Multi-Period Optimal Power Flow:
Brute Force & Temporal ADMM Formulations

1 Problem Overview

The Temporal ADMM (tADMM) algorithm decomposes the multi-period optimal power flow problem into T single-step subproblems, each corresponding to one time period. The hope is to enable parallel computation and improved scalability while still retaining solution optimality.

1.1 Variable Color Coding

Following the handwritten PDF formulation:

- $\mathbf{B^{t_0}}$: Local SOC variables for subproblem t_0 (blue)
- $\hat{\mathbf{B}}$: Global consensus SOC trajectory (red)
- $\mathbf{u^{t_0}}$: Local scaled dual variables for subproblem t_0 (green)

1.2 tADMM Algorithm Structure

The algorithm alternates between three update steps:

1.3 Step 1: Primal Update (Blue Variables) tADMM Optimization Model

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\min_{P_{\text{subs}}^{t_0}, P_B^{t_0}, \mathbf{B^{t_0}}} \quad C^{t_0} \cdot P_{\text{subs}}^{t_0} + C_B \cdot \left(P_B^{t_0}\right)^2 + \frac{\rho}{2} \left\| \mathbf{B^{t_0}} - \hat{\mathbf{B}} + \mathbf{u^{t_0}} \right\|_2^2 \tag{1}$$

Subject to SOC Dynamics for Entire Trajectory:

$$\mathbf{B^{t_0}[1]} = B_0 - P_B^{t_0} \cdot \Delta t \tag{2}$$

$$\mathbf{B^{t_0}[t]} = \mathbf{B^{t_0}[t-1]} - P_B^{t_0} \cdot \Delta t, \quad \forall t \in \{2, \ldots, T\} \tag{3}$$

$$P_{\text{subs}}^{t_0} + P_B^{t_0} = P_L[t_0] \tag{4}$$

$$-P_{B,R} \leq P_B^{t_0} \leq P_{B,R} \tag{5}$$

$$\underline{B} \leq \mathbf{B^{t_0}[t]} \leq \overline{B}, \quad \forall t \in \{1, \ldots, T\} \tag{6}$$

Key Formulation Notes:

- Each subproblem t_0 optimizes the battery power $P_B^{t_0}$ for *only* time step t_0

- However, the SOC trajectory $\mathbf{B^{t_0}[t]}$ is computed for *all* time steps $t \in \{1, \ldots, T\}$
- This ensures that the ADMM penalty term can compare the full trajectory $\mathbf{B^{t_0}}$ with the consensus $\hat{\mathbf{B}}$
- The power balance constraint is enforced only for the specific time t_0

1.4 Step 2: Consensus Update (Red Variables)

$$\hat{\mathbf{B}}[\mathbf{t}] = \text{clamp}\left(\frac{1}{T}\sum_{t_0=1}^{T}\left(\mathbf{B^{t_0}[t]} + \mathbf{u^{t_0}[t]}\right), \underline{B}, \overline{B}\right) \tag{7}$$
$$\forall t \in \{1, 2, \ldots, T-1\} \tag{8}$$
$$\hat{\mathbf{B}}[\mathbf{T}] = B_{T,\text{target}} \quad \text{(if terminal constraint exists)} \tag{9}$$

1.5 Step 3: Dual Update (Green Variables)

$$\mathbf{u^{t_0}[t]} := \mathbf{u^{t_0}[t]} + \left(\mathbf{B^{t_0}[t]} - \hat{\mathbf{B}}[\mathbf{t}]\right) \tag{10}$$
$$\forall t_0 \in \{1, \ldots, T\}, \forall t \in \{1, \ldots, T\} \tag{11}$$

2 Convergence Criteria

The algorithm terminates when both residuals fall below specified thresholds:

2.1 Primal Residual (Consensus Violation)

$$\|r^k\|_2 = \left\|\text{vec}\left(\left\{\mathbf{B^{t_0}} - \hat{\mathbf{B}}\right\}_{t_0=1}^{T}\right)\right\|_2 \le \epsilon_{\text{pri}} \tag{12}$$

2.2 Dual Residual (Consensus Change)

$$\|s^k\|_2 = \rho\left\|\hat{\mathbf{B}}^{\mathbf{k}} - \hat{\mathbf{B}}^{\mathbf{k-1}}\right\|_2 \le \epsilon_{\text{dual}} \tag{13}$$

3 Final Solution Recovery

After convergence, the final power schedules are reconstructed from the consensus trajectory:

$$P_B^t = \begin{cases} \frac{-\left(\hat{\mathbf{B}}[\mathbf{t}] - B_0\right)}{\Delta t} & \text{if } t = 1 \\ \frac{-\left(\hat{\mathbf{B}}[\mathbf{t}] - \hat{\mathbf{B}}[\mathbf{t-1}]\right)}{\Delta t} & \text{if } t > 1 \end{cases} \tag{14}$$
$$P_{\text{Subs}}^t = P_L[t] - P_B^t \tag{15}$$

The final objective value includes both energy and battery costs:

$$J_{\text{total}} = \sum_{t=1}^{T}\left[C^t \cdot P_{\text{Subs}}^t \cdot \Delta t + C_B \cdot \left(P_B^t\right)^2 \cdot \Delta t\right] \tag{16}$$

4 Algorithm Parameters

4.1 Objective Function Components

The tADMM objective function for each subproblem t_0 consists of three terms:

$$\text{Energy Cost:} \quad C^{t_0} \cdot P_{\text{subs}}^{t_0} \cdot \Delta t \tag{17}$$

$$\text{Battery Quadratic Cost:} \quad C_B \cdot \left(P_B^{t_0}\right)^2 \cdot \Delta t \tag{18}$$

$$\text{ADMM Penalty:} \quad \frac{\rho}{2} \left\| \mathbf{B^{t_0}} - \hat{\mathbf{B}} + \mathbf{u^{t_0}} \right\|_2^2 \tag{19}$$

Where:

- C^{t_0}: Energy price at time t_0 [$/kWh]
- C_B: Battery quadratic cost coefficient [$/kW^2/h] (typically $10^{-6} \times \min(C^t)$)
- ρ: ADMM penalty parameter

The battery quadratic cost term $C_B \cdot \left(P_B^{t_0}\right)^2$ serves as a regularization to:

1. Prevent excessive battery cycling
2. Encourage smoother power trajectories
3. Improve numerical conditioning of the optimization problem

4.2 Algorithmic Parameters

- **Penalty Parameter**: ρ (typically 0.1 to 10.0)
- **Primal Tolerance**: $\epsilon_{\text{pri}} = 10^{-3}$
- **Dual Tolerance**: $\epsilon_{\text{dual}} = 10^{-3}$
- **Maximum Iterations**: 1000

5 Appendix: Full Variable and Parameter Definitions

5.1 System Bases

$$\text{kV}_B = \frac{4.16}{\sqrt{3}} \text{ kV (phase-to-neutral)} \tag{20}$$

$$\text{kVA}_B = 1000 \text{ kVA} \tag{21}$$

$$P_{\text{BASE}} = 1000 \text{ kW} \tag{22}$$

$$E_{\text{BASE}} = 1000 \text{ kWh per hour} \tag{23}$$

5.2 SOC Bound Definitions

$$\underline{B} = \text{SOC}_{\min} \cdot E_{\text{Rated}} \tag{24}$$

$$\overline{B} = \text{SOC}_{\max} \cdot E_{\text{Rated}} \tag{25}$$

5.3 Physical Interpretation

- $P_B[t] > 0$: Battery discharging (providing power to the system)
- $P_B[t] < 0$: Battery charging (consuming power from the system)
- $B[t]$: Battery state of charge at the end of period t
- $\underline{B} = \mathrm{SOC}_{\mathrm{min}} \cdot E_{\mathrm{Rated}}$: Lower SOC bound
- $\overline{B} = \mathrm{SOC}_{\mathrm{max}} \cdot E_{\mathrm{Rated}}$: Upper SOC bound