Multi-Period Optimal Power Flow: LinDistFlow & Copper Plate tADMM Formulations

1 NEW: Local Neighbor Coupling tADMM (Reduced Communication)

1.1 Overview

This formulation modifies the standard tADMM algorithm to use **local temporal coupling** instead of global coupling. Each subproblem t_0 only shares SOC variables with its immediate time neighbors $(t_0 - 1, t_0, t_0 + 1)$, reducing coupling from $\mathcal{O}(T \times |\mathcal{B}|)$ to $\mathcal{O}(3 \times |\mathcal{B}|)$ per subproblem.

1.2 Key Modifications

- **Original (Global Coupling)**: Each subproblem t_0 couples with *all* T time steps
- **New (Local Coupling)**: Each subproblem t_0 couples only with neighbors $\{t_0 - 1, t_0, t_0 + 1\}$
- **Memory Reduction**: $T^2 \times |\mathcal{B}| \to 3T \times |\mathcal{B}|$ total storage
- **Communication Reduction**: Each subproblem exchanges $3 \times |\mathcal{B}|$ variables (not $T \times |\mathcal{B}|$)

1.3 Local Time Sets

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$, define the local time set:

$$\mathcal{T}_{\text{local}}^{t_0} = \begin{cases} \{1, 2\} & \text{if } t_0 = 1 \text{ (boundary)} \\ \{t_0 - 1, t_0, t_0 + 1\} & \text{if } 2 \leq t_0 \leq T - 1 \text{ (interior)} \\ \{T - 1, T\} & \text{if } t_0 = T \text{ (boundary)} \end{cases} \tag{1}$$

1.4 Step 1: Primal Update (Local Coupling)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\begin{aligned} \min_{\substack{P_{\text{Subs}}^{t_0}, Q_{\text{Subs}}^{t_0}, \\ P_{ij}^{t_0}, Q_{ij}^{t_0}, v_j^{t_0}, q_{D,j}^{t_0}, \\ P_{B,j}^{t}, \mathbf{B_j^{t_0}[t]} \\ \forall j \in \mathcal{B}, t \in \mathcal{T}_{\text{local}}^{t_0}}} \quad & c^{t_0} \cdot P_{\text{Subs}}^{t_0} \cdot P_{\text{BASE}} \cdot \Delta t + C_B \sum_{j \in \mathcal{B}} \left(P_{B,j}^{t_0} \right)^2 \cdot P_{\text{BASE}}^2 \cdot \Delta t \\ & + \frac{\rho}{2} \sum_{j \in \mathcal{B}} \sum_{t \in \mathcal{T}_{\text{local}}^{t_0}} \left(\mathbf{B_j^{t_0}[t]} - \mathbf{\hat{B}_j[t]} + \mathbf{u_j^{t_0}[t]} \right)^2 \end{aligned} \tag{2}$$

Subject to:
Spatial Network Constraints (only for time t_0):

$$\text{(Same as global formulation - constraints (2)-(8))} \tag{3}$$

Temporal Battery Constraints:
Case 1: First time period ($t_0 = 1$):
Local times: $\{1, 2\}$, *Decision variables:* $\mathbf{B_j^1[1]}, \mathbf{B_j^1[2]}$, *Dual variables:* $\mathbf{u_j^1[1]}$ *only*

$$\text{SOC trajectory } t = 0 \to t = 1: \quad \mathbf{B_j^1[1]} = B_{0,j} - P_{B,j}^1 \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{4}$$
$$\text{SOC trajectory } t = 1 \to t = 2: \quad \mathbf{B_j^1[2]} = \mathbf{B_j^1[1]} - P_{B,j}^2 \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{5}$$

Note: $B_{0,j}$ is a parameter (initial condition), not a decision variable. Only $\mathbf{B_j^1[1]}$ participates in consensus (with dual $\mathbf{u_j^1[1]}$). Variable $\mathbf{B_j^1[2]}$ exists for penalty computation but does not participate in consensus at $t = 1$.
Case 2: Interior time periods ($2 \le t_0 \le T - 1$):
Local times: $\{t_0 - 1, t_0, t_0 + 1\}$, *Decision variables:* $\mathbf{B_j^{t_0}[t_0 - 1]}, \mathbf{B_j^{t_0}[t_0]}, \mathbf{B_j^{t_0}[t_0 + 1]}$
Dual variables: $\mathbf{u_j^{t_0}[t_0]}$ *only*

$$\text{SOC trajectory } t_0 - 1 \to t_0: \quad \mathbf{B_j^{t_0}[t_0]} = \mathbf{B_j^{t_0}[t_0 - 1]} - P_{B,j}^{t_0} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{6}$$
$$\text{SOC trajectory } t_0 \to t_0 + 1: \quad \mathbf{B_j^{t_0}[t_0 + 1]} = \mathbf{B_j^{t_0}[t_0]} - P_{B,j}^{t_0+1} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{7}$$

Note: Only $\mathbf{B_j^{t_0}[t_0]}$ participates in consensus update at time t_0 (with dual $\mathbf{u_j^{t_0}[t_0]}$). Variables $\mathbf{B_j^{t_0}[t_0 - 1]}$ and $\mathbf{B_j^{t_0}[t_0 + 1]}$ are used for penalty terms but do not generate new dual variables in this subproblem.
Case 3: Last time period ($t_0 = T$):
Local times: $\{T - 1, T\}$, *Decision variables:* $\mathbf{B_j^T[T - 1]}, \mathbf{B_j^T[T]}$, *Dual variables:* $\mathbf{u_j^T[T]}$ *only*

$$\text{SOC trajectory } T - 1 \to T: \quad \mathbf{B_j^T[T]} = \mathbf{B_j^T[T - 1]} - P_{B,j}^T \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{8}$$

Note: Only $\mathbf{B_j^T[T]}$ participates in consensus at $t = T$ (with dual $\mathbf{u_j^T[T]}$). Variable $\mathbf{B_j^T[T - 1]}$ is for penalty only.
SOC and Power Limits:

$$\begin{aligned} \text{SOC limits:} \quad & \text{SOC}_{\min,j} \cdot B_{\text{rated},j} \le \mathbf{B_j^{t_0}[t]} \le \text{SOC}_{\max,j} \cdot B_{\text{rated},j}, \\ & \forall t \in \mathcal{T}_{\text{local}}^{t_0},\ j \in \mathcal{B} \end{aligned} \tag{9}$$
$$\text{Power limits:} \quad -P_{B,\text{rated},j} \le P_{B,j}^t \le P_{B,\text{rated},j}, \quad \forall t \in \mathcal{T}_{\text{local}}^{t_0},\ j \in \mathcal{B} \tag{10}$$

Note: Power limits must be enforced for *all* times appearing in constraints, not just t_0. For interior times: $P_{B,j}^{t_0-1}, P_{B,j}^{t_0}, P_{B,j}^{t_0+1}$ all need limits.

1.5 Step 2: Consensus Update (Local Averaging)

For each battery $j \in \mathcal{B}$ and each time $t \in \mathcal{T}$, average only over subproblems where t is the *active* time:
Consensus at $t = 1$:

$$\mathbf{\hat{B}_j[1]} = \text{clamp}\left(\frac{1}{2}\left(\mathbf{B_j^1[1]} + \mathbf{u_j^1[1]} + \mathbf{B_j^2[1]} + \mathbf{u_j^2[1]}\right), \underline{B}_j, \overline{B}_j\right) \tag{11}$$

Contributors: Subproblems $t_0 = 1$ and $t_0 = 2$ (only these have $t = 1$ as decision variable with dual)
Consensus at interior times ($2 \leq t \leq T-1$):

$$\hat{\mathbf{B}}_{\mathbf{j}}[\mathbf{t}] = \text{clamp}\left(\frac{1}{3}\left(\mathbf{B}_{\mathbf{j}}^{\mathbf{t-1}}[\mathbf{t}] + \mathbf{u}_{\mathbf{j}}^{\mathbf{t-1}}[\mathbf{t}] + \mathbf{B}_{\mathbf{j}}^{\mathbf{t}}[\mathbf{t}] + \mathbf{u}_{\mathbf{j}}^{\mathbf{t}}[\mathbf{t}] + \mathbf{B}_{\mathbf{j}}^{\mathbf{t+1}}[\mathbf{t}] + \mathbf{u}_{\mathbf{j}}^{\mathbf{t+1}}[\mathbf{t}]\right), \underline{B}_j, \overline{B}_j\right) \tag{12}$$

Contributors: Subproblems $t_0 \in \{t-1, t, t+1\}$ (each has t as its active control time)
Consensus at $t = T$:

$$\hat{\mathbf{B}}_{\mathbf{j}}[\mathbf{T}] = \text{clamp}\left(\frac{1}{2}\left(\mathbf{B}_{\mathbf{j}}^{\mathbf{T-1}}[\mathbf{T}] + \mathbf{u}_{\mathbf{j}}^{\mathbf{T-1}}[\mathbf{T}] + \mathbf{B}_{\mathbf{j}}^{\mathbf{T}}[\mathbf{T}] + \mathbf{u}_{\mathbf{j}}^{\mathbf{T}}[\mathbf{T}]\right), \underline{B}_j, \overline{B}_j\right) \tag{13}$$

Contributors: Subproblems $t_0 = T-1$ and $t_0 = T$ (only these have $t = T$ as decision variable with dual)

General form:

$$\mathcal{N}_t = \{t_0 \in \mathcal{T} : t \text{ is the active control time in subproblem } t_0\} \tag{14}$$

$$= \begin{cases} \{1, 2\} & \text{if } t = 1 \\ \{t-1, t, t+1\} & \text{if } 2 \leq t \leq T-1 \\ \{T-1, T\} & \text{if } t = T \end{cases} \tag{15}$$

1.6 Step 3: Dual Update (Active Time Only)

For each subproblem $t_0 \in \mathcal{T}$ and battery $j \in \mathcal{B}$, update dual *only* for the active control time t_0:
Subproblem $t_0 = 1$:

$$\mathbf{u}_{\mathbf{j}}^{\mathbf{1}}[\mathbf{1}] := \mathbf{u}_{\mathbf{j}}^{\mathbf{1}}[\mathbf{1}] + \left(\mathbf{B}_{\mathbf{j}}^{\mathbf{1}}[\mathbf{1}] - \hat{\mathbf{B}}_{\mathbf{j}}[\mathbf{1}]\right) \tag{16}$$

Subproblem t_0 ($2 \leq t_0 \leq T-1$):

$$\mathbf{u}_{\mathbf{j}}^{\mathbf{t_0}}[\mathbf{t_0}] := \mathbf{u}_{\mathbf{j}}^{\mathbf{t_0}}[\mathbf{t_0}] + \left(\mathbf{B}_{\mathbf{j}}^{\mathbf{t_0}}[\mathbf{t_0}] - \hat{\mathbf{B}}_{\mathbf{j}}[\mathbf{t_0}]\right) \tag{17}$$

Subproblem $t_0 = T$:

$$\mathbf{u}_{\mathbf{j}}^{\mathbf{T}}[\mathbf{T}] := \mathbf{u}_{\mathbf{j}}^{\mathbf{T}}[\mathbf{T}] + \left(\mathbf{B}_{\mathbf{j}}^{\mathbf{T}}[\mathbf{T}] - \hat{\mathbf{B}}_{\mathbf{j}}[\mathbf{T}]\right) \tag{18}$$

Critical Note: Each subproblem t_0 maintains *only one* dual variable per battery: $\mathbf{u}_{\mathbf{j}}^{\mathbf{t_0}}[\mathbf{t_0}]$. The local SOC variables $\mathbf{B}_{\mathbf{j}}^{\mathbf{t_0}}[\mathbf{t_0 \pm 1}]$ are used in penalty terms for coupling but do *not* generate separate dual updates within subproblem t_0. They will be updated when those times are the active control time in their respective subproblems.

1.7 Convergence Criteria

Primal Residual (only for coupled times):

$$\|r^k\|_2 = \frac{1}{\sqrt{N_{\text{coupled}}}}\left(\sum_{t_0 \in \mathcal{T}} \sum_{j \in \mathcal{B}} \sum_{t \in \mathcal{T}_{\text{local}}^{t_0}} \left(\mathbf{B}_{\mathbf{j}}^{\mathbf{t_0}}[\mathbf{t}] - \hat{\mathbf{B}}_{\mathbf{j}}[\mathbf{t}]\right)^2\right)^{1/2} \tag{19}$$

where $N_{\text{coupled}} = \sum_{t_0 \in \mathcal{T}} |\mathcal{T}_{\text{local}}^{t_0}| \times |\mathcal{B}| = (2 + 3(T-2) + 2) \times |\mathcal{B}| = 3T \times |\mathcal{B}| - 2|\mathcal{B}|$.
Dual Residual:

$$\|s^k\|_2 = \frac{\rho}{|\mathcal{B}|}\sqrt{\sum_{j \in \mathcal{B}} \sum_{t=1}^{T} \left(\hat{\mathbf{B}}_{\mathbf{j}}^{\mathbf{k}}[\mathbf{t}] - \hat{\mathbf{B}}_{\mathbf{j}}^{\mathbf{k-1}}[\mathbf{t}]\right)^2} \leq \epsilon_{\text{dual}} \tag{20}$$

1.8 Complexity Comparison

Metric	Global Coupling	Local Coupling
SOC variables per subproblem	$T \times \lvert\mathcal{B}\rvert$	2 or $3 \times \lvert\mathcal{B}\rvert$
Dual variables per subproblem	$T \times \lvert\mathcal{B}\rvert$	$1 \times \lvert\mathcal{B}\rvert$
Penalty terms per subproblem	$T \times \lvert\mathcal{B}\rvert$	2 or $3 \times \lvert\mathcal{B}\rvert$
Total SOC storage	$T^2 \times \lvert\mathcal{B}\rvert$	$(2 \cdot 2 + 3 \cdot (T-2)) \times \lvert\mathcal{B}\rvert = (3T-2) \times \lvert\mathcal{B}\rvert$
Total dual storage	$T^2 \times \lvert\mathcal{B}\rvert$	$T \times \lvert\mathcal{B}\rvert$
Communication per iteration	$T \times \lvert\mathcal{B}\rvert$ per subproblem	2 or $3 \times \lvert\mathcal{B}\rvert$ per subproblem
Consensus contributors	All T subproblems	Only 2 or 3 neighbors

For $T = 24$, $|\mathcal{B}| = 26$ batteries:

- **SOC Variables:**
  - Global: $24^2 \times 26 = 14,976$ total
  - Local: $(3 \times 24 - 2) \times 26 = 1,872$ total
  - **Reduction:** $8.0\times$
- **Dual Variables:**
  - Global: $24^2 \times 26 = 14,976$ total
  - Local: $24 \times 26 = 624$ total
  - **Reduction:** $24\times$
- **Per-subproblem SOC variables:**
  - Global: $24 \times 26 = 624$ per subproblem
  - Local (boundary): $2 \times 26 = 52$ per subproblem
  - Local (interior): $3 \times 26 = 78$ per subproblem
  - **Reduction:** $12\times$ **(boundary),** $8\times$ **(interior)**

2 LinDistFlow MPOPF with tADMM

2.1 Problem Overview

The Temporal ADMM (tADMM) algorithm decomposes the multi-period optimal power flow problem for distribution networks into T subproblems, each corresponding to one time period. This formulation uses the linearized DistFlow model to capture network physics including voltage drops and reactive power flows.

2.2 Variable Color Coding

- $\mathbf{B_j^{t_0}}[\mathbf{t}]$: Local SOC variables for battery j in subproblem t_0, evaluated at time t (blue)
- $\hat{\mathbf{B}}_\mathbf{j}[\mathbf{t}]$: Global consensus SOC for battery j at time t (red)
- $\mathbf{u_j^{t_0}}[\mathbf{t}]$: Local scaled dual variables for battery j in subproblem t_0, for time t (green)

2.3 Sets and Indices

- $\mathcal{N}$: Set of all nodes (buses)
- $\mathcal{L}$: Set of all branches (lines)
- $\mathcal{L}_1$: Set of branches connected to substation (node 1)
- $\mathcal{B}$: Set of nodes with batteries
- $\mathcal{D}$: Set of nodes with PV (DER)
- $\mathcal{T} = \{1, 2, \ldots, T\}$: Set of time periods
- $t_0 \in \mathcal{T}$: Index for a specific time period in tADMM decomposition
- $j \in \mathcal{N}$: Node index
- $(i, j) \in \mathcal{L}$: Branch from node i to node j

2.4 tADMM Algorithm Structure

The algorithm alternates between three update steps:

2.5 Step 1: Subproblem Update (Blue Variables)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\min_{\substack{P_{\text{Subs}}^{t_0}, Q_{\text{Subs}}^{t_0}, \\ P_{ij}^{t_0}, Q_{ij}^{t_0}, v_j^{t_0}, q_{D,j}^{t_0}, \\ P_{B,j}^{t}, \mathbf{B_j^{t_0}}[\mathbf{t}] \\ \forall j \in \mathcal{B}, t \in \mathcal{T}}} \quad c^{t_0} \cdot P_{\text{Subs}}^{t_0} \cdot P_{\text{BASE}} \cdot \Delta t + C_B \sum_{j \in \mathcal{B}} \left(P_{B,j}^{t_0}\right)^2 \cdot P_{\text{BASE}}^2 \cdot \Delta t$$
$$+ \frac{\rho}{2} \sum_{j \in \mathcal{B}} \sum_{t=1}^{T} \left(\mathbf{B_j^{t_0}}[\mathbf{t}] - \hat{\mathbf{B}}_\mathbf{j}[\mathbf{t}] + \mathbf{u_j^{t_0}}[\mathbf{t}]\right)^2 \tag{21}$$

Subject to:
Spatial Network Constraints (only for time t_0):

$$\text{Real power balance (substation):}\quad P_{\text{Subs}}^{t_0} - \sum_{(1,j)\in\mathcal{L}_1} P_{1j}^{t_0} = 0 \tag{22}$$

$$\text{Real power balance (nodes):}\quad P_{ij}^{t_0} - \sum_{(j,k)\in\mathcal{L}} P_{jk}^{t_0} = P_{B,j}^{t_0} + p_{D,j}^{t_0} - p_{L,j}^{t_0}, \quad \forall (i,j) \in \mathcal{L}, \tag{23}$$

$$\text{Reactive power balance (substation):}\quad Q_{\text{Subs}}^{t_0} - \sum_{(1,j)\in\mathcal{L}_1} Q_{1j}^{t_0} = 0 \tag{24}$$

$$\text{Reactive power balance (nodes):}\quad Q_{ij}^{t_0} - \sum_{(j,k)\in\mathcal{L}} Q_{jk}^{t_0} = q_{D,j}^{t_0} - q_{L,j}^{t_0}, \quad \forall (i,j) \in \mathcal{L}, \tag{25}$$

$$\text{KVL constraints:}\quad v_i^{t_0} - v_j^{t_0} = 2(r_{ij}P_{ij}^{t_0} + x_{ij}Q_{ij}^{t_0}), \quad \forall (i,j) \in \mathcal{L} \tag{26}$$

$$\text{Voltage limits:}\quad (V_{\min,j})^2 \le v_j^{t_0} \le (V_{\max,j})^2, \quad \forall j \in \mathcal{N} \tag{27}$$

$$\text{PV reactive limits:}\quad -\sqrt{(S_{D,j})^2 - (p_{D,j}^{t_0})^2} \le q_{D,j}^{t_0} \le \sqrt{(S_{D,j})^2 - (p_{D,j}^{t_0})^2}, \quad \forall j \in \mathcal{D} \tag{28}$$

Temporal Battery Constraints (entire horizon $t \in \{1, \dots, T\}$):

$$\text{Initial SOC:}\quad \mathbf{B_j^{t_0}[1]} = B_{0,j} - P_{B,j}^1 \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{29}$$

$$\text{SOC trajectory:}\quad \mathbf{B_j^{t_0}[t]} = \mathbf{B_j^{t_0}[t-1]} - P_{B,j}^t \cdot \Delta t, \quad \forall t \in \{2, \dots, T\},\ j \in \mathcal{B} \tag{30}$$

$$\text{SOC limits:}\quad \text{SOC}_{\min,j} \cdot B_{\text{rated},j} \le \mathbf{B_j^{t_0}[t]} \le \text{SOC}_{\max,j} \cdot B_{\text{rated},j}, \quad \forall t \in \mathcal{T},\ j \in \mathcal{B} \tag{31}$$

$$\text{Power limits:}\quad -P_{B,\text{rated},j} \le P_{B,j}^t \le P_{B,\text{rated},j}, \quad \forall t \in \mathcal{T},\ j \in \mathcal{B} \tag{32}$$

Key Formulation Notes:

- **Network variables** $(P_{\text{Subs}}^{t_0}, Q_{\text{Subs}}^{t_0}, P_{ij}^{t_0}, Q_{ij}^{t_0}, v_j^{t_0}, q_{D,j}^{t_0})$ are optimized *only* for time step t_0
- **Battery power** $P_{B,j}^t$ is optimized for the *entire* horizon $t \in \{1, \dots, T\}$
- **Local SOC trajectory** $\mathbf{B_j^{t_0}[t]}$ (blue) is computed for *all* time steps $t \in \{1, \dots, T\}$
- The ADMM consensus penalty compares the full local trajectory $\mathbf{B_j^{t_0}[t]}$ with the global master copy $\mathbf{\hat{B}_j[t]}$ (red)
- Each battery $j \in \mathcal{B}$ has its own local/global SOC variables and dual variables

2.6 Step 2: Consensus Update (Red Variables)

For each battery $j \in \mathcal{B}$ and each time period $t \in \mathcal{T}$:

$$\mathbf{\hat{B}_j[t]} = \text{clamp}\left(\frac{1}{T}\sum_{t_0=1}^{T}\left(\mathbf{B_j^{t_0}[t]} + \mathbf{u_j^{t_0}[t]}\right), \underline{B}_j, \overline{B}_j\right) \tag{33}$$

where $\underline{B}_j = \text{SOC}_{\min,j} \cdot B_{\text{rated},j}$ and $\overline{B}_j = \text{SOC}_{\max,j} \cdot B_{\text{rated},j}$.

2.7 Step 3: Dual Update (Green Variables)

For each battery $j \in \mathcal{B}$, each subproblem $t_0 \in \mathcal{T}$, and each time period $t \in \mathcal{T}$:

$$\mathbf{u_j^{t_0}}[\mathbf{t}] := \mathbf{u_j^{t_0}}[\mathbf{t}] + \left(\mathbf{B_j^{t_0}}[\mathbf{t}] - \mathbf{\hat{B}_j}[\mathbf{t}]\right) \tag{34}$$

2.8 Convergence Criteria

Primal Residual (Consensus Violation):

$$\|r^k\|_2 = \frac{1}{|\mathcal{B}|}\sqrt{\sum_{j\in\mathcal{B}}\sum_{t=1}^{T}\left(\frac{1}{T}\sum_{t_0=1}^{T}\mathbf{B_j^{t_0}}[\mathbf{t}] - \mathbf{\hat{B}_j}[\mathbf{t}]\right)^2} \leq \epsilon_{\text{pri}} \tag{35}$$

Dual Residual (Consensus Change):

$$\|s^k\|_2 = \frac{\rho}{|\mathcal{B}|}\sqrt{\sum_{j\in\mathcal{B}}\sum_{t=1}^{T}\left(\mathbf{\hat{B}_j^k}[\mathbf{t}] - \mathbf{\hat{B}_j^{k-1}}[\mathbf{t}]\right)^2} \leq \epsilon_{\text{dual}} \tag{36}$$

3 Copper Plate MPOPF with tADMM (Simplified Case)

3.1 Problem Overview

The Temporal ADMM (tADMM) algorithm decomposes the multi-period optimal power flow problem into T single-step subproblems, each corresponding to one time period. The hope is to enable parallel computation and improved scalability while still retaining solution optimality.

3.2 Variable Color Coding

Following the handwritten PDF formulation:

- $\mathbf{B^{t_0}}$: Local SOC variables for subproblem t_0 (blue)
- $\hat{\mathbf{B}}$: Global consensus SOC trajectory (red)
- $\mathbf{u^{t_0}}$: Local scaled dual variables for subproblem t_0 (green)

3.3 tADMM Algorithm Structure

The algorithm alternates between three update steps:

3.4 Step 1: Primal Update (Blue Variables) tADMM Optimization Model

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\min_{P_{\text{subs}}^{t_0}, P_B^{t_0}, \mathbf{B^{t_0}}} \quad C^{t_0} \cdot P_{\text{subs}}^{t_0} + C_B \cdot \left(P_B^{t_0}\right)^2 + \frac{\rho}{2} \left\| \mathbf{B^{t_0}} - \hat{\mathbf{B}} + \mathbf{u^{t_0}} \right\|_2^2 \tag{37}$$

Subject to SOC Dynamics for Entire Trajectory:

$$\mathbf{B^{t_0}[1]} = B_0 - P_B^{t_0} \cdot \Delta t \tag{38}$$

$$\mathbf{B^{t_0}[t]} = \mathbf{B^{t_0}[t-1]} - P_B^{t_0} \cdot \Delta t, \quad \forall t \in \{2, \ldots, T\} \tag{39}$$

$$P_{\text{subs}}^{t_0} + P_B^{t_0} = P_L[t_0] \tag{40}$$

$$-P_{B,R} \leq P_B^{t_0} \leq P_{B,R} \tag{41}$$

$$\underline{B} \leq \mathbf{B^{t_0}[t]} \leq \overline{B}, \quad \forall t \in \{1, \ldots, T\} \tag{42}$$

Key Formulation Notes:

- Each subproblem t_0 optimizes the battery power $P_B^{t_0}$ for *only* time step t_0
- However, the SOC trajectory $\mathbf{B^{t_0}[t]}$ is computed for *all* time steps $t \in \{1, \ldots, T\}$
- This ensures that the ADMM penalty term can compare the full trajectory $\mathbf{B^{t_0}}$ with the consensus $\hat{\mathbf{B}}$
- The power balance constraint is enforced only for the specific time t_0

3.5 Step 2: Consensus Update (Red Variables)

$$\hat{\mathbf{B}}[\mathbf{t}] = \text{clamp}\left(\frac{1}{T}\sum_{t_0=1}^{T}\left(\mathbf{B^{t_0}}[\mathbf{t}] + \mathbf{u^{t_0}}[\mathbf{t}]\right), \underline{B}, \overline{B}\right) \tag{43}$$

$$\forall t \in \{1, 2, \ldots, T-1\} \tag{44}$$

$$\hat{\mathbf{B}}[\mathbf{T}] = B_{T,\text{target}} \quad \text{(if terminal constraint exists)} \tag{45}$$

3.6 Step 3: Dual Update (Green Variables)

$$\mathbf{u^{t_0}}[\mathbf{t}] := \mathbf{u^{t_0}}[\mathbf{t}] + \left(\mathbf{B^{t_0}}[\mathbf{t}] - \hat{\mathbf{B}}[\mathbf{t}]\right) \tag{46}$$

$$\forall t_0 \in \{1, \ldots, T\}, \forall t \in \{1, \ldots, T\} \tag{47}$$

4 Convergence Criteria

The algorithm terminates when both residuals fall below specified thresholds:

4.1 Primal Residual (Consensus Violation)

$$\|r^k\|_2 = \left\|\text{vec}\left(\left\{\mathbf{B^{t_0}} - \hat{\mathbf{B}}\right\}_{t_0=1}^{T}\right)\right\|_2 \leq \epsilon_{\text{pri}} \tag{48}$$

4.2 Dual Residual (Consensus Change)

$$\|s^k\|_2 = \rho\left\|\hat{\mathbf{B}}^{\mathbf{k}} - \hat{\mathbf{B}}^{\mathbf{k-1}}\right\|_2 \leq \epsilon_{\text{dual}} \tag{49}$$

5 Algorithm Parameters

5.1 Objective Function Components

The tADMM objective function for each subproblem t_0 consists of three terms:

$$\text{Energy Cost:} \quad C^{t_0} \cdot P_{\text{subs}}^{t_0} \cdot \Delta t \tag{50}$$

$$\text{Battery Quadratic Cost:} \quad C_B \cdot \left(P_B^{t_0}\right)^2 \cdot \Delta t \tag{51}$$

$$\text{ADMM Penalty:} \quad \frac{\rho}{2}\left\|\mathbf{B^{t_0}} - \hat{\mathbf{B}} + \mathbf{u^{t_0}}\right\|_2^2 \tag{52}$$

Where:

- C^{t_0}: Energy price at time t_0 [\$/kWh]
- C_B: Battery quadratic cost coefficient [\$/kW^2/h] (typically $10^{-6} \times \min(C^t)$)
- ρ: ADMM penalty parameter (adaptive or fixed)

The battery quadratic cost term $C_B \cdot \left(P_B^{t_0}\right)^2$ serves as a regularization to:

1. Prevent excessive battery cycling
2. Encourage smoother power trajectories
3. Improve numerical conditioning of the optimization problem

5.2 Algorithmic Parameters

- **Penalty Parameter**: $\rho_{\text{init}} = 10.0$ (initial value if adaptive)
- **Primal Tolerance**: $\epsilon_{\text{pri}} = 10^{-5}$
- **Dual Tolerance**: $\epsilon_{\text{dual}} = 10^{-4}$
- **Maximum Iterations**: 1000

5.3 Adaptive Penalty Parameter (Boyd et al. 2011)

The penalty parameter ρ can be adjusted dynamically to balance convergence rates of primal and dual residuals. The adaptive scheme is activated by setting `adaptive_rho = true`.

Residual Balance Criterion:
At iteration k, if the residuals are imbalanced by more than factor μ:

$$\text{If } \|r^k\|_2 > \mu \cdot \|s^k\|_2 : \quad \text{Primal residual too large} \tag{53}$$

$$\text{If } \|s^k\|_2 > \mu \cdot \|r^k\|_2 : \quad \text{Dual residual too large} \tag{54}$$

Penalty Parameter Update (every N_{update} iterations):

$$\rho^{k+1} = \begin{cases} \min(\rho_{\max}, \tau_{\text{incr}} \cdot \rho^k) & \text{if } \|r^k\|_2 > \mu \cdot \|s^k\|_2 \\ \max(\rho_{\min}, \rho^k/\tau_{\text{decr}}) & \text{if } \|s^k\|_2 > \mu \cdot \|r^k\|_2 \\ \rho^k & \text{otherwise (residuals balanced)} \end{cases} \tag{55}$$

Dual Variable Rescaling:
When ρ changes from ρ^k to ρ^{k+1}, the scaled dual variables must be rescaled:

$$\mathbf{u}^{\mathbf{t_0}}[\mathbf{t}] \leftarrow \mathbf{u}^{\mathbf{t_0}}[\mathbf{t}] \cdot \frac{\rho^{k+1}}{\rho^k}, \quad \forall t_0, t \tag{56}$$

This maintains the relationship between scaled and unscaled dual variables: $\mathbf{u} = \lambda/\rho$.
Adaptive Parameters:

- **Balance factor**: $\mu = 10.0$ (higher for copper plate vs. 5.0 for SOCP)
- **Increase factor**: $\tau_{\text{incr}} = 2.0$
- **Decrease factor**: $\tau_{\text{decr}} = 2.0$
- **Bounds**: $\rho_{\min} = 0.1$, $\rho_{\max} = 10^6$
- **Update interval**: $N_{\text{update}} = 10$ iterations

Convergence Impact:
For the copper plate problem (24-hour horizon, 1000 kVA base):

- **Fixed** $\rho = 10.0$: Converges in ∼200–400 iterations
- **Adaptive** ρ: Converges in ∼98 iterations (2–4× speedup)
- Typical ρ trajectory: $10.0 \rightarrow 5.0 \rightarrow 2.5 \rightarrow 1.25$ (decreasing to balance residuals)

6 Copper Plate Localized tADMM (Reduced Network)

6.1 Overview

The copper plate formulation simplifies the multi-period OPF by removing network constraints entirely, reducing each time period's problem to pure energy arbitrage. Combined with localized temporal coupling (Section 1), this yields the most compact tADMM subproblems.

Key Simplifications:

- **Network**: No voltage, power flow, or line constraints
- **Spatial**: Single aggregated load, single aggregated battery
- **Temporal**: Localized coupling (2–3 time steps per subproblem)
- **Problem Size**: Each subproblem has only 3–5 decision variables

6.2 Local Time Sets (Copper Plate)

Same as LinDistFlow localized formulation:

$$\mathcal{T}_{\text{local}}^{t_0} = \begin{cases} \{1,2\} & \text{if } t_0 = 1 \\ \{t_0-1, t_0, t_0+1\} & \text{if } 2 \le t_0 \le T-1 \\ \{T-1, T\} & \text{if } t_0 = T \end{cases} \tag{57}$$

6.3 Primal Update (Copper Plate, Subproblem t_0)

For each subproblem $t_0 \in \{1, 2, \dots, T\}$:

$$\begin{aligned} \min_{\substack{P_{\text{subs}}^{t_0}, P_B^t, \mathbf{B^{t_0}[t]} \\ t \in \mathcal{T}_{\text{local}}^{t_0}}} \quad & c^{t_0} \cdot P_{\text{subs}}^{t_0} \cdot P_{\text{BASE}} \cdot \Delta t + C_B \cdot \left(P_B^{t_0}\right)^2 \cdot P_{\text{BASE}}^2 \cdot \Delta t \\ & + \frac{\rho}{2} \sum_{t \in \mathcal{T}_{\text{local}}^{t_0}} \left(\mathbf{B^{t_0}[t]} - \mathbf{\hat{B}[t]} + \mathbf{u^{t_0}[t]}\right)^2 \end{aligned} \tag{58}$$

Subject to:

Nodal Real Power Balance (only at t_0):

$$P_{\text{subs}}^{t_0} + P_B^{t_0} = P_L^{t_0} \tag{59}$$

Battery SOC Dynamics:

Case 1: $t_0 = 1$ (First period):

Local times: $\{1,2\}$, *Optimize:* $\mathbf{B^1[1]}, \mathbf{B^1[2]}, P_B^1, P_B^2, P_{subs}^1$

$$\mathbf{B^1[1]} = B_0 - P_B^1 \cdot \Delta t \tag{60}$$

$$\mathbf{B^1[2]} = \mathbf{B^1[1]} - P_B^2 \cdot \Delta t \tag{61}$$

Case 2: $2 \le t_0 \le T-1$ (Interior periods):

Local times: $\{t_0-1, t_0, t_0+1\}$, *Optimize:* $\mathbf{B^{t_0}[t_0-1]}, \mathbf{B^{t_0}[t_0]}, \mathbf{B^{t_0}[t_0+1]}, P_B^{t_0}, P_B^{t_0+1}, P_{subs}^{t_0}$

$$\mathbf{B^{t_0}[t_0]} = \mathbf{B^{t_0}[t_0-1]} - P_B^{t_0} \cdot \Delta t \tag{62}$$

$$\mathbf{B^{t_0}[t_0+1]} = \mathbf{B^{t_0}[t_0]} - P_B^{t_0+1} \cdot \Delta t \tag{63}$$

Case 3: $t_0 = T$ **(Last period):**

Local times: $\{T-1, T\}$, *Optimize:* $\mathbf{B^T[T-1]}, \mathbf{B^T[T]}, P_B^T, P_{subs}^T$

$$\mathbf{B^T[T]} = \mathbf{B^T[T-1]} - P_B^T \cdot \Delta t \tag{64}$$

Battery Bounds (all local times):

$$\underline{B} \leq \mathbf{B^{t_0}[t]} \leq \overline{B}, \quad \forall t \in \mathcal{T}_{\text{local}}^{t_0} \tag{65}$$

$$-P_{B,R} \leq P_B^t \leq P_{B,R}, \quad \forall t \in \{t_0, t_0+1\} \cap \mathcal{T}_{\text{local}}^{t_0} \tag{66}$$

6.4 Consensus Update (Localized Averaging)

For each time step $t \in \{1, 2, \ldots, T\}$, average over subproblems containing t:

$$\mathbf{\hat{B}[t]} = \begin{cases} \frac{1}{2}\left(\mathbf{B^1[1]} + \mathbf{u^1[1]} + \mathbf{B^2[1]} + \mathbf{u^2[1]}\right) & \text{if } t = 1 \\ \frac{1}{3}\sum_{t_0 \in \{t-1,t,t+1\}} \left(\mathbf{B^{t_0}[t]} + \mathbf{u^{t_0}[t]}\right) & \text{if } 2 \leq t \leq T-1 \\ \frac{1}{2}\left(\mathbf{B^{T-1}[T]} + \mathbf{u^{T-1}[T]} + \mathbf{B^T[T]} + \mathbf{u^T[T]}\right) & \text{if } t = T \end{cases} \tag{67}$$

Projection onto Feasible Set:

$$\mathbf{\hat{B}[t]} \leftarrow \max\left(\underline{B}, \min\left(\overline{B}, \mathbf{\hat{B}[t]}\right)\right) \tag{68}$$

6.5 Dual Update (Localized)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$ and $t \in \mathcal{T}_{\text{local}}^{t_0}$:

$$\mathbf{u^{t_0}[t]} \leftarrow \mathbf{u^{t_0}[t]} + \left(\mathbf{B^{t_0}[t]} - \mathbf{\hat{B}[t]}\right) \tag{69}$$

Number of Dual Updates per Iteration:

- Subproblem $t_0 = 1$: Update 2 dual variables ($\mathbf{u^1[1]}, \mathbf{u^1[2]}$)
- Subproblems $t_0 = 2, \ldots, T-1$: Update 3 dual variables each
- Subproblem $t_0 = T$: Update 2 dual variables ($\mathbf{u^T[T-1]}, \mathbf{u^T[T]}$)
- **Total**: $2 + 3(T-2) + 2 = 3T - 2$ dual variable updates

6.6 Convergence Criteria

Primal Residual (consensus violation):

$$\|r^k\|_2 = \frac{1}{T}\sqrt{\sum_{t_0=1}^{T}\sum_{t\in\mathcal{T}_{\text{local}}^{t_0}}\left(\mathbf{B^{t_0}[t]} - \mathbf{\hat{B}[t]}\right)^2} \tag{70}$$

Dual Residual (consensus change):

$$\|s^k\|_2 = \frac{\rho}{T}\sqrt{\sum_{t=1}^{T}\left(\mathbf{\hat{B}^k[t]} - \mathbf{\hat{B}^{k-1}[t]}\right)^2} \tag{71}$$

Stopping Criteria:

$$\text{Converged if:} \quad \|r^k\|_2 \le \epsilon_{\text{pri}} \quad \text{and} \quad \|s^k\|_2 \le \epsilon_{\text{dual}} \tag{72}$$

6.7 Complexity Comparison

Formulation	Variables/Subproblem	Coupling Variables	Total Storage
Global tADMM	T SOC + T power	T	T^2
Localized tADMM (LinDistFlow)	2–3 SOC + 2–3 power	2–3	$3T$
Localized Copper Plate	**2–3 SOC + 1–2 power**	**2–3**	**3T**

Table 1: Comparison of tADMM formulations (per-unit battery system)

Copper Plate Advantages:

- **Minimal subproblem size**: 3–5 decision variables per subproblem
- **No spatial coupling**: Pure temporal decomposition
- **Fast solves**: Each subproblem is a small LP/QP (<1ms with Gurobi)
- **Scalability**: Computational cost grows linearly with T (not T^2)

6.8 Example: 24-Hour Copper Plate Problem

Problem Data:

- Time periods: $T = 24$ (hourly intervals)
- Battery: $E_{\text{rated}} = 4000$ kWh, $P_{B,R} = 800$ kW
- Load: $P_L \in [800, 1200]$ kW (time-varying)
- Energy price: $c^t \in [0.08, 0.20]$ \$/kWh (sinusoidal)
- Quadratic battery cost: $C_B = 10^{-6} \times \min(c^t)$

Convergence Performance:

- **Fixed** $\rho = 10.0$: ∼200–300 iterations
- **Adaptive** ρ **(Boyd)**: ∼98 iterations (ρ: $10.0 \to 5.0 \to 2.5 \to 1.25$)
- **Solve time**: <5 seconds total (single-threaded Julia)
- **Objective gap vs. centralized**: $< 10^{-6}$ (numerically exact)

7 Appendix: Full Variable and Parameter Definitions

7.1 System Bases

$$\text{kV}_B = \frac{4.16}{\sqrt{3}} \text{ kV (phase-to-neutral)} \tag{73}$$
$$\text{kVA}_B = 1000 \text{ kVA} \tag{74}$$
$$P_{\text{BASE}} = 1000 \text{ kW} \tag{75}$$
$$E_{\text{BASE}} = 1000 \text{ kWh per hour} \tag{76}$$

7.2 SOC Bound Definitions

$$\underline{B} = \text{SOC}_{\min} \cdot E_{\text{Rated}} \tag{77}$$
$$\overline{B} = \text{SOC}_{\max} \cdot E_{\text{Rated}} \tag{78}$$

7.3 Physical Interpretation

- $P_B[t] > 0$: Battery discharging (providing power to the system)
- $P_B[t] < 0$: Battery charging (consuming power from the system)
- $B[t]$: Battery state of charge at the end of period t
- $\underline{B} = \text{SOC}_{\min} \cdot E_{\text{Rated}}$: Lower SOC bound
- $\overline{B} = \text{SOC}_{\max} \cdot E_{\text{Rated}}$: Upper SOC bound