# Differential Dynamic Programming for Multistage Uncertain Optimal Control

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*Abstract*—**When a multistage system is transferred from one stage to the next one, it could be affected by some disturbance. This kind of system is called a multistage uncertain system if the disturbance is an uncertain factor. Based on the uncertainty theory and the equation of optimality in uncertain optimal control, a differential dynamic programming algorithm for an optimal control problem subject to a multistage uncertain system is established. The differential dynamic programming algorithm is implemented from an initial nominal control, which is provided by the genetic algorithm, to an optimal control of the problem step by step. Finally, a numerical example that the system is nonlinear with the disturbance of linear uncertain variables is given to illustrate the feasibility of the algorithm.**

*Keywords*-**multistage uncertain system; uncertain optimal control; differential dynamic programming; genetic algorithm**

## I. Introduction

With the development of computer technology and applications, to study system control problems by numerical calculation methods becomes one of the research fields in control. For a continuous system, we can transform it into a discrete system by using the method of sampling. Therefore, study of discrete time systems is of importance. The dynamic programming, as a kind of numerical algorithm, has a good effect in dealing with the low-order deterministic optimal control problems of discrete systems. However, researchers have found that using dynamic programming algorithm to deal with high order systems will increase dramatically the computer's storage and computation time. This is the so-called "dimension disaster". In order to avoid this event, Mayne [8] in 1966 proposed a differential dynamic programming algorithm (DDP) to deal with discrete time systems. Jacobson [1], [2] used this method to study continuous systems. Mayne and Jacobson's book [9] gave us a comprehensive summary of the method in 1970. Some further research on differential dynamic programming may be seen in such as Li and Todorov [4], Liao and Shoemaker [5], and Ohno [10]. Todorov [13] introduced stochastic differential dynamic programming to study stochastic optimal control problems in 2010.

Based on the uncertainty theory founded by Liu [6], [7], Zhu [16] introduced uncertain optimal control problem in 2010. Xu and Zhu [14] studied a bang-bang control problem for continuous uncertain systems in 2012. While Kang and Zhu[3] studied a bang-bang control problem for multistage uncertain systems in 2012. Sheng and Zhu [11] introduced an optimistic value optimal control model for uncertain systems. Further research may be seen in [12].

In this paper, we will establish a differential dynamic programming algorithm to deal with an optimal control problem subject to a multistage uncertain system.

## II. Preliminary

For convenience, we introduce some useful concepts at first [6]. Let $\Gamma$ be a nonempty set and $\mathcal{L}$ be a $\sigma$-algebra over $\Gamma$. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $\mathcal{M}$ defined on the $\sigma$-algebra $\mathcal{L}$ is called an uncertain measure if it satisfies three axioms: (Axiom 1) $\mathcal{M}\{\Gamma\} = 1$; (Axiom 2) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$, for $\Lambda \in \mathcal{L}$; (Axiom 3) $\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$, for $\Lambda_i \in \mathcal{L}$, $i = 1, 2, \cdots$.

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be an uncertainty space. An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers $\mathcal{R}$. The distribution function $\Phi : \mathcal{R} \rightarrow [0, 1]$ of an uncertain variable $\xi$ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$. The expected value of $\xi$ is defined by $E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\}\mathrm{d}r$ provided that at least one of the two integrals is finite. The variance of $\xi$ is $V[\xi] = E[(\xi - E[\xi])^2]$. For a common uncertain linear variable $\xi \sim \mathcal{L}(-1, 1)$, the expected value of $\xi^2 + b\xi$ is provided in [17] by

$$E[\xi^2 + b\xi] = \begin{cases} 7/24, & b = 0 \\ \frac{1}{48}(|b|^3 + 12b^2 - 12|b| + 14), & 0 < |b| < 1 \\ \frac{1}{48}(|b|^3 - 6b^2 + 12|b| + 8), & 1 \leq |b| < 2 \\ 1/3, & |b| \geq 2. \end{cases} \quad (1)$$

Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots, n$. Then the product uncertainty measure $\mathcal{M}$ is an uncertain measure on the $\sigma$-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ satisfying $\mathcal{M}\{\prod_{k=1}^n \Lambda_k\} = \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\}$. The uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if $\mathcal{M}\{\bigcap_{i=1}^m \xi_i \in B_i\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}$ for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

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## III. Multistage Uncertain Optimal Control Model

Consider the following problem:

$$
\begin{cases}
J(x_0, 0) = \min\limits_{\substack{u_j \in U_j \\ 0 \le j \le N-1}} E\left[\sum_{j=0}^{N-1} L(x_j, u_j, j) + G(x_N, N)\right] \\
\text{s.t.} \\
\quad x_{j+1} = f(x_j, u_j, j) + \sigma_{j+1}\xi_{j+1} \\
\quad j = 0, 1, \ldots, N-1,
\end{cases}
\tag{2}
$$

where $x_j$ is the state at the stage $j$ with the crisp initial state $x_0$, $u_j \in U_j$ the control variable in the constraint set $U_j$; $L(x, u, j)$ and $f(x, u, j)$ are twice differentiable functions in $x$ and $u$, $G(x, N)$ a twice differentiable function in $x$; $\xi_1, \xi_2, \ldots, \xi_N$ are independent uncertain variables. For any $0 \le k \le N-1$, let $J(x_k, k)$ denote the optimal value obtained between the stage $k$ and the stage $N$ with the condition that at stage $k$ we are in state $x_k$.

*Theorem 1:* (Kang and Zhu [3]) For the model (2), we have:

$$
J(x_N, N) = G(x_N, N), \tag{3}
$$

$$
J(x_k, k) = \min_{u_k \in U_k} E\left[L(x_k, u_k, k) + J(x_{k+1}, k+1)\right], \tag{4}
$$

$$
k = N-1, N-2, \ldots, 1, 0.
$$

## IV. Differential Dynamic Programming

To begin with, for $\bar{u}_k$, $k = 0, 1, 2, \ldots, N-1$, nominal states $\bar{x}_{k+1}$ ($k = 0, 1, 2, \ldots, N-1$) follow from the system

$$
\bar{x}_{k+1} = f(\bar{x}_k, \bar{u}_k, k), \quad \bar{x}_0 = x_0. \tag{5}
$$

By Taylor formula, we have

$$
\begin{aligned}
f(x_k, &u_k, k) \\
&= f(\bar{x}_k + \delta x_k, \bar{u}_k + \delta u_k, k) \\
&\approx f(\bar{x}_k, \bar{u}_k, k) + f_x \delta x_k + f_u \delta u_k \\
&\quad + \frac{1}{2} f_{xx} \delta x_k^2 + f_{xu} \delta x_k \delta u_k + \frac{1}{2} f_{uu} \delta u_k^2,
\end{aligned}
\tag{6}
$$

where $\delta x_k = x_k - \bar{x}_k$, $\delta u_k = u_k - \bar{u}_k$, $f_x$ and $f_u$ are the first order differentials of the function $f(x, u, k)$ in $x$ and $u$, respectively, $f_{xx}$, $f_{xu}$ and $f_{uu}$ are the second order differentials of the function $f(x, u, k)$ in $x$ and $u$, respectively. It follows from (5), (6) and

$$
x_{k+1} = f(x_k, u_k, k) + \sigma_{k+1}\xi_{k+1} \tag{7}
$$

that

$$
\begin{aligned}
\delta x_{k+1} &= x_{k+1} - \bar{x}_{k+1} \\
&= f(x_k, u_k, k) + \sigma_{k+1}\xi_{k+1} - f(\bar{x}_k, \bar{u}_k, k) \\
&\approx f_x \delta x_k + f_u \delta u_k + \frac{1}{2} f_{xx} \delta x_k^2 \\
&\quad + f_{xu} \delta x_k \delta u_k + \frac{1}{2} f_{uu} \delta u_k^2 \\
&\quad + \sigma_{k+1}\xi_{k+1}.
\end{aligned}
\tag{8}
$$

Similarly, by Taylor formula, we have

$$
\begin{aligned}
L(x_k, &u_k, k) \\
&= L(\bar{x}_k + \delta x_k, \bar{u}_k + \delta u_k, k) \\
&\approx L(\bar{x}_k, \bar{u}_k, k) + L_x \delta x_k + L_u \delta u_k \\
&\quad + \frac{1}{2} L_{xx} \delta x_k^2 + L_{xu} \delta x_k \delta u_k + \frac{1}{2} L_{uu} \delta u_k^2,
\end{aligned}
\tag{9}
$$

and

$$
J(\bar{x}_k + \delta x_k, k) \approx J(\bar{x}_k, k) + J_x(\bar{x}_k, k)\delta x_k + \frac{1}{2} J_{xx}(\bar{x}_k, k)\delta x_k^2. \tag{10}
$$

Let

$$
\bar{J}(\bar{x}_k, k) = \sum_{j=k}^{N-1} L(\bar{x}_j, \bar{u}_j, j) + G(\bar{x}_N, N), \tag{11}
$$

and

$$
\Delta_k = J(\bar{x}_k, k) - \bar{J}(\bar{x}_k, k). \tag{12}
$$

It follows from (10), (11) and (12) that

$$
\begin{aligned}
J(\bar{x}_k + \delta x_k, k) &\approx J_x(\bar{x}_k, k)\delta x_k + \frac{1}{2} J_{xx}(\bar{x}_k, k)\delta x_k^2 \\
&\quad + \bar{J}(\bar{x}_k, k) + \Delta_k,
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
J(\bar{x}_{k+1} &+ \delta x_{k+1}, k+1) \\
&\approx J_x(\bar{x}_{k+1}, k+1)\delta x_{k+1} + \frac{1}{2} J_{xx}(\bar{x}_{k+1}, k+1)\delta x_{k+1}^2 \\
&\quad + \bar{J}(\bar{x}_{k+1}, k+1) + \Delta_{k+1}.
\end{aligned}
\tag{14}
$$

Approximating the left side of (4) by (13) and the right side by (9) and (14) yields that

$$
\begin{aligned}
J_x(\bar{x}_k, &k)\delta x_k + \frac{1}{2} J_{xx}(\bar{x}_k, k)\delta x_k^2 + \bar{J}(\bar{x}_k, k) + \Delta_k \\
&= \min_{\substack{\delta u_k \\ \bar{u}_k + \delta u_k \in U_k}} E\Big[ L(\bar{x}_k, \bar{u}_k, k) + L_x \delta x_k + L_u \delta u_k \\
&\quad + \frac{1}{2} L_{xx} \delta x_k^2 + L_{xu} \delta x_k \delta u_k + \frac{1}{2} L_{uu} \delta u_k^2 \\
&\quad + J_x(\bar{x}_{k+1}, k+1)\delta x_{k+1} + \frac{1}{2} J_{xx}(\bar{x}_{k+1}, k+1)\delta x_{k+1}^2 \\
&\quad + \bar{J}(\bar{x}_{k+1}, k+1) + \Delta_{k+1} \Big].
\end{aligned}
\tag{15}
$$

It follows from (11) that

$$
\begin{aligned}
\bar{J}(\bar{x}_k, &k) \\
&= \sum_{j=k}^{N-1} L(\bar{x}_j, \bar{u}_j, j) + G(\bar{x}_N, N) \\
&= L(\bar{x}_k, \bar{u}_k, k) + \left[\sum_{j=k+1}^{N-1} L(\bar{x}_j, \bar{u}_j, j) + G(\bar{x}_N, N)\right] \\
&= L(\bar{x}_k, \bar{u}_k, k) + \bar{J}(\bar{x}_{k+1}, k+1).
\end{aligned}
\tag{16}
$$

It follows from (16) that

$$\bar{J}(\bar{x}_k, k) = \min_{\substack{\delta u_k \\ \bar{u}_k + \delta u_k \in U_k}} E[L(\bar{x}_k, \bar{u}_k, k) + \bar{J}(\bar{x}_{k+1}, k+1)]. \tag{17}$$

It follows from (15) and (17) that

$$\Delta_k + J_x(\bar{x}_k, k)\delta x_k + \frac{1}{2}J_{xx}(\bar{x}_k, k)\delta x_k^2$$
$$= \min_{\substack{\delta u_k \\ \bar{u}_k + \delta u_k \in U_k}} E\Big[\Delta_{k+1} + L_x\delta x_k + L_u\delta u_k$$
$$+ \frac{1}{2}L_{xx}\delta x_k^2 + L_{xu}\delta x_k\delta u_k + \frac{1}{2}L_{uu}\delta u_k^2$$
$$+ J_x(\bar{x}_{k+1}, k+1)\delta x_{k+1} + \frac{1}{2}J_{xx}(\bar{x}_{k+1}, k+1)\delta x_{k+1}^2\Big]. \tag{18}$$

Substituting (8) into (18) yields

$$\Delta_k + J_x(\bar{x}_k, k)\delta x_k + \frac{1}{2}J_{xx}(\bar{x}_k, k)\delta x_k^2$$
$$= \min_{\substack{\delta u_k \\ \bar{u}_k + \delta u_k \in U_k}} E\Big[\Delta_{k+1} + L_x\delta x_k + L_u\delta u_k$$
$$+ \frac{1}{2}L_{xx}\delta x_k^2 + L_{xu}\delta x_k\delta u_k + \frac{1}{2}L_{uu}\delta u_k^2$$
$$+ J_x(\bar{x}_{k+1}, k+1)(f_x\delta x_k + f_u\delta u_k + \frac{1}{2}f_{xx}\delta x_k^2$$
$$+ f_{xu}\delta x_k\delta u_k + \frac{1}{2}f_{uu}\delta u_k^2 + \sigma_{k+1}\xi_{k+1})$$
$$+ \frac{1}{2}J_{xx}(\bar{x}_{k+1}, k+1)(f_x\delta x_k + f_u\delta u_k + \frac{1}{2}f_{xx}\delta x_k^2$$
$$+ f_{xu}\delta x_k\delta u_k + \frac{1}{2}f_{uu}\delta u_k^2 + \sigma_{k+1}\xi_{k+1})^2\Big]. \tag{19}$$

Denote

$$\delta f_k = f_x\delta x_k + f_u\delta u_k,$$
$$\delta^2 f_k = f_{xx}\delta x_k^2 + 2f_{xu}\delta x_k\delta u_k + f_{uu}\delta u_k^2.$$

The (19) is rewritten as

$$J_x(\bar{x}_k, k)\delta x_k + \frac{1}{2}J_{xx}(\bar{x}_k, k)\delta x_k^2 + \Delta_k$$
$$= \min_{\substack{\delta u_k \\ \bar{u}_k + \delta u_k \in U_k}} \Big\{[L_x + J_x(\bar{x}_{k+1}, k+1)f_x]\delta x_k$$
$$+ [L_u + J_x(\bar{x}_{k+1}, k+1)f_u]\delta u_k$$
$$+ \frac{1}{2}[L_{xx} + J_x(\bar{x}_{k+1}, k+1)f_{xx}$$
$$+ J_{xx}(\bar{x}_{k+1}, k+1)f_x^2]\delta x_k^2$$
$$+ [L_{xu} + J_x(\bar{x}_{k+1}, k+1)f_{xu}$$
$$+ J_{xx}(\bar{x}_{k+1}, k+1)f_x f_u]\delta x_k\delta u_k$$
$$+ \frac{1}{2}[L_{uu} + J_x(\bar{x}_{k+1}, k+1)f_{uu_k}$$
$$+ J_{xx}(\bar{x}_{k+1}, k+1)f_u^2]\delta u_k^2$$
$$+ E\Big[\big(J_x(\bar{x}_{k+1}, k+1)$$
$$+ J_{xx}(\bar{x}_{k+1}, k+1)(\delta f_k + \frac{1}{2}\delta^2 f_k)\big)\sigma_{k+1}\xi_{k+1}$$

$$+ \frac{1}{2}J_{xx}(\bar{x}_{k+1}, k+1))\sigma_{k+1}^2\xi_{k+1}^2\Big] + \Delta_{k+1}\Big\}. \tag{20}$$

Let

$$H(x_k, u_k, J_x(\bar{x}_{k+1}, k+1), k)$$
$$= L(x_k, u_k, k) + J_x(\bar{x}_{k+1}, k+1)f(x_k, u_k, k).$$

Then

$$H_{x_k} = L_x + J_x(\bar{x}_{k+1}, k+1)f_x, \tag{21}$$
$$H_{u_k} = L_u + J_x(\bar{x}_{k+1}, k+1)f_u. \tag{22}$$

Denote

$$A_k = H_{x_k x_k} + J_{xx}(\bar{x}_{k+1}, k+1)f_x^2 \tag{23}$$
$$B_k = H_{x_k u_k} + J_{xx}(\bar{x}_{k+1}, k+1)f_x f_u \tag{24}$$
$$C_k = H_{u_k u_k} + J_{xx}(\bar{x}_{k+1}, k+1)f_u^2 \tag{25}$$

Then (19) is rewritten as

$$\Delta_k + J_x(\bar{x}_k, k)\delta x_k + \frac{1}{2}J_{xx}(\bar{x}_k, k)\delta x_k^2$$
$$= \min_{\substack{\delta u_k \\ \bar{u}_k + \delta u_k \in U_k}} \Big\{\Delta_{k+1} + H_{x_k}\delta x_k + H_{u_k}\delta u_k + \frac{1}{2}A_k\delta x_k^2$$
$$+ B_k\delta x_k\delta u_k + \frac{1}{2}C_k\delta u_k^2 + E\Big[\big(J_x(\bar{x}_{k+1}, k+1)$$
$$+ J_{xx}(\bar{x}_{k+1}, k+1)(\delta f_k + \frac{1}{2}\delta^2 f_k)\big)\sigma_{k+1}\xi_{k+1}$$
$$+ \frac{1}{2}J_{xx}(\bar{x}_{k+1}, k+1)\sigma_{k+1}^2\xi_{k+1}^2\Big]\Big\}. \tag{26}$$

Let

$$d_{k+1}$$
$$= \frac{2[J_x(\bar{x}_{k+1}, k+1) + J_{xx}(\bar{x}_{k+1}, k+1)(\delta f_k + \frac{1}{2}\delta^2 f_k)]}{\sigma_{k+1}J_{xx}(\bar{x}_{k+1}, k+1)}. \tag{27}$$

Then

$$E\Big[\big(J_x(\bar{x}_{k+1}, k+1) + J_{xx}(\bar{x}_{k+1}, k+1)(\delta f_k$$
$$+ \frac{1}{2}\delta^2 f_k)\big)\sigma_{k+1}\xi_{k+1} + \frac{1}{2}J_{xx}(\bar{x}_{k+1}, k+1)\sigma_{k+1}^2\xi_{k+1}^2\Big]$$
$$= \frac{1}{2}J_{xx}(\bar{x}_{k+1}, k+1)\sigma_{k+1}^2 E[d_{k+1}\xi_{k+1} + \xi_{k+1}^2]. \tag{28}$$

It follows from (27) that

$$d_{k+1} = \frac{2J_{xx}(\bar{x}_{k+1}, k+1)(\delta f_k + \frac{1}{2}\delta^2 f_k)}{\sigma_{k+1}J_{xx}(\bar{x}_{k+1}, k+1)}$$
$$+ \frac{2J_x(\bar{x}_{k+1}, k+1)}{\sigma_{k+1}J_{xx}(\bar{x}_{k+1}, k+1)}, \tag{29}$$

where

$$\frac{2J_{xx}(\bar{x}_{k+1}, k+1)(\delta f_k + \frac{1}{2}\delta^2 f_k)}{\sigma_{k+1}J_{xx}(\bar{x}_{k+1}, k+1)} \to 0$$

as $\delta x_k$, $\delta u_k \to 0$. Denote

$$\tilde{d}_{k+1} = \frac{2J_x(\bar{x}_{k+1}, k+1)}{\sigma_{k+1}J_{xx}(\bar{x}_{k+1}, k+1)}. \tag{30}$$

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Then $d_{k+1} \approx \tilde{d}_{k+1}$. Thus

$$E[d_{k+1}\xi_{k+1} + \xi_{k+1}^2] \approx E[\tilde{d}_{k+1}\xi_{k+1} + \xi_{k+1}^2]$$

which is computed by (1) or uncertain simulation [15].

The (26) may be approximated by

$$\Delta_k + J_x(\bar{x}_k, k)\delta x_k + \frac{1}{2}J_{xx}(\bar{x}_k, k)\delta x_k^2$$
$$= \min_{\substack{\delta u_k \\ \bar{u}_k + \delta u_k \in U_k}} \left\{ \Delta_{k+1} + H_{x_k}\delta x_k + H_{u_k}\delta u_k \right.$$
$$+ \frac{1}{2}A_k\delta x_k^2 + B_k\delta x_k\delta u_k + \frac{1}{2}C_k\delta u_k^2$$
$$\left. + \frac{1}{2}J_{xx}(\bar{x}_{k+1}, k+1)\sigma_{k+1}^2 E[\tilde{d}_{k+1}\xi_{k+1} + \xi_{k+1}^2] \right\}. \quad (31)$$

Denote the term in the brace of (31) by $Z$:

$$Z = \Delta_{k+1} + H_{x_k}\delta x_k + H_{u_k}\delta u_k$$
$$+ \frac{1}{2}A_k\delta x_k^2 + B_k\delta x_k\delta u_k + \frac{1}{2}C_k\delta u_k^2$$
$$+ \frac{1}{2}J_{xx}(\bar{x}_{k+1}, k+1)\sigma_{k+1}^2 E[\tilde{d}_{k+1}\xi_{k+1} + \xi_{k+1}^2]$$

Letting $\partial Z / \partial(\delta u_k) = 0$ yields

$$H_{u_k} + B_k\delta x_k + C_k\delta u_k = 0. \quad (32)$$

Thus, when $C_k \neq 0$, we have

$$\delta u_k = -C_k^{-1}(H_{u_k} + B_k\delta x_k). \quad (33)$$

For $\epsilon > 0$, revise $\delta u_k$:

$$\delta u_k = -C_k^{-1}(\epsilon H_{u_k} + B_k\delta x_k) \quad (0 < \epsilon \leq 1). \quad (34)$$

Substituting (34) into the right side of (31) yields

$$\Delta_k + J_x(\bar{x}_k, k)\delta x_k + \frac{1}{2}J_{xx}(\bar{x}_k, k)\delta x_k^2$$
$$= \Delta_{k+1} - \frac{1}{2}(2\epsilon - \epsilon^2)C_k^{-1}H_{u_k}^2$$
$$+ \frac{1}{2}J_{xx}(\bar{x}_{k+1}, k+1)\sigma_{k+1}^2 E[\tilde{d}_{k+1}\xi_{k+1} + \xi_{k+1}^2]$$
$$+ \left(H_{x_k} - H_{u_k}C_k^{-1}B_k\right)\delta x_k + \left(A_k - B_k^2 C_k^{-1}\right)\delta x_k^2. \quad (35)$$

By the arbitrariness of $\delta x_k$, it follows from (35) that

$$\Delta_k = \Delta_{k+1} - \frac{1}{2}(2\epsilon - \epsilon^2)C_k^{-1}H_{u_k}^2$$
$$+ \frac{1}{2}J_{xx}(\bar{x}_{k+1})\sigma_{k+1}^2 E[\tilde{d}_{k+1}\xi_{k+1} + \xi_{k+1}^2] \quad (36)$$
$$J_x(\bar{x}_k, k) = H_{x_k} - H_{u_k}C_k^{-1}B_k \quad (37)$$
$$J_{xx}(\bar{x}_k, k) = A_k - B_k^2 C_k^{-1} \quad (38)$$

It follows from $\bar{J}(\bar{x}_N, N) = J(\bar{x}_N, N) = G(\bar{x}_N, N)$ and (3) that the boundary conditions of (36)-(38) are

$$\Delta_N = 0 \quad (39)$$
$$J_x(\bar{x}_N, N) = L_x(\bar{x}_N, N) \quad (40)$$
$$J_{xx}(\bar{x}_N, N) = L_{xx}(\bar{x}_N, N) \quad (41)$$

*Summary of the algorithm*

Step 1: For initial nominal control $\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1}$, compute $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N$ by (5) and compute $\bar{J}(x_0)$ by (11).

Step 2: Compute $J_x(\bar{x}_k)$ and $J_{xx}(\bar{x}_k)$ by (21)-(25) and (37)-(38) with the boundary conditions (40)-(41) for $k = N, N-1, \cdots, 1, 0$. Meanwhile compute $C_k^{-1}H_{u_k}$ and $C_k^{-1}B_k$. Compute $\tilde{d}_{k+1}$ by (30).

Step 3: For $\bar{u}_k$, compute $\delta u_k$ by (34). Compute $\Delta_k$ by (36) with (39), where $E[\tilde{d}_{k+1}\xi_{k+1} + \xi_{k+1}^2]$ is computed by (1) or simulation [17]. If $\Delta_k < 0$ and $\bar{u}_k + \delta u_k \in U_k$, then $\bar{u}_k + \delta u_k$ is used as the next initial control. Otherwise, set $\epsilon = \epsilon/2$ and repeat the step.

Step 4: Repeat the Step 1 to 3 until the difference of the objective values obtained in two iterative step is less than a precision $\eta$.

## V. NUMERICAL EXAMPLES

*Example 1:* Consider the following problem

$$\begin{cases} J(x_0, 0) = \min_{\substack{u_j \in [-5, 5] \\ 0 \leq j \leq 9}} E\left[\sum_{j=0}^{9} Ax_j^2 + Bu_j^2 + Cx_{10}^2\right] \\ \text{subject to} \\ x_{j+1} = ax_j + b\tanh(u_j) + \sigma\xi_{j+1} \\ x_0 = 5.0, \quad j = 0, 1, 2, \ldots, 9, \end{cases} \quad (42)$$

where $A = 5.0/11$, $B = 0.5/11$, $C = 10 + 0.5/11$, $a = 0.99$, $b = 0.5$, $\sigma = 0.1$, $-10 \leq x_k \leq 10$, $-5 \leq u_k \leq 5$; $\xi_1$, $\xi_2$, ..., and $\xi_{10}$ are independent linear uncertain variables $\xi_{j+1} \sim \mathcal{L}(-1, 1)$, $j = 0, 1, 2, \ldots, 9$.

The initial nominal controls provided by GA is listed in Table I.

Table I
INITIAL NOMINAL CONTROLS

| $\bar{u}_0$ | $\bar{u}_1$ | $\bar{u}_2$ | $\bar{u}_3$ | $\bar{u}_4$ |
|---|---|---|---|---|
| $-2.872491$ | $-3.261430$ | $-2.477211$ | $-2.590905$ | $-2.365457$ |
| $\bar{u}_5$ | $\bar{u}_6$ | $\bar{u}_7$ | $\bar{u}_8$ | $\bar{u}_9$ |
| $-2.257614$ | $-1.830452$ | $-1.489557$ | $-0.724979$ | $-0.454651$ |

It follows from (42) that

$$L(x_k, u_k, k) = Ax_k^2 + Bu_k^2, f(x_k, u_k, k) = ax_k + b\tanh u_k.$$

Then

$$H_{x_k} = 2A\bar{x}_k + aJ_x(\bar{x}_{k+1}) \quad (43)$$
$$H_{u_k} = 2B\bar{u}_k + b(1 - \tanh^2 \bar{u}_k)J_x(\bar{x}_{k+1}) \quad (44)$$
$$A_k = 2A + a^2 J_{xx}(\bar{x}_{k+1}) \quad (45)$$
$$B_k = ab(1 - \tanh^2 \bar{u}_k)J_{xx}(\bar{x}_{k+1}) \quad (46)$$
$$C_k = 2B - 2b\tanh \bar{u}_k(1 - \tanh^2 \bar{u}_k)J_x(\bar{x}_{k+1})$$
$$+ b^2(1 - \tanh^2 \bar{u}_k)^2 J_{xx}(\bar{x}_{k+1}) \quad (47)$$

with the boundary conditions

$$J_x(\bar{x}_{10}) = 2C\bar{x}_{10}, \ J_{xx}(\bar{x}_{10}) = 2C \quad (48)$$

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where $\bar{x}_k$ is provided by

$$\bar{x}_{k+1} = a\bar{x}_k + b\tanh\bar{u}_k, \quad \bar{x}_0 = 5.0$$

for $\bar{u}_k$. It follows from (43)-(47) and (48) that

$$J_x(\bar{x}_k, k) = H_{x_k} - H_{u_k}C_k^{-1}B_k \qquad (49)$$
$$J_{xx}(\bar{x}_k, k) = A_k - B_k^2 C_k^{-1} \qquad (50)$$

Compute $\delta u_k$ by

$$\delta u_k = -C_k^{-1}(\epsilon H_{u_k} + B_k\delta x_k) \quad (0 < \epsilon \le 1).$$

Set $\epsilon = 1$. Compute $\Delta_k$ by

$$\Delta_k = \Delta_{k+1} - \frac{1}{2}(2\epsilon - \epsilon^2)C_k^{-1}H_{u_k}^2$$
$$+ \frac{1}{2}J_{xx}(\bar{x}_{k+1})\sigma_{k+1}^2 E[\tilde{d}_{k+1}\xi_{k+1} + \xi_{k+1}^2] \qquad (51)$$
$$\Delta_{10} = 0, \qquad (52)$$

where $E[\tilde{d}_{k+1}\xi_{k+1} + \xi_{k+1}^2]$ is computed by (1). When $\Delta_k < 0$ and $\bar{u}_k + \delta u_k \in [-5, 5]$, then replace $\bar{u}_k$ with $\bar{u}_k + \delta u_k$ as the initial values in the next step. Otherwise set $\epsilon = \epsilon/2$. The optimal results for the problem is obtained for the precision $\eta = 0.00001$ in Table II.

Table II
NUMERICAL SOLUTIONS

| $k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $J(x_0)$ | 44.745713 | 43.720804 | 43.519055 | 43.500925 |
| $u_0$ | $-2.872491$ | $-2.499168$ | $-2.559466$ | $-2.558647$ |
| $u_1$ | $-3.261430$ | $-1.942619$ | $-2.288074$ | $-2.441568$ |
| $u_2$ | $-2.477211$ | $-2.390049$ | $-2.369627$ | $-2.362194$ |
| $u_3$ | $-2.590905$ | $-2.207609$ | $-2.253928$ | $-2.246153$ |
| $u_4$ | $-2.365457$ | $-2.139681$ | $-2.127905$ | $-2.114214$ |
| $u_5$ | $-2.257614$ | $-2.003139$ | $-1.981760$ | $-1.962221$ |
| $u_6$ | $-1.830452$ | $-1.909367$ | $-1.809367$ | $-1.785041$ |
| $u_7$ | $-1.489557$ | $-1.728166$ | $-1.620483$ | $-1.577627$ |
| $u_8$ | $-0.724979$ | $-1.173456$ | $-1.370916$ | $-1.350510$ |
| $u_9$ | $-0.454651$ | $-0.792086$ | $-0.978474$ | $-1.097452$ |

| $k$ | 4 | 5 | 6 |
|---|---|---|---|
| $J(x_0)$ | 43.500345 | 43.500343 | 43.500343 |
| $u_0$ | $-2.557997$ | $-2.557960$ | $-2.557959$ |
| $u_1$ | $-2.464264$ | $-2.464648$ | $-2.464647$ |
| $u_2$ | $-2.361047$ | $-2.360985$ | $-2.360985$ |
| $u_3$ | $-2.244633$ | $-2.244551$ | $-2.244550$ |
| $u_4$ | $-2.112200$ | $-2.112087$ | $-2.112087$ |
| $u_5$ | $-1.959386$ | $-1.959224$ | $-1.959223$ |
| $u_6$ | $-1.780666$ | $-1.780418$ | $-1.780417$ |
| $u_7$ | $-1.570985$ | $-1.570579$ | $-1.570577$ |
| $u_8$ | $-1.336178$ | $-1.335619$ | $-1.335616$ |
| $u_9$ | $-1.133524$ | $-1.135977$ | $-1.135979$ |

## VI. CONCLUSION

A differential dynamic programming algorithm was presented for solving an optimal control problem subject to a multistage uncertain system. It goes from a nominal control which is provided by GA to the optimal control step by step. A numerical example showed the efficiency of the proposed method.

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