Multi-Period Optimal Power Flow with tADMM: Temporally Localized ADMM for LinDistFlow SOCP

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1.1 Overview

The Temporal ADMM (tADMM) algorithm decomposes the multi-period optimal power flow problem into T subproblems, one for each time period. This formulation uses **localized temporal coupling** where each subproblem t_0 only couples with its immediate time neighbors $(t_0 - 1, t_0, t_0 + 1)$, rather than coupling with all T time steps.

Key Benefits:

- **Reduced Coupling**: Each subproblem exchanges 2–3 $\times |\mathcal{B}|$ variables (not $T \times |\mathcal{B}|$)
- **Memory Efficiency**: $(3T - 2) \times |\mathcal{B}|$ total storage vs. $T^2 \times |\mathcal{B}|$ for global coupling
- **Parallel Computation**: All T subproblems can be solved in parallel
- **Network Physics**: Captures voltage drops, reactive power, and SOC constraints via LinDistFlow SOCP

1.2 Problem Definition

Sets and Indices:

- $\mathcal{N}$: Set of all nodes (buses) in the distribution network
- $\mathcal{L}$: Set of all branches (lines), (i, j) denotes branch from node i to j
- $\mathcal{L}_1$: Set of branches connected to substation (node 1)
- $\mathcal{B} \subseteq \mathcal{N}$: Set of nodes with batteries (energy storage)
- $\mathcal{D} \subseteq \mathcal{N}$: Set of nodes with solar PV (distributed energy resources)
- $\mathcal{T} = \{1, 2, \ldots, T\}$: Set of time periods (e.g., 24 hours, 96 15-min intervals)
- $t_0 \in \mathcal{T}$: Index for the control time of subproblem t_0

Variable Color Coding:

- $\mathbf{B_j^{t_0}[t]}$: Local SOC variables (blue) - battery j in subproblem t_0, evaluated at time t
- $\mathbf{\hat{B}_j[t]}$: Global consensus SOC (red) - agreed-upon SOC for battery j at time t
- $\mathbf{u_j^{t_0}[t]}$: Scaled dual variables (green) - enforces consensus between local and global SOC

1.3 Local Time Sets

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$, define the local time set:

$$\mathcal{T}_{\text{local}}^{t_0} = \begin{cases} \{1, 2\} & \text{if } t_0 = 1 \text{ (boundary)} \\ \{t_0 - 1, t_0, t_0 + 1\} & \text{if } 2 \leq t_0 \leq T - 1 \text{ (interior)} \\ \{T - 1, T\} & \text{if } t_0 = T \text{ (boundary)} \end{cases} \tag{1}$$

1.4 Step 1: Primal Update (Subproblem Optimization)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\min_{\substack{P_{\text{Subs}}^{t_0}, Q_{\text{Subs}}^{t_0}, \\ P_{ij}^{t_0}, Q_{ij}^{t_0}, v_j^{t_0}, \ell_{ij}^{t_0}, \\ q_{D,j}^{t_0}, P_{B,j}^{t}, \mathbf{B_j^{t_0}[t]} \\ \forall (i,j)\in\mathcal{L}, j\in\mathcal{N}\cup\mathcal{D}\cup\mathcal{B}, \\ t\in\mathcal{T}_{\text{local}}^{t_0}}} \quad c^{t_0} \cdot P_{\text{Subs}}^{t_0} \cdot P_{\text{BASE}} \cdot \Delta t + C_B \sum_{j\in\mathcal{B}} \left(P_{B,j}^{t_0}\right)^2 \cdot P_{\text{BASE}}^2 \cdot \Delta t$$

$$+ \frac{\rho}{2} \sum_{j\in\mathcal{B}} \sum_{t\in\mathcal{T}_{\text{local}}^{t_0}} \left(\mathbf{B_j^{t_0}[t]} - \mathbf{\hat{B}_j[t]} + \mathbf{u_j^{t_0}[t]}\right)^2 \tag{2}$$

Subject to:

Spatial Network Constraints (LinDistFlow SOCP, only for control time t_0):

$$\text{Power balance (substation):} \quad P_{\text{Subs}}^{t_0} - \sum_{(1,j)\in\mathcal{L}_1} P_{1j}^{t_0} = 0 \tag{3}$$

$$Q_{\text{Subs}}^{t_0} - \sum_{(1,j)\in\mathcal{L}_1} Q_{1j}^{t_0} = 0 \tag{4}$$

$$\text{Power balance (nodes):} \quad P_{ij}^{t_0} - \sum_{(j,k)\in\mathcal{L}} P_{jk}^{t_0} = P_{B,j}^{t_0} + p_{D,j}^{t_0} - p_{L,j}^{t_0}, \quad \forall (i,j)\in\mathcal{L} \tag{5}$$

$$Q_{ij}^{t_0} - \sum_{(j,k)\in\mathcal{L}} Q_{jk}^{t_0} = q_{D,j}^{t_0} - q_{L,j}^{t_0}, \quad \forall (i,j)\in\mathcal{L} \tag{6}$$

$$\text{KVL (voltage drop):} \quad v_j^{t_0} = v_i^{t_0} - 2(r_{ij}P_{ij}^{t_0} + x_{ij}Q_{ij}^{t_0}) + \ell_{ij}^{t_0}, \quad \forall (i,j)\in\mathcal{L} \tag{7}$$

$$\text{SOCP relaxation:} \quad \ell_{ij}^{t_0} \cdot v_i^{t_0} \geq (P_{ij}^{t_0})^2 + (Q_{ij}^{t_0})^2, \quad \forall (i,j)\in\mathcal{L} \tag{8}$$

$$\text{Voltage limits:} \quad (V_{\min,j})^2 \leq v_j^{t_0} \leq (V_{\max,j})^2, \quad \forall j\in\mathcal{N} \tag{9}$$

$$\text{Substation voltage:} \quad v_1^{t_0} = (V_{\text{nom}})^2 \tag{10}$$

$$\text{PV reactive limits:} \quad (q_{D,j}^{t_0})^2 \leq (S_{D,j})^2 - (p_{D,j}^{t_0})^2, \quad \forall j\in\mathcal{D} \tag{11}$$

Note: Network variables ($P_{\text{Subs}}^{t_0}, Q_{\text{Subs}}^{t_0}, P_{ij}^{t_0}, Q_{ij}^{t_0}, v_j^{t_0}, \ell_{ij}^{t_0}, q_{D,j}^{t_0}$) are optimized only for the control time t_0, not for neighbor times.

Temporal Battery Constraints:

Case 1: First time period ($t_0 = 1$):

Local times: $\{1, 2\}$, *Decision variables:* $\mathbf{B_j^1[1]}, \mathbf{B_j^1[2]}$, *Dual variables:* $\mathbf{u_j^1[1]}$ *only*

$$\text{SOC trajectory } t = 0 \to t = 1: \quad \mathbf{B_j^1[1]} = B_{0,j} - P_{B,j}^1 \cdot \Delta t, \quad \forall j\in\mathcal{B} \tag{12}$$

$$\text{SOC trajectory } t = 1 \to t = 2: \quad \mathbf{B_j^1[2]} = \mathbf{B_j^1[1]} - P_{B,j}^2 \cdot \Delta t, \quad \forall j\in\mathcal{B} \tag{13}$$

Note: $B_{0,j}$ is a parameter (initial condition), not a decision variable. Only $\mathbf{B_j^1[1]}$ participates in consensus (with dual $\mathbf{u_j^1[1]}$). Variable $\mathbf{B_j^1[2]}$ exists for penalty computation but does not participate in consensus at $t = 1$.

Case 2: Interior time periods ($2 \leq t_0 \leq T - 1$):

Local times: $\{t_0 - 1, t_0, t_0 + 1\}$, *Decision variables:* $\mathbf{B_j^{t_0}[t_0 - 1]}, \mathbf{B_j^{t_0}[t_0]}, \mathbf{B_j^{t_0}[t_0 + 1]}$

Dual variables: $\mathbf{u_j^{t_0}[t_0]}$ *only*

$$\text{SOC trajectory } t_0 - 1 \to t_0: \quad \mathbf{B_j^{t_0}[t_0]} = \mathbf{B_j^{t_0}[t_0-1]} - P_{B,j}^{t_0} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{14}$$

$$\text{SOC trajectory } t_0 \to t_0 + 1: \quad \mathbf{B_j^{t_0}[t_0+1]} = \mathbf{B_j^{t_0}[t_0]} - P_{B,j}^{t_0+1} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{15}$$

Note: Only $\mathbf{B_j^{t_0}[t_0]}$ participates in consensus update at time t_0 (with dual $\mathbf{u_j^{t_0}[t_0]}$). Variables $\mathbf{B_j^{t_0}[t_0-1]}$ and $\mathbf{B_j^{t_0}[t_0+1]}$ are used for penalty terms but do not generate new dual variables in this subproblem.

Case 3: Last time period ($t_0 = T$):

Local times: $\{T-1, T\}$, *Decision variables:* $\mathbf{B_j^T[T-1]}, \mathbf{B_j^T[T]}$, *Dual variables:* $\mathbf{u_j^T[T]}$ *only*

$$\text{SOC trajectory } T - 1 \to T: \quad \mathbf{B_j^T[T]} = \mathbf{B_j^T[T-1]} - P_{B,j}^{T} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{16}$$

Note: Only $\mathbf{B_j^T[T]}$ participates in consensus at $t = T$ (with dual $\mathbf{u_j^T[T]}$). Variable $\mathbf{B_j^T[T-1]}$ is for penalty only.

SOC and Power Limits:

$$\begin{aligned}\text{SOC limits:} \quad & \text{SOC}_{\text{min},j} \cdot B_{\text{rated},j} \leq \mathbf{B_j^{t_0}[t]} \leq \text{SOC}_{\text{max},j} \cdot B_{\text{rated},j}, \\ & \forall t \in \mathcal{T}_{\text{local}}^{t_0},\ j \in \mathcal{B} \end{aligned}\tag{17}$$

$$\text{Power limits:} \quad -P_{B,\text{rated},j} \leq P_{B,j}^{t} \leq P_{B,\text{rated},j}, \quad \forall t \in \mathcal{T}_{\text{local}}^{t_0},\ j \in \mathcal{B} \tag{18}$$

Note: Power limits must be enforced for *all* times appearing in constraints, not just t_0. For interior times: $P_{B,j}^{t_0-1}, P_{B,j}^{t_0}, P_{B,j}^{t_0+1}$ all need limits.

1.5 Step 2: Consensus Update (Local Averaging)

For each battery $j \in \mathcal{B}$ and each time $t \in \mathcal{T}$, average only over subproblems where t is the *active* time:

Consensus at $t = 1$:

$$\mathbf{\hat{B}_j[1]} = \text{clamp}\left(\frac{1}{2}\left(\mathbf{B_j^1[1]} + \mathbf{u_j^1[1]} + \mathbf{B_j^2[1]} + \mathbf{u_j^2[1]}\right), \underline{B}_j, \overline{B}_j\right) \tag{19}$$

Contributors: Subproblems $t_0 = 1$ and $t_0 = 2$ (only these have $t = 1$ as decision variable with dual)

Consensus at interior times ($2 \leq t \leq T-1$):

$$\mathbf{\hat{B}_j[t]} = \text{clamp}\left(\frac{1}{3}\left(\mathbf{B_j^{t-1}[t]} + \mathbf{u_j^{t-1}[t]} + \mathbf{B_j^{t}[t]} + \mathbf{u_j^{t}[t]} + \mathbf{B_j^{t+1}[t]} + \mathbf{u_j^{t+1}[t]}\right), \underline{B}_j, \overline{B}_j\right) \tag{20}$$

Contributors: Subproblems $t_0 \in \{t-1, t, t+1\}$ (each has t as its active control time)

Consensus at $t = T$:

$$\mathbf{\hat{B}_j[T]} = \text{clamp}\left(\frac{1}{2}\left(\mathbf{B_j^{T-1}[T]} + \mathbf{u_j^{T-1}[T]} + \mathbf{B_j^{T}[T]} + \mathbf{u_j^{T}[T]}\right), \underline{B}_j, \overline{B}_j\right) \tag{21}$$

Contributors: Subproblems $t_0 = T-1$ and $t_0 = T$ (only these have $t = T$ as decision variable with dual)

General form:

$$\mathcal{N}_t = \{t_0 \in \mathcal{T} : t \text{ is the active control time in subproblem } t_0\} \tag{22}$$

$$= \begin{cases} \{1, 2\} & \text{if } t = 1 \\ \{t-1, t, t+1\} & \text{if } 2 \leq t \leq T-1 \\ \{T-1, T\} & \text{if } t = T \end{cases} \tag{23}$$

1.6 Step 3: Dual Update (All Local Times)

For each subproblem $t_0 \in \mathcal{T}$ and battery $j \in \mathcal{B}$, update dual variables for **all local times** $t \in \mathcal{T}_{\text{local}}^{t_0}$:
Subproblem $t_0 = 1$ (local times: $\{1, 2\}$):

$$\mathbf{u_j^1[1]} := \mathbf{u_j^1[1]} + \left(\mathbf{B_j^1[1]} - \mathbf{\hat{B}_j[1]}\right) \tag{24}$$

$$\mathbf{u_j^1[2]} := \mathbf{u_j^1[2]} + \left(\mathbf{B_j^1[2]} - \mathbf{\hat{B}_j[2]}\right) \tag{25}$$

Subproblem t_0 ($2 \le t_0 \le T-1$, local times: $\{t_0 - 1, t_0, t_0 + 1\}$):

$$\mathbf{u_j^{t_0}[t_0 - 1]} := \mathbf{u_j^{t_0}[t_0 - 1]} + \left(\mathbf{B_j^{t_0}[t_0 - 1]} - \mathbf{\hat{B}_j[t_0 - 1]}\right) \tag{26}$$

$$\mathbf{u_j^{t_0}[t_0]} := \mathbf{u_j^{t_0}[t_0]} + \left(\mathbf{B_j^{t_0}[t_0]} - \mathbf{\hat{B}_j[t_0]}\right) \tag{27}$$

$$\mathbf{u_j^{t_0}[t_0 + 1]} := \mathbf{u_j^{t_0}[t_0 + 1]} + \left(\mathbf{B_j^{t_0}[t_0 + 1]} - \mathbf{\hat{B}_j[t_0 + 1]}\right) \tag{28}$$

Subproblem $t_0 = T$ (local times: $\{T-1, T\}$):

$$\mathbf{u_j^T[T - 1]} := \mathbf{u_j^T[T - 1]} + \left(\mathbf{B_j^T[T - 1]} - \mathbf{\hat{B}_j[T - 1]}\right) \tag{29}$$

$$\mathbf{u_j^T[T]} := \mathbf{u_j^T[T]} + \left(\mathbf{B_j^T[T]} - \mathbf{\hat{B}_j[T]}\right) \tag{30}$$

General Form:

$$\text{For each } t \in \mathcal{T}_{\text{local}}^{t_0}: \quad \mathbf{u_j^{t_0}[t]} := \mathbf{u_j^{t_0}[t]} + \left(\mathbf{B_j^{t_0}[t]} - \mathbf{\hat{B}_j[t]}\right) \tag{31}$$

Critical Note: Each subproblem t_0 maintains dual variables for *all* its local times $\mathcal{T}_{\text{local}}^{t_0}$, which is 2 or 3 times per battery. This ensures that each time t has duals from all subproblems that include it in their penalty terms, enabling proper consensus averaging in Step 2.

1.7 Convergence Criteria (Exact Implementation)

Primal Residual (Consensus Violation):

Only count residuals at active control times (where dual variables exist):

$$r_{\text{values}} = \left\{\mathbf{B_j^{t_0,k}[t_0]} - \mathbf{\hat{B}_j^k[t_0]} : t_0 \in \mathcal{T},\ j \in \mathcal{B}\right\} \tag{32}$$

$$\|r^k\|_2 = \frac{\|\mathbf{r}_{\text{values}}\|_2}{\sqrt{|\mathbf{r}_{\text{values}}|}} = \frac{1}{\sqrt{T \cdot |\mathcal{B}|}} \left(\sum_{t_0=1}^{T} \sum_{j \in \mathcal{B}} \left(\mathbf{B_j^{t_0,k}[t_0]} - \mathbf{\hat{B}_j^k[t_0]}\right)^2\right)^{1/2} \tag{33}$$

Note: We only compute residual at time t_0 for each subproblem t_0 (the active control time where dual $\mathbf{u_j^{t_0}[t_0]}$ exists). The normalization is by $\sqrt{T \cdot |\mathcal{B}|}$, the number of active consensus variables.

Dual Residual (Consensus Variable Change):

Measures how much the consensus variables $\mathbf{\hat{B}}$ changed from previous iteration:

$$\mathbf{s}_j^k = \mathbf{\hat{B}_j^k} - \mathbf{\hat{B}_j^{k-1}} = \begin{bmatrix} \mathbf{\hat{B}_j^k[1]} - \mathbf{\hat{B}_j^{k-1}[1]} \\ \vdots \\ \mathbf{\hat{B}_j^k[T]} - \mathbf{\hat{B}_j^{k-1}[T]} \end{bmatrix} \in \mathbb{R}^T \tag{34}$$

$$\|s^k\|_2 = \frac{\rho^k}{|\mathcal{B}|} \left\| \begin{bmatrix} \mathbf{s}_1^k \\ \vdots \\ \mathbf{s}_{|\mathcal{B}|}^k \end{bmatrix} \right\|_2 = \frac{\rho^k}{|\mathcal{B}|} \left(\sum_{j \in \mathcal{B}} \sum_{t=1}^{T} \left(\mathbf{\hat{B}_j^k[t]} - \mathbf{\hat{B}_j^{k-1}[t]}\right)^2\right)^{1/2} \tag{35}$$

Note: The dual residual is scaled by ρ and normalized by $|\mathcal{B}|$ (number of batteries), not by $\sqrt{T \cdot |\mathcal{B}|}$.

Convergence Condition:

$$\text{Converged if } \|r^k\|_2 \leq \epsilon_{\text{pri}} \quad \text{and} \quad \|s^k\|_2 \leq \epsilon_{\text{dual}} \tag{36}$$

with $\epsilon_{\text{pri}} = 10^{-5}$ and $\epsilon_{\text{dual}} = 10^{-4}$.

1.8 Complexity Comparison

Metric	Global Coupling	Local Coupling
SOC variables per subproblem	$T \times \lvert\mathcal{B}\rvert$	2 or $3 \times \lvert\mathcal{B}\rvert$
Dual variables per subproblem	$T \times \lvert\mathcal{B}\rvert$	2 or $3 \times \lvert\mathcal{B}\rvert$ (same as SOC)
Penalty terms per subproblem	$T \times \lvert\mathcal{B}\rvert$	2 or $3 \times \lvert\mathcal{B}\rvert$
Total SOC storage	$T^2 \times \lvert\mathcal{B}\rvert$	$(2 \cdot 2 + 3 \cdot (T-2)) \times \lvert\mathcal{B}\rvert = (3T-2) \times \lvert\mathcal{B}\rvert$
Total dual storage	$T^2 \times \lvert\mathcal{B}\rvert$	$(3T-2) \times \lvert\mathcal{B}\rvert$ (same as SOC)
Communication per iteration	$T \times \lvert\mathcal{B}\rvert$ per subproblem	2 or $3 \times \lvert\mathcal{B}\rvert$ per subproblem
Consensus contributors per time	All T subproblems	Only 2 or 3 neighbors

For $T = 24$, $|\mathcal{B}| = 26$ batteries:

- **SOC Variables:**
  - Global: $24^2 \times 26 = 14,976$ total
  - Local: $(3 \times 24 - 2) \times 26 = 1,872$ total
  - **Reduction:** $8.0\times$
- **Dual Variables:**
  - Global: $24^2 \times 26 = 14,976$ total
  - Local: $24 \times 26 = 624$ total
  - **Reduction:** $24\times$
- **Per-subproblem SOC variables:**
  - Global: $24 \times 26 = 624$ per subproblem
  - Local (boundary): $2 \times 26 = 52$ per subproblem
  - Local (interior): $3 \times 26 = 78$ per subproblem
  - **Reduction:** $12\times$ **(boundary),** $8\times$ **(interior)**

2 Adaptive Penalty Parameter: Two-Phase Strategy with Watchdog

The penalty parameter ρ is adjusted dynamically using a sophisticated two-phase strategy that prioritizes primal convergence initially, then balances both residuals. The scheme is activated by setting `adaptive_rho = true`.

2.1 Residual Calculations (Exact Implementation)

Primal Residual (Consensus Violation):

$$\|r^k\|_2 = \frac{1}{\sqrt{N_{\text{active}}}} \left(\sum_{t_0=1}^{T} \sum_{j \in \mathcal{B}} \left(\mathbf{B}_{\mathbf{j}}^{\mathbf{t_0,k}}[\mathbf{t_0}] - \hat{\mathbf{B}}_{\mathbf{j}}^{\mathbf{k}}[\mathbf{t_0}] \right)^2 \right)^{1/2} \tag{37}$$

where $N_{\text{active}} = T \times |\mathcal{B}|$ is the number of active time-battery pairs (only counting where duals exist).

Dual Residual (Consensus Variable Change):

$$\|s^k\|_2 = \frac{\rho^k}{|\mathcal{B}|} \left(\sum_{j \in \mathcal{B}} \sum_{t=1}^{T} \left(\hat{\mathbf{B}}_{\mathbf{j}}^{\mathbf{k}}[\mathbf{t}] - \hat{\mathbf{B}}_{\mathbf{j}}^{\mathbf{k-1}}[\mathbf{t}] \right)^2 \right)^{1/2} \tag{38}$$

2.2 Phase 1: Aggressive Primal Convergence (Until $\|r\| \leq \epsilon_{\text{pri}}$)

Objective: Drive primal residual below threshold by only increasing ρ. Never decrease ρ to avoid slowing consensus formation.

Update Rules (every $N_{\text{update}} = 5$ iterations):

$$\rho^{k+1} = \begin{cases} \min(\rho_{\max}, \tau_{\text{incr}} \cdot \rho^k) & \text{if } \|r^k\| > \mu \cdot \|s^k\| \\ \min(\rho_{\max}, \tau_{\text{nudge}} \cdot \rho^k) & \text{if } \|r^k\| > \epsilon_{\text{pri}} \text{ and no } \rho \text{ change for } N_{\text{stall}} \text{ iters} \\ \rho^k & \text{otherwise (do not decrease)} \end{cases} \tag{39}$$

Phase Transition:

$$\text{Switch to Phase 2 if } \|r^k\| \leq \epsilon_{\text{pri}} \text{ (primal converged once)} \tag{40}$$

2.3 Phase 2: Bidirectional Adaptation (After primal convergence)

Objective: Balance primal and dual residuals, allowing both increases and decreases of ρ.

Update Rules (every $N_{\text{update}} = 5$ iterations):

$$\rho^{k+1} = \begin{cases} \min(\rho_{\max}, \tau_{\text{incr}} \cdot \rho^k) & \text{if } \|r^k\| > \mu \cdot \|s^k\| \\ \max(\rho_{\min}, \rho^k / \tau_{\text{decr}}) & \text{if } \|s^k\| > \mu \cdot \|r^k\| \textbf{ and } \|r^k\| \leq \epsilon_{\text{pri}} \\ \rho^k & \text{otherwise (residuals balanced)} \end{cases} \tag{41}$$

Watchdog Protection: The decrease condition includes $\|r^k\| \leq \epsilon_{\text{pri}}$ to prevent premature decreases when primal consensus breaks down after Phase 1 $\to$ 2 transition.

2.4 Primal Residual Watchdog (All Phases)

Motivation: Detects when primal residual stays elevated for many consecutive iterations, indicating insufficient penalty to enforce consensus.

Hysteresis-Based Counter (runs every iteration):

$$\text{counter}^{k+1} = \begin{cases} \text{counter}^k + 1 & \text{if } \|r^k\| > 2\epsilon_{\text{pri}} \\ 0 & \text{if } \|r^k\| < 0.5\epsilon_{\text{pri}} \\ \text{counter}^k & \text{if } 0.5\epsilon_{\text{pri}} \le \|r^k\| \le 2\epsilon_{\text{pri}} \text{ (hysteresis zone)} \end{cases} \tag{42}$$

Watchdog Trigger:

$$\text{If counter}^k \ge N_{\text{watchdog}} = 20 : \quad \rho^{k+1} = \min(\rho_{\max}, \tau_{\text{watchdog}} \cdot \rho^k), \quad \text{counter}^{k+1} = 0 \tag{43}$$

The hysteresis prevents false resets from oscillations around ϵ_{pri}, while the 20-iteration window ensures sustained high residuals trigger aggressive action.

2.5 Dual Variable Rescaling

When ρ changes from ρ^k to ρ^{k+1}, the scaled dual variables must be rescaled:

$$\mathbf{u_j^{t_0}[t]} \leftarrow \mathbf{u_j^{t_0}[t]} \cdot \frac{\rho^k}{\rho^{k+1}}, \quad \forall t_0 \in \mathcal{T},\, j \in \mathcal{B},\, t \in \mathcal{T}_{\text{local}}^{t_0} \tag{44}$$

This maintains the relationship between scaled and unscaled dual variables: $\mathbf{u} = \lambda/\rho$.

2.6 Complete Parameter Summary

Parameter	Value (SOCP tADMM)
Tolerances	
Primal tolerance ϵ_{pri}	1×10^{-5}
Dual tolerance ϵ_{dual}	1×10^{-4} (relaxed for faster convergence)
Standard Adaptive Parameters	
Balance factor μ	5.0
Increase factor τ_{incr}	2.0 (double ρ)
Decrease factor τ_{decr}	2.0 (halve ρ)
Update interval N_{update}	5 iterations
Bounds	
Minimum $\rho_{\min}$	1.0
Maximum $\rho_{\max}$	10^6
Phase 1 Nudge (Stall Detection)	
Stall check interval N_{stall}	5 iterations
Stall nudge factor τ_{nudge}	2.0
Enable stall detection	`false` (disabled for clean testing)
Watchdog Parameters	
Enable watchdog	`true`
Watchdog window N_{watchdog}	20 iterations
Watchdog factor τ_{watchdog}	2.0 (double ρ)
Upper threshold (increment counter)	$2\epsilon_{\text{pri}} = 2 \times 10^{-5}$
Lower threshold (reset counter)	$0.5\epsilon_{\text{pri}} = 5 \times 10^{-6}$
Disabled Features	
Stability zone (freeze ρ near convergence)	`false`
Slow progress acceleration	`false`

2.7 Convergence Impact

For LinDistFlow SOCP problems (ieee123A_1ph, T=96, 26 batteries):

- **Phase 1**: ρ increases rapidly ($40000 \rightarrow 80000 \rightarrow 160000$) driving primal convergence
- **Phase 1→2 Transition**: At $k \approx 15$ when $\|r\| < 10^{-5}$
- **Phase 2**: Bidirectional adaptation, but decreases blocked by watchdog protection if $\|r\|$ rises
- **Watchdog**: Triggers every 20 iterations if primal residual stays elevated, preventing slow drift
- **Total Iterations**: Typically 100–200 iterations for tight tolerances with threading enabled

3 Copper Plate Localized tADMM (Reduced Network)

3.1 Overview

The copper plate formulation simplifies the multi-period OPF by removing network constraints entirely, reducing each time period's problem to pure energy arbitrage. Combined with localized temporal coupling (Section 1), this yields the most compact tADMM subproblems.

Key Simplifications:

- **Network**: No voltage, power flow, or line constraints
- **Spatial**: Single aggregated load, single aggregated battery
- **Temporal**: Localized coupling (2–3 time steps per subproblem)
- **Problem Size**: Each subproblem has only 3–5 decision variables

3.2 Local Time Sets (Copper Plate)

Same as LinDistFlow localized formulation:

$$\mathcal{T}_{\text{local}}^{t_0} = \begin{cases} \{1,2\} & \text{if } t_0 = 1 \\ \{t_0-1, t_0, t_0+1\} & \text{if } 2 \leq t_0 \leq T-1 \\ \{T-1, T\} & \text{if } t_0 = T \end{cases} \tag{45}$$

3.3 Primal Update (Copper Plate, Subproblem t_0)

For each subproblem $t_0 \in \{1, 2, \dots, T\}$:

$$\begin{aligned} \min_{\substack{P_{\text{subs}}^{t_0}, P_B^t, \mathbf{B^{t_0}[t]} \\ t \in \mathcal{T}_{\text{local}}^{t_0}}} \quad & c^{t_0} \cdot P_{\text{subs}}^{t_0} \cdot P_{\text{BASE}} \cdot \Delta t + C_B \cdot \left(P_B^{t_0}\right)^2 \cdot P_{\text{BASE}}^2 \cdot \Delta t \\ & + \frac{\rho}{2} \sum_{t \in \mathcal{T}_{\text{local}}^{t_0}} \left(\mathbf{B^{t_0}[t]} - \mathbf{\hat{B}[t]} + \mathbf{u^{t_0}[t]} \right)^2 \end{aligned} \tag{46}$$

Subject to:

Nodal Real Power Balance (only at t_0):

$$P_{\text{subs}}^{t_0} + P_B^{t_0} = P_L^{t_0} \tag{47}$$

Battery SOC Dynamics:

Case 1: $t_0 = 1$ (First period):

Local times: $\{1, 2\}$, *Optimize:* $\mathbf{B^1[1]}, \mathbf{B^1[2]}, P_B^1, P_B^2, P_{subs}^1$

$$\mathbf{B^1[1]} = B_0 - P_B^1 \cdot \Delta t \tag{48}$$

$$\mathbf{B^1[2]} = \mathbf{B^1[1]} - P_B^2 \cdot \Delta t \tag{49}$$

Case 2: $2 \leq t_0 \leq T-1$ (Interior periods):

Local times: $\{t_0-1, t_0, t_0+1\}$, *Optimize:* $\mathbf{B^{t_0}[t_0-1]}, \mathbf{B^{t_0}[t_0]}, \mathbf{B^{t_0}[t_0+1]}, P_B^{t_0}, P_B^{t_0+1}, P_{subs}^{t_0}$

$$\mathbf{B^{t_0}[t_0]} = \mathbf{B^{t_0}[t_0-1]} - P_B^{t_0} \cdot \Delta t \tag{50}$$

$$\mathbf{B^{t_0}[t_0+1]} = \mathbf{B^{t_0}[t_0]} - P_B^{t_0+1} \cdot \Delta t \tag{51}$$

Case 3: $t_0 = T$ **(Last period):**

Local times: $\{T-1, T\}$, *Optimize:* $\mathbf{B^T[T-1]}, \mathbf{B^T[T]}, P_B^T, P_{subs}^T$

$$\mathbf{B^T[T]} = \mathbf{B^T[T-1]} - P_B^T \cdot \Delta t \tag{52}$$

Battery Bounds (all local times):

$$\underline{B} \le \mathbf{B^{t_0}[t]} \le \overline{B}, \quad \forall t \in \mathcal{T}_{\text{local}}^{t_0} \tag{53}$$

$$-P_{B,R} \le P_B^t \le P_{B,R}, \quad \forall t \in \{t_0, t_0+1\} \cap \mathcal{T}_{\text{local}}^{t_0} \tag{54}$$

3.4 Consensus Update (Localized Averaging)

For each time step $t \in \{1, 2, \ldots, T\}$, average over subproblems containing t:

$$\mathbf{\hat{B}[t]} = \begin{cases} \frac{1}{2}\left(\mathbf{B^1[1]} + \mathbf{u^1[1]} + \mathbf{B^2[1]} + \mathbf{u^2[1]}\right) & \text{if } t = 1 \\ \frac{1}{3}\sum_{t_0 \in \{t-1, t, t+1\}}\left(\mathbf{B^{t_0}[t]} + \mathbf{u^{t_0}[t]}\right) & \text{if } 2 \le t \le T-1 \\ \frac{1}{2}\left(\mathbf{B^{T-1}[T]} + \mathbf{u^{T-1}[T]} + \mathbf{B^T[T]} + \mathbf{u^T[T]}\right) & \text{if } t = T \end{cases} \tag{55}$$

Projection onto Feasible Set:

$$\mathbf{\hat{B}[t]} \leftarrow \max\left(\underline{B}, \min\left(\overline{B}, \mathbf{\hat{B}[t]}\right)\right) \tag{56}$$

3.5 Dual Update (Localized)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$ and $t \in \mathcal{T}_{\text{local}}^{t_0}$:

$$\mathbf{u^{t_0}[t]} \leftarrow \mathbf{u^{t_0}[t]} + \left(\mathbf{B^{t_0}[t]} - \mathbf{\hat{B}[t]}\right) \tag{57}$$

Number of Dual Updates per Iteration:

- Subproblem $t_0 = 1$: Update 2 dual variables ($\mathbf{u^1[1]}, \mathbf{u^1[2]}$)
- Subproblems $t_0 = 2, \ldots, T-1$: Update 3 dual variables each
- Subproblem $t_0 = T$: Update 2 dual variables ($\mathbf{u^T[T-1]}, \mathbf{u^T[T]}$)
- **Total**: $2 + 3(T-2) + 2 = 3T - 2$ dual variable updates

3.6 Convergence Criteria

Primal Residual (consensus violation):

$$\|r^k\|_2 = \frac{1}{T}\sqrt{\sum_{t_0=1}^{T}\sum_{t\in\mathcal{T}_{\text{local}}^{t_0}}\left(\mathbf{B^{t_0}[t]} - \hat{\mathbf{B}}[\mathbf{t}]\right)^2} \tag{58}$$

Dual Residual (consensus change):

$$\|s^k\|_2 = \frac{\rho}{T}\sqrt{\sum_{t=1}^{T}\left(\hat{\mathbf{B}}^{\mathbf{k}}[\mathbf{t}] - \hat{\mathbf{B}}^{\mathbf{k-1}}[\mathbf{t}]\right)^2} \tag{59}$$

Stopping Criteria:

$$\text{Converged if:} \quad \|r^k\|_2 \le \epsilon_{\text{pri}} \quad \text{and} \quad \|s^k\|_2 \le \epsilon_{\text{dual}} \tag{60}$$

3.7 Complexity Comparison

Formulation	Variables/Subproblem	Coupling Variables	Total Storage
Global tADMM	T SOC + T power	T	T^2
Localized tADMM (LinDistFlow)	2–3 SOC + 2–3 power	2–3	$3T$
Localized Copper Plate	**2–3 SOC + 1–2 power**	**2–3**	**3T**

Table 1: Comparison of tADMM formulations (per-unit battery system)

Copper Plate Advantages:

- **Minimal subproblem size**: 3–5 decision variables per subproblem
- **No spatial coupling**: Pure temporal decomposition
- **Fast solves**: Each subproblem is a small LP/QP (<1ms with Gurobi)
- **Scalability**: Computational cost grows linearly with T (not T^2)

3.8 Example: 24-Hour Copper Plate Problem

Problem Data:

- Time periods: $T = 24$ (hourly intervals)
- Battery: $E_{\text{rated}} = 4000$ kWh, $P_{B,R} = 800$ kW
- Load: $P_L \in [800, 1200]$ kW (time-varying)
- Energy price: $c^t \in [0.08, 0.20]$ \$/kWh (sinusoidal)
- Quadratic battery cost: $C_B = 10^{-6} \times \min(c^t)$

Convergence Performance:

- **Fixed** $\rho = 10.0$: $\sim$200–300 iterations
- **Adaptive** ρ **(Boyd)**: $\sim$98 iterations (ρ: $10.0 \to 5.0 \to 2.5 \to 1.25$)
- **Solve time**: <5 seconds total (single-threaded Julia)
- **Objective gap vs. centralized**: $< 10^{-6}$ (numerically exact)

4 Appendix: Full Variable and Parameter Definitions

4.1 System Bases

$$\mathrm{kV}_B = \frac{4.16}{\sqrt{3}} \text{ kV (phase-to-neutral)} \tag{61}$$

$$\mathrm{kVA}_B = 1000 \text{ kVA} \tag{62}$$

$$P_{\mathrm{BASE}} = 1000 \text{ kW} \tag{63}$$

$$E_{\mathrm{BASE}} = 1000 \text{ kWh per hour} \tag{64}$$

4.2 SOC Bound Definitions

$$\underline{B} = \mathrm{SOC}_{\min} \cdot E_{\mathrm{Rated}} \tag{65}$$

$$\overline{B} = \mathrm{SOC}_{\max} \cdot E_{\mathrm{Rated}} \tag{66}$$

4.3 Physical Interpretation

- $P_B[t] > 0$: Battery discharging (providing power to the system)
- $P_B[t] < 0$: Battery charging (consuming power from the system)
- $B[t]$: Battery state of charge at the end of period t
- $\underline{B} = \mathrm{SOC}_{\min} \cdot E_{\mathrm{Rated}}$: Lower SOC bound
- $\overline{B} = \mathrm{SOC}_{\max} \cdot E_{\mathrm{Rated}}$: Upper SOC bound