EE 507 HW05

ARYAN RITWAJEET JHA

1. (a) Check if $\boxed{f_X(x) \cdot f_Y(y) \cdot f_Z(z) = f_{X,Y,Z}(x,y,z)}$ ⓪

implies that ~~the three~~ X and Y are independent,

i.e. check if ⓪ implies $\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$ ⓘ

~~Given:~~ Integrating both sides of ⓪ w.r.t. ~~z~~ z:

$$\int_{\forall z} f_X(x) \cdot f_Y(y) \cdot f_Z(z)\, dz = \int_{\forall z} f_{X,Y,Z}(x,y,z)\, dz$$

or $$f_X(x) f_Y(y) \int_{\forall z} f_Z(z) dz = f_{X,Y}(x,y)$$ (marginal pdf of X, Y)

or $$f_X(x) f_Y(y) \cdot 1 = f_{X,Y}(x,y)$$

or $$\boxed{f_X(x) f_Y(y) = f_{X,Y}(x,y)}$$ which is ⓘ which is what we set to check for!

$\therefore$ 1(a) $\boxed{f_X(x) \cdot f_Y(y) f_Z(z) = f_{X,Y,Z}(x,y,z) \Rightarrow X \text{ and } Y \text{ are independent}}$ Ans

1(b) check if $\boxed{P(X) P(Y) P(Z) = P(X,Y,Z)}$ ⓪ implies that X and Y are independent i.e. $\boxed{P(X) \cdot P(Y) = P(X,Y)}$ ⓘ where X, Y, Z are events of a probabilistic experiment.

No.

Counterexample:

~~Let experiment~~

In an experiment, ~~let~~ we are picking a number from 1 to 8 randomly. Each number has equal probability of being picked. This experiment is like throwing a fair 8-sided die.

Now let us define three events:

$X = \{1, 2, 3, 4\} \Rightarrow P(X) = \frac{1}{2}$ ~~$P(X, Y)$~~

$Y = \{1, 3, 4, 5\} \Rightarrow P(Y) = \frac{1}{2}$

$Z = \{1, 6, 7, 8\} \Rightarrow P(Z) = \frac{1}{2}$

$X \cap Y \cap Z = \{1\} \Rightarrow P(X \cap Y \cap Z) = \frac{1}{8}$

~~So $P(X) \cdot P(Y)$~~

So equation ⓪ is followed by events X, Y, Z, as

$$P(X) \cdot P(Y) \cdot P(Z) = \left(\frac{1}{2}\right)^3 = \frac{1}{8} = P(X, Y, Z)$$

But $P(X) \cdot P(Y) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$P(X \cap Y) = P(\{1, 3, 4\}) = \frac{3}{8}$

X and Y are NOT independent.

In fact, $P(Y) \cdot P(Z) = \frac{1}{4}$

$P(Y \cap Z) = P(\{1\}) = \frac{1}{8}$

Y and Z are NOT independent either

And $P(X) \cdot P(Z) = \frac{1}{4}$

$P(X \cap Z) = P(\{1\}) = \frac{1}{8}$

X and Z are NOT independent either.

$\boxed{1(b)}$

$\therefore$ The analogous event $P(X) \cdot P(Y) \cdot P(Z) = P(X\,Y\,Z)$ does NOT imply independence of X and Y.

Ans

2. 2(a) $X, Y \sim \mathcal{N}(\mu_X = 0, \mu_Y = 0, \sigma_X^2 = 1, \sigma_Y^2 = 4, \rho = 0.5)$

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{1-\rho^2}\,\sigma_X\sqrt{2\pi}\,\sigma_Y\sqrt{2\pi}}\, e^{\underbrace{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right\}}_{\text{Only component (say } g(x,y)) \text{ dependent on } x \text{ and } y.}}$$

Taking out the component $g(x,y)$ and putting the given values of $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ_{XY}:

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - \frac{2(x)(y)}{(1)(2)}$$

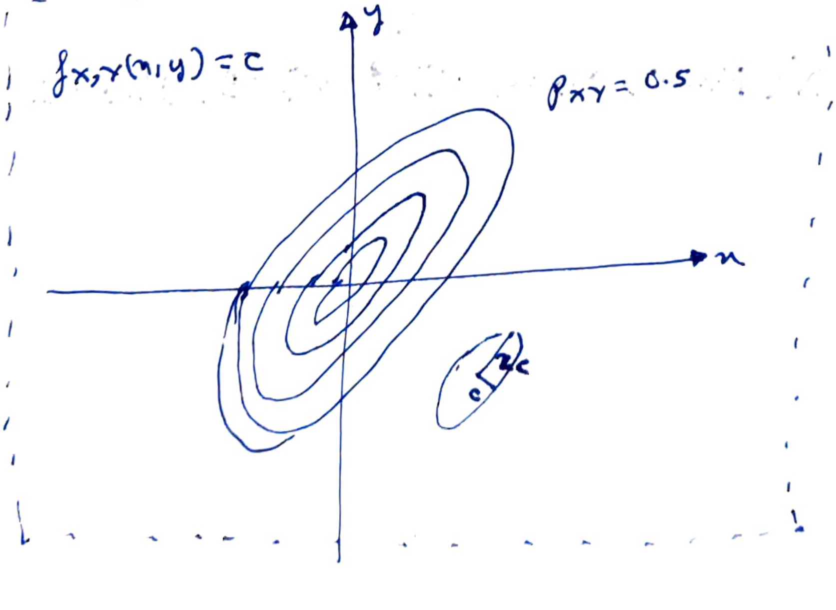
$\Rightarrow$ $f_{X,Y}(x,y) \xrightarrow[\text{i.e. is constant}]{} c$ for $g(x,y) \xrightarrow[\text{i.e. is constant}]{} c$

$\Rightarrow$ Contour of $f_{X,Y}(x,y)$ is

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - xy = c$$

which is the equation for an ellipse, tilted and with positive slope.

2(a)



Ans

2(b)

$$\boxed{\begin{aligned} f_X(x) &= \mathcal{N}(x; 0, 1^2) = \frac{1}{1.\sqrt{2\pi}}\, e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}} \quad \forall x \\ f_Y(y) &= \mathcal{N}(y, 0, 2^2) = \frac{1}{2.\sqrt{2\pi}}\, e^{-\frac{1}{2}\left\{\left(\frac{y}{2}\right)^2\right\}} \quad \forall y \end{aligned}}$$

Ans

2(c)

$$f_{Y|X}(y|X=x) = \frac{f_{X,Y}(X=x, Y)}{f_X(X=x)} \quad \forall x, y$$

or
$$f_{Y|X}(y|X=x) = \frac{\dfrac{1}{\sqrt{\left(1-\frac{1}{2^2}\right)}\, 1.\sqrt{2\pi}\,.\,2.\sqrt{2\pi}}\, e^{-\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2^2}}\cdot\left\{\left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - 2\times\frac{1}{2}\times\frac{x}{1}\times\frac{y}{2}\right\}} \quad \cancel{\forall x, y}}{\dfrac{1}{1.\sqrt{2\pi}}\, e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}}} \quad \forall (x, y)$$

or
$$f_{Y|X}(y|X=x) = \frac{1}{\sqrt{3}\,.\,\sqrt{2\pi}}\, e^{-\frac{1}{2}\cdot\left\{\left(\frac{x}{\sqrt{3}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 - 2.1.\left(\frac{x}{\sqrt{3}}\right)\left(\frac{y}{\sqrt{3}}\right)\right\}} \quad \forall (x, y)$$

~~2(c)~~

~~or~~ $\cancel{\boxed{f_{Y|X}(y|X=x) = \sqrt{2\pi}\,\mathcal{N}(\mu_X = 0, \mu_Y = 0, \sigma_X^2 = 3, \sigma_Y^2 = 3, \rho_{XY} = 1)} \quad \forall (x, y)}$ ~~Ans~~

~~or~~ $\cancel{f_{Y|X}(y|X=x) = \mathcal{N}_y(\mu_Y = , \sigma_Y^2 = 3) \quad \forall y}$

or $$f_{Y|X}(y|X=x) = \frac{1}{\sqrt{3}}\,.\,\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left\{\left(\frac{y-x}{\sqrt{3}}\right)^2\right\}} \quad \forall y$$

2(c)

or $$\boxed{f_{Y|X}(y|X=x) = \mathcal{N}(y, \mu_Y = x, \sigma_Y^2 = 3) \quad \forall y}$$ Ans

PTO.

~~2(d) Find x s.t. $E[Y|X=x] = -2$.~~

~~From 2(c), we know that $Y|X=x$ is fully correlated to $X=x$. ($\rho_{Y|X,X} = 1$).~~

~~$\because$ Y and X have the same support/domain of $x \in \mathbb{R}$ and $y \in \mathbb{R}$,~~

~~$\therefore$ $E_Y[Y|X=x] = -2 \equiv E_X[X=x] = -2$~~

~~But $E_X[X=x] = x$~~

~~$\Rightarrow$ $\boxed{x=-2}$ Ans~~

2(d) Find x s.t. $E[Y|X=x] = -2$

From 2(c), we know that $Y|X=x$ is a gaussian with mean x.

So $E[Y|X=x] = -2 \Rightarrow \boxed{x=-2}$ Ans

2(e) $Z = X + Y - 1$

Z is also a Gaussian, we need to only compute μ_Z and σ_Z^2 to find $f_Z(z)$:

$E(Z) = E(X) + E(Y) - E(1)$

or $\mu_Z = \mu_X + \mu_Y - 1$

or $\mu_Z = 0 + 0 - 2$

or $\boxed{\mu_Z = -1}$

$\sigma_Z^2 = E[\cancel{x-\mu_Z}\{(X+Y-1) - E[X+Y-1]\}^2]$

or $\sigma_Z^2 = E[\{(X-\mu_X) + (Y-\mu_Y) + (-1-E(-1))\}^2]$

or $\sigma_Z^2 = E[\{(X-\mu_X) + (Y-\mu_Y)\}^2]$

or $\sigma_Z^2 = E[(X-\mu_X)^2] + E[(Y-\mu_Y)^2] + \cancel{2\rho_{XY}}$ $2E[(X-\mu_X)(Y-\mu_Y)]$

or $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 1^2 + 2^2 + 2 \times 0.5 \times 1 \times 2$

or $\boxed{\sigma_Z^2 = 7}$

2(e)

$\therefore \boxed{Z \sim \mathcal{N}(z, \mu_Z = -1, \sigma_Z^2 = 7)}$ Ans

2(f) $Z = 2X + 3Y$ $W = X - Y$

$E(Z) = 2\mu_X + 3\mu_Y$ $E(W) = \mu_X - \mu_Y$

2(f)(i) or $\boxed{\mu_Z = 0}$ Ans 2(f)(iii) $\boxed{\mu_W = 0}$ Ans

$\sigma_Z^2 = E\left[\{(2X+3Y) - (2\mu_X + 3\mu_Y)\}^2\right]$

or $\sigma_Z^2 = E\left[\{2(X-\mu_X)\}^2 + \{3(Y-\mu_Y)\}^2 + 2.2.3.(X-\mu_X)(Y-\mu_Y)\right]$

or $\sigma_Z^2 = 4\sigma_X^2 + 9\sigma_Y^2 + 12\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 4.1^2 + 9.2^2 + 12.(0.5).1.2$

or $\sigma_Z^2 = 4 + 36 + 12$

2(f)(ii) or $\boxed{\sigma_Z^2 = 52}$ Ans

$\sigma_W^2 = E\left[\{(X-Y) - (\mu_X - \mu_Y)\}^2\right]$

or $\sigma_W^2 = E\left[\{X-\mu_X\}^2 + \{Y-\mu_Y\}^2 - \{2.1.1.(X-\mu_X)(Y-\mu_Y)\}\right]$

or $\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_W^2 = 1^2 + 2^2 - 2.(0.5).1.2$

or $\sigma_W^2 = 1 + 4 - 2$

2(f)(iv) $\boxed{\sigma_W^2 = 3}$ Ans

2(g) $Cov(Z,W) = E\left[(Z-\mu_Z)(W-\mu_W)\right]$

or $Cov(Z,W) = E\left[(2X+3Y-2\mu_X-3\mu_Y)(X-Y-\mu_X-(-\mu_Y))\right]$

or $Cov(Z,W) = E\left[\{2(X-\mu_X)+3(Y-\mu_Y)\}\{1(X-\mu_X)-1(Y-\mu_Y)\}\right]$

or $Cov(Z,W) = E\left[2(X-\mu_X)^2 - 3(Y-\mu_Y)^2 + 1(X-\mu_X)(Y-\mu_Y)\right]$

or $Cov(Z,W) = 2\sigma_X^2 - 3\sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y$

or $Cov(Z,W) = 2.1^2 - 3.2^2 + (0.5).1.2$

or $Cov(Z,W) = 2 - 12 + 1$

2(f)$^{(v)}$

or $\boxed{Cov(Z, W) = -9}$ Ans

$$\boxed{\rho_{Z,W} = \frac{Cov(Z, W)}{\sigma_Z \sigma_W} = \frac{-9}{\sqrt{52} \cdot \sqrt{3}} \approx -0.7206}$$

2(f)$^{(vi)}$

$$\boxed{f_{Z,W}(z, w) = \mathcal{N}(\mu_Z = 0, \mu_W = 0, \sigma_Z^2 = 52, \sigma_W^2 = 3, \rho_{Z,W} = -0.7206)}$$ Ans

2(g) $R = aX + bY \Rightarrow \sigma_R^2 = a^2 \cdot \sigma_X^2 + b^2 \sigma_Y^2$

or $\sigma_R^2 = a^2 \cdot 1^2 + b^2 \cdot 2^2$

or $\sigma_R^2 = a^2 + 4b^2$

$Cov(R, Y) = 0$ is a sufficient condition for R and Y to be independent, as both are gaussian.

$$Cov(R, Y) = E[(aX + bY - a\mu_X - b\mu_Y)(Y - \mu_Y)]$$

or $Cov(R, Y) = E[a(X - \mu_X)(Y - \mu_Y) + b(Y - \mu_Y)^2]$

or $Cov(R, Y) = a \rho_{X,Y} \sigma_X \sigma_Y + b \sigma_Y^2$

or $Cov(R, Y) = a(0.5) \cdot 1 \cdot 2 + b \cdot 2^2$

or $Cov(R, Y) = a + 4b$

2(g)

For R, Y to be independent distributions, $a + 4b = 0$

We can use any set of Real numbers to do so,

say $a = -4, b = 1$

Ans

2(b) $\quad Q = aX$

given $\quad p(a=2) = 0.5 \qquad f(Q \mid a=2) = \cancel{2} \, f_X(X=2x)$

$\qquad\qquad p(a=1) = 0.5 \qquad f(Q \mid a=1) = f_X(X=x)$

$$
\begin{tikzcd}
 & Q = X \\
Q \arrow[ur, "0.5"] \arrow[dr, "0.5"'] & \\
 & Q = 2X
\end{tikzcd}
$$

$f_Q(q) = f_{Q|a}(Q \mid a=1) \cdot p(a=1) + f_{Q|a}(Q \mid a=2) \cdot p(a=2)$ (LoTP)

or $f_Q(q) = 0.5\,\mathcal{N}(q, \mu_q = \mu_x, \sigma_q^2 = \sigma_x^2) + 0.5\,\mathcal{N}(q, \mu_q = 2\mu_x, \sigma_q^2 = 2^2\sigma_x^2)$

$\boxed{2(b)}$

or $\boxed{f_Q(q) = 0.5\,\mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 1) + 0.5\,\mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 4)}$ Ans

3. X, Y independent. $Z = X+Y$

$$F_Z(z=\alpha) = P(Z \le \alpha)$$

or $F_Z(z=\alpha) = P(X+Y \le \alpha)$

or $F_Z(z=\alpha) = P(\cancel{X} \le \alpha - X)$

or $$F_Z(z=\alpha) = \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\alpha-x} f_{X,Y}(x,y)\,dy\,dx$$

or $$F_Z(z=\alpha) = \int_{x=-\infty}^{x=\infty} f_X(x)\left(\int_{y=-\infty}^{y=\alpha-x} f_Y(y)\,dy\right)dx$$

or $$\left.\frac{d}{d\alpha}F_Z(z=\alpha)\right|_{\alpha=z} = \left.\frac{\partial}{\partial\alpha}\left[\int_{x=-\infty}^{x=\infty} f_X(x)\left(\int_{y=-\infty}^{y=\alpha-x} f_Y(y)\,dy\right)dx\right]\right|_{\alpha=z}$$

3(a)

or $$\boxed{f_Z(z) = \int_{x=-\infty}^{x=\infty} f_X(x)\cdot f_Y(z-x)\,dx}$$ Ans

By Leibniz rule

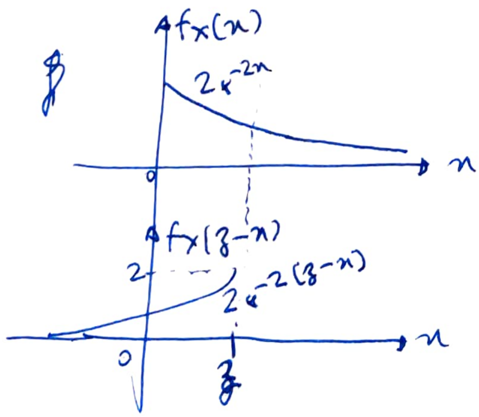
Hence Proved ☺

3(b)

Similarly $$\boxed{f_Z(z) = \int_{y=-\infty}^{y=\infty} f_Y(y)\cdot f_X(z-y)\,dy}$$ Ans

Hence Proved ☺

3 c) $f_X(x) = 2e^{-2x}$, $x \ge 0$. $f_Y(y) = 2e^{-2y}$, $y \ge 0$



$$f_Z(z) = \int_{x=0}^{x=z} 2e^{-2x}\cdot 2e^{-2(z-x)}\,dx$$

or $$f_Z(z) = \int_{x=0}^{x=z} 4e^{-2z}\,dx$$

3(c)

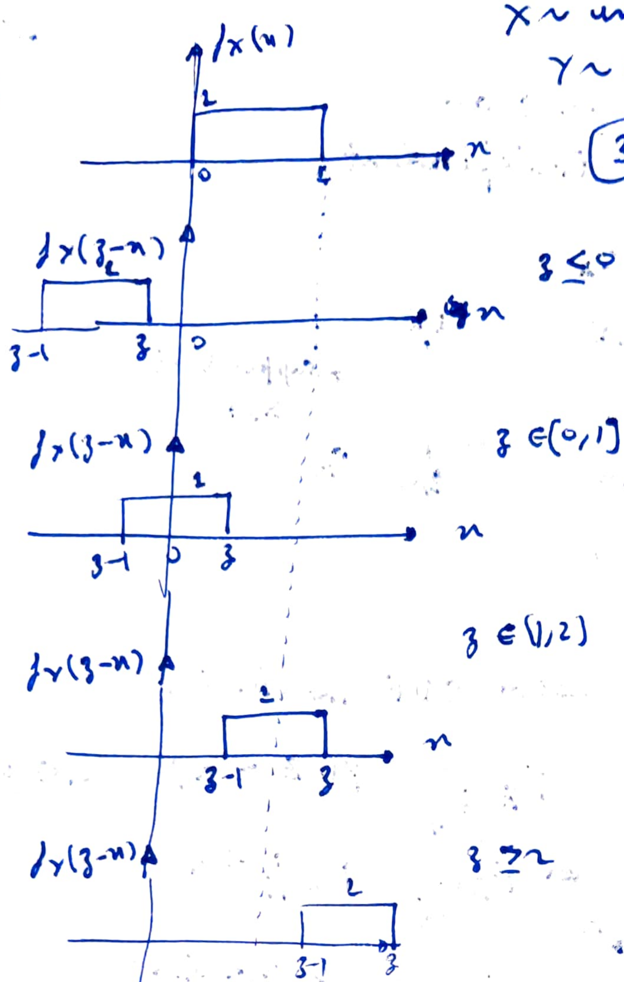
or $$\boxed{f_Z(z) = 4e^{-2z}\cdot z \qquad z\in(0,\infty)}$$ Ans

$\boxed{3.2}$

$X \sim unif([0,1])$

$Y \sim unif[0,2]$

3(d)



$\boxed{3(d)}$

$$f_Z(z) = \begin{cases} 0 & z \le 0 \\ z & z \in (0,1) \\ 2-z & z \in [1,2] \\ 0 & z \ge 2 \end{cases}$$

Ans

$\boxed{3(e)}$

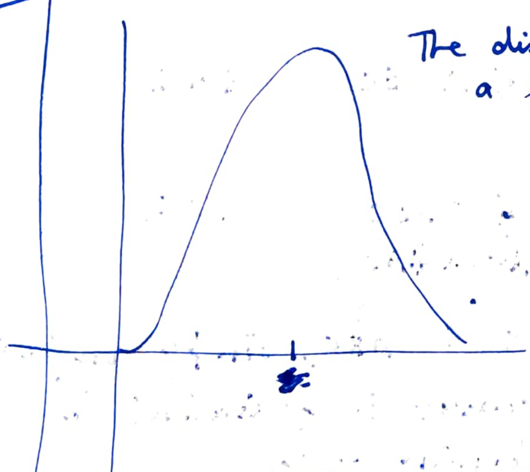
3(e)

$Z = X_1 + X_2 + X_3 + \dots + X_7$

$X_i \in unif[0,2]$

X_i, X_j are independent for $i \neq j$

The distribution resembles a Gaussian distribution.



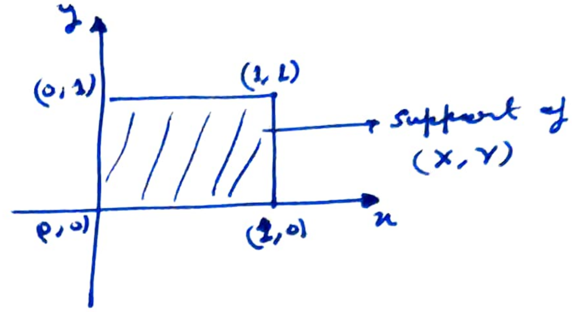
Ans

———×———×———×———×———

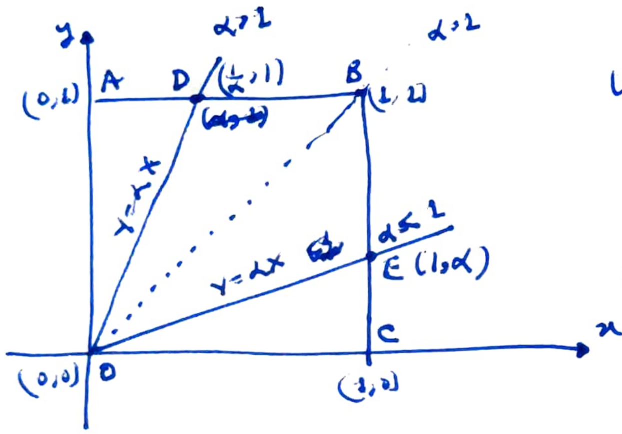
$\boxed{4.}$ $X \sim unif[0,1]$ $\quad$ X, Y are independent

$Y \sim unif[0,1]$

$Z = \dfrac{Y}{X}$



4(a) Find $f_Z(z) = ?$



Using CDF method:

$$F_Z(\underset{(=\alpha)}{z}) = P(Z \le \alpha) \qquad \alpha \in (0,\infty)$$

or $$F_Z(\underset{(=\alpha)}{z}) = P\left(\frac{Y}{X} \le \alpha\right)$$

or $$F_Z(\underset{(=\alpha)}{z}) = P(Y \le \alpha X)$$

or $$F_Z(z=\alpha) = \begin{cases} \text{Area of } \triangle OEC & \alpha \le 1 \\ \text{Area of } ODBC = 1 - \text{Area}(\triangle ADO) & \alpha \ge 1 \end{cases}$$

uniform

Note: $\because \underbrace{f_{X,Y}(x,y) = 1}$ $\forall (x,y) \in OABC$, simply computing the area under the curves is equivalent to computing the corresponding CDFs.

or $$F_Z(z=\alpha) = \begin{cases} \frac{1}{2} & 0 < \alpha \le 1 \\ \frac{1}{2\alpha^2} & \alpha \ge 1 \\ 0 & \alpha < 0 \end{cases}$$

$\boxed{4.2}$

$\boxed{4(a)}$

$$\Rightarrow \boxed{f_Z(z) = \frac{d}{d\alpha} F_Z(z=\alpha)\Big|_{\alpha \to z} = \begin{cases} 0 & z<0 \\ \frac{1}{2} & z \in (0,1] \\ \frac{1}{2z^2} & z \geq 1 \end{cases}}$$ Ans

4 b) $E[X^2+Y^2] = E[X^2] + E[Y^2]$

or $E[X^2+Y^2] = \int_{x=0}^{x=1} x^2 \cdot 1 \cdot dx + \int_{y=0}^{y=1} y^2 \cdot 1 \cdot dy$

or $E[X^2+Y^2] = \frac{1}{3} + \frac{1}{3}$

$\boxed{4(b)}$

or $\boxed{E[X^2+Y^2] = \frac{2}{3}}$ Ans

4 c) $f_{X,Z}(x,z) = \begin{cases} f_{Z|X}(z|X=x) \cdot f_X(X=x) & x \in (0,1], \ z \in (0,\frac{1}{x}] \\ 0 & \text{else} \end{cases}$

or $f_{X,Z}(x,z) = \begin{cases} \text{pdf of}\left(\frac{1}{x} \cdot Y\right) \cdot 1 & x \in (0,1], \ z \in (0,\frac{1}{x}] \\ 0 & \text{else} \end{cases}$

But $Y \sim unif(0,1)$

$Z|X=x \sim \frac{1}{x} \cdot unif(0,1)$

$\equiv Z|X=x \sim unif\left(0, \frac{1}{x}\right)$

$$\Rightarrow \boxed{pdf(Z|X=x) = \begin{cases} x & z \in (0,\frac{1}{x}) \\ 0 & \text{else} \end{cases}}$$

4.3

4(c)

So $\boxed{f_{X,Z}(x,z) = \begin{cases} x & x \in (0,1] \\ & z \in (0, \frac{1}{x}] \\ 0 & \text{else} \end{cases}}$ Ans

4(d) $E[XZ] = E[Y] = \frac{1}{2}$

~~But anyway using $E[XZ] = \iint f_{X,Z}(x,z) \cdot x \cdot z \, dy\, dx$~~

4(d)(i) $\boxed{E[XZ] = \frac{1}{2}}$ Ans

$$\mathrm{Cov}(X,Z) = E[XZ] - \mu_X \mu_Z$$

or $$\mathrm{Cov}(X,Z) = E[Y] - \frac{1}{2}\mu_Z$$

or $$\mathrm{Cov}(X,Z) = \frac{1}{2} - \frac{1}{2}\mu_Z$$

$$\mu_Z = \int f_Z(z) \cdot z \cdot dz$$

or $$\mu_Z = \int_{z=0}^{z=1} \left(\frac{1}{2}\right) \cdot z \cdot dz + \int_{z=1}^{z=\infty} \left(\frac{1}{2z^2}\right) z \, dz$$

or $$\mu_Z = \frac{1}{4} z^2 \Big|_{z=0}^{z=1} + \frac{1}{2} \ln z \Big|_{z=1}^{z=\infty}$$

or $\boxed{\mu_Z \to \infty}$

4(d)(ii) $\boxed{\mathrm{Cov}(X,Z) \to -\infty}$ Ans

$\boxed{4.4}$

$$\rho_{X,Z}(x,z) = \frac{Cov(X,Z)}{\sigma_X \sigma_Z}$$

~~σ_Z~~ $\sigma_X = \sqrt{\frac{1}{12}}$

$$\sigma_Z = \int (z-\mu_Z)^2 \cdot f_Z(z)\,dz \to \infty$$

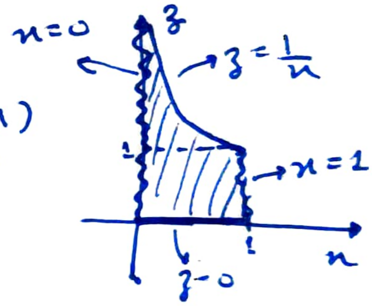
$$\mu_Z \to \infty$$

So $\rho_{X,Z}(x,z) \sim \dfrac{-\infty}{\sqrt{\frac{1}{12}}\cdot\infty} \approx -1$

4(d)(iii)

or $\boxed{\rho_{X,Z}(x,z) = -1}$ Ans

support of (X,Z):



4(e) $f_{X|Z}(x|Z=z) = \dfrac{f_{Z|X}(z|X=x)\cdot f_X(X=x)}{f_Z(Z=z)}$

or
$$f_{X|Z}(x|Z=z) = \begin{cases} n.d. & z<0 \text{ or } \cancel{z \le \frac{1}{12}} \\ \dfrac{x\cdot 1}{\frac{1}{2}} & z\in(0,1],\ x\in(0,1] \\ \dfrac{x\cdot 1}{\frac{1}{2}z^{-2}} & z\in[1,\infty),\ x\in(0,\frac{1}{z}] \\ 0 & \cancel{x<\frac{1}{z}} \text{ else} \end{cases}$$

$\boxed{4.5}$

$\boxed{4(e)(i)}$

or
$$\boxed{f_{X|Z}(x|Z=z) = \begin{cases} \text{n.d.} & z<0 \\ 2x & z\in(0,1] \\ & x\in(0,1] \\ 2z^2x & z\in[1,\infty) \\ & x\in(0,\tfrac{1}{z}] \\ 0 & \text{else} \end{cases}}$$
Ans

$$E[X|Z=z] = \int_{x=0}^{x=2} x\cdot f_{X|Z}(X|Z=z)\,dx$$

or
$$E[X|Z=z] = \begin{cases} \displaystyle\int_{x=0}^{x=1} x\cdot 2x\,dx & z\in(0,1] \\ \displaystyle\int_{x=0}^{x=\frac{1}{z}} x\cdot 2z^2x\,dx & z\in[1,\infty) \end{cases}$$

$\boxed{4(e)(ii)}$

or
$$\boxed{E[X|Z=z] = \begin{cases} \dfrac{2}{3} & z\in(0,1] \\ \dfrac{2}{3z} & z\in[1,\infty) \end{cases}}$$
Ans

4(f) $E_Z[E_X[X|Z=z]]$

$$= \int_{z} E_X[X|Z=z] \cdot f_Z(z)\, dz$$

or $E_Z[E_X[X|Z=z]] = \int_{z=0}^{z=1} \frac{2}{3} \cdot \frac{1}{2}\, dz + \int_{z=1}^{z=\infty} \frac{2}{3z} \cdot \frac{1}{2z^2}\, dz$

or $E_Z[E_X[X|Z=z]] = \int_{z=0}^{z=1} \frac{1}{3}\, dz + \int_{z=1}^{z=\infty} \frac{1}{3} z^{-3}\, dz$

or $E_Z[E_X[X|Z=z]] = \frac{1}{3} + \left[\frac{1}{6} z^{-2}\right]_{z=\infty}^{z=1}$

or $E_Z[E_X[X|Z=z]] = \frac{1}{3} + \frac{1}{6}$

4(f) or $\boxed{E_Z[E_X[X|Z=z]] = \frac{1}{2} = E[X]}$ Hence Verified! ☺

Ans

4(g) $E[XZ|Z=z] = E[zX|Z=z]$

or $E[XZ|Z=z] = z\,E[X|Z=z]$

4(g) or

$$\boxed{E[XZ|Z=z] = \begin{cases} \frac{2}{3}z & z \in [0,1],\ x \in [0,z] \\ \frac{2}{3} & z \in [1,\infty),\ x \in [0,1] \end{cases}}$$

$\downarrow$

or $E[Y|Z=z]$

Ans

Verify (optional) if $E_Z[E_Y[Y|Z=z]] = E_Y[Y] \rightarrow \frac{1}{2}$

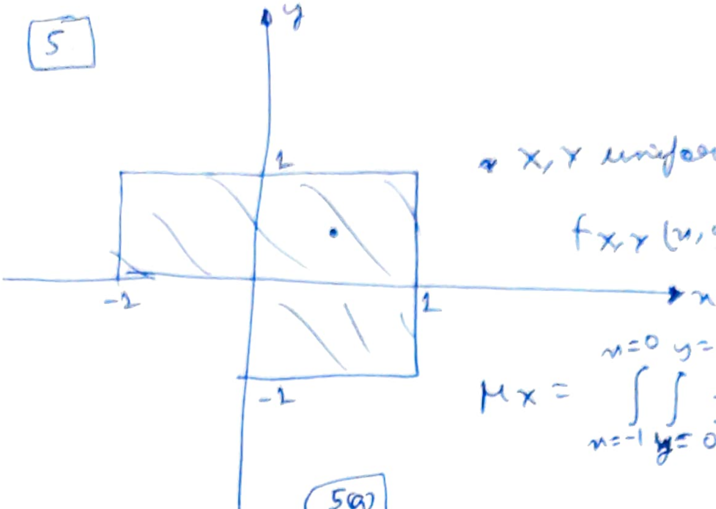
$E_Z[E_Y[Y|Z=z]] = \int_{z=0}^{z=1} \frac{2}{3} z \cdot \frac{1}{2}\, dz + \int_{z=1}^{z=\infty} \frac{2}{3} \cdot \frac{1}{2z^2}\, dz$

or $E_Z[E_Y[Y|Z=z]] = \frac{1}{3} \frac{z^2}{2}\Big|_0^1 + \frac{1}{3z}\Big|_\infty^1$

or $E_Z[E_Y[Y|Z=z]] = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} = E[Y]$ Hence Verified! ☺

——— × ——— × ——— × ——— × ———

5



X, Y uniform in shaded region

$f_{X,Y}(x,y) = \frac{1}{3}$ in shaded region

$$\mu_X = \int_{x=-1}^{x=0}\int_{y=0}^{y=1} \frac{1}{3} x \, dy \, dx + \int_{x=0}^{x=1}\int_{y=-1}^{y=1} \frac{1}{3} x \, dy \, dx$$

5(a) or $\boxed{\mu_X = \frac{1}{6}}$ Ans $\Rightarrow$ 5(b) $\boxed{\mu_Y = \frac{1}{6}}$ (by symmetry) Ans

$$\mu_{XY} = \int_{-1}^{0}\int_{0}^{1} \frac{1}{3} xy \, dy \, dx + \int_{0}^{1}\int_{-1}^{1} \frac{1}{3} xy \, dy \, dx$$

5(c) or $\boxed{\mu_{XY} = \frac{-1}{12}}$ Ans

5(d) $\sigma_X^2 = \mu_{X^2} - (\mu_X)^2$

$$\mu_{X^2} = \iint \frac{1}{3} x^2 \, dy \, dx = \frac{1}{3}$$

$$\sigma_X^2 = \frac{1}{3} - \left(\frac{1}{6}\right)^2$$

5(d) or $\boxed{\sigma_X^2 = \frac{11}{36}}$ Ans $\Rightarrow$ 5(e) $\boxed{\sigma_X^2 = \frac{11}{36}}$ (By symmetry) Ans

5(f) $\mathrm{Cov}(X,Y) = E[XY] - E[X].E[Y]$

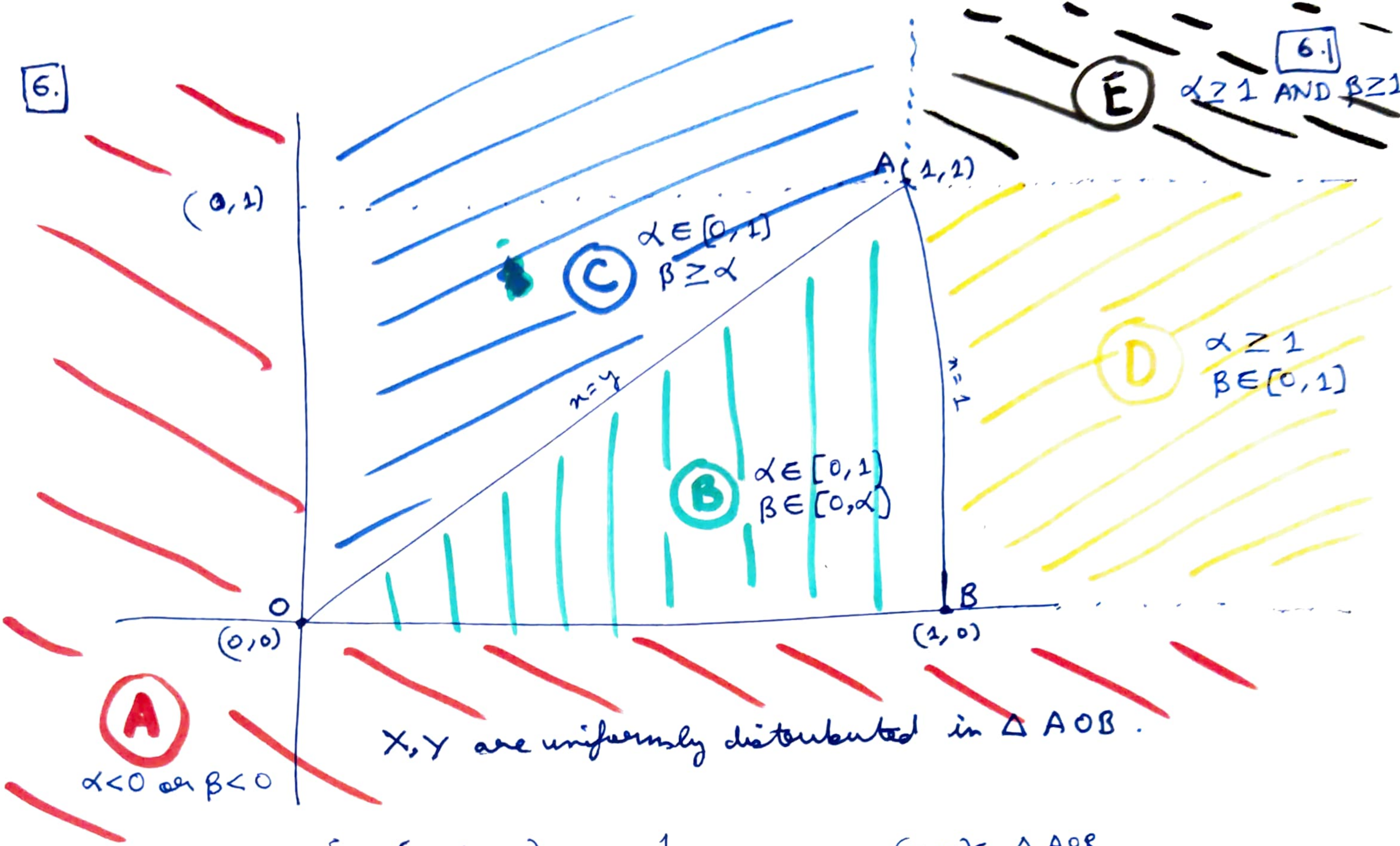
or $\mathrm{Cov}(X,Y) = \frac{-1}{12} - \left(\frac{1}{6}\right)^2$

5(f) or $\boxed{\mathrm{Cov}(X,Y) = \frac{-1}{9}}$ Ans $\Rightarrow$ 5(g) $\boxed{\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_X} = \frac{-4}{11}}$ Ans

or $\rho_{XY} \approx -0.3636$

——— X ——— X ——— X ——— X ———

6.



X, Y are uniformly distributed in Δ AOB.

So $f_{X,Y}(x,y) = \dfrac{1}{\text{Area of } \Delta AOB}$ $\quad (x,y) \in \Delta AOB$

or $f_{X,Y}(x,y) = \dfrac{1}{\frac{1}{2}\times 1 \times 1}$ $\quad y \in [0,1]$ or $x \in [y,1]$

6(a)

or

$f_{X,Y}(x,y) =$	2	$y \in [0,1]$, $x \in [y,1]$	
$=$	0	else	Ans

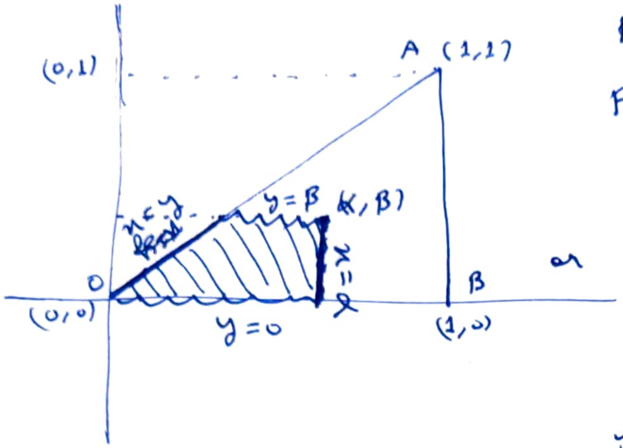
6(b) (Calculation on next pages.)

$$F_{X,Y}(x=\alpha, y=\beta) = \begin{cases} 0 & \alpha < 0 \text{ or } \beta < 0 \quad \text{A} \\ 2\alpha\beta - \beta^2 & \alpha \in [0,1],\ \beta \in [0,\alpha] \quad \text{B} \\ \alpha^2 & \alpha \in [0,1],\ \beta \geq \alpha \quad \text{C} \\ 2\beta - \beta^2 & \alpha \geq 1,\ \beta \in [0,\alpha] \quad \text{D} \\ 1 & \alpha \geq 1,\ \beta \geq 1 \quad \text{E} \end{cases}$$

Ans

For (α, β) in B:

i.e. $\alpha \in (0,1]$, $\beta \in [0, \alpha]$



~~$F_{X,Y}(x,y)$~~

$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\,dx\,dy \quad (\alpha,\beta) \in \text{B}$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\beta}\int_{x=y}^{x=\alpha} 2\,dx\,dy \quad (\alpha,\beta) \in \text{B}$$

or $$F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2x\Big|_{y}^{\alpha} dy \quad (\alpha,\beta) \in \text{B}$$

or $$F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2(\alpha - y)\,dy \quad (\alpha,\beta) \in \text{B}$$

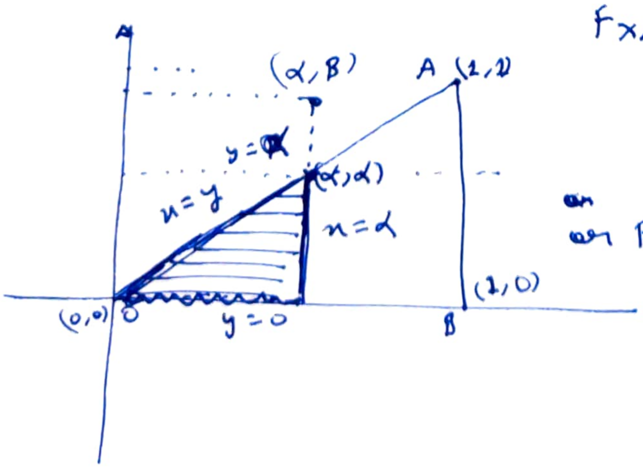
or $$F_{X,Y}(\alpha,\beta) = 2\alpha y - y^2\Big|_0^\beta \quad (\alpha,\beta) \in \text{B}$$

or $$\boxed{F_{X,Y}(\alpha,\beta) = 2\alpha\beta - \beta^2 \quad (\alpha,\beta) \in \text{B}}$$

or $\alpha \in (0,1]$, $\beta \in [0,\alpha]$

For (α, β) in C:

i.e. $\alpha \in [0,1]$, $\beta \geq \alpha$



$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\,dx\,dy \quad (\alpha,\beta) \in \text{C}$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha}\int_{x=y}^{x=\alpha} 2\,dx\,dy \quad (\alpha,\beta) \in \text{C}$$

or $F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha} 2(\alpha - y)\,dy \qquad (\alpha,\beta) \in C$

or $F_{X,Y}(x=\alpha, y=\beta) = 2\alpha^2 - \alpha^2 \qquad (\alpha,\beta) \in C$

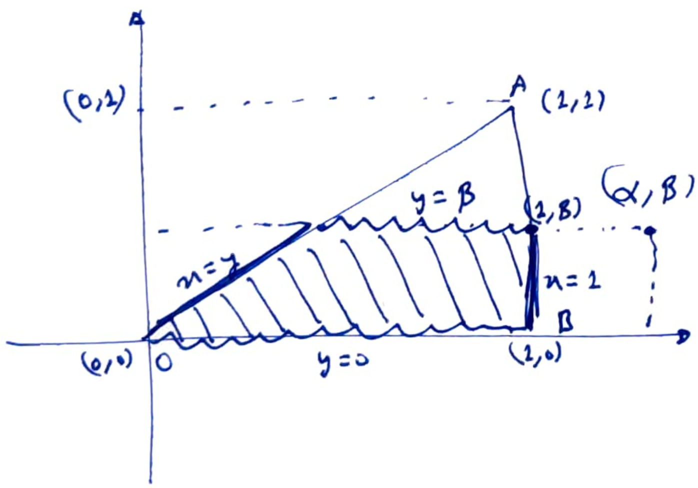
or $\boxed{F_{X,Y}(x=\alpha, y=\beta) = \alpha^2 \qquad (\alpha,\beta) \in C \quad \text{or} \quad \alpha \in [0,1],\ \beta \geq \alpha}$

Area of the shaded Δ. * $f_{X,Y}(x,y)$ in the region.

For (α,β) in D

i.e. $\alpha \geq 1$, $\beta \in [0,1]$

$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\,dx\,dy \qquad (\alpha,\beta) \in D$



or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} \int_{x=y}^{x=1} 2\,dx\,dy \qquad (\alpha,\beta) \in D$

or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2(1-y)\,dy \qquad (\alpha,\beta) \in D$

or $F_{X,Y}(\alpha,\beta) = 2y - y^2 \Big|_0^\beta \qquad (\alpha,\beta) \in D$

or $\boxed{F_{X,Y}(\alpha,\beta) = 2\beta - \beta^2 \qquad (\alpha,\beta) \in D \quad \text{or} \quad \alpha \geq 1,\ \beta \in [0,1)}$

$$f_X(x) = \int_{\forall y} f_{X,Y}(x,y)\,dy$$

or
$$f_X(x) = \begin{cases} 0 & x<0 \\ \int_{y=0}^{y=x} 2\,dy & x \in [0,1] \\ 0 & x \geq 1 \end{cases}$$

6(c)

or
$$\boxed{f_X(x) = \begin{cases} 0 & x<0 \\ 2x & x \in [0,1] \\ 0 & x>1 \end{cases}}$$

Ans

$$f_Y(y) = \int_{\forall x} f_{X,Y}(x,y)\,dx$$

or
$$f_Y(y) = \begin{cases} 0 & y<0 \\ \int_{x=y}^{x=1} 2\,dx & y \in [0,1] \\ 0 & y>1 \end{cases}$$

6(d)

or
$$\boxed{f_Y(y) = \begin{cases} 0 & y<0 \\ 2(1-y) & y \in [0,1] \\ 0 & y>1 \end{cases}}$$

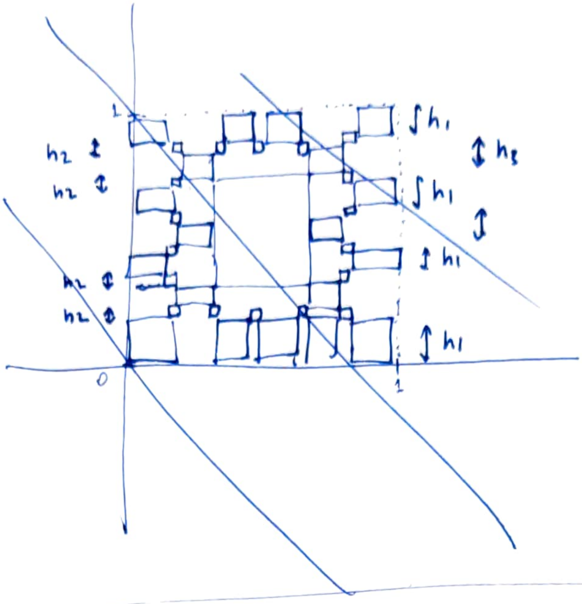
Ans

$$f_{X|Y}(x \mid Y=y) = \frac{f_{X,Y}(x, y=y)}{f_Y(y)}$$

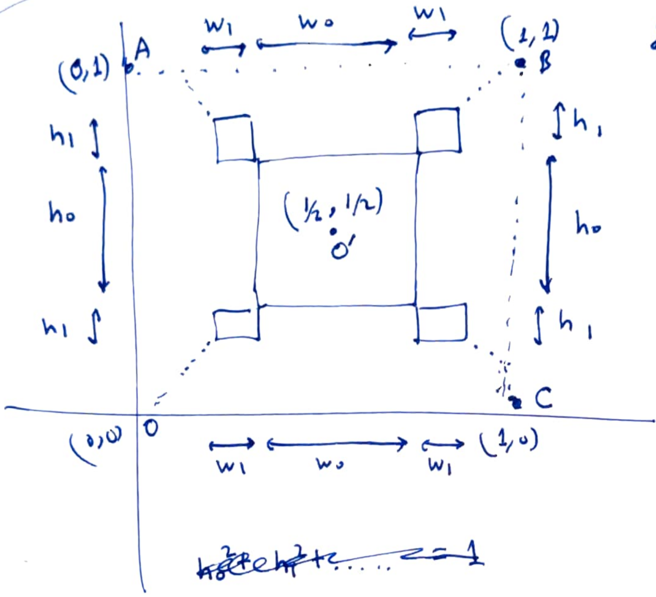
6(e)

or
$$\boxed{f_{X|Y}(x \mid Y=y) = \begin{cases} \text{not defined} & y < 0 \\ \frac{1}{1-y} & y \in (0,1) \\ & x \in (y,1) \\ \text{not defined.} & y \geq 1 \\ \hline 0 & x < 0 \\ & \text{or } x > 1 \end{cases}}$$ Ans

$\boxed{7.}$



Ans



~~$h_0^2 + h_1^2 + \ldots = 1$~~

~~$h_0 + h_1 + h_2 + \ldots = 1$~~

where $h_0 + h_1 + h_2 + \ldots = 1$

$w_0 + w_1 + w_2 + \ldots = 1$

$(\text{uniform pdf}) \times \{h_0 w_0 + h_1 w_1 + h_2 w_2 + \ldots\} = 1$

In ~~rectangle~~ square OABC of size 1×1 unit squares, any pdf distribution which is

1) symmetric about $\left(\frac{1}{2}, \frac{1}{2}\right)$.
2) NOT a single rectangle.
3) ~~But~~ Filled with a set of more than one (possibly infinite) rectangles which are touching together (to maintain continuity in both x and y) as well as are symmetrically placed around $\left(\frac{1}{2}, \frac{1}{2}\right)$ (to maintain zero covariance), ~~and~~

can qualify for a distribution ~~val~~ in x, y s.t.

1) X and Y are uncorrelated.
2) X and Y are NOT independent.
3) $X \sim \text{unif}[0,1)$ and $Y \sim \text{unif}[0,1]$.

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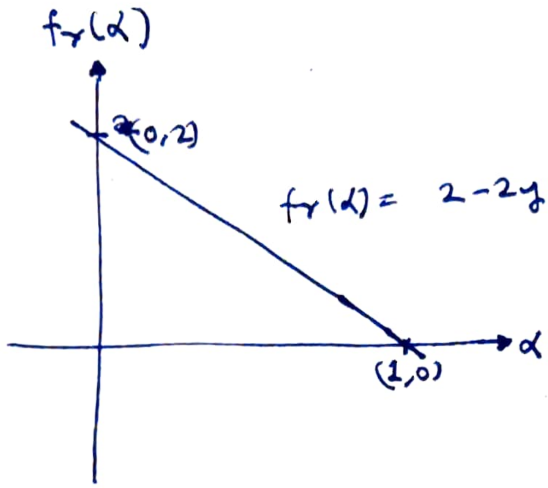
8.



8 (a) $\hat{Y}_{MP} = \arg\max_{\alpha} \left(f_Y(\alpha) \right) \Big|_{\alpha \in [0,1]}$

or $\hat{Y}_{MP} = \arg\max_{\alpha} \left(2 - 2\alpha \right) \Big|_{\alpha \in [0,1]}$

8(a)

or $\boxed{\hat{Y}_{MP} = 0}$ Ans for which $f_Y(\alpha = 0) = 2$

Ans

8(b) $\hat{Y}_{MMSE} = \arg\min_{\alpha} \left[E[(Y - \alpha)^2] \right]$

We know that $\hat{Y}_{MMSE} = \mu_Y = E(Y)$

or $\hat{Y}_{MMSE} = \int_{y=0}^{y=1} y \cdot f_Y(y)\, dy$

or $\hat{y}_{MMSE} = \int_{y=0}^{y=1} y \cdot (2 - 2y)\, dy$

or $\hat{y}_{MMSE} = \int_{y=0}^{y=1} 2y - 2y^2 \, dy$

or $\hat{y}_{MMSE} = y^2 - \frac{2}{3}y^3 \Big|_{y=0}^{y=1}$

8(b)

or $\boxed{\hat{y}_{MMSE} = \frac{1}{3}}$ Ans

PTO

None, the less, let us derive $\hat{Y}_{MMSE}$ using only first principles.

$$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[E\left[(Y-\alpha)^2\right] \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[\int_{y=0}^{y=1} (y-\alpha)^2 \cdot (2-2y)\, dy \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2\int_{y=0}^{y=1} \{ y^2 - 2\alpha y + \alpha^2 \}\{1-y\}\, dy \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2\int_{y=0}^{y=1} \{ -y^3 + (1+2\alpha)y^2 - (\alpha^2+2\alpha)y + \alpha^2 \} dy \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2 \left[-\frac{y^4}{4} + \frac{(1+2\alpha)}{3} y^3 - \frac{(\alpha^2+2\alpha)}{2} y^2 + \alpha^2 y \right]_{y=0}^{y=1} \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2 \left[-\frac{1}{4} + \frac{(1+2\alpha)}{3} - \frac{(\alpha^2+2\alpha)}{2} + \alpha^2 \right] \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2 \left[\frac{\alpha^2}{2} - \frac{1}{3}\alpha + \frac{1}{12} \right] \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[\left(\alpha - \frac{1}{3}\right)^2 + \frac{1}{18} \right]$$

$\boxed{8(b)}$

or $$\boxed{\hat{y}_{MMSE} = \frac{1}{3} \quad \text{with error} = \frac{1}{18}}$$ Ans

8(c) $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(E(|y-\alpha|)\right) = \underset{\alpha}{\arg\min}\left(\int_{y=0}^{y=\alpha}(\alpha-y)(2-2y)\,dy + \int_{y=\alpha}^{y=1}(y-\alpha)(2-2y)\,dy\right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(2\int_{y=0}^{y=\alpha}(y-\alpha)(y-1)\,dy + 2\int_{y=1}^{y=\alpha}(y-\alpha)(y-1)\,dy\right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(2\int_{y=0}^{y=\alpha}\{y^2-(\alpha+1)y+\alpha\}\,dy + 2\int_{y=1}^{y=\alpha}\{y^2-(\alpha+1)y+\alpha\}\,dy\right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\times\left[\frac{y^3}{3}-\frac{(\alpha+1)}{2}y^2+\alpha y\right]\Bigg|_{y=\alpha} - 2\left[\frac{y^3}{3}-\frac{(\alpha+1)}{2}y^2+\alpha y\right]\Bigg|_{y=1} \cancel{- 2\left[\frac{y^3}{3}-\frac{(\alpha+1)}{2}y^2+\alpha y\right]\Bigg|_{y=0}}\right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\left[\frac{\alpha^3}{3}-\frac{(\alpha+1)}{2}\alpha^2+\alpha^2\right] - 2\left[\frac{1}{3}-\frac{(\alpha+1)}{2}+\alpha\right]\right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\left[-\frac{\alpha^3}{6}+\frac{\alpha^2}{2}\right] - 2\left[\frac{\alpha}{2}-\frac{1}{6}\right]\right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(-\frac{4}{6}\alpha^3+2\alpha^2-\alpha+\frac{1}{3}\right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(\frac{d}{d\alpha}\left[-\frac{2}{3}\alpha^3+2\alpha^2-\alpha+\frac{1}{3}\right]=0\right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(-2\alpha^2+4\alpha-1=0\right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(-2(\alpha-1)^2+1=0\right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left((\alpha-1)^2=\frac{1}{2}\right)$$

or $$\hat{y}_{MMAE} = 1-\frac{1}{\sqrt{2}}$$

or 8(c) $\boxed{\hat{y}_{MMAE} \approx 0.293}$ Ans

$\boxed{9.1}$

~~For DC~~ ~~for~~ F~~...~~, ~~we require~~ :

$\boxed{9.}$ $Y|X \sim \exp\left(\frac{1}{x^2}\right) \Rightarrow$ ~~$f_X(y)$~~ $f_{Y|X}(y|X=x) = \frac{1}{x^2} e^{-\frac{1}{x^2}y}$ $\quad y \geq 0$

$X \sim \text{unif}(0,1) \Rightarrow f_X(x) = 1 \quad x \in [0,1]$

~~$\hat{Y}_{MMSE}$~~

9(a)(i) $\boxed{\hat{Y}|X=x_{\,MMSE} = \mu_{Y|X} = x^2}$ Ans

9(a)(ii) $\boxed{\text{Error } \hat{Y}|X = \sigma^2_{Y|X} = x^4 \quad \text{or std. error } \hat{Y}|X = \sigma_{Y|X} = x^2}$ Ans

$$\text{Overall error} = \int_{x=0}^{x=1} \sigma^2_{Y|X} \cdot f_X(x)\, dx =$$

$$\text{or Overall error} = \int_{x=0}^{x=1} x^4 \cdot 1 \cdot dx$$

9(a)(iii) or $\boxed{\text{Overall error} = \frac{1}{5}}$ Ans

$$\hat{Y^2}|X_{\,MMSE} = E(Y^2|X=x)$$

$$\text{or } \hat{Y^2}|X_{\,MMSE} = \int_{y=0}^{y=\infty} \frac{1}{x^2} e^{-\frac{1}{x^2}y} \cdot y^2\, dy$$

$$\text{or } \hat{Y^2}|X_{\,MMSE} = \left(y^2 + 2x^2 y + 2x^4\right) e^{-\frac{1}{x^2}y} \Big|_{y=\infty}^{y=0}$$

9(b). or $\boxed{\hat{Y^2}|X_{\,MMSE} = 2x^4}$ Ans $\neq \left(\hat{Y}|X_{MMSE}\right)^2$

$\boxed{\text{Mean of squares} \neq \text{Square of mean}}$ Ans

Rough

$\int A y^2 e^{-Ay}$

$= \frac{d}{dy}\left[\left(Cy^2 + Dy + E\right)e^{-Ay}\right]$

$\Rightarrow -CA = A$

$2C - AD = 0$

$D - AE = 0$

$\Rightarrow C = -1$

$D = -2x^2$

$E = 2x^4$

$\Rightarrow \int A y^2 e^{-Ay}$

$= (-y^2 - 2x^2 y - 2x^4) \cdot e^{-\frac{1}{x^2}y}$

9.2

9 (c)

$$f_{X|Y}(x|Y=y) = \frac{f_{Y|X}(y|X=x) \cdot f_X(x)}{f_Y(y)}$$

or $$\boxed{f_{X|Y}(x|Y=y) = \frac{\left(\frac{1}{x^2} e^{-\frac{1}{x^2}y}\right) \cdot (1)}{\int_{y=0}^{y=\infty} \frac{1}{x^2} e^{-\frac{1}{x^2}y}\, dy}}$$

$y \in [0, \infty)$
$x \in [0, 1]$

9(c) $$\boxed{\mu_{X|Y} = \int_{x=0}^{x=1} x \cdot f_{X|Y}(x|Y=y)\, dx}$$ Ans

9 (d) $\hat{y}|X=x_{LMMSE} = a^* x + b^*$ where,

9(d) $$a^* = \frac{Cov(X, Y)}{\sigma_X^2}$$

or $$a^* = \frac{\mu_{XY} - \mu_X \cdot \mu_Y}{\sigma_X^2}$$

or $$a^* = \frac{\int_{x=0}^{x=1}\int_{y=0}^{y=\infty} xy \cdot \frac{1}{x^2} e^{-\frac{1}{x^2}y}\, dy\, dx - \frac{1}{2}\int_{x=0}^{x=1}\int_{y=0}^{y=\infty} y \cdot \frac{1}{x^2} e^{-\frac{1}{x^2}y}\, dy\, dx}{\left(\frac{1}{12}\right)}$$

or ~~$a^* = \int_{x=0}^{x=1} \frac{1}{x} \ldots$~~

or $$a^* = \frac{\int_{x=0}^{x=1} \frac{1}{x}\left(\int_{y=0}^{y=\infty} y \cdot e^{-\frac{1}{x^2}y}\, dy\right) dx - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{x^2}\left(\int_{y=0}^{y=\infty} y\, e^{-\frac{1}{x^2}y}\, dy\right) dx}{1/12}$$

or $$a^* = \frac{\int_{x=0}^{x=1} \frac{1}{x} \cdot \left(\frac{x^2 y + x^4}{e^{\frac{1}{x^2}y}} \Big|_{y=\infty}^{y=0} \right) dx - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{x^2}\left[\left(x^2 y + x^4\right) e^{-\frac{1}{x^2}y}\right]_{y=\infty}^{y=0} dx}{1/12}$$

~~Hi Rudrayat~~

$$\Rightarrow a^* = \frac{\int_{x=0}^{x=1} \frac{1}{x} \cdot (x^4)\, dx \quad - \frac{1}{2} \int_{x=0}^{x=1} \frac{1}{x} \cdot (x^4)\, dx}{1/12}$$

$$\Rightarrow a^* = \frac{\frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{12}}$$

$$\Rightarrow \boxed{a^* = 1}$$

$$b^* = \mu_Y - a^* \mu_X$$

$$\Rightarrow b^* = \frac{1}{3} - 1 \cdot \frac{1}{2}$$

$$\Rightarrow \boxed{b^* = -\frac{1}{6}}$$

$\boxed{9(d)}$ $\boxed{\hat{Y}|X_{LMMSE} = 1 \cdot x - \frac{1}{6}}$ Ans

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10. 10(a)

10(a) Mapping of outcomes of an experiment to signals. Ans

Ω: A, B, C

$X(A,t) = \cos(t)$

$X(C,t) = \begin{cases} 7 & \lfloor t \rfloor \text{ is even} \\ 0 & \lfloor t \rfloor \text{ is odd} \end{cases}$

$X(B,t) = 2e^{-2t}$

where, $\lfloor x \rfloor$: Floor function or greatest integer less than or equal to the x.

10(b) Virtually any physical phenomenon or simulated phenomenon can be represented as a Random Process.

Eg1: Bus voltages $V_i(t)$ are time ~~varying signals~~ time varying signals, and there are multiple buses in a power system.

Ω: $i=1, \vdots, i=14$ $\rightarrow$ $V_1(t)$ ... $V_i(t)$ $i \in 1$ to 14

Ω: {IEEE 14 Bus system}

Being able to estimate/predict these processes is beneficial to humans.

& in most cases

Eg2: if tomorrow's weather could be predicted using local temperatures, humidity and trends, with reasonable accuracy, it would help us all in determining the optimal clothes to wear, time to travel, mode of transport (bus if snowing, car on a sunny day, etc.).

Ans

10(c) 10.2

Random Process:

10(c) US Daily Power Generation Mix.

Percentage values of biggest 5 contributors:

Ω	09 Nov 2022	10 Nov 2022	11 Nov 2022
Natural gas	40	38	37
Nuclear	19	22	20
Coal	16	14	16
Wind	14	12	16
Hydro	6	7	6

Ans

10(d) Some advantages of modelling $x_i(t)$ instead of $x_i(t=T)$ separately for several T:

> Our interest is in determining how a particular process varies with time. ~~It makes sense for the sampling to be done in a single experiment with identical conditions.~~

$x_i(t)$ as $t \uparrow$ as.

Eg.. Theory of Critical Slowing Down. says that the autocorrelation and variance of the state variables of a dynamical system $\uparrow$ as the dynamical system approaches a catastrophical point w.r.t. its stability.

> Similarly, we may be interested in cross correlation b/w different signals w.r.t. time as a parameter

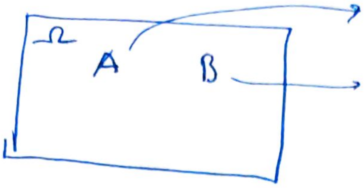
Eg. Electricity prices vs Load demand, which themselves are functions of time.

Ans

— X — X — X —

11.2

11



$X(A,t) = \cos(t)$

$X(B,t) = \sin(t)$

$P(A) = 0.5 \quad P(B) = 0.5$

11 (a) $f_{X(t)}(x) = \cancel{X(A,t)}\ P(A) \cdot \delta(x - X(A,t)) + P(B) \cdot \delta(x - X(B,t))$

11 (a)

or $\boxed{f_{X(t)}(x) = 0.5\,\delta(x - \cos(t)) + 0.5\,\delta(x - \sin(t))}$ Ans

—— x —— x —— x ——

12.

$x(n)$:

Ω 1   0

$P(X(n)=0) = 0.6$

$P(X(n)=1) = 0.4$

$$f_{X(K)}(n,k) = 0.6\,\delta(n) + 0.4\,\delta(n-1) \quad k = 1, 2, \ldots$$

12 (c)

$$Y[k] = \sum_{i=0}^{k-1} X[i]$$

$$Y[k] = \begin{array}{lll} 0 & \text{w.p.} & {}^{k}C_{k}(0.6)^{k}(0.4)^{0} \\ 1 & \text{w.p.} & {}^{k}C_{k-1}(0.6)^{k-1}(0.4)^{1} \\ \vdots & & \\ k & \text{w.p.} & {}^{k}C_{0}(0.6)^{k-k}(0.4)^{k} \end{array}$$

i.e. $Y[k] = i$ w.p. ${}^{k}C_{i}(0.6)^{k-i}(0.4)^{i}$

$i = 0, 1, \ldots k$

12 (c)

$$\Rightarrow \boxed{f_{Y(k)}(y,k) = \sum_{i=0}^{k} {}^{k}C_{i}(0.6)^{k-i}(0.4)^{i}\,\delta(y-i)} \quad \text{Ans}$$

—x—x—x—x—