EE 507 HW05

ARYAN RITWAJEET JHA

1.(a) Check if $\boxed{f_X(x) \cdot f_Y(y) \cdot f_Z(z) = f_{X,Y,Z}(x,y,z)}$ ⓪
implies that ~~the three~~ X and Y are independent,
i.e. check if ⓪ implies $\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$ ⓵

~~given~~ Integrating both sides of ⓪ w.r.t. ~~z~~ z:

$$\int_{\forall z} f_X(x) \cdot f_Y(y) \cdot f_Z(z)\, dz = \int_{\forall z} f_{X,Y,Z}(x,y,z)\, dz$$

or $$f_X(x) f_Y(y) \int_{\forall z} f_Z(z) dz = f_{X,Y}(x,y)$$ (marginal pdf of X, Y)

or $$f_X(x) \cdot f_Y(y) \cdot 1 = f_{X,Y}(x,y)$$

or $$\boxed{f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)}$$ which is ⓵ which is what we set to check for!

∴ 1(a) $\boxed{f_X(x) \cdot f_Y(y) f_Z(z) = f_{X,Y,Z}(x,y,z) \Rightarrow X \text{ and } Y \text{ are independent}}$ Ans

1(b) check if $\boxed{P(X)P(Y)P(Z) = P(X,Y,Z)}$ ⓪ implies that X and Y are independent i.e. $\boxed{P(X) \cdot P(Y) = P(X,Y)}$ ⓵
where X, Y, Z are events of a probabilistic experiment.

No.

Counterexample:

~~Let experiment~~

In an experiment, ~~let~~ we are picking a number from 1 to 8 randomly: Each number has equal probability of being picked. This experiment is like throwing a fair 8-sided die.

Now let us define three events:

$X = \{1, 2, 3, 4\} \Rightarrow P(X) = \frac{1}{2}$ ~~$P(X, Y)$~~

$Y = \{1, 3, 4, 5\} \Rightarrow P(Y) = \frac{1}{2}$

$Z = \{1, 6, 7, 8\} \Rightarrow P(Z) = \frac{1}{2}$

$X \cap Y \cap Z = \{1\} \Rightarrow P(X \cap Y \cap Z) = \frac{1}{8}$

~~So $P(X) \cdot P(Y)$~~

So equation 0 is followed by events X, Y, Z, as

$$P(X) \cdot P(Y) \cdot P(Z) = \left(\frac{1}{2}\right)^3 = \frac{1}{8} = P(X, Y, Z)$$

But $P(X) \cdot P(Y) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$P(X \cap Y) = P(\{1, 3, 4\}) = \frac{3}{8}$

X and Y are NOT independent.

In fact, $P(Y) \cdot P(Z) = \frac{1}{4}$

$P(Y \cap Z) = P(\{1\}) = \frac{1}{8}$

Y and Z are NOT independent either

And $P(X) \cdot P(Z) = \frac{1}{4}$

$P(X \cap Z) = P(\{1\}) = \frac{1}{8}$

X and Z are NOT independent either.

$\boxed{1(b)}$

$\therefore$ The analogous event $P(X) \cdot P(Y) \cdot P(Z) = P(XYZ)$ does NOT imply independence of X and Y.

Ans

---

2. 2(a) $X, Y \sim \mathcal{N}(\mu_X = 0, \mu_Y = 0, \sigma_X^2 = 1, \sigma_Y^2 = 4, \rho = 0.5)$

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{1-\rho^2}\,\sigma_X\sqrt{2\pi}\,\sigma_Y\sqrt{2\pi}}\, e^{\underbrace{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right\}}_{\text{Only component (say } g(x,y)) \text{ dependent on } x \text{ and } y.}}$$

Taking out the component $g(x,y)$ and putting the given values of $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ_{XY}:

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - \frac{2(x)(y)}{(1)(2)}$$

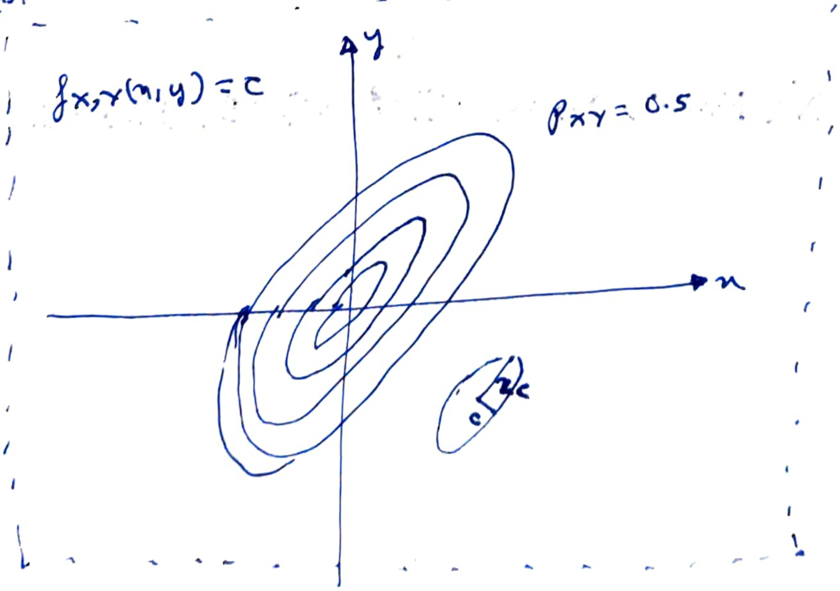
$\Rightarrow f_{X,Y}(x,y) \underset{\substack{\text{i.e.}\\ \text{is}\\ \text{constant}}}{\longrightarrow} c$ for $g(x,y) \underset{\substack{\text{i.e.}\\ \text{is}\\ \text{constant}}}{\longrightarrow} c$

$\Rightarrow$ Contour of $f_{X,Y}(x,y)$ is

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - \cancel{2}\, xy = c$$

which is the equation for an ellipse, tilted and with positive slope.

2(a)



Ans

2(b)

$$\boxed{\begin{aligned} f_X(x) &= \mathcal{N}(x, 0, 1^2) = \frac{1}{1.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}} \quad \forall x \\ f_Y(y) &= \mathcal{N}(y, 0, 2^2) = \frac{1}{2.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y}{2}\right)^2\right\}} \quad \forall y \end{aligned}}$$

Ans

2(c)

$$f_{Y|X}(y|X=x) = \frac{f_{X,Y}(X=x, Y)}{f_X(X=x)} \quad \forall x, y$$

or
$$f_{Y|X}(y|X=x) = \frac{\dfrac{1}{\sqrt{(1-\frac{1}{2}^2)}\,1.\sqrt{2\pi}\,.\,2.\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}^2}\cdot\left\{\left(\frac{x}{1}\right)^2+\left(\frac{y}{2}\right)^2 - 2\times\frac{1}{2}\times\frac{x}{1}\times\frac{y}{2}\right\}} \quad \cancel{\forall x, y}}{\dfrac{1}{1.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}}} \quad \forall (x, y)$$

or
$$f_{Y|X}(y|X=x) = \frac{1}{\sqrt{3}.\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\left\{\left(\frac{x}{\sqrt{3}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 - 2.1.\left(\frac{x}{\sqrt{3}}\right)\left(\frac{y}{\sqrt{3}}\right)\right\}} \quad \forall (x, y)$$

2(c)

or ~~$f_{Y|X}(y|X=x) = \sqrt{2\pi}\,\mathcal{N}(\mu_X=0, \mu_Y=0, \sigma_X^2=3, \sigma_Y^2=3, \rho_{XY}=1) \quad \forall (x,y)$~~ Ans

or ~~$f_{Y|X}(y|X=x) = \mathcal{N}_y(\mu_Y=x, \sigma_Y^2=3) \quad \forall y$~~

or
$$f_{Y|X}(y|X=x) = \frac{1}{\sqrt{3}\,.\,\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y-x}{\sqrt{3}}\right)^2\right\}} \quad \forall y$$

2(c)

or
$$\boxed{f_{Y|X}(y|X=x) = \mathcal{N}(y, \mu_Y=x, \sigma_Y^2=3) \quad \forall y}$$

Ans

PTO.

2(d) Find x s.t. $E[Y \mid X = x] = -2$.

From 2(c), we know that $Y \mid X = x$ is fully correlated to $X = x$. ($\rho_{Y|X, X} = 1$).

$\because$ Y and X have ~~share~~ the same support/domain of $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

$\therefore$ $E_Y[Y \mid X = x] = -2 \equiv E_X[X = x] = -2$

But $E_X[X = x] = x$

~~$\Rightarrow x =$~~ $\Rightarrow$ $\boxed{x = -2}$ $\boxed{2(d)}$ Ans

2(d) Find x s.t. $E[Y \mid X = x] = -2$

From 2(c), we know that $Y \mid X = x$ is a gaussian with mean x.

So $E[Y \mid X = x] = -2 \Rightarrow \boxed{x = -2}$ Ans

2(e) $Z = X + Y - 1$

Z is also a gaussian, we need to only compute μ_Z and σ_Z^2 to find $f_Z(z)$:

$E(Z) = E(X) + E(Y) - E(1)$

or $\mu_Z = \mu_X + \mu_Y - 1$

or $\mu_Z = 0 + 0 - 2$

or $\boxed{\mu_Z = -1}$

$\sigma_Z^2 = E\left[\cancel{X - \mu_Z}\left\{(X + Y - 1) - E[X + Y - 1]\right\}^2\right]$

or $\sigma_Z^2 = E\left[\left\{(X - \mu_X) + (Y - \mu_Y) + (-1 - E(-1))\right\}^2\right]$

or $\sigma_Z^2 = E\left[\left\{(X - \mu_X) + (Y - \mu_Y)\right\}^2\right]$

or $\sigma_Z^2 = E[(X - \mu_X)^2] + E[(Y - \mu_Y)^2] + \cancel{2\rho_{XY}}\ 2E[(X - \mu_X)(Y - \mu_Y)]$

or $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 1^2 + 2^2 + 2 \times 0.5 \times 1 \times 2$

or $\boxed{\sigma_Z^2 = 7}$

2(e)

$\therefore \boxed{Z \sim \mathcal{N}(z, \mu_Z = -1, \sigma_Z^2 = 7)}$ Ans

2(f) $\boxed{2.5}$

$Z = 2X + 3Y$

$E(Z) = 2\mu_X + 3\mu_Y$

2(f)(i)

or $\boxed{\mu_Z = 0}$ Ans

$W = X - Y$

$E(W) = \mu_X - \mu_Y$

2(f)(iii)

$\boxed{\mu_W = 0}$ Ans

$\sigma^2_Z = E\left[\{(2X+3Y) - (2\mu_X + 3\mu_Y)\}^2\right]$

or $\sigma^2_Z = E\left[\{2(X-\mu_X)\}^2 + \{3(Y-\mu_Y)\}^2 + 2.2.3.(X-\mu_X)(Y-\mu_Y)\right]$

or $\sigma^2_Z = 4\sigma_X^2 + 9\sigma_Y^2 + 12\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 4.1^2 + 9.2^2 + 12.(0.5).1.2$

or $\sigma_Z^2 = 4 + 36 + 12$

2(f)(ii)

or $\boxed{\sigma_Z^2 = 52}$ Ans

$\sigma_W^2 = E\left[\{(X-Y) - (\mu_X - \mu_Y)\}^2\right]$

or $\sigma_W^2 = E\left[\{X-\mu_X\}^2 + \{Y-\mu_Y\}^2 - \{2.1.1.(X-\mu_X)(Y-\mu_Y)\}\right]$

or $\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_W^2 = 1^2 + 2^2 - 2.(0.5).1.2$

or $\sigma_W^2 = 1 + 4 - 2$

2(f)(iv)

$\boxed{\sigma_W^2 = 3}$ Ans

2(g)

$Cov(Z, W) = E\left[(Z - \mu_Z)(W - \mu_W)\right]$

or $Cov(Z, W) = E\left[(2X + 3Y - 2\mu_X - 3\mu_Y)(X - Y - \mu_X - (-\mu_Y))\right]$

or $Cov(Z, W) = E\left[\{2(X-\mu_X) + 3(Y-\mu_Y)\}\{1(X-\mu_X) - 1(Y-\mu_Y)\}\right]$

or $Cov(Z, W) = E\left[2(X-\mu_X)^2 - 3(Y-\mu_Y)^2 + 1(X-\mu_X)(Y-\mu_Y)\right]$

or $Cov(Z, W) = 2\sigma_X^2 - 3\sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y$

or $Cov(Z, W) = 2.1^2 - 3.2^2 + (0.5).1.2$

or $Cov(Z, W) = 2 - 12 + 1$

2(f)$^{(v)}$

or $\boxed{Cov(Z, W) = -9}$ Ans

$$\boxed{\rho_{Z,W} = \frac{Cov(Z, W)}{\sigma_Z \sigma_W} = \frac{-9}{\sqrt{52}\cdot\sqrt{3}} \approx -0.7206}$$

2(f)$^{(vi)}$

2(f) $$\boxed{f_{Z,W}(z, w) = \mathcal{N}(\mu_Z = 0, \mu_W = 0, \sigma_Z^2 = 52, \sigma_W^2 = 3, \rho_{Z,W} = -0.7206)}$$ Ans

2(g) $R = aX + bY \Rightarrow \sigma_R^2 = a^2 \cdot \sigma_X^2 + b^2 \sigma_Y^2$

or $\sigma_R^2 = a^2 \cdot 1^2 + b^2 \cdot 2^2$

or $\sigma_R^2 = a^2 + 4b^2$

$Cov(R, Y) = 0$ is a sufficient condition for R and Y to be independent, as both are gaussian.

$$Cov(R, Y) = E[(aX + bY - a\mu_X - b\mu_Y)(Y - \mu_Y)]$$

or $Cov(R, Y) = E[a(X - \mu_X)(Y - \mu_Y) + b(Y - \mu_Y)^2]$

or $Cov(R, Y) = a\rho_{X,Y}\sigma_X\sigma_Y + b\sigma_Y^2$

or $Cov(R, Y) = a(0.5) \cdot 1 \cdot 2 + b \cdot 2^2$

or $Cov(R, Y) = a + 4b$

2(g) For R, Y to be independent distributions, $a + 4b = 0$

we can use any set of Real numbers to do so,

say $a = -4, b = 1$

Ans

2(b) $Q = aX$

given $p(a=2) = 0.5$ $\quad f(Q \mid a=2) = \cancel{2} \, f_X(X = 2x)$

$p(a=1) = 0.5$ $\quad f(Q \mid a=1) = f_X(X = x)$

$$
\begin{tikzcd}
 & Q = X \\
Q \arrow[ur, "0.5"] \arrow[dr, "0.5"'] & \\
 & Q = 2X
\end{tikzcd}
$$

$$f_Q(q) = f_{Q|a}(Q \mid a=1) \cdot P(a=1) + f_{Q|a}(Q \mid a=2) \cdot P(a=2) \quad \text{(LoTP)}$$

or $f_Q(q) = 0.5\,\mathcal{N}(q, \mu_q = \mu_x, \sigma_q^2 = \sigma_X^2) + 0.5\,\mathcal{N}(q, \mu_q = 2\mu_x, \sigma_q^2 = 2^2\sigma_X^2)$

$\boxed{2(b)}$

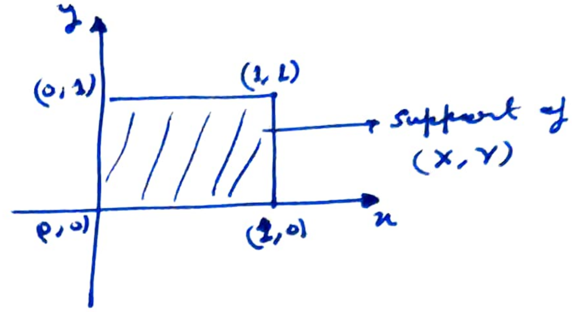
or $\boxed{f_Q(q) = 0.5\,\mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 1) + 0.5\,\mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 4)}$ Ans

———— × ———— × ———— × ———— × ————

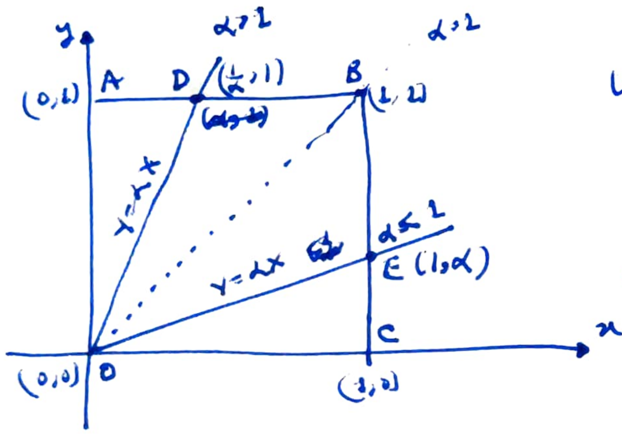
4. $X \sim unif[0,1]$ $\quad$ X, Y are independent

$Y \sim unif[0,1]$

$Z = \dfrac{Y}{X}$



4(a) Find $f_Z(z) = ?$



Using CDF method:

$F_Z(z = \alpha) = P(Z \le \alpha) \quad \alpha \in (0, \infty)$

or $F_Z(z = \alpha) = P\left(\dfrac{Y}{X} \le \alpha\right)$

or $F_Z(z = \alpha) = P(Y \le \alpha X)$

or $F_Z(z=\alpha) = \begin{cases} \text{Area of } \triangle OEC & \alpha \le 1 \\ \text{Area of } ODBC = 1 - \text{Area}(\triangle ADO) & \alpha \ge 1 \end{cases}$

uniform

Note: $\because f_{X,Y}(x,y) = 1 \ \forall (x,y) \in OABC$, simply computing the area under the curves is equivalent to computing the corresponding CDFs.

or $F_Z(z=\alpha) = \begin{cases} \dfrac{1}{2} & 0 < \alpha \le 1 \\ \dfrac{1}{2\alpha^2} & \alpha \ge 1 \\ 0 & \alpha < 0 \end{cases}$

4.2

4(a)

$$\Rightarrow \boxed{f_Z(z) = \frac{d}{d\alpha} F_Z(z=\alpha)\Big|_{\alpha\to z} = \begin{cases} 0 & z<0 \\ \frac{1}{2} & z \in (0,1] \\ \frac{1}{2z^2} & z \geq 1 \end{cases}}$$

Ans

4 b) $E[X^2+Y^2] = E[X^2] + E[Y^2]$

or $E[X^2+Y^2] = \int_{x=0}^{x=1} x^2 \cdot 1 \cdot dx + \int_{y=0}^{y=1} y^2 \cdot 1 \cdot dy$

or $E[X^2+Y^2] = \frac{1}{3} + \frac{1}{3}$

4(b)

or $\boxed{E[X^2+Y^2] = \frac{2}{3}}$ Ans

4 c) $f_{X,Z}(x,z) = \begin{cases} f_{Z|X}(z|X=x) \cdot f_X(X=x) & x \in (0,1],\ z \in (0, \frac{1}{x}] \\ 0 & \text{else} \end{cases}$

or $f_{X,Z}(x,z) = \begin{cases} \text{pdf of } \left(\frac{1}{x} \cdot Y\right) \cdot 1 & x \in (0,1],\ z \in (0,\frac{1}{x}] \\ 0 & \text{else} \end{cases}$

But $Y \sim unif(0,1)$

$Z|X=x \sim \frac{1}{x} \cdot unif(0,1)$

$\equiv Z|X=x \sim unif(0, \frac{1}{x})$

$$\Rightarrow \boxed{pdf(Z|X=x) = \begin{cases} x & z \in (0,\frac{1}{x}) \\ 0 & \text{else} \end{cases}}$$

4(c)

So $\boxed{f_{X,Z}(x,z) = \begin{cases} x & x \in (0,1] \\ & z \in (0, \frac{1}{x}] \\ 0 & \text{else} \end{cases}}$ Ans

4(d) $E[XZ] = E[Y] = \frac{1}{2}$

~~But anyway using $E[XZ] = \iint f_{X,Z}(x,z) \cdot x \cdot z \cdot dy\,dx$~~

4(d)(i) $\boxed{E[XZ] = \frac{1}{2}}$ Ans

$Cov(X,Z) = E[XZ] - \mu_X \mu_Z$

or $Cov(X,Z) = E[Y] - \frac{1}{2}\mu_Z$

or $Cov(X,Z) = \frac{1}{2} - \frac{1}{2}\mu_Z$

$\mu_Z = \int f_Z(z) \cdot z \cdot dz$

or $\mu_Z = \int_{z=0}^{z=1} \left(\frac{1}{2}\right) \cdot z \cdot dz + \int_{z=1}^{z=\infty} \left(\frac{1}{2z^2}\right) z \, dz$

or $\mu_Z = \frac{1}{4} z^2 \Big|_{z=0}^{z=1} + \frac{1}{2} \ln z \Big|_{z=1}^{z=\infty}$

or $\boxed{\mu_Z \to \infty}$

4(d)(ii) $\boxed{Cov(X,Z) \to -\infty}$ Ans

$\boxed{4.4}$

$$\rho_{X,Z}(x,z) = \frac{\text{Cov}(X,Z)}{\sigma_X \sigma_Z}$$

$\cancel{\sigma_Z}$ $\sigma_X = \sqrt{\frac{1}{12}}$

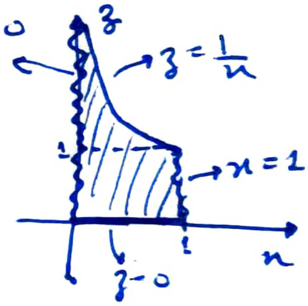
$$\sigma_Z = \int \underbrace{(z - \mu_Z)^2}_{\mu_Z \to \infty} \cdot f_Z(z)\, dz \to \infty$$

So $\rho_{X,Z}(x,z) \sim \dfrac{-\infty}{\sqrt{\frac{1}{12}} \cdot \infty} \approx -1$

$\boxed{4(d)^{\text{iii)}}}$

or $\boxed{\rho_{X,Z}(x,z) = -1}$ Ans

support of (X,Z):



4(e) $$f_{X|Z}(x \mid Z=z) = \frac{f_{Z|X}(z \mid X=x) \cdot f_X(X=x)}{f_Z(Z=z)}$$

or $$f_{X|Z}(x \mid Z=z) = \begin{cases} \text{n.d.} & z<0 \text{ or } \cancel{z \ldots \frac{1}{12}} \\ \dfrac{x \cdot 1}{\frac{1}{2}} & z \in (0,1],\ x \in (0,1] \\ \dfrac{x \cdot 1}{\frac{1}{2} z^{-2}} & z \in [1,\infty),\ x \in (0, \frac{1}{z}] \\ 0 & \cancel{x<\frac{1}{z}} \text{ else} \end{cases}$$

4(e)(i)

or $f_{X|Z}(x|Z=z) = \begin{cases} \text{n.d.} & z<0 \\ 2x & z\in(0,1],\ x\in(0,1] \\ 2z^2x & z\in[1,\infty),\ x\in(0,\frac{1}{z}] \\ 0 & \text{else} \end{cases}$ Ans

$$E[X|Z=z] = \int_{x=0}^{x=1} x \cdot f_{X|Z}(x|Z=z)\,dx$$

$$\text{or } E[X|Z=z] = \begin{cases} \int_{x=0}^{x=1} x\cdot 2x\,dx & z\in(0,1] \\ \int_{x=0}^{x=\frac{1}{z}} x\cdot 2z^2x\,dx & z\in[1,\infty) \end{cases}$$

4(e)(ii)

or $E[X|Z=z] = \begin{cases} \frac{2}{3} & z\in(0,1] \\ \frac{2}{3z} & z\in[1,\infty) \end{cases}$ Ans

4(f) $E_Z[E_X[X|Z=z]]$

$$= \int_{z} E_X[X|Z=z] \cdot f_Z(z)\, dz$$

or $$E_Z[E_X[X|Z=z]] = \int_{z=0}^{z=1} \frac{2}{3}\cdot\frac{1}{2}\, dz + \int_{z=1}^{z=\infty} \frac{2}{3z}\cdot\frac{1}{2z^2}\, dz$$

or $$E_Z[E_X[X|Z=z]] = \int_{z=0}^{z=1} \frac{1}{3}\, dz + \int_{z=1}^{z=\infty} \frac{1}{3} z^{-3}\, dz$$

or $$E_Z[E_X[X|Z=z]] = \frac{1}{3} + \left[\frac{1}{6} z^{-2}\right]_{z=\infty}^{z=1}$$

or $$E_Z[E_X[X|Z=z]] = \frac{1}{3} + \frac{1}{6}$$

4(f)

or $$\boxed{E_Z[E_X[X|Z=z]] = \frac{1}{2} = E[X]}$$ Ans

Hence Verified! ☺

4(g) $$E[XZ|Z=z] = E[zX|Z=z]$$

or $$E[XZ|Z=z] = z\,E[X|Z=z]$$

4(g)

or $$E[XZ|Z=z] = \begin{cases} \frac{2}{3}z & z\in[0,1],\ x\in[0,z] \\ \frac{2}{3} & z\in[1,\infty),\ x\in[0,1] \end{cases}$$

$\downarrow$

or $E[Y|Z=z]$

Ans

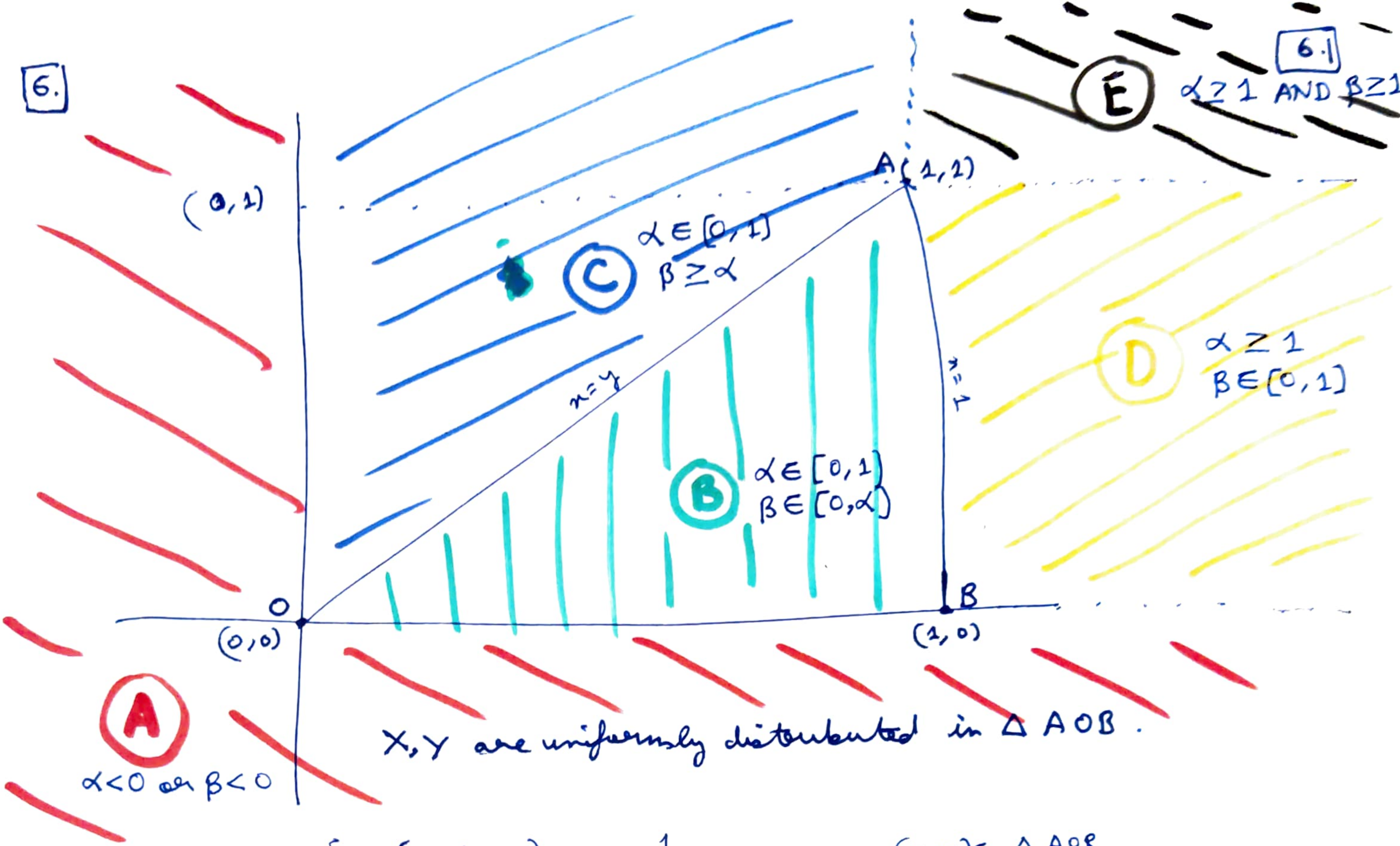
Verify (optional) if $E_Z[E_Y[Y|Z=z]] = E_Y[Y] \to \frac{1}{2}$

$$E_Z[E_Y[Y|Z=z]] = \int_{z=0}^{z=1} \frac{2}{3}z\cdot\frac{1}{2}\, dz + \int_{z=1}^{z=\infty} \frac{2}{3}\cdot\frac{1}{2z^2}\, dz$$

or $$E_Z[E_Y[Y|Z=z]] = \frac{1}{3}\frac{z^2}{2}\Big|_0^1 + \frac{1}{3z}\Big|_\infty^1$$

or $$E_Z[E_Y[Y|Z=z]] = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} = E[Y]$$ Hence Verified! ☺

——— × ——— × ——— × ——— × ———



X, Y are uniformly distributed in Δ AOB.

So $f_{X,Y}(x,y) = \dfrac{1}{\text{Area of } \Delta AOB}$ $\quad (x,y) \in \Delta AOB$

or $f_{X,Y}(x,y) = \dfrac{1}{\frac{1}{2}\times 1\times 1}$ $\quad y\in[0,1]$ and $x\in[y,1)$

6(a)

or
$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{as } y\in[0,1] \\ & x\in[y,1) \\ 0 & \text{else} \end{cases}$$

Ans

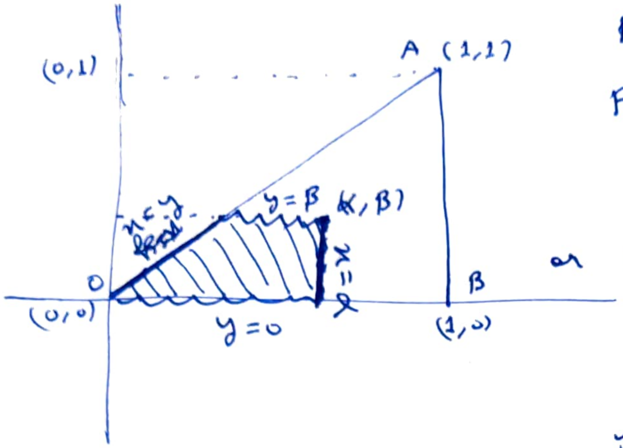
6(b) (Calculations on next pages.)

$$F_{X,Y}(x=\alpha, y=\beta) = \begin{cases} 0 & \alpha<0 \text{ or } \beta<0 \quad \text{A} \\ 2\alpha\beta-\beta^2 & \alpha\in[0,1),\ \beta\in[0,\alpha] \quad \text{B} \\ \alpha^2 & \alpha\in[0,1),\ \beta\geq\alpha \quad \text{C} \\ 2\beta-\beta^2 & \alpha\geq 1,\ \beta\in[0,\alpha] \quad \text{D} \\ 1 & \alpha\geq 1,\ \beta\geq 1 \quad \text{E} \end{cases}$$

Ans

For (α, β) in (B):

i.e. $\alpha \in (0,1]$, $\beta \in [0, \alpha]$



~~$F_{X,Y}(x,y)$~~

$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{Shaded region}} 2\,dx\,dy \qquad (\alpha,\beta) \in (B)$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\beta}\int_{x=y}^{x=\alpha} 2\,dx\,dy \qquad (\alpha,\beta) \in (B)$$

or $$F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2x\Big|_{y}^{\alpha}\,dy \qquad (\alpha,\beta) \in (B)$$

or $$F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2(\alpha - y)\,dy \qquad (\alpha,\beta) \in (B)$$

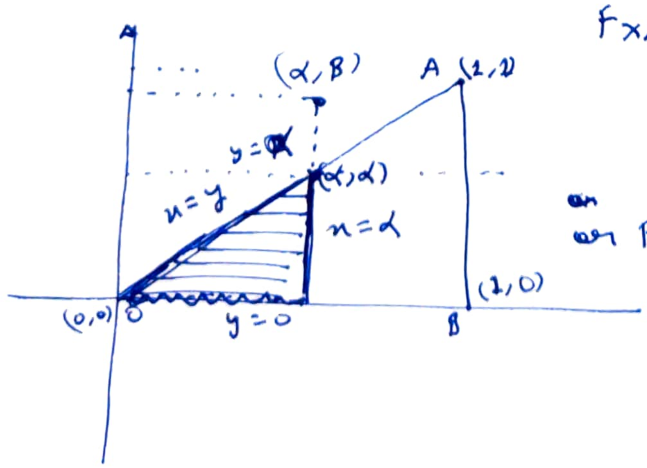
or $$F_{X,Y}(\alpha,\beta) = 2\alpha y - y^2\Big|_{0}^{\beta} \qquad (\alpha,\beta) \in (B)$$

or $$\boxed{F_{X,Y}(\alpha,\beta) = 2\alpha\beta - \beta^2 \qquad (\alpha,\beta) \in (B)}$$

$\boxed{\text{or } \alpha \in (0,1],\ \beta \in [0,\alpha]}$

For (α, β) in (C):

i.e. $\alpha \in [0,1]$, $\beta \geq \alpha$



$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{Shaded region}} 2\,dx\,dy \qquad (\alpha,\beta) \in (C)$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha}\int_{x=y}^{x=\alpha} 2\,dx\,dy \qquad (\alpha,\beta) \in (C)$$

or $F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha} 2(\alpha - y)\, dy \qquad (\alpha,\beta) \in$ C

or $F_{X,Y}(x=\alpha, y=\beta) = 2\alpha^2 - \alpha^2 \qquad (\alpha,\beta) \in$ C

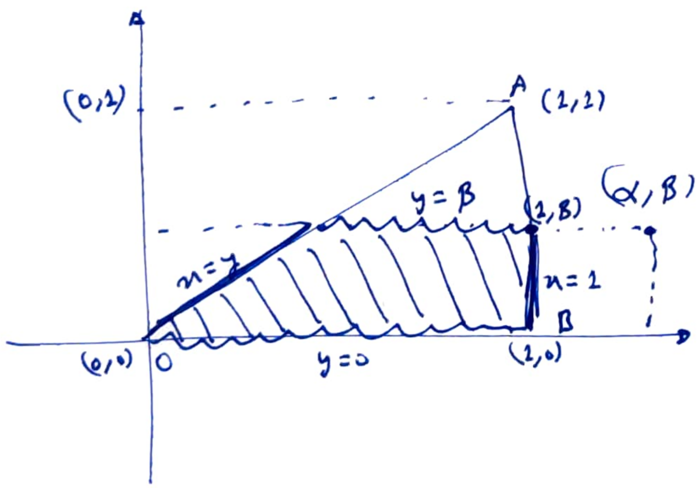
or $\boxed{F_{X,Y}(x=\alpha, y=\beta) = \alpha^2 \qquad (\alpha,\beta) \in \text{C} \quad \text{or} \quad \alpha \in [0,1],\ \beta \geq \alpha}$

Area of the shaded Δ.
$\times f_{X,Y}(x,y)$ in the region.

For (α, β) in D

i.e. $\alpha \geq 1$, $\beta \in [0,1)$

$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\, dx\, dy \qquad (\alpha,\beta) \in$ D



or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} \int_{x=y}^{x=1} 2\, dx\, dy \qquad (\alpha,\beta) \in$ D

or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2(1-y)\, dy \qquad (\alpha,\beta) \in$ D

or $F_{X,Y}(\alpha,\beta) = 2y - y^2 \Big|_0^\beta \qquad (\alpha,\beta) \in$ D

or $\boxed{F_{X,Y}(\alpha,\beta) = 2\beta - \beta^2 \qquad (\alpha,\beta) \in \text{D} \quad \text{or} \quad \alpha \geq 1,\ \beta \in [0,1)}$

$$f_X(x) = \int_{\forall y} f_{X,Y}(x,y)\,dy$$

or
$$f_X(x) = \begin{cases} 0 & x < 0 \\ \int_{y=0}^{y=x} 2\,dy & x \in [0,1] \\ 0 & x > 1 \end{cases}$$

6(c)

or
$$\boxed{f_X(x) = \begin{cases} 0 & x < 0 \\ 2x & x \in [0,1] \\ 0 & x > 1 \end{cases}}$$ Ans

$$f_Y(y) = \int_{\forall x} f_{X,Y}(x,y)\,dx$$

or
$$f_Y(y) = \begin{cases} 0 & y < 0 \\ \int_{x=y}^{x=1} 2\,dx & y \in [0,1] \\ 0 & y > 1 \end{cases}$$

6(d)

or
$$\boxed{f_Y(y) = \begin{cases} 0 & y < 0 \\ 2(1-y) & y \in [0,1] \\ 0 & y > 1 \end{cases}}$$ Ans

$$f_{X|Y}(x|Y=y) = \frac{f_{X,Y}(x, y=y)}{f_Y(y)}$$

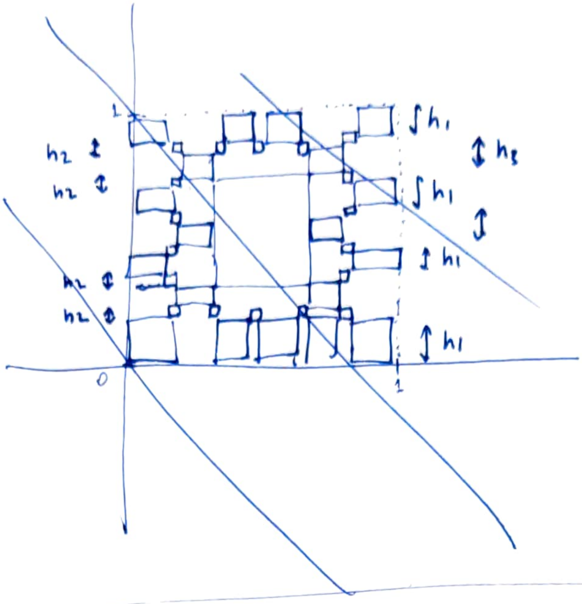
6(e)

or

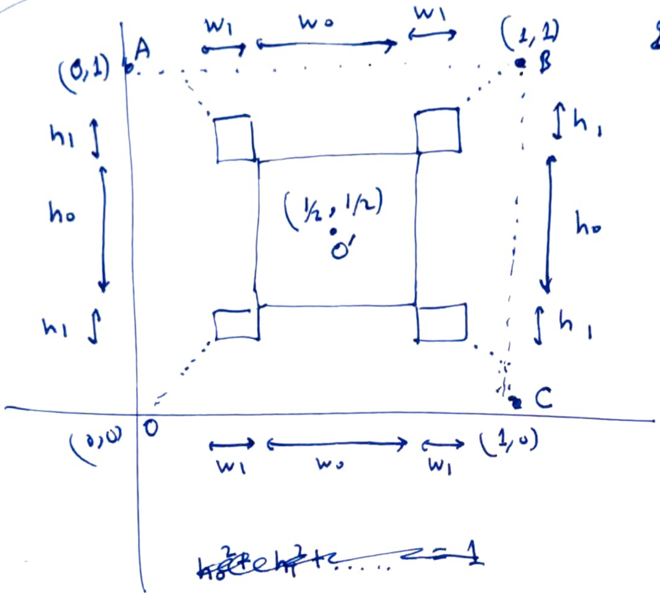
$$f_{X|Y}(x|Y=y) = \begin{cases} \text{not defined} & y < 0 \\ \dfrac{1}{1-y} & y \in (0,1),\ x \in (y,1) \\ \text{not defined} & y \geq 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Ans

$\boxed{7.}$



Ans



In ~~rectangle~~ square OABC of size 1×1 unit squares, any pdf distribution which is

1) symmetric about $\left(\frac{1}{2}, \frac{1}{2}\right)$.
2) NOT a single rectangle.
3) ~~But~~ Filled with a set of more than one (possibly infinite) rectangles which are touching together (to maintain continuity in both x and y) as well as are symmetrically placed around $\left(\frac{1}{2}, \frac{1}{2}\right)$ (to maintain zero covariance), ~~and~~

can qualify for a distribution ~~rel~~ in x, y s.t.

1) X and Y are uncorrelated.
2) X and Y are NOT independent.
3) $X \sim unif[0,1]$ and $Y \sim unif[0,1]$.

~~$h_0^2 + h_1^2 + \ldots = 1$~~

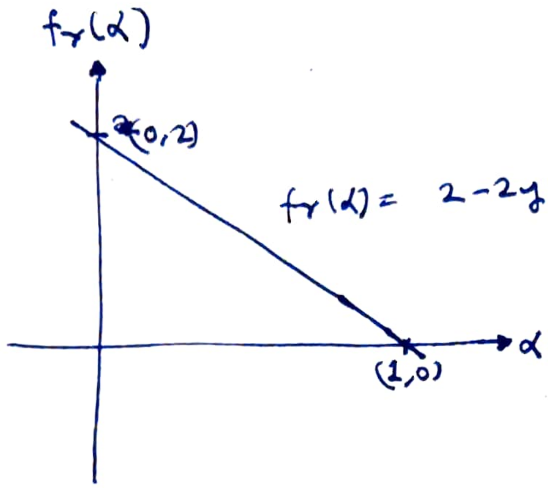
~~$h_0 + h_1 + h_2 + \ldots = 1$~~

where $h_0 + h_1 + h_2 + \ldots = 1$

$w_0 + w_1 + w_2 + \ldots = 1$

$(\text{uniform pdf}) \times \{h_0 w_0 + h_1 w_1 + h_2 w_2 + \ldots\} = 1$

8.



8 (a) $\hat{Y}_{MP} = \arg\max_{\alpha} \left(f_Y(\alpha) \right) \Big|_{\alpha \in [0,1]}$

or $\hat{Y}_{MP} = \arg\max_{\alpha} \left(2 - 2\alpha \right) \Big|_{\alpha \in [0,1]}$

8(a)

or $\boxed{\hat{Y}_{MP} = 0}$ Ans for which $f_Y(\alpha = 0) = 2$

Ans

8(b) $\hat{Y}_{MMSE} = \arg\min_{\alpha} . \left[E\left[(Y - \alpha)^2 \right] \right]$

We know that $\hat{Y}_{MMSE} = \mu_Y = E(Y)$

or $\hat{Y}_{MMSE} = \int_{y=0}^{y=1} y . f_Y(y) \, dy$

or $\hat{y}_{MMSE} = \int_{y=0}^{y=1} y . (2 - 2y) \, dy$

or $\hat{y}_{MMSE} = \int_{y=0}^{y=1} 2y - 2y^2 \, dy$

or $\hat{y}_{MMSE} = y^2 - \frac{2}{3} y^3 \Big|_{y=0}^{y=1}$

8(b)

or $\boxed{\hat{y}_{MMSE} = \frac{1}{3}}$ Ans

PTO

None the less, let us derive $\hat{Y}_{MMSE}$ using only first principles.

$$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[E\left[(Y-\alpha)^2 \right] \right]$$

$$\text{or} \quad \hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[\int_{y=0}^{y=1} (y-\alpha)^2 \cdot (2-2y)\, dy \right]$$

$$\text{or} \quad \hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2\int_{y=0}^{y=1} \{ y^2 - 2\alpha y + \alpha^2 \}\{1-y\}\, dy \right]$$

$$\text{or} \quad \hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2\int_{y=0}^{y=1} \{ -y^3 + (1+2\alpha)y^2 - (\alpha^2+2\alpha)y + \alpha^2 \} dy \right]$$

$$\text{or} \quad \hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2 \left[-\frac{y^4}{4} + \frac{(1+2\alpha)}{3}y^3 - \frac{(\alpha^2+2\alpha)}{2}y^2 + \alpha^2 y \right]_{y=0}^{y=1} \right]$$

$$\text{or} \quad \hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2 \left[-\frac{1}{4} + \frac{(1+2\alpha)}{3} - \frac{(\alpha^2+2\alpha)}{2} + \alpha^2 \right] \right]$$

$$\text{or} \quad \hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2 \left[\frac{\alpha^2}{2} - \frac{1}{3}\alpha + \frac{1}{12} \right] \right]$$

$$\text{or} \quad \hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[\left(\alpha - \frac{1}{3}\right)^2 + \frac{1}{18} \right]$$

8(b)

$$\text{or} \quad \boxed{\hat{y}_{MMSE} = \frac{1}{3} \quad \text{with error} = \frac{1}{18}}$$

Ans

$\boxed{8.3}$

8(c)

$$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(E(|y-\alpha|) \right) = \underset{\alpha}{\arg\min}\left(\int_{y=0}^{y=\alpha} (\alpha-y)(2-2y)\,dy + \int_{y=\alpha}^{y=1} (y-\alpha)(2-2y)\,dy \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(2\int_{y=0}^{y=\alpha} (y-\alpha)(y-1)\,dy + 2\int_{y=1}^{y=\alpha} (y-\alpha)(y-1)\,dy \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(2\int_{y=0}^{y=\alpha} \{y^2-(\alpha+1)y+\alpha\}\,dy + 2\int_{y=1}^{y=\alpha} \{y^2-(\alpha+1)y+\alpha\}\,dy \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\times\left[\frac{y^3}{3} - \frac{(\alpha+1)}{2}y^2 + \alpha y\right]\Bigg|_{y=\alpha} - 2\left[\frac{y^3}{3} - \frac{(\alpha+1)}{2}y^2 + \alpha y\right]\Bigg|_{y=1} \cancel{- 2\left[\frac{y^3}{3} - \frac{(\alpha+1)}{2}y^2 + \alpha y\right]\Bigg|_{y=0}} \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\left[\frac{\alpha^3}{3} - \frac{(\alpha+1)}{2}\alpha^2 + \alpha^2\right] - 2\left[\frac{1}{3} - \frac{(\alpha+1)}{2} + \alpha\right] \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\left[-\frac{\alpha^3}{6} + \frac{\alpha^2}{2}\right] - 2\left[\frac{\alpha}{2} - \frac{1}{6}\right] \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(-\frac{4}{6}\alpha^3 + 2\alpha^2 - \alpha + \frac{1}{3} \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(\frac{d}{d\alpha}\left[-\frac{2}{3}\alpha^3 + 2\alpha^2 - \alpha + \frac{1}{3} \right] = 0 \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(-2\alpha^2 + 4\alpha - 1 = 0 \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(-2(\alpha-1)^2 + 1 = 0 \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left((\alpha-1)^2 = \frac{1}{2} \right)$$

or $$\hat{y}_{MMAE} = 1 - \frac{1}{\sqrt{2}}$$

or 8(c) $\boxed{\hat{y}_{MMAE} \approx 0.293}$ Ans

9.1

~~For BC for Floor, we require:~~

9. $Y|X \sim \exp\left(\frac{1}{x^2}\right) \Rightarrow \cancel{f_Y(y)}\; f_{Y|X}(y|X=x) = \frac{1}{x^2} e^{-\frac{1}{x^2} y}$, $y \geq 0$

$X \sim \text{unif}(0,1) \Rightarrow f_X(x) = 1 \quad x \in [0,1]$

9(a)(i)

$$\boxed{\hat{Y}|X=x_{MMSE} = \mu_{Y|X} = x^2}$$ Ans

9(a)(ii)

$$\text{Error } \hat{Y}|X = \sigma^2_{Y|X} = x^4$$
$$\text{or std. error } \hat{Y}|X = \sigma_{Y|X} = x^2$$ Ans

$$\text{Overall error} = \int_{x=0}^{x=1} \sigma^2_{Y|X} \cdot f_X(x)\, dx =$$

$$\text{or Overall error} = \int_{x=0}^{x=1} x^4 \cdot 1 \cdot dx$$

9(a)(iii)

$$\text{or } \boxed{\text{Overall error} = \frac{1}{5}}$$ Ans

$$\hat{Y^2}|X_{MMSE} = E(Y^2|X=x)$$

$$\text{or } \hat{Y^2}|X_{MMSE} = \int_{y=0}^{y=\infty} \frac{1}{x^2} e^{-\frac{1}{x^2}y} \cdot y^2\, dy$$

$$\text{or } \hat{Y^2}|X_{MMSE} = \left(y^2 + 2x^2 y + 2x^4\right) e^{-\frac{1}{x^2}y} \Bigg|_{y=\infty}^{y=0}$$

9(b)

$$\text{or } \boxed{\hat{Y^2}|X_{MMSE} = 2x^4} \neq \left(\hat{Y}|X_{MMSE}\right)^2$$ Ans

Mean of squares ≠ Square of mean Ans

Rough

$\int A y^2 e^{-Ay}$

$= \frac{d}{dy}\left[(Cy^2 + Dy + E) e^{-Ay}\right]$

$\Rightarrow -CA = A$

$2C - AD = 0$

$D - AE = 0$

$\Rightarrow C = -1$

$D = -2x^2$

$E = -2x^4$

$\Rightarrow \int A y^2 e^{-Ay}$

$= (-y^2 - 2x^2 y - 2x^4) \cdot e^{-\frac{1}{x^2}y}$

9.2

9(c)

$$f_{X|Y}(x|Y=y) = \frac{f_{Y|X}(Y|X=x) \cdot f_X(x)}{f_Y(y)}$$

$y \in [0,\infty)$
$x \in [0,1]$

or $$\boxed{f_{X|Y}(X|Y=y) = \frac{\left(\frac{1}{x^2} e^{-\frac{1}{x^2}y}\right) \cdot (1)}{\int_{y=0}^{y=\infty} \frac{1}{x^2} e^{-\frac{1}{x^2}y}\, dy}}$$

9(c) $$\boxed{\mu_{X|Y} = \int_{x=0}^{x=1} x \cdot f_{X|Y}(X|Y=y)\, dx}$$ Ans

9(d) $\hat{y}|X=x_{\,LMMSE} = a^* x + b^*$ where,

9(d) $$a^* = \frac{\mathrm{Cov}(X,Y)}{\sigma_X^2}$$

$$\text{or } a^* = \frac{\mu_{XY} - \mu_X \cdot \mu_Y}{\sigma_X^2}$$

$$\text{or } a^* = \frac{\int_{x=0}^{x=1}\int_{y=0}^{y=\infty} xy \cdot \frac{1}{x^2} e^{-\frac{1}{x^2}y}\, dy\, dx - \frac{1}{2}\int_{x=0}^{x=1}\int_{y=0}^{y=\infty} y \cdot \frac{1}{x^2} e^{-\frac{1}{x^2}y}\, dy\, dx}{\left(\frac{1}{12}\right)}$$

$$\text{or } a^* = \frac{\int_{x=0}^{x=1} \frac{1}{x}\left(\int_{y=0}^{y=\infty} y \cdot e^{-\frac{1}{x^2}y}\, dy\right) dx - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{x^2}\left(\int_{y=0}^{y=\infty} y\, e^{-\frac{1}{x^2}y}\, dy\right) dx}{1/12}$$

$$\text{or } a^* = \frac{\int_{x=0}^{x=1} \frac{1}{x} \cdot \left(\frac{x^2 y + x^4}{e^{\frac{1}{x^2}y}} \Big|_{y=\infty}^{y=0} \right) dx - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{x^2}\left(\left(x^2 y + x^4\right) e^{-\frac{1}{x^2}y}\right)\Big|_{y=\infty}^{y=0} dx}{1/12}$$

~~Hi Rudrayat~~

$$\Rightarrow a^* = \frac{\int_{x=0}^{x=1} \frac{1}{x} \cdot (x^4)\, dx - \frac{1}{2} \int_{x=0}^{x=1} \frac{1}{x} \cdot (x^4)\, dx}{1/12}$$

$$\Rightarrow a^* = \frac{\frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{12}}$$

$$\Rightarrow \boxed{a^* = 1}$$

$$b^* = \mu_Y - a^* \mu_X$$

$$\Rightarrow b^* = \frac{1}{3} - 1 \cdot \frac{1}{2}$$

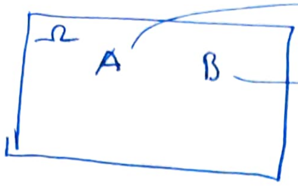
$$\Rightarrow \boxed{b^* = -\frac{1}{6}}$$

9(d) $\boxed{\hat{Y}|X_{LMMSE} = 1 \cdot x - \frac{1}{6}}$ Ans

— x — x — x — x —

11.1

11



$X(A,t) = \cos(t)$

$X(B,t) = \sin(t)$

$P(A) = 0.5 \quad P(B) = 0.5$

11 (a) $$f_{X(t)}(x) = \cancel{X(A,t)}\, P(A) \cdot \delta(x - X(A,t)) + P(B) \cdot \delta(x - X(B,t))$$

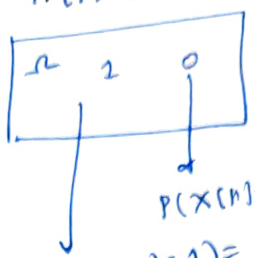
11 (a)

or $$\boxed{f_{X(t)}(x) = 0.5\,\delta(x - \cos(t)) + 0.5\,\delta(x - \sin(t))}$$ Ans

—x—x—x—

12.

$x[n]$:



$P(X[n]=0) = 0.6$

$P(X[n]=1) = 0.4$

$$f_{X(k)}(n,k) = 0.6\,\delta(n) + 0.4\,\delta(n-1) \qquad k = 1, 2, 3 \ldots$$

12 (c) $Y[k] = \sum_{i=0}^{k-1} X[i]$

$$Y[k] = \begin{cases} 0 & \text{w.p. } {}^{k}C_{k}\,(0.6)^{k}(0.4)^{0} \\ 1 & \text{w.p. } {}^{k}C_{k-1}\,(0.6)^{k-1}(0.4)^{1} \\ \vdots & \\ k & \text{w.p. } {}^{k}C_{0}\,(0.6)^{k-k}(0.4)^{k} \end{cases}$$

$\therefore$ $Y[k] = i$ w.p. ${}^{k}C_{i}\,(0.6)^{k-i}(0.4)^{i}$

$i = 0, 1, \ldots k$

12 (c)

$$\Rightarrow \boxed{f_{Y(k)}(y,k) = \sum_{i=0}^{k} {}^{k}C_{i}\,(0.6)^{k-i}(0.4)^{i}\,\delta(y-i)}$$ Ans

— x — x — x — x —