Multi-Period Optimal Power Flow:
LinDistFlow & Copper Plate tADMM Formulations

# 1 LinDistFlow MPOPF with tADMM

## 1.1 Problem Overview

The Temporal ADMM (tADMM) algorithm decomposes the multi-period optimal power flow problem for distribution networks into $T$ subproblems, each corresponding to one time period. This formulation uses the linearized DistFlow model to capture network physics including voltage drops and reactive power flows.

## 1.2 Variable Color Coding

- $\mathbf{B_j^{t_0}[t]}$: Local SOC variables for battery $j$ in subproblem $t_0$, evaluated at time $t$ (blue)
- $\mathbf{\hat{B}_j[t]}$: Global consensus SOC for battery $j$ at time $t$ (red)
- $\mathbf{u_j^{t_0}[t]}$: Local scaled dual variables for battery $j$ in subproblem $t_0$, for time $t$ (green)

## 1.3 Sets and Indices

- $\mathcal{N}$: Set of all nodes (buses)
- $\mathcal{L}$: Set of all branches (lines)
- $\mathcal{L}_1$: Set of branches connected to substation (node 1)
- $\mathcal{B}$: Set of nodes with batteries
- $\mathcal{D}$: Set of nodes with PV (DER)
- $\mathcal{T} = \{1, 2, \ldots, T\}$: Set of time periods
- $t_0 \in \mathcal{T}$: Index for a specific time period in tADMM decomposition
- $j \in \mathcal{N}$: Node index
- $(i, j) \in \mathcal{L}$: Branch from node $i$ to node $j$

## 1.4 tADMM Algorithm Structure

The algorithm alternates between three update steps:

### 1.5 Step 1: Subproblem Update (Blue Variables)

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\min_{\substack{P_{\text{Subs}}^{t_0}, Q_{\text{Subs}}^{t_0}, \\ P_{ij}^{t_0}, Q_{ij}^{t_0}, v_j^{t_0}, q_{D,j}^{t_0}, \\ P_{B,j}^{t}, \mathbf{B_j^{t_0}[t]} \\ \forall j \in \mathcal{B}, t \in \mathcal{T}}} \quad c^{t_0} \cdot P_{\text{Subs}}^{t_0} \cdot P_{\text{BASE}} \cdot \Delta t + C_B \sum_{j \in \mathcal{B}} \left(P_{B,j}^{t_0}\right)^2 \cdot P_{\text{BASE}}^2 \cdot \Delta t$$

$$+ \frac{\rho}{2} \sum_{j \in \mathcal{B}} \sum_{t=1}^{T} \left(\mathbf{B_j^{t_0}[t]} - \mathbf{\hat{B}_j[t]} + \mathbf{u_j^{t_0}[t]}\right)^2 \tag{1}$$

**Subject to:**
**Spatial Network Constraints (only for time $t_0$):**

$$\text{Real power balance (substation):} \quad P_{\text{Subs}}^{t_0} - \sum_{(1,j) \in \mathcal{L}_1} P_{1j}^{t_0} = 0 \tag{2}$$

$$\text{Real power balance (nodes):} \quad P_{ij}^{t_0} - \sum_{(j,k) \in \mathcal{L}} P_{jk}^{t_0} = P_{B,j}^{t_0} + p_{D,j}^{t_0} - p_{L,j}^{t_0}, \quad \forall (i,j) \in \mathcal{L}, \tag{3}$$

$$\text{Reactive power balance (substation):} \quad Q_{\text{Subs}}^{t_0} - \sum_{(1,j) \in \mathcal{L}_1} Q_{1j}^{t_0} = 0 \tag{4}$$

$$\text{Reactive power balance (nodes):} \quad Q_{ij}^{t_0} - \sum_{(j,k) \in \mathcal{L}} Q_{jk}^{t_0} = q_{D,j}^{t_0} - q_{L,j}^{t_0}, \quad \forall (i,j) \in \mathcal{L}, \tag{5}$$

$$\text{KVL constraints:} \quad v_i^{t_0} - v_j^{t_0} = 2(r_{ij} P_{ij}^{t_0} + x_{ij} Q_{ij}^{t_0}), \quad \forall (i,j) \in \mathcal{L} \tag{6}$$

$$\text{Voltage limits:} \quad (V_{\min,j})^2 \leq v_j^{t_0} \leq (V_{\max,j})^2, \quad \forall j \in \mathcal{N} \tag{7}$$

$$\text{PV reactive limits:} \quad -\sqrt{(S_{D,j})^2 - (p_{D,j}^{t_0})^2} \leq q_{D,j}^{t_0} \leq \sqrt{(S_{D,j})^2 - (p_{D,j}^{t_0})^2}, \quad \forall j \in \mathcal{D} \tag{8}$$

**Temporal Battery Constraints (entire horizon $t \in \{1, \ldots, T\}$):**

$$\text{Initial SOC:} \quad \mathbf{B_j^{t_0}[1]} = B_{0,j} - P_{B,j}^{1} \cdot \Delta t, \quad \forall j \in \mathcal{B} \tag{9}$$

$$\text{SOC trajectory:} \quad \mathbf{B_j^{t_0}[t]} = \mathbf{B_j^{t_0}[t-1]} - P_{B,j}^{t} \cdot \Delta t, \quad \forall t \in \{2, \ldots, T\}, \ j \in \mathcal{B} \tag{10}$$

$$\text{SOC limits:} \quad \text{SOC}_{\min,j} \cdot B_{\text{rated},j} \leq \mathbf{B_j^{t_0}[t]} \leq \text{SOC}_{\max,j} \cdot B_{\text{rated},j}, \quad \forall t \in \mathcal{T}, \ j \in \mathcal{B} \tag{11}$$

$$\text{Power limits:} \quad -P_{B,\text{rated},j} \leq P_{B,j}^{t} \leq P_{B,\text{rated},j}, \quad \forall t \in \mathcal{T}, \ j \in \mathcal{B} \tag{12}$$

**Key Formulation Notes:**

- **Network variables** ($P_{\text{Subs}}^{t_0}, Q_{\text{Subs}}^{t_0}, P_{ij}^{t_0}, Q_{ij}^{t_0}, v_j^{t_0}, q_{D,j}^{t_0}$) are optimized *only* for time step $t_0$
- **Battery power** $P_{B,j}^{t}$ is optimized for the *entire* horizon $t \in \{1, \ldots, T\}$
- **Local SOC trajectory** $\mathbf{B_j^{t_0}[t]}$ (blue) is computed for *all* time steps $t \in \{1, \ldots, T\}$

- The ADMM consensus penalty compares the full local trajectory $\mathbf{B_j^{to}[t]}$ with the global master copy $\mathbf{\hat{B}_j[t]}$ (red)
- Each battery $j \in \mathcal{B}$ has its own local/global SOC variables and dual variables

### 1.6 Step 2: Consensus Update (Red Variables)

For each battery $j \in \mathcal{B}$ and each time period $t \in \mathcal{T}$:

$$\mathbf{\hat{B}_j[t]} = \text{clamp}\left(\frac{1}{T}\sum_{t_0=1}^{T}\left(\mathbf{B_j^{to}[t]} + \mathbf{u_j^{to}[t]}\right), \underline{B}_j, \overline{B}_j\right) \tag{13}$$

where $\underline{B}_j = \text{SOC}_{\text{min},j} \cdot B_{\text{rated},j}$ and $\overline{B}_j = \text{SOC}_{\text{max},j} \cdot B_{\text{rated},j}$.

### 1.7 Step 3: Dual Update (Green Variables)

For each battery $j \in \mathcal{B}$, each subproblem $t_0 \in \mathcal{T}$, and each time period $t \in \mathcal{T}$:

$$\mathbf{u_j^{to}[t]} := \mathbf{u_j^{to}[t]} + \left(\mathbf{B_j^{to}[t]} - \mathbf{\hat{B}_j[t]}\right) \tag{14}$$

### 1.8 Convergence Criteria

**Primal Residual (Consensus Violation):**

$$\|r^k\|_2 = \frac{1}{|\mathcal{B}|}\sqrt{\sum_{j\in\mathcal{B}}\sum_{t=1}^{T}\left(\frac{1}{T}\sum_{t_0=1}^{T}\mathbf{B_j^{to}[t]} - \mathbf{\hat{B}_j[t]}\right)^2} \leq \epsilon_{\text{pri}} \tag{15}$$

**Dual Residual (Consensus Change):**

$$\|s^k\|_2 = \frac{\rho}{|\mathcal{B}|}\sqrt{\sum_{j\in\mathcal{B}}\sum_{t=1}^{T}\left(\mathbf{\hat{B}_j^k[t]} - \mathbf{\hat{B}_j^{k-1}[t]}\right)^2} \leq \epsilon_{\text{dual}} \tag{16}$$

# 2 Copper Plate MPOPF with tADMM (Simplified Case)

## 2.1 Problem Overview

The Temporal ADMM (tADMM) algorithm decomposes the multi-period optimal power flow problem into $T$ single-step subproblems, each corresponding to one time period. The hope is to enable parallel computation and improved scalability while still retaining solution optimality.

## 2.2 Variable Color Coding

Following the handwritten PDF formulation:

- $\mathbf{B^{t_0}}$: Local SOC variables for subproblem $t_0$ (blue)
- $\hat{\mathbf{B}}$: Global consensus SOC trajectory (red)
- $\mathbf{u^{t_0}}$: Local scaled dual variables for subproblem $t_0$ (green)

## 2.3 tADMM Algorithm Structure

The algorithm alternates between three update steps:

## 2.4 Step 1: Primal Update (Blue Variables) tADMM Optimization Model

For each subproblem $t_0 \in \{1, 2, \ldots, T\}$:

$$\min_{P_{\text{subs}}^{t_0}, P_B^{t_0}, \mathbf{B^{t_0}}} \quad C^{t_0} \cdot P_{\text{subs}}^{t_0} + C_B \cdot \left(P_B^{t_0}\right)^2 + \frac{\rho}{2} \left\| \mathbf{B^{t_0}} - \hat{\mathbf{B}} + \mathbf{u^{t_0}} \right\|_2^2 \tag{17}$$

**Subject to SOC Dynamics for Entire Trajectory:**

$$\mathbf{B^{t_0}[1]} = B_0 - P_B^{t_0} \cdot \Delta t \tag{18}$$

$$\mathbf{B^{t_0}[t]} = \mathbf{B^{t_0}[t-1]} - P_B^{t_0} \cdot \Delta t, \quad \forall t \in \{2, \ldots, T\} \tag{19}$$

$$P_{\text{subs}}^{t_0} + P_B^{t_0} = P_L[t_0] \tag{20}$$

$$-P_{B,R} \le P_B^{t_0} \le P_{B,R} \tag{21}$$

$$\underline{B} \le \mathbf{B^{t_0}[t]} \le \overline{B}, \quad \forall t \in \{1, \ldots, T\} \tag{22}$$

**Key Formulation Notes:**

- Each subproblem $t_0$ optimizes the battery power $P_B^{t_0}$ for *only* time step $t_0$
- However, the SOC trajectory $\mathbf{B^{t_0}[t]}$ is computed for *all* time steps $t \in \{1, \ldots, T\}$
- This ensures that the ADMM penalty term can compare the full trajectory $\mathbf{B^{t_0}}$ with the consensus $\hat{\mathbf{B}}$
- The power balance constraint is enforced only for the specific time $t_0$

### 2.5 Step 2: Consensus Update (Red Variables)

$$\hat{\mathbf{B}}[\mathbf{t}] = \text{clamp}\left(\frac{1}{T}\sum_{t_0=1}^{T}\left(\mathbf{B^{t_0}}[\mathbf{t}] + \mathbf{u^{t_0}}[\mathbf{t}]\right), \underline{B}, \overline{B}\right) \tag{23}$$

$$\forall t \in \{1, 2, \ldots, T-1\} \tag{24}$$

$$\hat{\mathbf{B}}[\mathbf{T}] = B_{T,\text{target}} \quad \text{(if terminal constraint exists)} \tag{25}$$

### 2.6 Step 3: Dual Update (Green Variables)

$$\mathbf{u^{t_0}}[\mathbf{t}] := \mathbf{u^{t_0}}[\mathbf{t}] + \left(\mathbf{B^{t_0}}[\mathbf{t}] - \hat{\mathbf{B}}[\mathbf{t}]\right) \tag{26}$$

$$\forall t_0 \in \{1, \ldots, T\}, \forall t \in \{1, \ldots, T\} \tag{27}$$

## 3 Convergence Criteria

The algorithm terminates when both residuals fall below specified thresholds:

### 3.1 Primal Residual (Consensus Violation)

$$\|r^k\|_2 = \left\|\text{vec}\left(\left\{\mathbf{B^{t_0}} - \hat{\mathbf{B}}\right\}_{t_0=1}^{T}\right)\right\|_2 \leq \epsilon_{\text{pri}} \tag{28}$$

### 3.2 Dual Residual (Consensus Change)

$$\|s^k\|_2 = \rho\left\|\hat{\mathbf{B}}^{\mathbf{k}} - \hat{\mathbf{B}}^{\mathbf{k-1}}\right\|_2 \leq \epsilon_{\text{dual}} \tag{29}$$

## 4 Final Solution Recovery

After convergence, the final power schedules are reconstructed from the consensus trajectory:

$$P_B^t = \begin{cases} \frac{-\left(\hat{\mathbf{B}}[\mathbf{t}] - B_0\right)}{\Delta t} & \text{if } t = 1 \\ \frac{-\left(\hat{\mathbf{B}}[\mathbf{t}] - \hat{\mathbf{B}}[\mathbf{t-1}]\right)}{\Delta t} & \text{if } t > 1 \end{cases} \tag{30}$$

$$P_{\text{Subs}}^t = P_L[t] - P_B^t \tag{31}$$

The final objective value includes both energy and battery costs:

$$J_{\text{total}} = \sum_{t=1}^{T}\left[C^t \cdot P_{\text{Subs}}^t \cdot \Delta t + C_B \cdot \left(P_B^t\right)^2 \cdot \Delta t\right] \tag{32}$$

## 5 Algorithm Parameters

### 5.1 Objective Function Components

The tADMM objective function for each subproblem $t_0$ consists of three terms:

$$\text{Energy Cost:} \quad C^{t_0} \cdot P_{\text{subs}}^{t_0} \cdot \Delta t \tag{33}$$

$$\text{Battery Quadratic Cost:} \quad C_B \cdot \left(P_B^{t_0}\right)^2 \cdot \Delta t \tag{34}$$

$$\text{ADMM Penalty:} \quad \frac{\rho}{2} \left\| \mathbf{B^{t_0}} - \hat{\mathbf{B}} + \mathbf{u^{t_0}} \right\|_2^2 \tag{35}$$

Where:

- $C^{t_0}$: Energy price at time $t_0$ [$/kWh]
- $C_B$: Battery quadratic cost coefficient [$/kW$^2$/h] (typically $10^{-6} \times \min(C^t)$)
- $\rho$: ADMM penalty parameter

The battery quadratic cost term $C_B \cdot \left(P_B^{t_0}\right)^2$ serves as a regularization to:

1. Prevent excessive battery cycling
2. Encourage smoother power trajectories
3. Improve numerical conditioning of the optimization problem

### 5.2 Algorithmic Parameters

- **Penalty Parameter**: $\rho$ (typically 0.1 to 10.0)
- **Primal Tolerance**: $\epsilon_{\text{pri}} = 10^{-3}$
- **Dual Tolerance**: $\epsilon_{\text{dual}} = 10^{-3}$
- **Maximum Iterations**: 1000

## 6 Appendix: Full Variable and Parameter Definitions

### 6.1 System Bases

$$\text{kV}_B = \frac{4.16}{\sqrt{3}} \text{ kV (phase-to-neutral)} \tag{36}$$

$$\text{kVA}_B = 1000 \text{ kVA} \tag{37}$$

$$P_{\text{BASE}} = 1000 \text{ kW} \tag{38}$$

$$E_{\text{BASE}} = 1000 \text{ kWh per hour} \tag{39}$$

### 6.2 SOC Bound Definitions

$$\underline{B} = \text{SOC}_{\min} \cdot E_{\text{Rated}} \tag{40}$$

$$\overline{B} = \text{SOC}_{\max} \cdot E_{\text{Rated}} \tag{41}$$

### 6.3 Physical Interpretation

- $P_B[t] > 0$: Battery discharging (providing power to the system)
- $P_B[t] < 0$: Battery charging (consuming power from the system)
- $B[t]$: Battery state of charge at the end of period $t$
- $\underline{B} = \text{SOC}_{\min} \cdot E_{\text{Rated}}$: Lower SOC bound
- $\overline{B} = \text{SOC}_{\max} \cdot E_{\text{Rated}}$: Upper SOC bound