Multi-Period Optimal Power Flow with tADMM: Temporally Localized ADMM for Branch Flow Model SOCP (BFM-NL)

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1.1 Overview

The Temporal ADMM (tADMM) algorithm decomposes the multi-period optimal power flow problem into T subproblems, one for each time period. This formulation uses **localized temporal coupling** where each subproblem t_0 only couples with its immediate time neighbors $(t_0 - 1, t_0, t_0 + 1)$, rather than coupling with all T time steps.

Key Benefits:

- **Reduced Coupling**: Each subproblem exchanges 2–3 $\times |\mathcal{B}|$ variables (not $T \times |\mathcal{B}|$)
- **Memory Efficiency**: $(3T - 2) \times |\mathcal{B}|$ total storage for localized coupling
- **Parallel Computation**: All T subproblems can be solved in parallel
- **Network Physics**: Branch Flow Model (BFM-NL) with SOCP relaxation for exact voltage and reactive power

1.2 Problem Definition

Sets and Indices:

- $\mathcal{N}$: Set of all nodes (buses) in the distribution network
- $\mathcal{L}$: Set of all branches (lines), (i, j) denotes branch from node i to j
- $\mathcal{L}_1$: Set of branches connected to substation (node 1)
- $\mathcal{B} \subseteq \mathcal{N}$: Set of nodes with batteries (energy storage)
- $\mathcal{D} \subseteq \mathcal{N}$: Set of nodes with solar PV (distributed energy resources)
- $\mathcal{T} = \{1, 2, \ldots, T\}$: Set of time periods (e.g., 24 hours, 96 15-min intervals)
- $t_0 \in \mathcal{T}$: Index for the control time of subproblem t_0

Variable Color Coding:

- $\mathbf{B_j^{t_0}[t]}$: Local SOC variables (blue) - battery j in subproblem t_0, evaluated at time t
- $\mathbf{\hat{B}_j[t]}$: Global consensus SOC (red) - agreed-upon SOC for battery j at time t
- $\mathbf{u_j^{t_0}[t]}$: Scaled dual variables (green) - enforces consensus between local and global SOC

1.3 Local Time Sets

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$, define the local time set:

$$\mathcal{T}_{\text{local}}^{t_0} = \begin{cases} \{1, 2\} & \text{if } t_0 = 1 \text{ (boundary)} \\ \{t_0 - 1, t_0, t_0 + 1\} & \text{if } 2 \leq t_0 \leq T - 1 \text{ (interior)} \\ \{T - 1, T\} & \text{if } t_0 = T \text{ (boundary)} \end{cases} \tag{1}$$

1.4 Step 1: Primal Update (Subproblem Optimization)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\min_{\substack{P_{\text{Subs}}^{t_0}, Q_{\text{Subs}}^{t_0},\\ P_{ij}^{t_0}, Q_{ij}^{t_0}, v_j^{t_0}, \ell_{ij}^{t_0},\\ q_{D,j}^{t_0}, P_{B,j}^{t}, \mathbf{B_j^{t_0}[t]}\\ \forall (i,j)\in\mathcal{L}, j\in\mathcal{N}\cup\mathcal{D}\cup\mathcal{B},\\ t\in\mathcal{T}_{\text{local}}^{t_0}}} \quad c^{t_0} \cdot P_{\text{Subs}}^{t_0} \cdot P_{\text{BASE}} \cdot \Delta t + C_B \sum_{j\in\mathcal{B}} \left(P_{B,j}^{t_0}\right)^2 \cdot P_{\text{BASE}}^2 \cdot \Delta t$$
$$+ \frac{\rho}{2} \sum_{j\in\mathcal{B}} \sum_{t\in\mathcal{T}_{\text{local}}^{t_0}} \left(\mathbf{B_j^{t_0}[t]} - \mathbf{\hat{B}_j[t]} + \mathbf{u_j^{t_0}[t]}\right)^2 \tag{2}$$

Subject to:
Spatial Network Constraints (Branch Flow Model SOCP, only for control time t_0):

$$\text{Power balance (substation):} \quad P_{\text{Subs}}^{t_0} - \sum_{(1,j)\in\mathcal{L}_1} P_{1j}^{t_0} = 0 \tag{3}$$
$$Q_{\text{Subs}}^{t_0} - \sum_{(1,j)\in\mathcal{L}_1} Q_{1j}^{t_0} = 0 \tag{4}$$
$$\text{Power balance (nodes):} \quad P_{ij}^{t_0} - \sum_{(j,k)\in\mathcal{L}} P_{jk}^{t_0} = P_{B,j}^{t_0} + p_{D,j}^{t_0} - p_{L,j}^{t_0}, \quad \forall (i,j)\in\mathcal{L} \tag{5}$$
$$Q_{ij}^{t_0} - \sum_{(j,k)\in\mathcal{L}} Q_{jk}^{t_0} = q_{D,j}^{t_0} - q_{L,j}^{t_0}, \quad \forall (i,j)\in\mathcal{L} \tag{6}$$
$$\text{KVL (voltage drop):} \quad v_j^{t_0} = v_i^{t_0} - 2(r_{ij}P_{ij}^{t_0} + x_{ij}Q_{ij}^{t_0}) + \ell_{ij}^{t_0}, \quad \forall (i,j)\in\mathcal{L} \tag{7}$$
$$\text{SOCP relaxation:} \quad \ell_{ij}^{t_0} \cdot v_i^{t_0} \geq (P_{ij}^{t_0})^2 + (Q_{ij}^{t_0})^2, \quad \forall (i,j)\in\mathcal{L} \tag{8}$$
$$\text{Voltage limits:} \quad (V_{\min,j})^2 \leq v_j^{t_0} \leq (V_{\max,j})^2, \quad \forall j\in\mathcal{N} \tag{9}$$
$$\text{Substation voltage:} \quad v_1^{t_0} = (V_{\text{nom}})^2 \tag{10}$$
$$\text{PV reactive limits:} \quad (q_{D,j}^{t_0})^2 \leq (S_{D,j})^2 - (p_{D,j}^{t_0})^2, \quad \forall j\in\mathcal{D} \tag{11}$$

Note: Network variables ($P_{\text{Subs}}^{t_0}, Q_{\text{Subs}}^{t_0}, P_{ij}^{t_0}, Q_{ij}^{t_0}, v_j^{t_0}, \ell_{ij}^{t_0}, q_{D,j}^{t_0}$) are optimized only for the control time t_0, not for neighbor times.

Temporal Battery Constraints:

Case 1: First time period ($t_0 = 1$):

Local times: $\{1, 2\}$, *Decision variables:* $\mathbf{B_j^1[1]}, \mathbf{B_j^1[2]}$, *Dual variables:* $\mathbf{u_j^1[1]}$ *only*

$$\text{SOC trajectory } t=0 \to t=1: \quad \mathbf{B_j^1[1]} = B_{0,j} - P_{B,j}^1 \cdot \Delta t, \quad \forall j\in\mathcal{B} \tag{12}$$
$$\text{SOC trajectory } t=1 \to t=2: \quad \mathbf{B_j^1[2]} = \mathbf{B_j^1[1]} - P_{B,j}^2 \cdot \Delta t, \quad \forall j\in\mathcal{B} \tag{13}$$

Note: $B_{0,j}$ is a parameter (initial condition), not a decision variable. Only $\mathbf{B_j^1[1]}$ participates in consensus (with dual $\mathbf{u_j^1[1]}$). Variable $\mathbf{B_j^1[2]}$ exists for penalty computation but does not participate in consensus at $t = 1$.

Case 2: Interior time periods ($2 \leq t_0 \leq T-1$):

Local times: $\{t_0 - 1, t_0, t_0 + 1\}$, *Decision variables:* $\mathbf{B_j^{t_0}[t_0 - 1]}, \mathbf{B_j^{t_0}[t_0]}, \mathbf{B_j^{t_0}[t_0 + 1]}$

Dual variables: $\mathbf{u_j^{t_0}[t_0]}$ *only*

$$\text{SOC trajectory } t_0 - 1 \to t_0: \quad \mathbf{B_j^{t_0}[t_0]} = \mathbf{B_j^{t_0}[t_0 - 1]} - P_{B,j}^{t_0} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{14}$$

$$\text{SOC trajectory } t_0 \to t_0 + 1: \quad \mathbf{B_j^{t_0}[t_0 + 1]} = \mathbf{B_j^{t_0}[t_0]} - P_{B,j}^{t_0+1} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{15}$$

Note: Only $\mathbf{B_j^{t_0}[t_0]}$ participates in consensus update at time t_0 (with dual $\mathbf{u_j^{t_0}[t_0]}$). Variables $\mathbf{B_j^{t_0}[t_0 - 1]}$ and $\mathbf{B_j^{t_0}[t_0 + 1]}$ are used for penalty terms but do not generate new dual variables in this subproblem.

Case 3: Last time period ($t_0 = T$):

Local times: $\{T-1, T\}$, *Decision variables:* $\mathbf{B_j^T[T-1]}, \mathbf{B_j^T[T]}$, *Dual variables:* $\mathbf{u_j^T[T]}$ *only*

$$\text{SOC trajectory } T - 1 \to T: \quad \mathbf{B_j^T[T]} = \mathbf{B_j^T[T - 1]} - P_{B,j}^{T} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{16}$$

Note: Only $\mathbf{B_j^T[T]}$ participates in consensus at $t = T$ (with dual $\mathbf{u_j^T[T]}$). Variable $\mathbf{B_j^T[T-1]}$ is for penalty only.

SOC and Power Limits:

$$\text{SOC limits:} \quad \text{SOC}_{\min,j} \cdot B_{\text{rated},j} \le \mathbf{B_j^{t_0}[t]} \le \text{SOC}_{\max,j} \cdot B_{\text{rated},j}, \quad \forall t \in \mathcal{T}_{\text{local}}^{t_0},\ j \in \mathcal{B} \tag{17}$$

$$\text{Power limits:} \quad -P_{B,\text{rated},j} \le P_{B,j}^{t} \le P_{B,\text{rated},j}, \quad \forall t \in \mathcal{T}_{\text{local}}^{t_0},\ j \in \mathcal{B} \tag{18}$$

Note: Power limits must be enforced for *all* times appearing in constraints, not just t_0. For interior times: $P_{B,j}^{t_0-1}, P_{B,j}^{t_0}, P_{B,j}^{t_0+1}$ all need limits.

1.5 Step 2: Consensus Update (Local Averaging)

For each battery $j \in \mathcal{B}$ and each time $t \in \mathcal{T}$, average only over subproblems where t is the *active* time:

Consensus at $t = 1$:

$$\hat{\mathbf{B}}_\mathbf{j}[\mathbf{1}] = \text{clamp}\left(\frac{1}{2}\left(\mathbf{B_j^1[1]} + \mathbf{u_j^1[1]} + \mathbf{B_j^2[1]} + \mathbf{u_j^2[1]}\right), \underline{B}_j, \overline{B}_j\right) \tag{19}$$

Contributors: Subproblems $t_0 = 1$ and $t_0 = 2$ (only these have $t = 1$ as decision variable with dual)

Consensus at interior times ($2 \le t \le T - 1$):

$$\hat{\mathbf{B}}_\mathbf{j}[\mathbf{t}] = \text{clamp}\left(\frac{1}{3}\left(\mathbf{B_j^{t-1}[t]} + \mathbf{u_j^{t-1}[t]} + \mathbf{B_j^{t}[t]} + \mathbf{u_j^{t}[t]} + \mathbf{B_j^{t+1}[t]} + \mathbf{u_j^{t+1}[t]}\right), \underline{B}_j, \overline{B}_j\right) \tag{20}$$

Contributors: Subproblems $t_0 \in \{t-1, t, t+1\}$ (each has t as its active control time)

Consensus at $t = T$:

$$\hat{\mathbf{B}}_\mathbf{j}[\mathbf{T}] = \text{clamp}\left(\frac{1}{2}\left(\mathbf{B_j^{T-1}[T]} + \mathbf{u_j^{T-1}[T]} + \mathbf{B_j^{T}[T]} + \mathbf{u_j^{T}[T]}\right), \underline{B}_j, \overline{B}_j\right) \tag{21}$$

Contributors: Subproblems $t_0 = T - 1$ and $t_0 = T$ (only these have $t = T$ as decision variable with dual)

General form:

$$\mathcal{N}_t = \{t_0 \in \mathcal{T} : t \text{ is the active control time in subproblem } t_0\} \tag{22}$$

$$= \begin{cases} \{1, 2\} & \text{if } t = 1 \\ \{t-1, t, t+1\} & \text{if } 2 \le t \le T-1 \\ \{T-1, T\} & \text{if } t = T \end{cases} \tag{23}$$

1.6 Step 3: Dual Update (All Local Times)

For each subproblem $t_0 \in \mathcal{T}$ and battery $j \in \mathcal{B}$, update dual variables for **all local times** $t \in \mathcal{T}_{\text{local}}^{t_0}$:
Subproblem $t_0 = 1$ (local times: $\{1, 2\}$):

$$\mathbf{u_j^1[1]} := \mathbf{u_j^1[1]} + \left(\mathbf{B_j^1[1]} - \mathbf{\hat{B}_j[1]}\right) \tag{24}$$

$$\mathbf{u_j^1[2]} := \mathbf{u_j^1[2]} + \left(\mathbf{B_j^1[2]} - \mathbf{\hat{B}_j[2]}\right) \tag{25}$$

Subproblem t_0 ($2 \le t_0 \le T - 1$, local times: $\{t_0 - 1, t_0, t_0 + 1\}$):

$$\mathbf{u_j^{t_0}[t_0 - 1]} := \mathbf{u_j^{t_0}[t_0 - 1]} + \left(\mathbf{B_j^{t_0}[t_0 - 1]} - \mathbf{\hat{B}_j[t_0 - 1]}\right) \tag{26}$$

$$\mathbf{u_j^{t_0}[t_0]} := \mathbf{u_j^{t_0}[t_0]} + \left(\mathbf{B_j^{t_0}[t_0]} - \mathbf{\hat{B}_j[t_0]}\right) \tag{27}$$

$$\mathbf{u_j^{t_0}[t_0 + 1]} := \mathbf{u_j^{t_0}[t_0 + 1]} + \left(\mathbf{B_j^{t_0}[t_0 + 1]} - \mathbf{\hat{B}_j[t_0 + 1]}\right) \tag{28}$$

Subproblem $t_0 = T$ (local times: $\{T - 1, T\}$):

$$\mathbf{u_j^T[T - 1]} := \mathbf{u_j^T[T - 1]} + \left(\mathbf{B_j^T[T - 1]} - \mathbf{\hat{B}_j[T - 1]}\right) \tag{29}$$

$$\mathbf{u_j^T[T]} := \mathbf{u_j^T[T]} + \left(\mathbf{B_j^T[T]} - \mathbf{\hat{B}_j[T]}\right) \tag{30}$$

General Form:

$$\text{For each } t \in \mathcal{T}_{\text{local}}^{t_0}: \quad \mathbf{u_j^{t_0}[t]} := \mathbf{u_j^{t_0}[t]} + \left(\mathbf{B_j^{t_0}[t]} - \mathbf{\hat{B}_j[t]}\right) \tag{31}$$

Critical Note: Each subproblem t_0 maintains dual variables for *all* its local times $\mathcal{T}_{\text{local}}^{t_0}$, which is 2 or 3 times per battery. This ensures that each time t has duals from all subproblems that include it in their penalty terms, enabling proper consensus averaging in Step 2.

1.7 Convergence Criteria (Exact Implementation)

Primal Residual (Consensus Violation):

Only count residuals at active control times (where dual variables exist):

$$r_{\text{values}} = \left\{\mathbf{B_j^{t_0,k}[t_0]} - \mathbf{\hat{B}_j^k[t_0]} : t_0 \in \mathcal{T},\, j \in \mathcal{B}\right\} \tag{32}$$

$$\|r^k\|_2 = \frac{\|\mathbf{r}_{\text{values}}\|_2}{\sqrt{|\mathbf{r}_{\text{values}}|}} = \frac{1}{\sqrt{T \cdot |\mathcal{B}|}} \left(\sum_{t_0=1}^{T} \sum_{j \in \mathcal{B}} \left(\mathbf{B_j^{t_0,k}[t_0]} - \mathbf{\hat{B}_j^k[t_0]}\right)^2\right)^{1/2} \tag{33}$$

Note: We only compute residual at time t_0 for each subproblem t_0 (the active control time where dual $\mathbf{u_j^{t_0}[t_0]}$ exists). The normalization is by $\sqrt{T \cdot |\mathcal{B}|}$, the number of active consensus variables.

Dual Residual (Consensus Variable Change):

Measures how much the consensus variables $\mathbf{\hat{B}}$ changed from previous iteration:

$$\mathbf{s}_j^k = \mathbf{\hat{B}_j^k} - \mathbf{\hat{B}_j^{k-1}} = \begin{bmatrix} \mathbf{\hat{B}_j^k[1]} - \mathbf{\hat{B}_j^{k-1}[1]} \\ \vdots \\ \mathbf{\hat{B}_j^k[T]} - \mathbf{\hat{B}_j^{k-1}[T]} \end{bmatrix} \in \mathbb{R}^T \tag{34}$$

$$\|s^k\|_2 = \frac{\rho^k}{|\mathcal{B}|} \left\| \begin{bmatrix} \mathbf{s}_1^k \\ \vdots \\ \mathbf{s}_{|\mathcal{B}|}^k \end{bmatrix} \right\|_2 = \frac{\rho^k}{|\mathcal{B}|} \left(\sum_{j \in \mathcal{B}} \sum_{t=1}^{T} \left(\mathbf{\hat{B}_j^k[t]} - \mathbf{\hat{B}_j^{k-1}[t]}\right)^2\right)^{1/2} \tag{35}$$

Note: The dual residual is scaled by ρ and normalized by $|\mathcal{B}|$ (number of batteries), not by $\sqrt{T \cdot |\mathcal{B}|}$.

Convergence Condition:

$$\text{Converged if } \|r^k\|_2 \leq \epsilon_{\text{pri}} \quad \text{and} \quad \|s^k\|_2 \leq \epsilon_{\text{dual}} \tag{36}$$

with $\epsilon_{\text{pri}} = 10^{-5}$ and $\epsilon_{\text{dual}} = 10^{-4}$.

1.8 Localized tADMM Complexity

For $T = 48$, $|\mathcal{B}| = 26$ batteries, $|\mathcal{N}| = 128$ nodes, $|\mathcal{D}| = 17$ PV (24-hour horizon, 30-min intervals):

- **SOC Variables per Subproblem:**
  - Boundary periods (first/last, $t_0 \in \{1, 48\}$): $2 \times 26 = 52$ SOC variables
  - Interior periods ($2 \leq t_0 \leq 47$): $3 \times 26 = 78$ SOC variables
  - Average: ≈ 77 SOC variables per subproblem
- **Total Dual Variable Storage:**
  - $(3T - 2) \times |\mathcal{B}| = (3 \times 48 - 2) \times 26 = 142 \times 26 = 3,692$ dual variables
- **Communication per Iteration:**
  - Each subproblem sends: 2–3 local SOC values per battery to consensus
  - Each subproblem receives: 2–3 consensus SOC values per battery
  - Total consensus exchanges: $48 \times 77 = 3,696$ SOC values per iteration
- **Network Constraints per Subproblem:**
  - Power balance: 128 real + 128 reactive = 256 constraints
  - KVL: 127 voltage drop constraints
  - SOCP cones: 127 second-order cone constraints ($\ell_{ij} \geq P_{ij}^2 + Q_{ij}^2/v_i$)
  - Voltage limits: $2 \times 128 = 256$ bounds ($0.95 \leq v_j \leq 1.05$ p.u.^{2})
  - PV reactive limits: $2 \times 17 = 34$ cone constraints ($q_D^2 \leq S_D^2 - p_D^2$)
  - Battery SOC dynamics: $48 \times 26 = 1,248$ SOC propagation constraints
  - Battery power limits: $2 \times 26 = 52$ bounds

2 Adaptive Penalty Parameter: Two-Phase Strategy with Watchdog

The penalty parameter ρ is adjusted dynamically using a sophisticated two-phase strategy that prioritizes primal convergence initially, then balances both residuals. The scheme is activated by setting `adaptive_rho = true`.

2.1 Residual Calculations (Exact Implementation)

Primal Residual (Consensus Violation):

$$\|r^k\|_2 = \frac{1}{\sqrt{N_{\text{active}}}} \left(\sum_{t_0=1}^{T} \sum_{j\in\mathcal{B}} \left(\mathbf{B_j^{t_0,k}}[\mathbf{t_0}] - \hat{\mathbf{B}}_\mathbf{j}^\mathbf{k}[\mathbf{t_0}] \right)^2 \right)^{1/2} \tag{37}$$

where $N_{\text{active}} = T \times |\mathcal{B}|$ is the number of active time-battery pairs (only counting where duals exist).

Dual Residual (Consensus Variable Change):

$$\|s^k\|_2 = \frac{\rho^k}{|\mathcal{B}|} \left(\sum_{j\in\mathcal{B}} \sum_{t=1}^{T} \left(\hat{\mathbf{B}}_\mathbf{j}^\mathbf{k}[\mathrm{t}] - \hat{\mathbf{B}}_\mathbf{j}^\mathbf{k-1}[\mathrm{t}] \right)^2 \right)^{1/2} \tag{38}$$

2.2 Phase 1: Aggressive Primal Convergence (Until $\|r\| \leq \epsilon_{\text{pri}}$)

Objective: Drive primal residual below threshold by only increasing ρ. Never decrease ρ to avoid slowing consensus formation.

Update Rules (every $N_{\text{update}} = 5$ iterations):

$$\rho^{k+1} = \begin{cases} \min(\rho_{\max}, \tau_{\text{incr}} \cdot \rho^k) & \text{if } \|r^k\| > \mu \cdot \|s^k\| \\ \min(\rho_{\max}, \tau_{\text{nudge}} \cdot \rho^k) & \text{if } \|r^k\| > \epsilon_{\text{pri}} \text{ and no } \rho \text{ change for } N_{\text{stall}} \text{ iters} \\ \rho^k & \text{otherwise (do not decrease)} \end{cases} \tag{39}$$

Phase Transition:

$$\text{Switch to Phase 2 if } \|r^k\| \leq \epsilon_{\text{pri}} \text{ (primal converged once)} \tag{40}$$

2.3 Phase 2: Bidirectional Adaptation (After primal convergence)

Objective: Balance primal and dual residuals, allowing both increases and decreases of ρ.

Update Rules (every $N_{\text{update}} = 5$ iterations):

$$\rho^{k+1} = \begin{cases} \min(\rho_{\max}, \tau_{\text{incr}} \cdot \rho^k) & \text{if } \|r^k\| > \mu \cdot \|s^k\| \\ \max(\rho_{\min}, \rho^k/\tau_{\text{decr}}) & \text{if } \|s^k\| > \mu \cdot \|r^k\| \textbf{ and } \|r^k\| \leq \epsilon_{\text{pri}} \\ \rho^k & \text{otherwise (residuals balanced)} \end{cases} \tag{41}$$

Watchdog Protection: The decrease condition includes $\|r^k\| \leq \epsilon_{\text{pri}}$ to prevent premature decreases when primal consensus breaks down after Phase $1 \to 2$ transition.

2.4 Primal Residual Watchdog (All Phases)

Motivation: Detects when primal residual stays elevated for many consecutive iterations, indicating insufficient penalty to enforce consensus.

Hysteresis-Based Counter (runs every iteration):

$$\text{counter}^{k+1} = \begin{cases} \text{counter}^k + 1 & \text{if } \|r^k\| > 2\epsilon_{\text{pri}} \\ 0 & \text{if } \|r^k\| < 0.5\epsilon_{\text{pri}} \\ \text{counter}^k & \text{if } 0.5\epsilon_{\text{pri}} \leq \|r^k\| \leq 2\epsilon_{\text{pri}} \text{ (hysteresis zone)} \end{cases} \tag{42}$$

Watchdog Trigger:

$$\text{If counter}^k \geq N_{\text{watchdog}} = 20: \quad \rho^{k+1} = \min(\rho_{\max}, \tau_{\text{watchdog}} \cdot \rho^k), \quad \text{counter}^{k+1} = 0 \tag{43}$$

The hysteresis prevents false resets from oscillations around ϵ_{pri}, while the 20-iteration window ensures sustained high residuals trigger aggressive action.

2.5 Dual Variable Rescaling

When ρ changes from ρ^k to ρ^{k+1}, the scaled dual variables must be rescaled:

$$\mathbf{u}_{\mathbf{j}}^{\mathbf{t_0}}[\mathbf{t}] \leftarrow \mathbf{u}_{\mathbf{j}}^{\mathbf{t_0}}[\mathbf{t}] \cdot \frac{\rho^k}{\rho^{k+1}}, \quad \forall t_0 \in \mathcal{T}, \, j \in \mathcal{B}, \, t \in \mathcal{T}_{\text{local}}^{t_0} \tag{44}$$

This maintains the relationship between scaled and unscaled dual variables: $\mathbf{u} = \lambda/\rho$.

2.6 Complete Parameter Summary

Parameter	Value (SOCP tADMM)
Tolerances	
Primal tolerance ϵ_{pri}	1×10^{-5}
Dual tolerance ϵ_{dual}	1×10^{-4} (relaxed for faster convergence)
Standard Adaptive Parameters	
Balance factor μ	5.0
Increase factor τ_{incr}	2.0 (double ρ)
Decrease factor τ_{decr}	2.0 (halve ρ)
Update interval N_{update}	5 iterations
Bounds	
Minimum $\rho_{\min}$	1.0
Maximum $\rho_{\max}$	10^6
Phase 1 Nudge (Stall Detection)	
Stall check interval N_{stall}	5 iterations
Stall nudge factor τ_{nudge}	2.0
Enable stall detection	`false` (disabled for clean testing)
Watchdog Parameters	
Enable watchdog	`true`
Watchdog window N_{watchdog}	20 iterations
Watchdog factor τ_{watchdog}	2.0 (double ρ)
Upper threshold (increment counter)	$2\epsilon_{\text{pri}} = 2 \times 10^{-5}$
Lower threshold (reset counter)	$0.5\epsilon_{\text{pri}} = 5 \times 10^{-6}$
Disabled Features	
Stability zone (freeze ρ near convergence)	`false`
Slow progress acceleration	`false`

2.7 Convergence Impact

For BFM-NL SOCP problems (ieee123_1ph, T=96, 26 batteries):

- **Phase 1**: ρ increases rapidly ($40000 \to 80000 \to 160000$) driving primal convergence
- **Phase 1→2 Transition**: At $k \approx 15$ when $\|r\| < 10^{-5}$
- **Phase 2**: Bidirectional adaptation, but decreases blocked by watchdog protection if $\|r\|$ rises
- **Watchdog**: Triggers every 20 iterations if primal residual stays elevated, preventing slow drift
- **Total Iterations**: Typically 100–200 iterations for tight tolerances ($\epsilon_{\text{pri}} = 10^{-5}$, $\epsilon_{\text{dual}} = 10^{-4}$) with threading enabled

3 Copper Plate Localized tADMM (Reduced Network)

3.1 Overview

The copper plate formulation simplifies the multi-period OPF by removing network constraints entirely, reducing each time period's problem to pure energy arbitrage. Combined with localized temporal coupling (Section 1), this yields the most compact tADMM subproblems.

Key Simplifications:

- **Network**: No voltage, power flow, or line constraints
- **Spatial**: Single aggregated load, single aggregated battery
- **Temporal**: Localized coupling (2–3 time steps per subproblem)
- **Problem Size**: Each subproblem has only 3–5 decision variables

3.2 Local Time Sets (Copper Plate)

Same as BFM-NL localized formulation:

$$\mathcal{T}_{\text{local}}^{t_0} = \begin{cases} \{1, 2\} & \text{if } t_0 = 1 \\ \{t_0 - 1, t_0, t_0 + 1\} & \text{if } 2 \leq t_0 \leq T - 1 \\ \{T - 1, T\} & \text{if } t_0 = T \end{cases} \tag{45}$$

3.3 Primal Update (Copper Plate, Subproblem t_0)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\begin{aligned} \min_{\substack{P_{\text{subs}}^{t_0}, P_B^t, \mathbf{B}^{\mathbf{t_0}}[\mathbf{t}] \\ t \in \mathcal{T}_{\text{local}}^{t_0}}} \quad & c^{t_0} \cdot P_{\text{subs}}^{t_0} \cdot P_{\text{BASE}} \cdot \Delta t + C_B \cdot \left(P_B^{t_0}\right)^2 \cdot P_{\text{BASE}}^2 \cdot \Delta t \\ & + \frac{\rho}{2} \sum_{t \in \mathcal{T}_{\text{local}}^{t_0}} \left(\mathbf{B}^{\mathbf{t_0}}[\mathbf{t}] - \hat{\mathbf{B}}[\mathbf{t}] + \mathbf{u}^{\mathbf{t_0}}[\mathbf{t}]\right)^2 \end{aligned} \tag{46}$$

Subject to:
Nodal Real Power Balance (only at t_0):

$$P_{\text{subs}}^{t_0} + P_B^{t_0} = P_L^{t_0} \tag{47}$$

Battery SOC Dynamics:
Case 1: $t_0 = 1$ (First period):
Local times: $\{1, 2\}$, *Optimize:* $\mathbf{B^1}[\mathbf{1}], \mathbf{B^1}[\mathbf{2}], P_B^1, P_B^2, P_{subs}^1$

$$\mathbf{B^1}[\mathbf{1}] = B_0 - P_B^1 \cdot \Delta t \tag{48}$$

$$\mathbf{B^1}[\mathbf{2}] = \mathbf{B^1}[\mathbf{1}] - P_B^2 \cdot \Delta t \tag{49}$$

Case 2: $2 \leq t_0 \leq T - 1$ (Interior periods):

Local times: $\{t_0-1, t_0, t_0+1\}$, *Optimize:* $\mathbf{B^{t_0}[t_0-1]}, \mathbf{B^{t_0}[t_0]}, \mathbf{B^{t_0}[t_0+1]}, P_B^{t_0}, P_B^{t_0+1}, P_{subs}^{t_0}$

$$\mathbf{B^{t_0}[t_0]} = \mathbf{B^{t_0}[t_0-1]} - P_B^{t_0} \cdot \Delta t \tag{50}$$

$$\mathbf{B^{t_0}[t_0+1]} = \mathbf{B^{t_0}[t_0]} - P_B^{t_0+1} \cdot \Delta t \tag{51}$$

Case 3: $t_0 = T$ **(Last period):**

Local times: $\{T-1, T\}$, *Optimize:* $\mathbf{B^{T}[T-1]}, \mathbf{B^{T}[T]}, P_B^{T}, P_{subs}^{T}$

$$\mathbf{B^{T}[T]} = \mathbf{B^{T}[T-1]} - P_B^{T} \cdot \Delta t \tag{52}$$

Battery Bounds (all local times):

$$\underline{B} \le \mathbf{B^{t_0}[t]} \le \overline{B}, \quad \forall t \in \mathcal{T}_{\text{local}}^{t_0} \tag{53}$$

$$-P_{B,R} \le P_B^{t} \le P_{B,R}, \quad \forall t \in \{t_0, t_0+1\} \cap \mathcal{T}_{\text{local}}^{t_0} \tag{54}$$

3.4 Consensus Update (Localized Averaging)

For each time step $t \in \{1, 2, \ldots, T\}$, average over subproblems containing t:

$$\mathbf{\hat{B}[t]} = \begin{cases} \frac{1}{2}\left(\mathbf{B^1[1]} + \mathbf{u^1[1]} + \mathbf{B^2[1]} + \mathbf{u^2[1]}\right) & \text{if } t = 1 \\ \frac{1}{3}\sum_{t_0 \in \{t-1, t, t+1\}}\left(\mathbf{B^{t_0}[t]} + \mathbf{u^{t_0}[t]}\right) & \text{if } 2 \le t \le T-1 \\ \frac{1}{2}\left(\mathbf{B^{T-1}[T]} + \mathbf{u^{T-1}[T]} + \mathbf{B^{T}[T]} + \mathbf{u^{T}[T]}\right) & \text{if } t = T \end{cases} \tag{55}$$

Projection onto Feasible Set:

$$\mathbf{\hat{B}[t]} \leftarrow \max\left(\underline{B}, \min\left(\overline{B}, \mathbf{\hat{B}[t]}\right)\right) \tag{56}$$

3.5 Dual Update (Localized)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$ and $t \in \mathcal{T}_{\text{local}}^{t_0}$:

$$\mathbf{u^{t_0}[t]} \leftarrow \mathbf{u^{t_0}[t]} + \left(\mathbf{B^{t_0}[t]} - \mathbf{\hat{B}[t]}\right) \tag{57}$$

Number of Dual Updates per Iteration:

- Subproblem $t_0 = 1$: Update 2 dual variables ($\mathbf{u^1[1]}, \mathbf{u^1[2]}$)
- Subproblems $t_0 = 2, \ldots, T-1$: Update 3 dual variables each
- Subproblem $t_0 = T$: Update 2 dual variables ($\mathbf{u^T[T-1]}, \mathbf{u^T[T]}$)
- **Total**: $2 + 3(T-2) + 2 = 3T - 2$ dual variable updates

3.6 Convergence Criteria

Primal Residual (consensus violation):

$$\|r^k\|_2 = \frac{1}{T}\sqrt{\sum_{t_0=1}^{T}\sum_{t\in\mathcal{T}_{\text{local}}^{t_0}}\left(\mathbf{B^{t_0}[t]} - \hat{\mathbf{B}}[\mathbf{t}]\right)^2} \tag{58}$$

Dual Residual (consensus change):

$$\|s^k\|_2 = \frac{\rho}{T}\sqrt{\sum_{t=1}^{T}\left(\hat{\mathbf{B}}^{\mathbf{k}}[\mathbf{t}] - \hat{\mathbf{B}}^{\mathbf{k-1}}[\mathbf{t}]\right)^2} \tag{59}$$

Stopping Criteria:

$$\text{Converged if:} \quad \|r^k\|_2 \le \epsilon_{\text{pri}} \quad \text{and} \quad \|s^k\|_2 \le \epsilon_{\text{dual}} \tag{60}$$

3.7 Complexity Comparison

Formulation	Variables/Subproblem	Coupling Variables	Total Storage
Localized tADMM (BFM-NL)	2–3 SOC + 2–3 power + network	2–3	$3T$
Localized Copper Plate	**2–3 SOC + 1–2 power**	**2–3**	**3T**

Table 1: tADMM formulations with localized temporal coupling

Copper Plate Advantages:

- **Minimal subproblem size**: 3–5 decision variables per subproblem
- **No spatial coupling**: Pure temporal decomposition
- **Fast solves**: Each subproblem is a small LP/QP (<1ms with Gurobi)
- **Scalability**: Computational cost grows linearly with T (not T^2)

3.8 Example: 24-Hour Copper Plate Problem

Problem Data:

- Time periods: $T = 24$ (hourly intervals)
- Battery: $E_{\text{rated}} = 4000$ kWh, $P_{B,R} = 800$ kW
- Load: $P_L \in [800, 1200]$ kW (time-varying)
- Energy price: $c^t \in [0.08, 0.20]$ \$/kWh (sinusoidal)
- Quadratic battery cost: $C_B = 10^{-6} \times \min(c^t)$

Convergence Performance:

- **Fixed** $\rho = 10.0$: ∼200–300 iterations

- **Adaptive ρ (Boyd)**: $\sim$98 iterations (ρ: $10.0 \to 5.0 \to 2.5 \to 1.25$)
- **Solve time**: <5 seconds total (single-threaded Julia)
- **Objective gap vs. centralized**: $< 10^{-6}$ (numerically exact)

4 Numerical Results and Simulation Details

4.1 Test System Specifications

Network Topology: IEEE 123-bus (Single-phase Equivalent - ieee123A_1ph)

- **Buses**: $|\mathcal{N}| = 128$ nodes (including virtual regulator nodes)
- **Branches**: $|\mathcal{L}| = 127$ distribution lines
- **Network Type**: Radial distribution feeder
- **Base Voltage**: $V_{\text{base}} = 2.4018$ kV (line-to-neutral)
- **Base Power**: $S_{\text{base}} = 1000$ kVA
- **Total Load**: 1,163 kW peak demand (85 load points)
- **Substation**: Node 1, voltage set to 1.03 p.u. (2.474 kV)

4.2 DER Penetration and Battery Placement

Photovoltaic (PV) Systems:

- **Number of PV nodes**: $|\mathcal{D}| = 17$ nodes with PV
- **PV Penetration**: 13.3% of buses (17/128 nodes), 20% of loads (17/85 loads)
- **Total PV Capacity**: 77.0 kW (peak rated capacity)
- **Individual PV Size**: 2.33–16.33 kW per installation
  - Most common: 4.67 kW (11 units)
  - Small: 2.33 kW (5 units)
  - Large: 16.33 kW (1 unit at node 67)
- **PV-to-Load Ratio**: 6.6% (77 kW PV / 1,163 kW load)
- **Power Factor**: Inverter-coupled, capable of reactive power dispatch (0.85–1.0)

Battery Energy Storage Systems (BESS):

- **Number of batteries**: $|\mathcal{B}| = 26$ battery nodes
- **Battery Penetration**: 20.3% of buses (26/128 nodes), 30.6% of loads (26/85 loads)
- **Co-location**: Batteries placed at load/PV buses for local energy management
- **Individual Battery Capacity**:
  - Energy: E_{rated} ranges from 9.33–98.0 kWh per battery
  - Power: P_{rated} ranges from 2.33–24.5 kW per battery
  - Duration: Mostly 4-hour duration (typical: 18.66 kWh / 4.67 kW)
  - SOC limits: $[20\%, 90\%]$ of rated capacity (70% usable range)

- Efficiency: 95% charge/discharge efficiency
- Initial SOC: 62.5% (mid-range start)

- **Total System Storage**: 527.3 kWh energy, 131.8 kW power
- **Battery-to-Load Ratio**: 11.3% power (132/1,163 kW), 45.3% energy (527/1,163 kWh)

4.3 Temporal Resolution and Simulation Horizon

- **Time Periods**: $T = 48$ intervals (24-hour horizon)
- **Time Step**: $\Delta t = 0.5$ hours (30 minutes)
- **Load Profile**: Sinusoidal with morning/evening peaks (LoadShapeLoad)
  - Base load: 80% of peak
  - Peak variation: ±20% sinusoidal modulation
- **PV Profile**: Bell curve (active 25%–75% of horizon, zeros at night)
  - Nighttime (0%–25%, 75%–100%): Zero generation
  - Daytime (25%–75%): Half-sine wave (0 at sunrise/sunset, 1 at noon)
- **Energy Prices**: Sinusoidal time-varying tariff
  - Base price: \$0.08/kWh
  - Peak surcharge: ±\$0.12/kWh (range: \$0.02–\$0.20/kWh)

4.4 Optimization Problem Size

Per-Subproblem Variables (for each $t_0 \in \{1, \ldots, 48\}$):

- **Network State Variables (single time t_0):**
  - Real power flows: $P_{ij}^{t_0}$ for $|\mathcal{L}| = 127$ branches
  - Reactive power flows: $Q_{ij}^{t_0}$ for $|\mathcal{L}| = 127$ branches
  - Squared voltage magnitudes: $v_j^{t_0}$ for $|\mathcal{N}| = 128$ nodes
  - Squared current magnitudes: $\ell_{ij}^{t_0}$ for $|\mathcal{L}| = 127$ branches (SOCP)
  - PV reactive power: $q_{D,j}^{t_0}$ for $|\mathcal{D}| = 17$ PV nodes
  - Substation power: $P_{\text{subs}}^{t_0}, Q_{\text{subs}}^{t_0}$
  - **Subtotal**: $127 + 127 + 128 + 127 + 17 + 2 = 528$ network variables
- **Battery Variables (full horizon):**
  - Battery power: $P_{B,j}^t$ for all $t \in \{1, \ldots, 48\}$, $j \in \mathcal{B}$ (optimized across full horizon)
  - Local SOC trajectory: $B_j^{t_0}[t]$ for $t \in \mathcal{T}_{\text{local}}^{t_0}$ (2–3 time steps per battery)
  - **Subtotal**: $48 \times 26 + (2\text{–}3) \times 26 = 1248 + 52\text{–}78 = 1300\text{–}1326$ battery variables
- **Total per subproblem**: $528 + 1313 \approx 1841$ decision variables (avg)

Global Consensus Variables:

- Battery SOC consensus: $\hat{B}_j[t]$ for all $j \in \mathcal{B}$, $t \in \mathcal{T}$
- **Total**: $26 \times 48 = 1,248$ consensus variables

Total ADMM Problem:

- **Subproblems**: $T = 48$ parallel optimization problems
- **Total local variables**: $48 \times 1841 \approx 88,400$ variables (distributed)
- **Coupling variables**: $1,248$ consensus SOC variables (centralized averaging)
- **Dual variables**: $(3T - 2) \times |\mathcal{B}| = 142 \times 26 = 3,692$ scaled duals

4.5 Convergence Performance

Typical Convergence (ieee123A_1ph, T=48, 26 batteries):

- **Convergence Tolerances**:
  - Primal tolerance: $\epsilon_{\text{pri}} = 10^{-5}$
  - Dual tolerance: $\epsilon_{\text{dual}} = 10^{-4}$ (relaxed for faster convergence)
- **Iterations to convergence**: Typically 100–200 iterations with adaptive ρ and watchdog
- **Final primal residual**: $\|r^*\| \approx 10^{-6}$–10^{-5} (satisfies ϵ_{pri})
- **Final dual residual**: $\|s^*\| \approx 10^{-5}$–10^{-4} (satisfies ϵ_{dual})
- **Penalty parameter evolution**: ρ typically increases from
  $approx 20,000$ to
  $approx 160,000$ in Phase 1, then stabilizes with watchdog protection

Computational Performance:

- **Solver**: Gurobi 11.0.3 (commercial SOCP solver, NonConvex=2 mode)
- **Threading**: Julia multi-threading (Threads.@threads) for parallel subproblem solves
- **Hardware**: Typical workstation with 8–16 CPU cores
- **Subproblem solve time**:
  $approx 50$–150 ms per subproblem (SOCP with
  $approx 1,841$ variables,
  $approx 800$ constraints)
- **Per-iteration time**:
  $approx 100$–300 ms/iteration (parallel solve + consensus + dual update)
- **Total wall-clock time**: 15–60 seconds (48 subproblems, 100–200 iterations, parallel execution)

Physical Results and Validation
Voltage Regulation:

- **Voltage limits**: $[0.95, 1.05]$ p.u. (enforced as squared: $[0.9025, 1.1025]$ p.u.^{2})
- **Substation voltage**: 1.03 p.u. (2.474 kV, fixed at node 1)
- **Voltage drop**: Varies with load and PV injection across 128 nodes
- **Voltage constraint satisfaction**: All nodes within limits for optimal dispatch

Battery and PV Operation:

- **Battery dispatch**: Charge during low-cost periods, discharge during peak price
- **SOC coordination**: Consensus algorithm ensures smooth SOC trajectories across all 26 batteries
- **PV reactive power**: Optimized within inverter capability curve ($q_D^2 + p_D^2 \leq S_D^2$)
- **Network losses**: Reduced through coordinated battery and PV dispatch

Algorithm Robustness:

- **Adaptive ρ with watchdog**: Two-phase strategy with hysteresis-based watchdog ensures robust convergence
- **Fixed ρ**: Requires manual tuning and often slower convergence or divergence
- **Phase 1 → Phase 2 transition**: Typically occurs around iteration 15–30 when $\|r\| < \epsilon_{\text{pri}}$
- **Watchdog activation**: Triggers when primal residual stays elevated for 20 consecutive iterations, doubling ρ to enforce consensus

5 Appendix: Full Variable and Parameter Definitions

5.1 System Bases

$$\mathrm{kV}_B = \frac{4.16}{\sqrt{3}} \text{ kV (phase-to-neutral)} \tag{61}$$

$$\mathrm{kVA}_B = 1000 \text{ kVA} \tag{62}$$

$$P_{\mathrm{BASE}} = 1000 \text{ kW} \tag{63}$$

$$E_{\mathrm{BASE}} = 1000 \text{ kWh per hour} \tag{64}$$

5.2 SOC Bound Definitions

$$\underline{B} = \mathrm{SOC}_{\min} \cdot E_{\mathrm{Rated}} \tag{65}$$

$$\overline{B} = \mathrm{SOC}_{\max} \cdot E_{\mathrm{Rated}} \tag{66}$$

5.3 Physical Interpretation

- $P_B[t] > 0$: Battery discharging (providing power to the system)
- $P_B[t] < 0$: Battery charging (consuming power from the system)
- $B[t]$: Battery state of charge at the end of period t
- $\underline{B} = \mathrm{SOC}_{\min} \cdot E_{\mathrm{Rated}}$: Lower SOC bound
- $\overline{B} = \mathrm{SOC}_{\max} \cdot E_{\mathrm{Rated}}$: Upper SOC bound