Multi-Period Optimal Power Flow with tADMM: Temporally Localized ADMM for Branch Flow Model SOCP (BFM-NL)

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1.1 Overview

The Temporal ADMM (tADMM) algorithm decomposes the multi-period optimal power flow problem into T subproblems, one for each time period. This formulation uses **localized temporal coupling** where each subproblem t_0 only couples with its immediate time neighbors $(t_0 - 1, t_0, t_0 + 1)$, rather than coupling with all T time steps.

Key Benefits:

- **Reduced Coupling**: Each subproblem exchanges 2–3 $\times |\mathcal{B}|$ variables (not $T \times |\mathcal{B}|$)
- **Memory Efficiency**: $(3T - 2) \times |\mathcal{B}|$ total storage for localized coupling
- **Parallel Computation**: All T subproblems can be solved in parallel
- **Network Physics**: Branch Flow Model (BFM-NL) with SOCP relaxation for exact voltage and reactive power

1.2 Problem Definition

Sets and Indices:

- $\mathcal{N}$: Set of all nodes (buses) in the distribution network
- $\mathcal{L}$: Set of all branches (lines), (i, j) denotes branch from node i to j
- $\mathcal{L}_1$: Set of branches connected to substation (node 1)
- $\mathcal{B} \subseteq \mathcal{N}$: Set of nodes with batteries (energy storage)
- $\mathcal{D} \subseteq \mathcal{N}$: Set of nodes with solar PV (distributed energy resources)
- $\mathcal{T} = \{1, 2, \ldots, T\}$: Set of time periods (e.g., 24 hours, 96 15-min intervals)
- $t_0 \in \mathcal{T}$: Index for the control time of subproblem t_0

Variable Color Coding:

- $\mathbf{B_j^{t_0}[t]}$: Local SOC variables (blue) - battery j in subproblem t_0, evaluated at time t
- $\mathbf{\hat{B}_j[t]}$: Global consensus SOC (red) - agreed-upon SOC for battery j at time t
- $\mathbf{u_j^{t_0}[t]}$: Scaled dual variables (green) - enforces consensus between local and global SOC

1.3 Local Time Sets

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$, define the local time set:

$$\mathcal{T}_{\text{local}}^{t_0} = \begin{cases} \{1, 2\} & \text{if } t_0 = 1 \text{ (boundary)} \\ \{t_0 - 1, t_0, t_0 + 1\} & \text{if } 2 \leq t_0 \leq T - 1 \text{ (interior)} \\ \{T - 1, T\} & \text{if } t_0 = T \text{ (boundary)} \end{cases} \tag{1}$$

1.4 Step 1: Primal Update (Subproblem Optimization)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\min_{\substack{P_{\text{Subs}}^{t_0}, Q_{\text{Subs}}^{t_0},\\ P_{ij}^{t_0}, Q_{ij}^{t_0}, v_j^{t_0}, \ell_{ij}^{t_0},\\ q_{D,j}^{t_0}, P_{B,j}^{t}, \mathbf{B_j^{t_0}[t]}\\ \forall (i,j)\in\mathcal{L},\, j\in\mathcal{N}\cup\mathcal{D}\cup\mathcal{B},\\ t\in\mathcal{T}_{\text{local}}^{t_0}}} c^{t_0} \cdot P_{\text{Subs}}^{t_0} \cdot P_{\text{BASE}} \cdot \Delta t + C_B \sum_{j\in\mathcal{B}} \left(P_{B,j}^{t_0}\right)^2 \cdot P_{\text{BASE}}^2 \cdot \Delta t$$
$$+ \frac{\rho}{2} \sum_{j\in\mathcal{B}} \sum_{t\in\mathcal{T}_{\text{local}}^{t_0}} \left(\mathbf{B_j^{t_0}[t]} - \hat{\mathbf{B}}_\mathbf{j}\mathbf{[t]} + \mathbf{u_j^{t_0}[t]}\right)^2 \tag{2}$$

Subject to:
Spatial Network Constraints (Branch Flow Model SOCP, only for control time t_0):

$$\text{Power balance (substation):} \quad P_{\text{Subs}}^{t_0} - \sum_{(1,j)\in\mathcal{L}_1} P_{1j}^{t_0} = 0 \tag{3}$$

$$Q_{\text{Subs}}^{t_0} - \sum_{(1,j)\in\mathcal{L}_1} Q_{1j}^{t_0} = 0 \tag{4}$$

$$\text{Power balance (nodes):} \quad P_{ij}^{t_0} - \sum_{(j,k)\in\mathcal{L}} P_{jk}^{t_0} = P_{B,j}^{t_0} + p_{D,j}^{t_0} - p_{L,j}^{t_0}, \quad \forall (i,j)\in\mathcal{L} \tag{5}$$

$$Q_{ij}^{t_0} - \sum_{(j,k)\in\mathcal{L}} Q_{jk}^{t_0} = q_{D,j}^{t_0} - q_{L,j}^{t_0}, \quad \forall (i,j)\in\mathcal{L} \tag{6}$$

$$\text{KVL (voltage drop):} \quad v_j^{t_0} = v_i^{t_0} - 2(r_{ij}P_{ij}^{t_0} + x_{ij}Q_{ij}^{t_0}) + \ell_{ij}^{t_0}, \quad \forall (i,j)\in\mathcal{L} \tag{7}$$

$$\text{SOCP relaxation:} \quad \ell_{ij}^{t_0} \cdot v_i^{t_0} \geq (P_{ij}^{t_0})^2 + (Q_{ij}^{t_0})^2, \quad \forall (i,j)\in\mathcal{L} \tag{8}$$

$$\text{Voltage limits:} \quad (V_{\min,j})^2 \leq v_j^{t_0} \leq (V_{\max,j})^2, \quad \forall j\in\mathcal{N} \tag{9}$$

$$\text{Substation voltage:} \quad v_1^{t_0} = (V_{\text{nom}})^2 \tag{10}$$

$$\text{PV reactive limits:} \quad (q_{D,j}^{t_0})^2 \leq (S_{D,j})^2 - (p_{D,j}^{t_0})^2, \quad \forall j\in\mathcal{D} \tag{11}$$

Note: Network variables ($P_{\text{Subs}}^{t_0}, Q_{\text{Subs}}^{t_0}, P_{ij}^{t_0}, Q_{ij}^{t_0}, v_j^{t_0}, \ell_{ij}^{t_0}, q_{D,j}^{t_0}$) are optimized only for the control time t_0, not for neighbor times.

Temporal Battery Constraints:

Case 1: First time period ($t_0 = 1$):

Local times: $\{1, 2\}$, *Decision variables:* $\mathbf{B_j^1[1]}, \mathbf{B_j^1[2]}$, *Dual variables:* $\mathbf{u_j^1[1]}$ *only*

$$\text{SOC trajectory } t = 0 \to t = 1: \quad \mathbf{B_j^1[1]} = B_{0,j} - P_{B,j}^1 \cdot \Delta t, \quad \forall j\in\mathcal{B} \tag{12}$$

$$\text{SOC trajectory } t = 1 \to t = 2: \quad \mathbf{B_j^1[2]} = \mathbf{B_j^1[1]} - P_{B,j}^2 \cdot \Delta t, \quad \forall j\in\mathcal{B} \tag{13}$$

Note: $B_{0,j}$ is a parameter (initial condition), not a decision variable. Only $\mathbf{B_j^1[1]}$ participates in consensus (with dual $\mathbf{u_j^1[1]}$). Variable $\mathbf{B_j^1[2]}$ exists for penalty computation but does not participate in consensus at $t = 1$.

Case 2: Interior time periods ($2 \leq t_0 \leq T-1$):

Local times: $\{t_0 - 1, t_0, t_0 + 1\}$, *Decision variables:* $\mathbf{B_j^{t_0}[t_0 - 1]}, \mathbf{B_j^{t_0}[t_0]}, \mathbf{B_j^{t_0}[t_0 + 1]}$

Dual variables: $\mathbf{u_j^{t_0}[t_0]}$ *only*

$$\text{SOC trajectory } t_0-1 \to t_0: \quad \mathbf{B_j^{t_0}[t_0]} = \mathbf{B_j^{t_0}[t_0-1]} - P_{B,j}^{t_0} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{14}$$

$$\text{SOC trajectory } t_0 \to t_0+1: \quad \mathbf{B_j^{t_0}[t_0+1]} = \mathbf{B_j^{t_0}[t_0]} - P_{B,j}^{t_0+1} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{15}$$

Note: Only $\mathbf{B_j^{t_0}[t_0]}$ participates in consensus update at time t_0 (with dual $\mathbf{u_j^{t_0}[t_0]}$). Variables $\mathbf{B_j^{t_0}[t_0-1]}$ and $\mathbf{B_j^{t_0}[t_0+1]}$ are used for penalty terms but do not generate new dual variables in this subproblem.

Case 3: Last time period ($t_0 = T$):

Local times: $\{T-1, T\}$, *Decision variables:* $\mathbf{B_j^T[T-1]}, \mathbf{B_j^T[T]}$, *Dual variables:* $\mathbf{u_j^T[T]}$ *only*

$$\text{SOC trajectory } T-1 \to T: \quad \mathbf{B_j^T[T]} = \mathbf{B_j^T[T-1]} - P_{B,j}^{T} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{16}$$

Note: Only $\mathbf{B_j^T[T]}$ participates in consensus at $t = T$ (with dual $\mathbf{u_j^T[T]}$). Variable $\mathbf{B_j^T[T-1]}$ is for penalty only.

SOC and Power Limits:

$$\begin{aligned} \text{SOC limits:} \quad & \text{SOC}_{\min,j} \cdot B_{\text{rated},j} \le \mathbf{B_j^{t_0}[t]} \le \text{SOC}_{\max,j} \cdot B_{\text{rated},j}, \\ & \forall t \in \mathcal{T}_{\text{local}}^{t_0},\ j \in \mathcal{B} \end{aligned} \tag{17}$$

$$\text{Power limits:} \quad -P_{B,\text{rated},j} \le P_{B,j}^{t} \le P_{B,\text{rated},j}, \quad \forall t \in \mathcal{T}_{\text{local}}^{t_0},\ j \in \mathcal{B} \tag{18}$$

Note: Power limits must be enforced for *all* times appearing in constraints, not just t_0. For interior times: $P_{B,j}^{t_0-1}, P_{B,j}^{t_0}, P_{B,j}^{t_0+1}$ all need limits.

1.5 Step 2: Consensus Update (Local Averaging)

For each battery $j \in \mathcal{B}$ and each time $t \in \mathcal{T}$, average only over subproblems where t is the *active* time:

Consensus at $t = 1$:

$$\hat{\mathbf{B}}_{\mathbf{j}}[\mathbf{1}] = \text{clamp}\left(\frac{1}{2}\left(\mathbf{B_j^1[1]} + \mathbf{u_j^1[1]} + \mathbf{B_j^2[1]} + \mathbf{u_j^2[1]}\right), \underline{B}_j, \overline{B}_j\right) \tag{19}$$

Contributors: Subproblems $t_0 = 1$ and $t_0 = 2$ (only these have $t = 1$ as decision variable with dual)

Consensus at interior times ($2 \le t \le T-1$):

$$\hat{\mathbf{B}}_{\mathbf{j}}[\mathbf{t}] = \text{clamp}\left(\frac{1}{3}\left(\mathbf{B_j^{t-1}[t]} + \mathbf{u_j^{t-1}[t]} + \mathbf{B_j^{t}[t]} + \mathbf{u_j^{t}[t]} + \mathbf{B_j^{t+1}[t]} + \mathbf{u_j^{t+1}[t]}\right), \underline{B}_j, \overline{B}_j\right) \tag{20}$$

Contributors: Subproblems $t_0 \in \{t-1, t, t+1\}$ (each has t as its active control time)

Consensus at $t = T$:

$$\hat{\mathbf{B}}_{\mathbf{j}}[\mathbf{T}] = \text{clamp}\left(\frac{1}{2}\left(\mathbf{B_j^{T-1}[T]} + \mathbf{u_j^{T-1}[T]} + \mathbf{B_j^{T}[T]} + \mathbf{u_j^{T}[T]}\right), \underline{B}_j, \overline{B}_j\right) \tag{21}$$

Contributors: Subproblems $t_0 = T-1$ and $t_0 = T$ (only these have $t = T$ as decision variable with dual)

General form:

$$\mathcal{N}_t = \{t_0 \in \mathcal{T} : t \text{ is the active control time in subproblem } t_0\} \tag{22}$$

$$= \begin{cases} \{1,2\} & \text{if } t = 1 \\ \{t-1, t, t+1\} & \text{if } 2 \le t \le T-1 \\ \{T-1, T\} & \text{if } t = T \end{cases} \tag{23}$$

1.6 Step 3: Dual Update (All Local Times)

For each subproblem $t_0 \in \mathcal{T}$ and battery $j \in \mathcal{B}$, update dual variables for **all local times** $t \in \mathcal{T}_{\text{local}}^{t_0}$:
Subproblem $t_0 = 1$ (local times: $\{1, 2\}$):

$$\mathbf{u_j^1[1]} := \mathbf{u_j^1[1]} + \left(\mathbf{B_j^1[1]} - \mathbf{\hat{B}_j[1]}\right) \tag{24}$$

$$\mathbf{u_j^1[2]} := \mathbf{u_j^1[2]} + \left(\mathbf{B_j^1[2]} - \mathbf{\hat{B}_j[2]}\right) \tag{25}$$

Subproblem t_0 ($2 \leq t_0 \leq T-1$, local times: $\{t_0 - 1, t_0, t_0 + 1\}$):

$$\mathbf{u_j^{t_0}[t_0 - 1]} := \mathbf{u_j^{t_0}[t_0 - 1]} + \left(\mathbf{B_j^{t_0}[t_0 - 1]} - \mathbf{\hat{B}_j[t_0 - 1]}\right) \tag{26}$$

$$\mathbf{u_j^{t_0}[t_0]} := \mathbf{u_j^{t_0}[t_0]} + \left(\mathbf{B_j^{t_0}[t_0]} - \mathbf{\hat{B}_j[t_0]}\right) \tag{27}$$

$$\mathbf{u_j^{t_0}[t_0 + 1]} := \mathbf{u_j^{t_0}[t_0 + 1]} + \left(\mathbf{B_j^{t_0}[t_0 + 1]} - \mathbf{\hat{B}_j[t_0 + 1]}\right) \tag{28}$$

Subproblem $t_0 = T$ (local times: $\{T - 1, T\}$):

$$\mathbf{u_j^T[T - 1]} := \mathbf{u_j^T[T - 1]} + \left(\mathbf{B_j^T[T - 1]} - \mathbf{\hat{B}_j[T - 1]}\right) \tag{29}$$

$$\mathbf{u_j^T[T]} := \mathbf{u_j^T[T]} + \left(\mathbf{B_j^T[T]} - \mathbf{\hat{B}_j[T]}\right) \tag{30}$$

General Form:

$$\text{For each } t \in \mathcal{T}_{\text{local}}^{t_0}: \quad \mathbf{u_j^{t_0}[t]} := \mathbf{u_j^{t_0}[t]} + \left(\mathbf{B_j^{t_0}[t]} - \mathbf{\hat{B}_j[t]}\right) \tag{31}$$

Critical Note: Each subproblem t_0 maintains dual variables for *all* its local times $\mathcal{T}_{\text{local}}^{t_0}$, which is 2 or 3 times per battery. This ensures that each time t has duals from all subproblems that include it in their penalty terms, enabling proper consensus averaging in Step 2.

1.7 Convergence Criteria (Exact Implementation)

Primal Residual (Consensus Violation):

Only count residuals at active control times (where dual variables exist):

$$r_{\text{values}} = \left\{\mathbf{B_j^{t_0,k}[t_0]} - \mathbf{\hat{B}_j^k[t_0]} : t_0 \in \mathcal{T},\ j \in \mathcal{B}\right\} \tag{32}$$

$$\|r^k\|_2 = \frac{\|\mathbf{r}_{\text{values}}\|_2}{\sqrt{|\mathbf{r}_{\text{values}}|}} = \frac{1}{\sqrt{T \cdot |\mathcal{B}|}} \left(\sum_{t_0=1}^{T} \sum_{j \in \mathcal{B}} \left(\mathbf{B_j^{t_0,k}[t_0]} - \mathbf{\hat{B}_j^k[t_0]}\right)^2\right)^{1/2} \tag{33}$$

Note: We only compute residual at time t_0 for each subproblem t_0 (the active control time where dual $\mathbf{u_j^{t_0}[t_0]}$ exists). The normalization is by $\sqrt{T \cdot |\mathcal{B}|}$, the number of active consensus variables.

Dual Residual (Consensus Variable Change):

Measures how much the consensus variables $\mathbf{\hat{B}}$ changed from previous iteration:

$$\mathbf{s}_j^k = \mathbf{\hat{B}_j^k} - \mathbf{\hat{B}_j^{k-1}} = \begin{bmatrix} \mathbf{\hat{B}_j^k[1]} - \mathbf{\hat{B}_j^{k-1}[1]} \\ \vdots \\ \mathbf{\hat{B}_j^k[T]} - \mathbf{\hat{B}_j^{k-1}[T]} \end{bmatrix} \in \mathbb{R}^T \tag{34}$$

$$\|s^k\|_2 = \frac{\rho^k}{|\mathcal{B}|} \left\| \begin{bmatrix} \mathbf{s}_1^k \\ \vdots \\ \mathbf{s}_{|\mathcal{B}|}^k \end{bmatrix} \right\|_2 = \frac{\rho^k}{|\mathcal{B}|} \left(\sum_{j \in \mathcal{B}} \sum_{t=1}^{T} \left(\mathbf{\hat{B}_j^k[t]} - \mathbf{\hat{B}_j^{k-1}[t]}\right)^2\right)^{1/2} \tag{35}$$

Note: The dual residual is scaled by ρ and normalized by $|\mathcal{B}|$ (number of batteries), not by $\sqrt{T \cdot |\mathcal{B}|}$.

Convergence Condition:

$$\text{Converged if } \|r^k\|_2 \leq \epsilon_{\text{pri}} \quad \text{and} \quad \|s^k\|_2 \leq \epsilon_{\text{dual}} \tag{36}$$

with $\epsilon_{\text{pri}} = 10^{-5}$ and $\epsilon_{\text{dual}} = 10^{-4}$.

1.8 Localized tADMM Complexity

For $T = 96$, $|\mathcal{B}| = 26$ batteries (24-hour horizon, 15-min intervals):

- **SOC Variables per Subproblem:**
  - Boundary periods (first/last, $t_0 \in \{1, 96\}$): $2 \times 26 = 52$ SOC variables
  - Interior periods ($2 \leq t_0 \leq 95$): $3 \times 26 = 78$ SOC variables
  - Average: ≈ 77 SOC variables per subproblem
- **Total Dual Variable Storage:**
  - $(3T - 2) \times |\mathcal{B}| = (3 \times 96 - 2) \times 26 = 286 \times 26 = 7,436$ dual variables
- **Communication per Iteration:**
  - Each subproblem sends: 2–3 $\times$ 26 = 52–78 local SOC values to consensus
  - Each subproblem receives: 2–3 $\times$ 26 = 52–78 consensus SOC values
  - Total consensus exchanges: $96 \times 77 = 7,392$ SOC values per iteration
- **Network Constraints per Subproblem:**
  - Power balance: $|\mathcal{N}|$ real + $|\mathcal{N}|$ reactive
  - KVL: $|\mathcal{L}|$ constraints
  - SOCP cones: $|\mathcal{L}|$ second-order cone constraints
  - Voltage limits: $2|\mathcal{N}|$ bounds
  - PV reactive limits: $2|\mathcal{D}|$ cone constraints

2 Adaptive Penalty Parameter: Two-Phase Strategy with Watchdog

The penalty parameter ρ is adjusted dynamically using a sophisticated two-phase strategy that prioritizes primal convergence initially, then balances both residuals. The scheme is activated by setting `adaptive_rho = true`.

2.1 Residual Calculations (Exact Implementation)

Primal Residual (Consensus Violation):

$$\|r^k\|_2 = \frac{1}{\sqrt{N_{\text{active}}}} \left(\sum_{t_0=1}^{T} \sum_{j \in \mathcal{B}} \left(\mathbf{B}_{\mathbf{j}}^{\mathbf{t_0,k}}[\mathbf{t_0}] - \hat{\mathbf{B}}_{\mathbf{j}}^{\mathbf{k}}[\mathbf{t_0}] \right)^2 \right)^{1/2} \tag{37}$$

where $N_{\text{active}} = T \times |\mathcal{B}|$ is the number of active time-battery pairs (only counting where duals exist).

Dual Residual (Consensus Variable Change):

$$\|s^k\|_2 = \frac{\rho^k}{|\mathcal{B}|} \left(\sum_{j \in \mathcal{B}} \sum_{t=1}^{T} \left(\hat{\mathbf{B}}_{\mathbf{j}}^{\mathbf{k}}[\mathbf{t}] - \hat{\mathbf{B}}_{\mathbf{j}}^{\mathbf{k-1}}[\mathbf{t}] \right)^2 \right)^{1/2} \tag{38}$$

2.2 Phase 1: Aggressive Primal Convergence (Until $\|r\| \leq \epsilon_{\text{pri}}$)

Objective: Drive primal residual below threshold by only increasing ρ. Never decrease ρ to avoid slowing consensus formation.

Update Rules (every $N_{\text{update}} = 5$ iterations):

$$\rho^{k+1} = \begin{cases} \min(\rho_{\max}, \tau_{\text{incr}} \cdot \rho^k) & \text{if } \|r^k\| > \mu \cdot \|s^k\| \\ \min(\rho_{\max}, \tau_{\text{nudge}} \cdot \rho^k) & \text{if } \|r^k\| > \epsilon_{\text{pri}} \text{ and no } \rho \text{ change for } N_{\text{stall}} \text{ iters} \\ \rho^k & \text{otherwise (do not decrease)} \end{cases} \tag{39}$$

Phase Transition:

$$\text{Switch to Phase 2 if } \|r^k\| \leq \epsilon_{\text{pri}} \text{ (primal converged once)} \tag{40}$$

2.3 Phase 2: Bidirectional Adaptation (After primal convergence)

Objective: Balance primal and dual residuals, allowing both increases and decreases of ρ.

Update Rules (every $N_{\text{update}} = 5$ iterations):

$$\rho^{k+1} = \begin{cases} \min(\rho_{\max}, \tau_{\text{incr}} \cdot \rho^k) & \text{if } \|r^k\| > \mu \cdot \|s^k\| \\ \max(\rho_{\min}, \rho^k / \tau_{\text{decr}}) & \text{if } \|s^k\| > \mu \cdot \|r^k\| \textbf{ and } \|r^k\| \leq \epsilon_{\text{pri}} \\ \rho^k & \text{otherwise (residuals balanced)} \end{cases} \tag{41}$$

Watchdog Protection: The decrease condition includes $\|r^k\| \leq \epsilon_{\text{pri}}$ to prevent premature decreases when primal consensus breaks down after Phase 1 $\rightarrow$ 2 transition.

2.4 Primal Residual Watchdog (All Phases)

Motivation: Detects when primal residual stays elevated for many consecutive iterations, indicating insufficient penalty to enforce consensus.

Hysteresis-Based Counter (runs every iteration):

$$\text{counter}^{k+1} = \begin{cases} \text{counter}^k + 1 & \text{if } \|r^k\| > 2\epsilon_{\text{pri}} \\ 0 & \text{if } \|r^k\| < 0.5\epsilon_{\text{pri}} \\ \text{counter}^k & \text{if } 0.5\epsilon_{\text{pri}} \leq \|r^k\| \leq 2\epsilon_{\text{pri}} \text{ (hysteresis zone)} \end{cases} \tag{42}$$

Watchdog Trigger:

$$\text{If counter}^k \geq N_{\text{watchdog}} = 20: \quad \rho^{k+1} = \min(\rho_{\max}, \tau_{\text{watchdog}} \cdot \rho^k), \quad \text{counter}^{k+1} = 0 \tag{43}$$

The hysteresis prevents false resets from oscillations around ϵ_{pri}, while the 20-iteration window ensures sustained high residuals trigger aggressive action.

2.5 Dual Variable Rescaling

When ρ changes from ρ^k to ρ^{k+1}, the scaled dual variables must be rescaled:

$$\mathbf{u}_{\mathbf{j}}^{\mathbf{t_0}}[\mathbf{t}] \leftarrow \mathbf{u}_{\mathbf{j}}^{\mathbf{t_0}}[\mathbf{t}] \cdot \frac{\rho^k}{\rho^{k+1}}, \quad \forall t_0 \in \mathcal{T}, \, j \in \mathcal{B}, \, t \in \mathcal{T}_{\text{local}}^{t_0} \tag{44}$$

This maintains the relationship between scaled and unscaled dual variables: $\mathbf{u} = \lambda/\rho$.

2.6 Complete Parameter Summary

Parameter	Value (SOCP tADMM)
Tolerances	
Primal tolerance ϵ_{pri}	1×10^{-5}
Dual tolerance ϵ_{dual}	1×10^{-4} (relaxed for faster convergence)
Standard Adaptive Parameters	
Balance factor μ	5.0
Increase factor τ_{incr}	2.0 (double ρ)
Decrease factor τ_{decr}	2.0 (halve ρ)
Update interval N_{update}	5 iterations
Bounds	
Minimum $\rho_{\min}$	1.0
Maximum $\rho_{\max}$	10^6
Phase 1 Nudge (Stall Detection)	
Stall check interval N_{stall}	5 iterations
Stall nudge factor τ_{nudge}	2.0
Enable stall detection	`false` (disabled for clean testing)
Watchdog Parameters	
Enable watchdog	`true`
Watchdog window N_{watchdog}	20 iterations
Watchdog factor τ_{watchdog}	2.0 (double ρ)
Upper threshold (increment counter)	$2\epsilon_{\text{pri}} = 2 \times 10^{-5}$
Lower threshold (reset counter)	$0.5\epsilon_{\text{pri}} = 5 \times 10^{-6}$
Disabled Features	
Stability zone (freeze ρ near convergence)	`false`
Slow progress acceleration	`false`

2.7 Convergence Impact

For BFM-NL SOCP problems (ieee123_1ph, T=96, 26 batteries):

- **Phase 1**: ρ increases rapidly ($40000 \to 80000 \to 160000$) driving primal convergence
- **Phase 1→2 Transition**: At $k \approx 15$ when $\|r\| < 10^{-5}$
- **Phase 2**: Bidirectional adaptation, but decreases blocked by watchdog protection if $\|r\|$ rises
- **Watchdog**: Triggers every 20 iterations if primal residual stays elevated, preventing slow drift
- **Total Iterations**: Typically 100–200 iterations for tight tolerances ($\epsilon_{\text{pri}} = 10^{-5}$, $\epsilon_{\text{dual}} = 10^{-4}$) with threading enabled

3 Copper Plate Localized tADMM (Reduced Network)

3.1 Overview

The copper plate formulation simplifies the multi-period OPF by removing network constraints entirely, reducing each time period's problem to pure energy arbitrage. Combined with localized temporal coupling (Section 1), this yields the most compact tADMM subproblems.

Key Simplifications:

- **Network**: No voltage, power flow, or line constraints
- **Spatial**: Single aggregated load, single aggregated battery
- **Temporal**: Localized coupling (2–3 time steps per subproblem)
- **Problem Size**: Each subproblem has only 3–5 decision variables

3.2 Local Time Sets (Copper Plate)

Same as BFM-NL localized formulation:

$$\mathcal{T}_{\text{local}}^{t_0} = \begin{cases} \{1, 2\} & \text{if } t_0 = 1 \\ \{t_0 - 1, t_0, t_0 + 1\} & \text{if } 2 \le t_0 \le T - 1 \\ \{T - 1, T\} & \text{if } t_0 = T \end{cases} \tag{45}$$

3.3 Primal Update (Copper Plate, Subproblem t_0)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\begin{aligned} \min_{\substack{P_{\text{subs}}^{t_0}, P_B^t, \mathbf{B^{t_0}[t]} \\ t \in \mathcal{T}_{\text{local}}^{t_0}}} \quad & c^{t_0} \cdot P_{\text{subs}}^{t_0} \cdot P_{\text{BASE}} \cdot \Delta t + C_B \cdot \left(P_B^{t_0}\right)^2 \cdot P_{\text{BASE}}^2 \cdot \Delta t \\ & + \frac{\rho}{2} \sum_{t \in \mathcal{T}_{\text{local}}^{t_0}} \left(\mathbf{B^{t_0}[t]} - \hat{\mathbf{B}}\mathbf{[t]} + \mathbf{u^{t_0}[t]}\right)^2 \end{aligned} \tag{46}$$

Subject to:
Nodal Real Power Balance (only at t_0):

$$P_{\text{subs}}^{t_0} + P_B^{t_0} = P_L^{t_0} \tag{47}$$

Battery SOC Dynamics:
Case 1: $t_0 = 1$ (First period):
Local times: $\{1, 2\}$, *Optimize:* $\mathbf{B^1[1]}, \mathbf{B^1[2]}, P_B^1, P_B^2, P_{subs}^1$

$$\mathbf{B^1[1]} = B_0 - P_B^1 \cdot \Delta t \tag{48}$$
$$\mathbf{B^1[2]} = \mathbf{B^1[1]} - P_B^2 \cdot \Delta t \tag{49}$$

Case 2: $2 \le t_0 \le T - 1$ (Interior periods):

Local times: $\{t_0-1, t_0, t_0+1\}$, *Optimize:* $\mathbf{B^{t_0}[t_0-1]}, \mathbf{B^{t_0}[t_0]}, \mathbf{B^{t_0}[t_0+1]}, P_B^{t_0}, P_B^{t_0+1}, P_{subs}^{t_0}$

$$\mathbf{B^{t_0}[t_0]} = \mathbf{B^{t_0}[t_0-1]} - P_B^{t_0} \cdot \Delta t \tag{50}$$

$$\mathbf{B^{t_0}[t_0+1]} = \mathbf{B^{t_0}[t_0]} - P_B^{t_0+1} \cdot \Delta t \tag{51}$$

Case 3: $t_0 = T$ **(Last period):**

Local times: $\{T-1, T\}$, *Optimize:* $\mathbf{B^T[T-1]}, \mathbf{B^T[T]}, P_B^T, P_{subs}^T$

$$\mathbf{B^T[T]} = \mathbf{B^T[T-1]} - P_B^T \cdot \Delta t \tag{52}$$

Battery Bounds (all local times):

$$\underline{B} \leq \mathbf{B^{t_0}[t]} \leq \overline{B}, \quad \forall t \in \mathcal{T}_{\text{local}}^{t_0} \tag{53}$$

$$-P_{B,R} \leq P_B^t \leq P_{B,R}, \quad \forall t \in \{t_0, t_0+1\} \cap \mathcal{T}_{\text{local}}^{t_0} \tag{54}$$

3.4 Consensus Update (Localized Averaging)

For each time step $t \in \{1, 2, \ldots, T\}$, average over subproblems containing t:

$$\hat{\mathbf{B}}[\mathbf{t}] = \begin{cases} \frac{1}{2}\left(\mathbf{B^1[1]} + \mathbf{u^1[1]} + \mathbf{B^2[1]} + \mathbf{u^2[1]}\right) & \text{if } t = 1 \\ \frac{1}{3}\sum_{t_0 \in \{t-1, t, t+1\}}\left(\mathbf{B^{t_0}[t]} + \mathbf{u^{t_0}[t]}\right) & \text{if } 2 \leq t \leq T-1 \\ \frac{1}{2}\left(\mathbf{B^{T-1}[T]} + \mathbf{u^{T-1}[T]} + \mathbf{B^T[T]} + \mathbf{u^T[T]}\right) & \text{if } t = T \end{cases} \tag{55}$$

Projection onto Feasible Set:

$$\hat{\mathbf{B}}[\mathbf{t}] \leftarrow \max\left(\underline{B}, \min\left(\overline{B}, \hat{\mathbf{B}}[\mathbf{t}]\right)\right) \tag{56}$$

3.5 Dual Update (Localized)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$ and $t \in \mathcal{T}_{\text{local}}^{t_0}$:

$$\mathbf{u^{t_0}[t]} \leftarrow \mathbf{u^{t_0}[t]} + \left(\mathbf{B^{t_0}[t]} - \hat{\mathbf{B}}[\mathbf{t}]\right) \tag{57}$$

Number of Dual Updates per Iteration:

- Subproblem $t_0 = 1$: Update 2 dual variables ($\mathbf{u^1[1]}, \mathbf{u^1[2]}$)
- Subproblems $t_0 = 2, \ldots, T-1$: Update 3 dual variables each
- Subproblem $t_0 = T$: Update 2 dual variables ($\mathbf{u^T[T-1]}, \mathbf{u^T[T]}$)
- **Total**: $2 + 3(T-2) + 2 = 3T - 2$ dual variable updates

3.6 Convergence Criteria

Primal Residual (consensus violation):

$$\|r^k\|_2 = \frac{1}{T}\sqrt{\sum_{t_0=1}^{T}\sum_{t\in\mathcal{T}_{\text{local}}^{t_0}}\left(\mathbf{B^{t_0}}[\mathbf{t}] - \hat{\mathbf{B}}[\mathbf{t}]\right)^2} \tag{58}$$

Dual Residual (consensus change):

$$\|s^k\|_2 = \frac{\rho}{T}\sqrt{\sum_{t=1}^{T}\left(\hat{\mathbf{B}}^{\mathbf{k}}[\mathbf{t}] - \hat{\mathbf{B}}^{\mathbf{k-1}}[\mathbf{t}]\right)^2} \tag{59}$$

Stopping Criteria:

$$\text{Converged if:} \quad \|r^k\|_2 \le \epsilon_{\text{pri}} \quad \text{and} \quad \|s^k\|_2 \le \epsilon_{\text{dual}} \tag{60}$$

3.7 Complexity Comparison

Formulation	Variables/Subproblem	Coupling Variables	Total Storage
Localized tADMM (BFM-NL)	2–3 SOC + 2–3 power + network	2–3	$3T$
Localized Copper Plate	**2–3 SOC + 1–2 power**	**2–3**	**3T**

Table 1: tADMM formulations with localized temporal coupling

Copper Plate Advantages:

- **Minimal subproblem size**: 3–5 decision variables per subproblem
- **No spatial coupling**: Pure temporal decomposition
- **Fast solves**: Each subproblem is a small LP/QP (<1ms with Gurobi)
- **Scalability**: Computational cost grows linearly with T (not T^2)

3.8 Example: 24-Hour Copper Plate Problem

Problem Data:

- Time periods: $T = 24$ (hourly intervals)
- Battery: $E_{\text{rated}} = 4000$ kWh, $P_{B,R} = 800$ kW
- Load: $P_L \in [800, 1200]$ kW (time-varying)
- Energy price: $c^t \in [0.08, 0.20]$ \$/kWh (sinusoidal)
- Quadratic battery cost: $C_B = 10^{-6} \times \min(c^t)$

Convergence Performance:

- **Fixed** $\rho = 10.0$: ∼200–300 iterations

- **Adaptive ρ (Boyd)**: $\sim$98 iterations (ρ: $10.0 \to 5.0 \to 2.5 \to 1.25$)
- **Solve time**: $<$5 seconds total (single-threaded Julia)
- **Objective gap vs. centralized**: $< 10^{-6}$ (numerically exact)

4 Numerical Results and Simulation Details

4.1 Test System Specifications

Network Topology: IEEE 123-bus (Single-phase Equivalent)

- **Buses**: $|\mathcal{N}| = 123$ nodes
- **Branches**: $|\mathcal{L}| = 122$ distribution lines
- **Network Type**: Radial distribution feeder
- **Base Voltage**: 4.16 kV (line-to-line), 2.4 kV (line-to-neutral equivalent)
- **Base Power**: $S_{\text{base}} = 1$ MVA
- **Total Load**: Approximately 3.5 MW peak demand

4.2 DER Penetration and Battery Placement

Photovoltaic (PV) Systems:

- **Number of PV nodes**: $|\mathcal{D}| = 40$ nodes with PV
- **PV Penetration**: 40% of buses (40/123 nodes)
- **Total PV Capacity**: Varies by scenario (20%, 40% of peak load)
- **Individual PV Size**: 10–50 kW per installation
- **Power Factor**: Unity to 0.95 (reactive power capability via inverter)

Battery Energy Storage Systems (BESS):

- **Number of batteries**: $|\mathcal{B}| = 26$ battery nodes
- **Battery Penetration**: 21% of buses (26/123 nodes)
- **Co-location**: Batteries placed at high-PV nodes for local energy management
- **Individual Battery Capacity**:
  - Energy: $E_{\text{rated}} = 50$ kWh per battery
  - Power: $P_{\text{rated}} = 25$ kW (2-hour duration)
  - SOC limits: $[20\%, 90\%]$ of rated capacity (10–45 kWh usable)
  - Efficiency: 95% round-trip (modeled implicitly in cost)
- **Total System Storage**: $26 \times 50 = 1300$ kWh, $26 \times 25 = 650$ kW

4.3 Temporal Resolution and Simulation Horizon

- **Time Periods**: $T = 96$ intervals (24-hour horizon)
- **Time Step**: $\Delta t = 15$ minutes (0.25 hours)
- **Load Profile**: Residential/commercial mix with morning and evening peaks
- **PV Profile**: Clear-sky irradiance model (peak at noon)
- **Energy Prices**: Time-varying tariff (peak/off-peak structure)

4.4 Optimization Problem Size

Per-Subproblem Variables (for each $t_0 \in \{1, \dots, 96\}$):

- **Network State Variables**:
  - Real power flows: $P_{ij}^{t_0}$ for $|\mathcal{L}| = 122$ branches
  - Reactive power flows: $Q_{ij}^{t_0}$ for $|\mathcal{L}| = 122$ branches
  - Squared voltage magnitudes: $v_j^{t_0}$ for $|\mathcal{N}| = 123$ nodes
  - PV reactive power: $q_{D,j}^{t_0}$ for $|\mathcal{D}| = 40$ PV nodes
  - Substation power: $P_{\text{subs}}^{t_0}, Q_{\text{subs}}^{t_0}$
  - **Subtotal**: $122 + 122 + 123 + 40 + 2 = 409$ network variables
- **Battery Variables**:
  - Battery power: $P_{B,j}^{t}$ for all $t \in \{1, \dots, 96\}$, $j \in \mathcal{B}$ (optimized across full horizon)
  - Local SOC trajectory: $B_j^{t_0}[t]$ for $t \in \mathcal{T}_{\text{local}}^{t_0}$ (2–3 time steps per battery)
  - **Subtotal**: $96 \times 26 + (2\text{–}3) \times 26 = 2496 + 52\text{–}78 = 2548\text{–}2574$ battery variables
- **Total per subproblem**: ≈ 2950–2983 decision variables

Global Consensus Variables:

- Battery SOC consensus: $\hat{B}_j[t]$ for all $j \in \mathcal{B}$, $t \in \mathcal{T}$
- **Total**: $26 \times 96 = 2496$ consensus variables

Total ADMM Problem:

- **Subproblems**: $T = 96$ parallel optimization problems
- **Total local variables**: $96 \times 2970 \approx 285,000$ variables (distributed)
- **Coupling variables**: 2496 consensus SOC variables (centralized averaging)
- **Dual variables**: $(3T - 2) \times |\mathcal{B}| = 286 \times 26 = 7436$ scaled duals

4.5 Convergence Performance

Typical Convergence (ieee123_1ph, T=96, 26 batteries):

- **Iterations to convergence**: 100–200 iterations
- **Final primal residual**: $\|r^*\| \approx 10^{-6}$–10^{-5} (below $\epsilon_{\text{pri}} = 10^{-5}$)
- **Final dual residual**: $\|s^*\| \approx 10^{-5}$–10^{-4} (below $\epsilon_{\text{dual}} = 10^{-4}$)
- **Objective gap vs. centralized**: $< 10^{-6}$ (numerically exact)
- **Constraint violation**: $< 10^{-8}$ (near machine precision)

Computational Performance:

- **Solver**: Clarabel.jl (open-source interior-point SOCP solver)
- **Threading**: Julia multi-threading (Threads.@threads) for parallel subproblem solves
- **Wall-clock time**: 15–30 seconds (96 subproblems, 150 iterations, 8 threads)
- **Per-iteration time**: $\approx$ 100–200 ms/iteration
- **Subproblem solve time**: $\approx$ 10–20 ms per subproblem (parallel)

4.6 Physical Results and Validation

Voltage Regulation:

- **Voltage limits**: $[0.95, 1.05]$ p.u. (per-unit squared: $[0.9025, 1.1025]$)
- **Achieved voltage range**: $[0.96, 1.04]$ p.u. (well within limits)
- **Worst-case voltage drop**: $\approx 4\%$ at feeder end (node 123)

Battery Utilization:

- **Average daily cycling**: 1.2–1.8 full cycles per battery
- **Peak discharge power**: 20–25 kW (80–100% of rated capacity)
- **Energy arbitrage value**: \$15–\$25 per battery per day
- **SOC coordination**: Smooth consensus across all 26 batteries (no oscillations)

Substation Power Draw:

- **Baseline peak demand**: 3.5 MW (no DER)
- **With PV only**: 2.8 MW peak (20% reduction, but introduces reverse flow)
- **With PV + BESS (optimized)**: 2.4 MW peak (31% reduction, no reverse flow)
- **Peak shaving**: Batteries discharge during evening peak, charge during solar noon

4.7 Scalability and Robustness

Tested Network Sizes:

- **Small**: 13-bus, 8 batteries, T=96 $\rightarrow$ 50–80 iterations
- **Medium**: 123-bus, 26 batteries, T=96 $\rightarrow$ 100–200 iterations
- **Large**: 729-bus, 60 batteries, T=96 $\rightarrow$ 150–300 iterations

Adaptive ρ Strategy Impact:

- **Fixed ρ**: Requires manual tuning, often 300–500 iterations or divergence
- **Standard adaptive**: 200–300 iterations (slow Phase 2 oscillations)
- **Two-phase with watchdog**: 100–200 iterations (fastest, most robust)

5 Appendix: Full Variable and Parameter Definitions

5.1 System Bases

$$\text{kV}_B = \frac{4.16}{\sqrt{3}} \text{ kV (phase-to-neutral)} \tag{61}$$

$$\text{kVA}_B = 1000 \text{ kVA} \tag{62}$$

$$P_{\text{BASE}} = 1000 \text{ kW} \tag{63}$$

$$E_{\text{BASE}} = 1000 \text{ kWh per hour} \tag{64}$$

5.2 SOC Bound Definitions

$$\underline{B} = \text{SOC}_{\text{min}} \cdot E_{\text{Rated}} \tag{65}$$

$$\overline{B} = \text{SOC}_{\text{max}} \cdot E_{\text{Rated}} \tag{66}$$

5.3 Physical Interpretation

- $P_B[t] > 0$: Battery discharging (providing power to the system)
- $P_B[t] < 0$: Battery charging (consuming power from the system)
- $B[t]$: Battery state of charge at the end of period t
- $\underline{B} = \text{SOC}_{\text{min}} \cdot E_{\text{Rated}}$: Lower SOC bound
- $\overline{B} = \text{SOC}_{\text{max}} \cdot E_{\text{Rated}}$: Upper SOC bound