Multi-Period Optimal Power Flow:
LinDistFlow & Copper Plate tADMM Formulations

1 LinDistFlow MPOPF with tADMM

1.1 Problem Overview

The Temporal ADMM (tADMM) algorithm decomposes the multi-period optimal power flow problem for distribution networks into T subproblems, each corresponding to one time period. This formulation uses the linearized DistFlow model to capture network physics including voltage drops and reactive power flows.

1.2 Variable Color Coding

- $\mathbf{B_j^{t_0}[t]}$: Local SOC variables for battery j in subproblem t_0, evaluated at time t (blue)
- $\mathbf{\hat{B}_j[t]}$: Global consensus SOC for battery j at time t (red)
- $\mathbf{u_j^{t_0}[t]}$: Local scaled dual variables for battery j in subproblem t_0, for time t (green)

1.3 Sets and Indices

- $\mathcal{N}$: Set of all nodes (buses)
- $\mathcal{L}$: Set of all branches (lines)
- $\mathcal{L}_1$: Set of branches connected to substation (node 1)
- $\mathcal{B}$: Set of nodes with batteries
- $\mathcal{D}$: Set of nodes with PV (DER)
- $\mathcal{T} = \{1, 2, \ldots, T\}$: Set of time periods
- $t_0 \in \mathcal{T}$: Index for a specific time period in tADMM decomposition
- $j \in \mathcal{N}$: Node index
- $(i, j) \in \mathcal{L}$: Branch from node i to node j

1.4 tADMM Algorithm Structure

The algorithm alternates between three update steps:

1.5 Step 1: Subproblem Update (Blue Variables)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\min_{\substack{P^{t_0}_{\text{Subs}}, Q^{t_0}_{\text{Subs}},\\ P^{t_0}_{ij}, Q^{t_0}_{ij}, v^{t_0}_j, q^{t_0}_{D,j},\\ P^t_{B,j}, \mathbf{B}^{\mathbf{t_0}}_{\mathbf{j}}[\mathbf{t}]\\ \forall j \in \mathcal{B}, t \in \mathcal{T}}} \quad c^{t_0} \cdot P^{t_0}_{\text{Subs}} \cdot P_{\text{BASE}} \cdot \Delta t + C_B \sum_{j \in \mathcal{B}} \left(P^{t_0}_{B,j}\right)^2 \cdot P^2_{\text{BASE}} \cdot \Delta t$$

$$+ \frac{\rho}{2} \sum_{j \in \mathcal{B}} \sum_{t=1}^{T} \left(\mathbf{B}^{\mathbf{t_0}}_{\mathbf{j}}[\mathbf{t}] - \hat{\mathbf{B}}_{\mathbf{j}}[\mathbf{t}] + \mathbf{u}^{\mathbf{t_0}}_{\mathbf{j}}[\mathbf{t}]\right)^2 \tag{1}$$

Subject to:
Spatial Network Constraints (only for time t_0):

$$\text{Real power balance (substation):} \quad P^{t_0}_{\text{Subs}} - \sum_{(1,j) \in \mathcal{L}_1} P^{t_0}_{1j} = 0 \tag{2}$$

$$\text{Real power balance (nodes):} \quad P^{t_0}_{ij} - \sum_{(j,k) \in \mathcal{L}} P^{t_0}_{jk} = P^{t_0}_{B,j} + p^{t_0}_{D,j} - p^{t_0}_{L,j}, \quad \forall (i,j) \in \mathcal{L}, \tag{3}$$

$$\text{Reactive power balance (substation):} \quad Q^{t_0}_{\text{Subs}} - \sum_{(1,j) \in \mathcal{L}_1} Q^{t_0}_{1j} = 0 \tag{4}$$

$$\text{Reactive power balance (nodes):} \quad Q^{t_0}_{ij} - \sum_{(j,k) \in \mathcal{L}} Q^{t_0}_{jk} = q^{t_0}_{D,j} - q^{t_0}_{L,j}, \quad \forall (i,j) \in \mathcal{L}, \tag{5}$$

$$\text{KVL constraints:} \quad v^{t_0}_i - v^{t_0}_j = 2(r_{ij} P^{t_0}_{ij} + x_{ij} Q^{t_0}_{ij}), \quad \forall (i,j) \in \mathcal{L} \tag{6}$$

$$\text{Voltage limits:} \quad (V_{\min,j})^2 \leq v^{t_0}_j \leq (V_{\max,j})^2, \quad \forall j \in \mathcal{N} \tag{7}$$

$$\text{PV reactive limits:} \quad -\sqrt{(S_{D,j})^2 - (p^{t_0}_{D,j})^2} \leq q^{t_0}_{D,j} \leq \sqrt{(S_{D,j})^2 - (p^{t_0}_{D,j})^2}, \quad \forall j \in \mathcal{D} \tag{8}$$

Temporal Battery Constraints (entire horizon $t \in \{1, \ldots, T\}$):

$$\text{Initial SOC:} \quad \mathbf{B}^{\mathbf{t_0}}_{\mathbf{j}}[\mathbf{1}] = B_{0,j} - P^1_{B,j} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{9}$$

$$\text{SOC trajectory:} \quad \mathbf{B}^{\mathbf{t_0}}_{\mathbf{j}}[\mathbf{t}] = \mathbf{B}^{\mathbf{t_0}}_{\mathbf{j}}[\mathbf{t-1}] - P^t_{B,j} \cdot \Delta t, \quad \forall t \in \{2, \ldots, T\}, j \in \mathcal{B} \tag{10}$$

$$\text{SOC limits:} \quad \text{SOC}_{\min,j} \cdot B_{\text{rated},j} \leq \mathbf{B}^{\mathbf{t_0}}_{\mathbf{j}}[\mathbf{t}] \leq \text{SOC}_{\max,j} \cdot B_{\text{rated},j}, \quad \forall t \in \mathcal{T}, j \in \mathcal{B} \tag{11}$$

$$\text{Power limits:} \quad -P_{B,\text{rated},j} \leq P^t_{B,j} \leq P_{B,\text{rated},j}, \quad \forall t \in \mathcal{T}, j \in \mathcal{B} \tag{12}$$

Key Formulation Notes:

- **Network variables** ($P^{t_0}_{\text{Subs}}, Q^{t_0}_{\text{Subs}}, P^{t_0}_{ij}, Q^{t_0}_{ij}, v^{t_0}_j, q^{t_0}_{D,j}$) are optimized *only* for time step t_0
- **Battery power** $P^t_{B,j}$ is optimized for the *entire* horizon $t \in \{1, \ldots, T\}$
- **Local SOC trajectory** $\mathbf{B}^{\mathbf{t_0}}_{\mathbf{j}}[\mathbf{t}]$ (blue) is computed for *all* time steps $t \in \{1, \ldots, T\}$

- The ADMM consensus penalty compares the full local trajectory $\mathbf{B_j^{to}[t]}$ with the global master copy $\mathbf{\hat{B}_j[t]}$ (red)
- Each battery $j \in \mathcal{B}$ has its own local/global SOC variables and dual variables

1.6 Step 2: Consensus Update (Red Variables)

For each battery $j \in \mathcal{B}$ and each time period $t \in \mathcal{T}$:

$$\mathbf{\hat{B}_j[t]} = \text{clamp}\left(\frac{1}{T}\sum_{t_0=1}^{T}\left(\mathbf{B_j^{to}[t]} + \mathbf{u_j^{to}[t]}\right), \underline{B}_j, \overline{B}_j\right) \tag{13}$$

where $\underline{B}_j = \text{SOC}_{\text{min},j} \cdot B_{\text{rated},j}$ and $\overline{B}_j = \text{SOC}_{\text{max},j} \cdot B_{\text{rated},j}$.

1.7 Step 3: Dual Update (Green Variables)

For each battery $j \in \mathcal{B}$, each subproblem $t_0 \in \mathcal{T}$, and each time period $t \in \mathcal{T}$:

$$\mathbf{u_j^{to}[t]} := \mathbf{u_j^{to}[t]} + \left(\mathbf{B_j^{to}[t]} - \mathbf{\hat{B}_j[t]}\right) \tag{14}$$

1.8 Convergence Criteria

Primal Residual (Consensus Violation):

$$\|r^k\|_2 = \sqrt{\sum_{j\in\mathcal{B}}\sum_{t=1}^{T}\left(\frac{1}{T}\sum_{t_0=1}^{T}\mathbf{B_j^{to}[t]} - \mathbf{\hat{B}_j[t]}\right)^2} \leq \epsilon_{\text{pri}} \tag{15}$$

Dual Residual (Consensus Change):

$$\|s^k\|_2 = \rho\sqrt{\sum_{j\in\mathcal{B}}\sum_{t=1}^{T}\left(\mathbf{\hat{B}_j^k[t]} - \mathbf{\hat{B}_j^{k-1}[t]}\right)^2} \leq \epsilon_{\text{dual}} \tag{16}$$

2 Copper Plate MPOPF with tADMM (Simplified Case)

2.1 Problem Overview

The Temporal ADMM (tADMM) algorithm decomposes the multi-period optimal power flow problem into T single-step subproblems, each corresponding to one time period. The hope is to enable parallel computation and improved scalability while still retaining solution optimality.

2.2 Variable Color Coding

Following the handwritten PDF formulation:

- $\mathbf{B^{t_0}}$: Local SOC variables for subproblem t_0 (blue)
- $\hat{\mathbf{B}}$: Global consensus SOC trajectory (red)
- $\mathbf{u^{t_0}}$: Local scaled dual variables for subproblem t_0 (green)

2.3 tADMM Algorithm Structure

The algorithm alternates between three update steps:

2.4 Step 1: Primal Update (Blue Variables) tADMM Optimization Model

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\min_{P_{\text{subs}}^{t_0}, P_B^{t_0}, \mathbf{B^{t_0}}} \quad C^{t_0} \cdot P_{\text{subs}}^{t_0} + C_B \cdot \left(P_B^{t_0}\right)^2 + \frac{\rho}{2} \left\| \mathbf{B^{t_0}} - \hat{\mathbf{B}} + \mathbf{u^{t_0}} \right\|_2^2 \tag{17}$$

Subject to SOC Dynamics for Entire Trajectory:

$$\mathbf{B^{t_0}[1]} = B_0 - P_B^{t_0} \cdot \Delta t \tag{18}$$

$$\mathbf{B^{t_0}[t]} = \mathbf{B^{t_0}[t-1]} - P_B^{t_0} \cdot \Delta t, \quad \forall t \in \{2, \ldots, T\} \tag{19}$$

$$P_{\text{subs}}^{t_0} + P_B^{t_0} = P_L[t_0] \tag{20}$$

$$-P_{B,R} \leq P_B^{t_0} \leq P_{B,R} \tag{21}$$

$$\underline{B} \leq \mathbf{B^{t_0}[t]} \leq \overline{B}, \quad \forall t \in \{1, \ldots, T\} \tag{22}$$

Key Formulation Notes:

- Each subproblem t_0 optimizes the battery power $P_B^{t_0}$ for *only* time step t_0
- However, the SOC trajectory $\mathbf{B^{t_0}[t]}$ is computed for *all* time steps $t \in \{1, \ldots, T\}$
- This ensures that the ADMM penalty term can compare the full trajectory $\mathbf{B^{t_0}}$ with the consensus $\hat{\mathbf{B}}$
- The power balance constraint is enforced only for the specific time t_0

2.5 Step 2: Consensus Update (Red Variables)

$$\hat{\mathbf{B}}[\mathbf{t}] = \text{clamp}\left(\frac{1}{T}\sum_{t_0=1}^{T}\left(\mathbf{B^{t_0}}[\mathbf{t}] + \mathbf{u^{t_0}}[\mathbf{t}]\right), \underline{B}, \overline{B}\right) \tag{23}$$

$$\forall t \in \{1, 2, \ldots, T-1\} \tag{24}$$

$$\hat{\mathbf{B}}[\mathbf{T}] = B_{T,\text{target}} \quad \text{(if terminal constraint exists)} \tag{25}$$

2.6 Step 3: Dual Update (Green Variables)

$$\mathbf{u^{t_0}}[\mathbf{t}] := \mathbf{u^{t_0}}[\mathbf{t}] + \left(\mathbf{B^{t_0}}[\mathbf{t}] - \hat{\mathbf{B}}[\mathbf{t}]\right) \tag{26}$$

$$\forall t_0 \in \{1, \ldots, T\}, \forall t \in \{1, \ldots, T\} \tag{27}$$

3 Convergence Criteria

The algorithm terminates when both residuals fall below specified thresholds:

3.1 Primal Residual (Consensus Violation)

$$\|r^k\|_2 = \left\|\text{vec}\left(\left\{\mathbf{B^{t_0}} - \hat{\mathbf{B}}\right\}_{t_0=1}^{T}\right)\right\|_2 \leq \epsilon_{\text{pri}} \tag{28}$$

3.2 Dual Residual (Consensus Change)

$$\|s^k\|_2 = \rho\left\|\hat{\mathbf{B}}^{\mathbf{k}} - \hat{\mathbf{B}}^{\mathbf{k-1}}\right\|_2 \leq \epsilon_{\text{dual}} \tag{29}$$

4 Final Solution Recovery

After convergence, the final power schedules are reconstructed from the consensus trajectory:

$$P_B^t = \begin{cases} \frac{-\left(\hat{\mathbf{B}}[\mathbf{t}] - B_0\right)}{\Delta t} & \text{if } t = 1 \\ \frac{-\left(\hat{\mathbf{B}}[\mathbf{t}] - \hat{\mathbf{B}}[\mathbf{t-1}]\right)}{\Delta t} & \text{if } t > 1 \end{cases} \tag{30}$$

$$P_{\text{Subs}}^t = P_L[t] - P_B^t \tag{31}$$

The final objective value includes both energy and battery costs:

$$J_{\text{total}} = \sum_{t=1}^{T}\left[C^t \cdot P_{\text{Subs}}^t \cdot \Delta t + C_B \cdot \left(P_B^t\right)^2 \cdot \Delta t\right] \tag{32}$$

5 Algorithm Parameters

5.1 Objective Function Components

The tADMM objective function for each subproblem t_0 consists of three terms:

$$\text{Energy Cost:} \quad C^{t_0} \cdot P_{\text{subs}}^{t_0} \cdot \Delta t \tag{33}$$

$$\text{Battery Quadratic Cost:} \quad C_B \cdot \left(P_B^{t_0}\right)^2 \cdot \Delta t \tag{34}$$

$$\text{ADMM Penalty:} \quad \frac{\rho}{2} \left\| \mathbf{B^{t_0}} - \hat{\mathbf{B}} + \mathbf{u^{t_0}} \right\|_2^2 \tag{35}$$

Where:

- C^{t_0}: Energy price at time t_0 [\$/kWh]
- C_B: Battery quadratic cost coefficient [\$/kW^2/h] (typically $10^{-6} \times \min(C^t)$)
- ρ: ADMM penalty parameter

The battery quadratic cost term $C_B \cdot \left(P_B^{t_0}\right)^2$ serves as a regularization to:

1. Prevent excessive battery cycling
2. Encourage smoother power trajectories
3. Improve numerical conditioning of the optimization problem

5.2 Algorithmic Parameters

- **Penalty Parameter**: ρ (typically 0.1 to 10.0)
- **Primal Tolerance**: $\epsilon_{\text{pri}} = 10^{-3}$
- **Dual Tolerance**: $\epsilon_{\text{dual}} = 10^{-3}$
- **Maximum Iterations**: 1000

6 Appendix: Full Variable and Parameter Definitions

6.1 System Bases

$$\text{kV}_B = \frac{4.16}{\sqrt{3}} \text{ kV (phase-to-neutral)} \tag{36}$$

$$\text{kVA}_B = 1000 \text{ kVA} \tag{37}$$

$$P_{\text{BASE}} = 1000 \text{ kW} \tag{38}$$

$$E_{\text{BASE}} = 1000 \text{ kWh per hour} \tag{39}$$

6.2 SOC Bound Definitions

$$\underline{B} = \text{SOC}_{\min} \cdot E_{\text{Rated}} \tag{40}$$

$$\overline{B} = \text{SOC}_{\max} \cdot E_{\text{Rated}} \tag{41}$$

6.3 Physical Interpretation

- $P_B[t] > 0$: Battery discharging (providing power to the system)
- $P_B[t] < 0$: Battery charging (consuming power from the system)
- $B[t]$: Battery state of charge at the end of period t
- $\underline{B} = \text{SOC}_{\text{min}} \cdot E_{\text{Rated}}$: Lower SOC bound
- $\overline{B} = \text{SOC}_{\text{max}} \cdot E_{\text{Rated}}$: Upper SOC bound