstructed signal

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{x}_i)^2$$

High compression would cause high MSE which gives low quality and the opposite low compression causes low MSE which gives higher quality

4.3 Exploration Question

JPEG2000 is an advanced image compression standard that replaces the Discrete Cosine Transform (DCT) used in JPEG2000 with a wavelet-based approach.
(1) JPEG2000 employs 2 different wavelets:

irreversible: the CDF 9/7 wavelet transform (developed by Ingrid Daubechies). It differs from Haar wavelet because it introduces quantization noise that depends on the precision of the decoder thus making it irreversible.

reversible: a rounded version of the biorthogonal Le Gall–Tabatabai (LGT) 5/3 wavelet transform(developed by Didier Le Gall and Ali J. Tabatabai).
It differs from Haar wavelet because it uses 2 filters, low-pass and high-pass, unlike the Haar which uses functions.

(2) JPEG 2000 decomposes the image into a multiple-resolution representation during compression. This pyramid representation can be used in Progressive transmission. These features are more commonly known as progressive decoding and signal-to-noise ratio scalability. JPEG 2000 provides efficient code-stream organizations that are progressive by pixel accuracy and by image resolution (or by image size). This allows the viewer to see a lower-quality version of the final picture before the whole file has been downloaded. The quality improves progressively as more data is downloaded from the source.
This is similar to Haar wavelet since Haar wavelet can transmit all $d_{j',k}$ progressively but different in the sense that each layer provides 2x the accuracy.

(3) The threshold in JPEG2000 is similar to ours because both thresholds are designed to throw away small details to save space

(4) We can make our low-pass and high-pass filters into kernels but that might be difficult to reverse.
Alternatively, we could make our image a series of rows and apply the 1D Haar wavelet to each row.

Bibliography

https://en.wikipedia.org/wiki/JPEG_2000

https://tim.cogan.dev/lgt-wavelet/

https://mdav.ece.gatech.edu/ece-6250-fall2019/notes/10-notes-6250-f19.pdf