

- 1(a) Check if: $f_x(n) \cdot f_y(y) \cdot f_z(z) = f_{x,y,z}(n,y,z)$ ①
 implies that the three X and Y are independent.
 i.e. check if ① implies $f_{x,y}(n,y) = f_x(n) \cdot f_y(y)$?

Answer: Integrating both sides of ① w.r.t. Z :

$$\int f_x(n) \cdot f_y(y) \cdot f_z(z) dz = \int f_{x,y,z}(n,y,z) dz$$

on $\forall z$: since $f_z(z) = 1$ (constant) \Rightarrow must integrate w.r.t. z

$$\text{or } f_x(n) \cdot f_y(y) \int f_z(z) dz = f_{x,y}(n,y) \quad (\text{marginal pdf of } x,y)$$

$$\text{or } f_x(n) \cdot f_y(y) = 1 \quad (\text{if } f_{x,y}(n,y) = 1 \text{ then})$$

$$\text{or } [f_x(n) \cdot f_y(y) = f_{x,y}(n,y)] \text{ which is } ② \text{ which is what we set to check for!}$$

$$\therefore \boxed{f_x(n) \cdot f_y(y) \cdot f_z(z) = f_{x,y,z}(n,y,z)} \Rightarrow X \text{ and } Y \text{ are independent} \quad \text{Ans}$$

$$\text{2(b)} \quad \text{check if } P(X)P(Y)P(Z) = P(X,Y,Z) \quad ③$$

implies that X and Y are independent i.e. $P(X)P(Y) = P(X,Y)$?
 where X, Y, Z are events of a probabilistic experiment.

No.

Counterexample:

Set experiment

In an experiment, let we are picking a number from 1 to 8 randomly: Each number has equal probability of being picked. This experiment is like throwing a fair 8-sided die.

Now let us define three events:

$$X = \{1, 2, 3, 4\} \Rightarrow P(X) = \frac{1}{2}$$
$$Y = \{1, 3, 4, 5\} \Rightarrow P(Y) = \frac{1}{2}$$
$$Z = \{1, 6, 7, 8\} \Rightarrow P(Z) = \frac{1}{2}$$

~~$$X \cap Y \cap Z = \{1\}$$~~
$$\Rightarrow P(X \cap Y \cap Z) = \frac{1}{8}$$

So $P(X) \neq P(\bar{X})$

So equation ① is followed by events X, Y, Z , as

$$P(X) \cdot P(Y) \cdot P(Z) = \left(\frac{1}{2}\right)^3 = \frac{1}{8} = P(X, Y, Z)$$

But $P(X) \cdot P(Y) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$$P(X \cap Y) = P(\{1, 3, 4\}) = \frac{3}{8} \quad \boxed{\text{X and Y are NOT independent.}}$$

In fact, $P(Y) \cdot P(Z) = \frac{1}{4}$

$$P(Y \cap Z) = P(\{1\}) = \frac{1}{8} \quad \boxed{\text{Y and Z are NOT independent either}}$$

And $P(X) \cdot P(Z) = \frac{1}{4}$

$$P(X \cap Z) = P(\{1\}) = \frac{1}{8} \quad \boxed{\text{X and Z are NOT independent either.}}$$

∴ The analogous event $P(X) \cdot P(Y) \cdot P(Z) = P(X, Y, Z)$ does NOT imply independence of X and Y. Ans

∴ $P(X) \cdot P(Y) \cdot P(Z) = P(X, Y, Z)$ does NOT imply independence of X and Y. Ans

2.1

2. $x, y \sim \mathcal{N}(\mu_x=0, \mu_y=0, \sigma_x^2=1, \sigma_y^2=4, \rho_{xy}=0.5)$

$$f_{x,y}(n, y) = \frac{1}{\sqrt{1-\rho^2} \sigma_x \sqrt{2\pi} \sigma_y \sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{1-(0.5)^2}} \left\{ \frac{(n-\mu_x)^2 + (\frac{y-\mu_y}{2})^2}{\sigma_x^2} - \frac{2(n-\mu_x)(y-\mu_y)}{\sigma_x \sigma_y} \right\}$$

\downarrow
only component (say $g(n, y)$) dependent on n and y .

Taking out the component $g(n, y)$ and putting the given values of $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2$ and ρ_{xy} :

$$g(n, y) = \left(\frac{n}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - \frac{2(n)(y)}{(1)(2)}$$

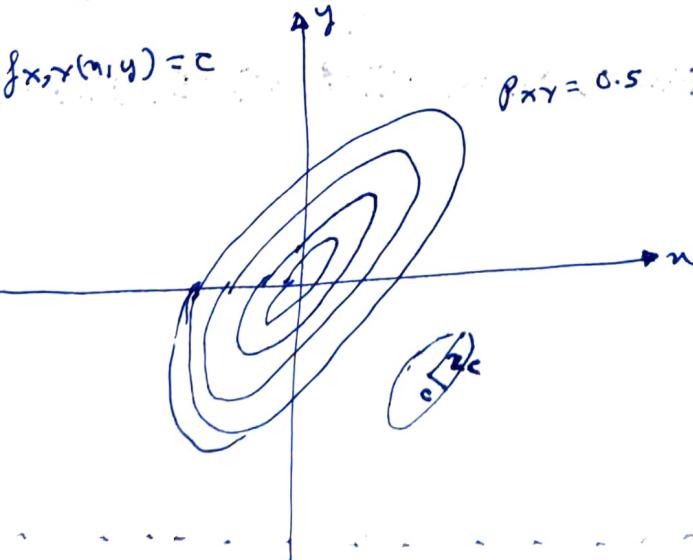
$\Rightarrow f_{x,y}(n, y) \rightarrow c$ for $g(n, y) \rightarrow c$
i.e. is constant is constant

\Rightarrow Contours of $f_{x,y}(n, y)$ is

$$g(n, y) = \left(\frac{n}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - ny = c$$

which is the equation for an ellipse, tilted and with positive slope.

2(a)

Ans

2(b)

$$f_X(n) = \mathcal{N}(x, 0, 1^2) = \frac{1}{1 \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \left(\frac{n}{1}\right)^2 \right\}} \quad \forall n$$

$$f_Y(y) = \mathcal{N}(y, 0, 2^2) = \frac{1}{2 \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \left(\frac{y}{2}\right)^2 \right\}} \quad \forall y$$

2(c)

$$f_{Y|X}(y|x=x) = \frac{f_{X,Y}(x=n, y)}{f_X(x=n)} \quad \forall n, y$$

$$\text{or } f_{Y|X}(y|x=x) = \frac{1}{\sqrt{(1-\frac{1}{2})} \cdot 1 \cdot \sqrt{2n} \cdot 2 \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} \cdot \left\{ \frac{n}{2} \left(\frac{y}{1}\right)^2 + \left(\frac{y}{2}\right)^2 \right\}} - 2 \times \frac{1}{2} \pi \frac{n}{1} \times \frac{y}{2}$$

Am

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \left(\frac{n}{1}\right)^2 \right\}} \quad \forall (n, y)$$

$$\text{or } f_{Y|X}(y|x=x) = \frac{1}{\sqrt{3} \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \cdot \left\{ \left(\frac{n}{\sqrt{3}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 - 2 \cdot 1 \cdot \frac{n}{\sqrt{3}} \cdot \frac{y}{\sqrt{3}} \right\}}$$

Am

2(c)

$$\text{or } f_{Y|X}(y|x=x) = \sqrt{2\pi} \mathcal{N}(\mu_x=0, \mu_y=0, \sigma_x^2=3, \sigma_y^2=3, \rho_{xy}=1) \quad \text{Am } (m, y)$$

$$\text{or } f_{Y|X}(y|x=x) = \mathcal{N}(y=\cancel{\text{m}}, \sigma_y^2=3) \quad \forall y$$

$$\text{or } f_{Y|X}(y|x=x) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \left(\frac{y-n}{\sqrt{3}}\right)^2 \right\}} \quad \forall y$$

2(c)

$$\text{or } f_{Y|X}(y|x=x) = \mathcal{N}(y=\cancel{n}, \mu_y=n, \sigma_y^2=3) \quad \text{Am}$$

PBO.

2(d) Find n s.t. $E[Y|X=n] = -2$.

From 2(c), we know that $Y|X=n$ is fully correlated to $X=n$. ($P_{Y|X=n}=1$).

$\therefore Y$ and X have the same support/domain of $n \in \mathbb{R}$ and $y \in \mathbb{R}$,

$$\therefore E_Y[Y|X=n] = -2 \equiv E_X[X=n] = -2$$

But $E_X[X=n] = n$

$$\therefore n = -2 \quad \boxed{n = -2} \text{ Ans}$$

2(d) Find n s.t. $E[Y|X=n] = -2$

From 2(c), we know that $Y|X=n$ is a gaussian

with mean n .

$$\therefore E[Y|X=n] = -2 \Rightarrow \boxed{n = -2} \text{ Ans}$$

$$2(e) \quad Z = X + Y - 1$$

Z is also a Gaussian, we need to only compute μ_Z and σ_Z^2

To find $f_Z(z)$: Find μ_Z and σ_Z^2

$$E(Z) = E(X) + E(Y) - E(1)$$

$$\text{or } \mu_Z = \mu_X + \mu_Y - 1$$

$$\text{or } \mu_Z = 0 + 0 - 1$$

$$\text{or } \boxed{\mu_Z = -1}$$

$$\sigma_Z^2 = E[(X + Y - 1) - E[X + Y - 1]]^2$$

$$\text{or } \sigma_Z^2 = E[(X - \mu_X + Y - \mu_Y + (-1 - E(-1)))^2]$$

$$\text{or } \sigma_Z^2 = E[(X - \mu_X)^2 + (Y - \mu_Y)^2]$$

$$\text{or } \sigma_Z^2 = E[(X - \mu_X)^2] + E[(Y - \mu_Y)^2] + \cancel{2\text{cov}}[X, Y]$$

$$\text{or } \sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y$$

$$\text{or } \sigma_Z^2 = 1^2 + 2^2 + 2 \times 0.5 \times 1 \times 2$$

$$\text{or } \boxed{\sigma_Z^2 = 7}$$

$$\begin{aligned} &2(e) \\ \therefore &Z \sim \mathcal{N}(3, \mu_Z = -1, \sigma_Z^2 = 7) \end{aligned}$$

2(f)

$$Z = 2X + 3Y$$

2.5

$$W = X - Y$$

$$E(Z) = 2\mu_X + 3\mu_Y$$

$$E(W) = \mu_X - \mu_Y$$

or $\boxed{2(f)(ii)} \boxed{\mu_Z = 0} \text{ Ans}$

or $\boxed{2(f)(ii)} \boxed{\mu_W = 0} \text{ Ans}$

$$\sigma_Z^2 = E[(Z - \mu_Z)^2] = E[(2X + 3Y - (2\mu_X + 3\mu_Y))^2]$$

$$\sigma_W^2 = E[(W - \mu_W)^2] = E[(X - Y - (\mu_X - \mu_Y))^2]$$

$$\text{or } \sigma_Z^2 = E[2(X - \mu_X)^2 + 3(Y - \mu_Y)^2 + 2 \cdot 3 \cdot (X - \mu_X)(Y - \mu_Y)]$$

$$\text{or } \sigma_W^2 = E[(X - \mu_X)^2 + (Y - \mu_Y)^2 - 2 \cdot 1 \cdot (X - \mu_X)(Y - \mu_Y)]$$

$$\text{or } \sigma_Z^2 = 4\sigma_X^2 + 9\sigma_Y^2 + 2 \cdot 2 \cdot 12\rho_{XY}\sigma_X\sigma_Y$$

$$\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y$$

$$\text{or } \sigma_Z^2 = 4 \cdot 1^2 + 9 \cdot 2^2 + 12 \cdot (0.5) \cdot 1 \cdot 2$$

$$\sigma_W^2 = 1^2 + 2^2 - 2 \cdot (0.5) \cdot 1 \cdot 2$$

$$\text{or } \sigma_Z^2 = 4 + 36 + 12$$

$$\sigma_W^2 = 1 + 4 - 2$$

or $\boxed{2(f)(ii)} \boxed{\sigma_Z^2 = 52} \text{ Ans}$

or $\boxed{2(f)(iv)} \boxed{\sigma_W^2 = 3} \text{ Ans}$

or $\text{Cor}(Z, W) = E[(Z - \mu_Z)(W - \mu_W)]$

$$\text{or } \text{Cor}(Z, W) = E[(2X + 3Y - 2\mu_X - 3\mu_Y)(X - Y - \mu_X - \mu_Y)]$$

$$\text{or } \text{Cor}(Z, W) = E[2(X - \mu_X) + 3(Y - \mu_Y) \{ 2(X - \mu_X) - 1(Y - \mu_Y) \}]$$

$$\text{or } \text{Cor}(Z, W) = E[2(X - \mu_X)^2 - 3(Y - \mu_Y)^2 + 2(X - \mu_X)(Y - \mu_Y)]$$

$$\text{or } \text{Cor}(Z, W) = 2\sigma_X^2 - 3\sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y$$

$$\text{or } \text{Cor}(Z, W) = 2 \cdot 1^2 - 3 \cdot 2^2 + (0.5) \cdot 1 \cdot 2$$

$$\text{or } \text{Cor}(Z, W) = 2 - 12 + 1$$

2(f)(v)

$$\text{con}(z, w) = -9 \quad \text{Ans}$$

$$\rho_{z,w} = \frac{\text{con}(z, w)}{\sigma_z \sigma_w} = \frac{-9}{\sqrt{52} \cdot \sqrt{3}} \approx -0.7206$$

2(f)(vi)

$$f_{z,w}(z, w) = N(\mu_z=0, \mu_w=0, \sigma_z^2=52, \sigma_w^2=3, \rho_{zw}=-0.7206) \quad \text{Ans}$$

$$2(g) R = ax + by \quad \text{and} \quad \sigma_R^2 = a^2 \cdot \sigma_x^2 + b^2 \cdot \sigma_y^2 \\ \text{or} \quad \sigma_R^2 = a^2 \cdot 1^2 + b^2 \cdot 2^2 \\ \therefore \sigma_R^2 = a^2 + 4b^2$$

$\text{con}(R, Y) = 0$ is a sufficient condition for R and Y to be independent, as both are gaussian.

$$\text{con}(R, Y) = E[(ax + by) - a\mu_x - b\mu_y](Y - \mu_y)$$

$$\text{or} \quad \text{con}(R, Y) = E[a(X - \mu_x)(Y - \mu_y) + b(Y - \mu_y)^2]$$

$$\text{or} \quad \text{con}(R, Y) = a \rho_{x,y} \sigma_x \sigma_y + b \sigma_y^2$$

$$\text{or} \quad \text{con}(R, Y) = a(0.5) \cdot 1 \cdot 2 + b \cdot 2^2$$

$$\text{or} \quad \text{con}(R, Y) = a + 4b$$

2(g)

For R, Y to be independent distributions, $a + 4b = 0$

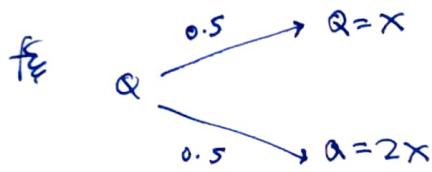
we can use any set of real numbers to do so;

$$\text{say } a = -4, b = 1$$

$$2(b) \quad Q = aX$$

given $P(a=2) = 0.5 \quad f(Q|a=2) = \cancel{2} \cdot \cancel{f_X}(x=2n)$

$$P(a=1) = 0.5 \quad f(Q|a=1) = f_X(x=n)$$



$$f_Q(q) = f_{Q|a}(Q|a=1) \cdot P(a=1) + f_{Q|a}(Q|a=2) \cdot P(a=2) \quad (\text{LTP})$$

$$\text{or } f_Q(q) = 0.5 N(q, \mu_q = \mu_n, \sigma_q^2 = \sigma_x^2) + 0.5 N(q, \mu_q = 2\mu_n, \sigma_q^2 = 2^2 \sigma_x^2)$$

2(b)

$$\text{or } f_Q(q) = 0.5 N(q, \mu_q = 0, \sigma_q^2 = 1) + 0.5 N(q, \mu_q = 0, \sigma_q^2 = 4) \quad \text{Ans}$$

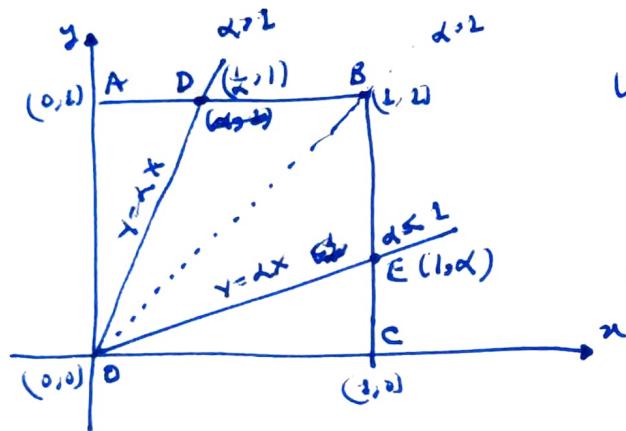
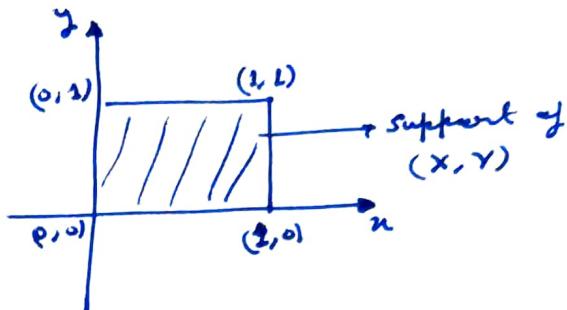
→ X → X → X → X → X

4. $X \sim \text{unif}[0, 1]$
 $Y \sim \text{unif}[0, 1]$

$$Z = \frac{Y}{X}$$

Ques $\rightarrow f_Z(z) = ?$

X, Y are independent



Using CDF method:

$$F_Z(z) = P(Z \leq z) \quad z \in (0, \infty)$$

$$\text{or } F_Z(z) = P\left(\frac{Y}{X} \leq z\right)$$

$$\text{or } F_Z(z) = P(Y \leq zx)$$

or $F_Z(z) = \begin{cases} \text{Area of } \triangle OEC & z \leq 1 \\ \text{Area of } ODBC \\ = 1 - \text{Area } (AOB) & z \geq 1 \end{cases}$

Note: $\because f_{X,Y}(x,y) = 1 \quad \forall (x,y) \in OABC$, simply computing the area under the curves is equivalent to computing the corresponding CDFs.

or $F_Z(z) = \begin{cases} \frac{1}{2} & 0 < z \leq 1 \\ \frac{1}{2z^2} & z \geq 1 \\ 0 & z < 0 \end{cases}$

4(a)

$$\Rightarrow f_Z(z) = \frac{d}{dz} F_Z(z=d) \Big|_{d=z} = \begin{cases} 0 & z < 0 \\ \frac{1}{2} & z \in (0, 1) \\ \frac{1}{2z^2} & z \geq 1 \end{cases}$$

4.2

Ans

4(b) $E[X^2 + Y^2] = E[X^2] + E[Y^2]$

or $E[X^2 + Y^2] = \sum_{n=0}^{n=1} n^2 \cdot 2 \cdot d_n + \int_{y=0}^{y=1} y^2 \cdot 1 \cdot dy$

or $E[X^2 + Y^2] = \frac{1}{3} + \frac{1}{3}$

4(b) or $E[X^2 + Y^2] = \frac{2}{3}$ Ans

4(c) $f_{X,Z}(n, z) = \begin{cases} f_Z|_{X=n}(z) \cdot f_X(x=n) & n \in (0, 1) \\ 0 & z \in (0, \frac{1}{n}) \end{cases}$

else.

or $f_{X,Z}(n, z) = \begin{cases} \text{pdf of } \text{pdf of } \left(\frac{1}{n} \cdot Y \right) \cdot 1 & n \in (0, 1) \\ 0 & z \in (0, \frac{1}{n}) \end{cases}$

else

But $Y \sim \text{unif}(0, 1)$

$$Z|_{X=n} \sim \frac{1}{n} \cdot \text{unif}(0, 1)$$

$$\therefore Z|_{X=n} \sim \text{unif}\left(0, \frac{1}{n}\right)$$

$\Rightarrow \text{pdf}(Z|_{X=n}) = \begin{cases} n & z \in (0, \frac{1}{n}) \\ 0 & \text{else} \end{cases}$

4.3

4(c) f_{x,z} $f_{x,z}(n, z) = \begin{cases} n & z \in (0, \frac{1}{n}] \\ 0 & \text{else} \end{cases}$ Ans

4(d) $E[XZ] = E[Y] = \frac{1}{2}$

~~Berechnung von $E[XZ]$ mit $\int f_{x,z}(x, z) x z dx dz$~~

4(d) ii E[XZ] = $\frac{1}{2}$ Ans

$\text{Cov}(X, Z) = E[XZ] - \mu_X \mu_Z$

or $\text{Cov}(X, Z) = E[Y] - \frac{1}{2} \mu_Z$

or $\text{Cov}(X, Z) = \frac{1}{2} - \frac{1}{2} \mu_Z$

$\mu_Z = \int f_Z(z) \cdot z \cdot dz$

or $\mu_Z = \int_0^1 \left(\frac{1}{z}\right) \cdot z \cdot dz + \int_1^\infty \left(\frac{1}{2z^2}\right) z \cdot dz$

or $\mu_Z = \frac{1}{4} z^2 \Big|_{z=0}^{z=1} + \frac{1}{2} \ln z \Big|_{z=1}^{z=\infty}$

or | $\mu_Z \rightarrow \infty$

4(d) iii Cov(X, Z) $\rightarrow -\infty$ Ans

$$P_{X,Z}(n, z) = \frac{\text{Cov}(X, Z)}{\sigma_x \sigma_z}$$

$$\sigma_x = \sigma_x = \sqrt{\frac{1}{12}}$$

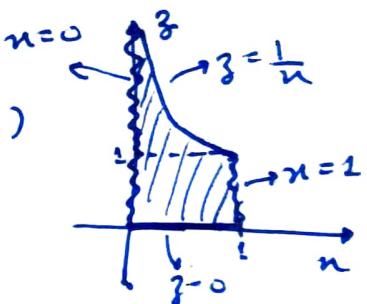
$$\sigma_z = \int (z - \mu_z)^2 \cdot f_z(z) dz \rightarrow \infty$$

\downarrow
 $\mu_z \rightarrow \infty$

So $P_{X,Z}(n, z) \sim \frac{-\infty}{\sqrt{n} \cdot \infty} \approx -1$

or 4(d) iii) P_{X,Z}(n, z) = -1 Ans

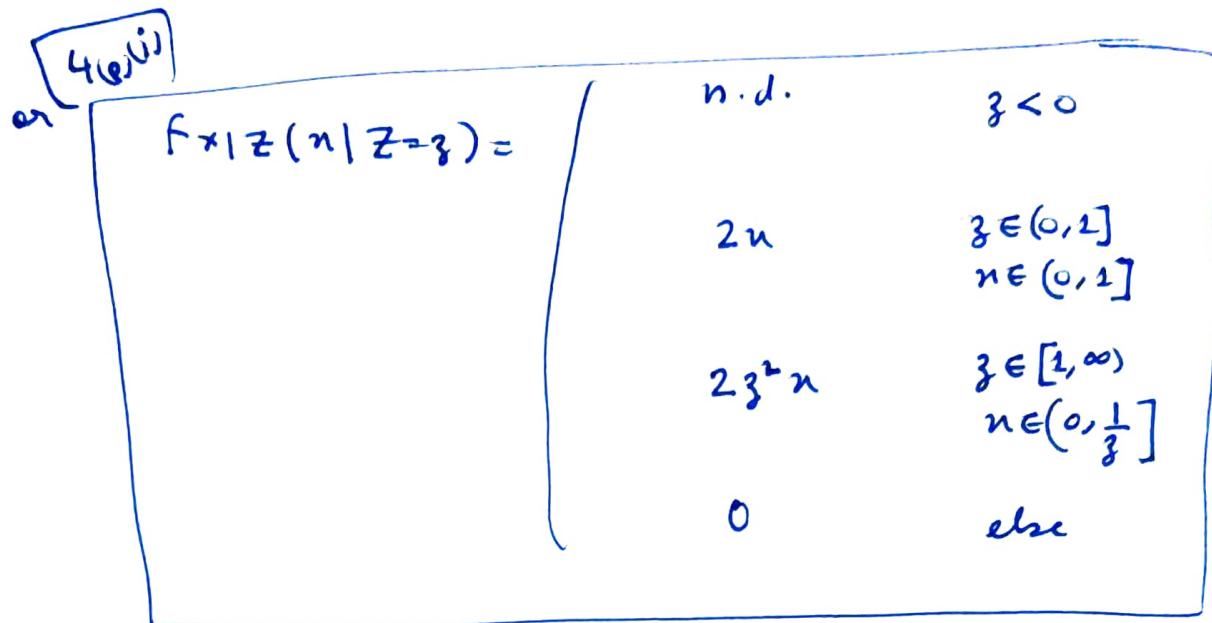
support of (X, Z) :



4(e) $f_{X|Z}(n | Z=z) = \frac{f_{Z|X}(z | X=n) \cdot f_X(X=n)}{f_Z(z)}$

or $f_{X|Z}(n | Z=z) = \begin{cases} n.d. & z < 0 \text{ or } z \geq \frac{1}{n} \\ \frac{n+1}{2} & z \in (0, 1] \\ \frac{n+1}{2z^2} & n \in (0, \frac{1}{z}] \\ 0 & \text{else} \end{cases}$

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$$\mathbb{E}[X|Z=g] = \int_{n=0}^{n=2} n \cdot f_{x|Z}(x|Z=g) dn$$

or $\mathbb{E}[X|Z=g] = \begin{cases} \int_{n=0}^{n=2} n \cdot 2n dn & g \in (0, 1] \\ \int_{n=0}^{n=\frac{1}{g}} n \cdot 2g^2 n dn & g \in [1, \infty) \end{cases}$

4(iv)

or $\mathbb{E}[X|Z=g] = \begin{cases} \frac{2}{3} & g \in (0, 1] \\ \frac{2}{3g} & g \in [1, \infty) \end{cases}$

Ans

$$4(f) \quad E_Z [E_X [x | Z=3]]$$

$$= \int_{-\infty}^{\infty} E_X [x | Z=3] \cdot f_Z(z) dz$$

$$\text{or } E_Z [E_X [x | Z=3]] = \int_{z=2}^{z=1} \frac{2}{3} \cdot \frac{1}{2} dz + \int_{z=1}^{z=\infty} \frac{2}{3z} \cdot \frac{1}{2z^2} dz$$

$$\text{or } E_Z [E_X [x | Z=3]] = \int_{z=0}^{z=1} \frac{1}{3} dz + \int_{z=1}^{z=\infty} \frac{1}{3} z^{-3} dz$$

$$\text{or } E_Z [E_X [x | Z=3]] = \frac{1}{3} + \int_{z=\infty}^{z=1} \frac{1}{6} z^{-2} dz$$

$$\text{or } E_Z [E_X [x | Z=3]] = \frac{1}{3} + \frac{1}{6}$$

4(f)

$$\text{or } E_Z [E_X [x | Z=3]] = \frac{1}{2} = E[X]$$

Hence Verified!



Ans

$$4(g) \quad E[xz | Z=3] = E[zx | Z=3]$$

$$\text{or } E[xz | Z=3] = 3 E[x | Z=3]$$

$$\text{or } E[xz | Z=3] = \begin{cases} \frac{2}{3} z & z \in [0, 1] \\ \frac{2}{3} & z \in [1, \infty) \end{cases} \quad \text{Ans}$$

$$\text{Verify (optional) by } E_Z [E_Y [y | Z=3]] = E_Y [y]$$

$$E_Z [E_Y [y | Z=3]] = \int_{z=0}^{z=2} \frac{2}{3} z \cdot \frac{1}{2} dz + \int_{z=2}^{z=\infty} \frac{2}{3} \cdot \frac{1}{2z^2} dz$$

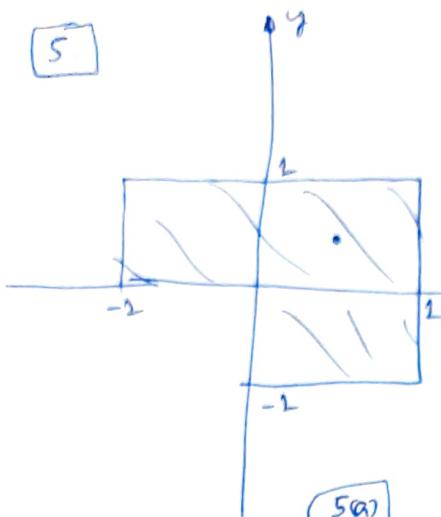
$$\text{or } E_Z [E_Y [y | Z=3]] = \left. \frac{1}{3} \frac{z^2}{2} \right|_0^1 + \left. \frac{1}{3z} \right|_2^\infty$$

$$\text{or } E_Z [E_Y [y | Z=3]] = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} = E[y] \quad \text{Hence Verified!}$$

X — X — X — X — X — ☺

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* X, Y uniform in shaded region.

$$f_{X,Y}(x,y) = \frac{1}{3} \text{ in shaded region}$$

$$\mu_x = \int_{n=-1}^{n=0} \int_{y=0}^{y=1} \frac{1}{3} n dy dn + \int_{n=0}^{n=1} \int_{y=-1}^{y=1} \frac{1}{3} n dy dn$$

5(a)
or $\mu_x = \frac{1}{6}$

5(b)
 $\Rightarrow \mu_y = \frac{1}{6}$ (by symmetry)

$$\mu_{xy} = \int_{-1}^0 \int_{y=0}^1 \frac{1}{3} ny dy dn + \int_0^1 \int_{y=-1}^1 \frac{1}{3} ny dy dn$$

5(c)
or $\mu_{xy} = -\frac{1}{12}$

5(d)
 $\sigma_x^2 = \mu_{x^2} - (\mu_x)^2$

$$\mu_{x^2} = \int \int \frac{1}{3} n^2 dy dn = \frac{1}{3}$$

$$\hat{\sigma}_x^2 = \frac{1}{3} - \left(\frac{1}{6}\right)^2$$

5(e)
or $\hat{\sigma}_x^2 = \frac{11}{36}$

5(f)
 $\Rightarrow \hat{\sigma}_x^2 = \frac{11}{36}$ (by symmetry)

5(f) $\text{Corr}(X, Y) = E[XY] - E[X] \cdot E[Y]$

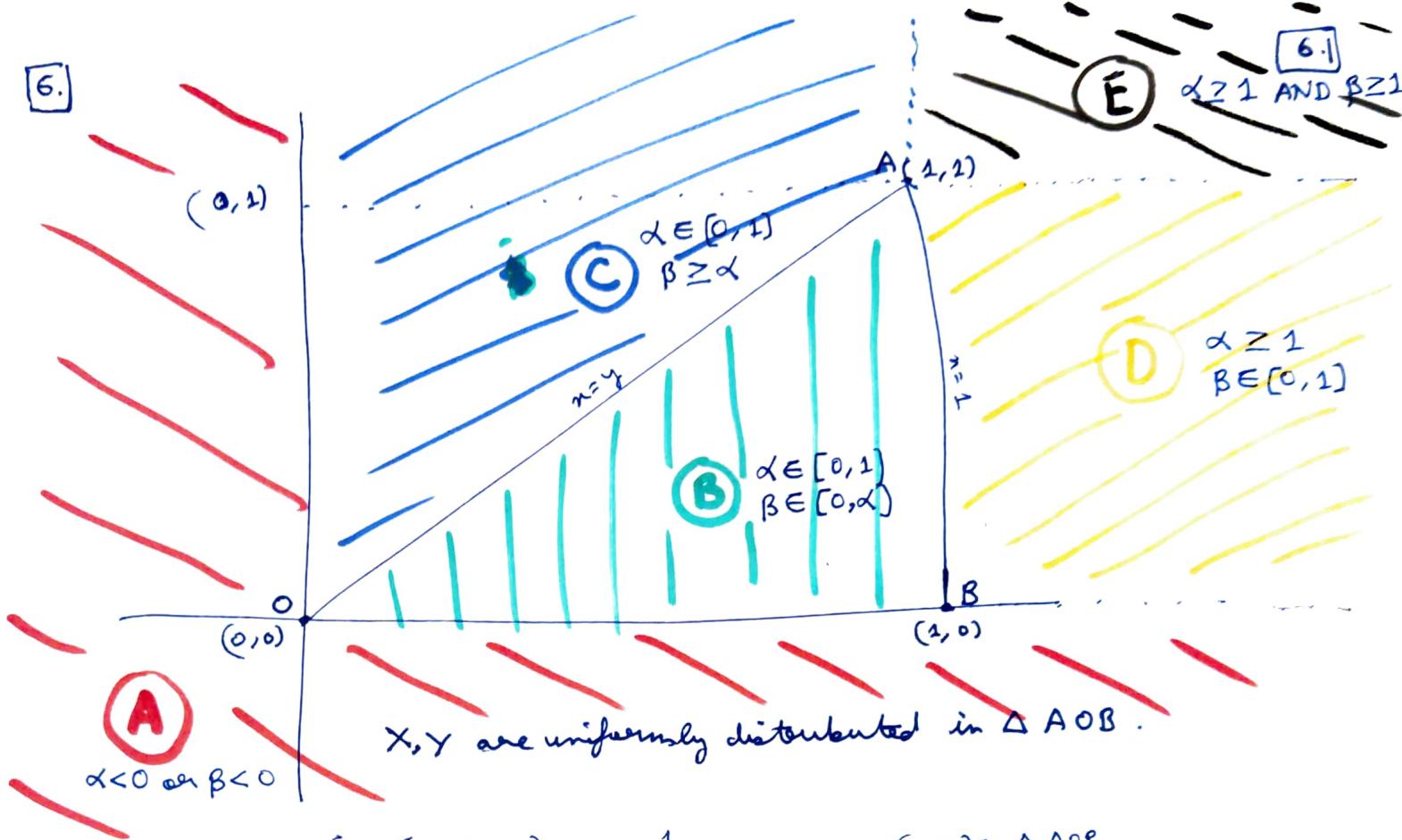
$$\text{or } \text{Corr}(X, Y) = -\frac{1}{12} - \frac{1}{36} \left(\frac{1}{6}\right)^2$$

5(g)
or $\text{Corr}(X, Y) = -\frac{1}{9}$

$$\Rightarrow \rho_{XX} = \frac{\text{Corr}(X, X)}{\hat{\sigma}_x \hat{\sigma}_x} = -\frac{4}{11}$$

or $\rho_{XX} \approx -0.3636$

6.



$$\text{So } f_{x,y}(n, y) = \frac{1}{\text{Area of } \triangle AOB} \quad (n, y) \in \triangle AOB$$

$$\text{or } f_{x,y}(n, y) = \frac{1}{\frac{1}{2} \times 1 \times 1} \quad \begin{matrix} \cancel{n \in [0,1]} \\ y \in [0,1] \\ n \in [y,1] \end{matrix}$$

6(a)

$$f_{x,y}(n, y) = \begin{cases} 2 & \text{if } y \in [0,1] \\ & n \in [y,1] \\ 0 & \text{else} \end{cases}$$

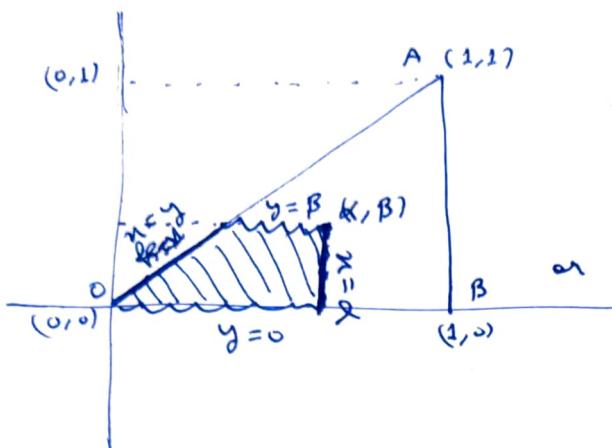
Ans

6(b) (Calculations on next pages.)

$f_{x,y}(n=\alpha, y=\beta) =$	$\left\{ \begin{array}{ll} 0 & \alpha < 0 \text{ and } \beta < 0 \\ 2\alpha\beta - \beta^2 & \alpha \in [0,1], \beta \in [0,\alpha] \\ \alpha^2 & \alpha \in [0,1], \beta = \alpha \\ 2\beta - \beta^2 & \alpha \geq 1, \beta \in [0,\alpha] \\ 1 & \alpha \geq 1, \beta \geq 1 \end{array} \right.$	A
		C
		D
		E

For (α, β) in **B**:

i.e. $\alpha \in [0, 1], \beta \in [0, \alpha]$



~~$F_{X,Y}(x,y)$~~

$$F_{X,Y}(n=\alpha, y=\beta) = \iint_{\text{shaded region}} 2 dndy \quad (\alpha, \beta) \in \textcircled{B}$$

$$\text{or } F_{X,Y}(n=\alpha, y=\beta) = \iint_{y=0}^{y=\beta} \iint_{n=0}^{n=\alpha} 2 dndy \quad (\alpha, \beta) \in \textcircled{B}$$

$$\text{or } F_{X,Y}(\alpha, \beta) = \int_{y=0}^{y=\beta} 2n \Big|_0^{\alpha} dy \quad (\alpha, \beta) \in \textcircled{B}$$

$$\text{or } F_{X,Y}(\alpha, \beta) = \int_{y=0}^{y=\beta} 2(\alpha - y) dy \quad (\alpha, \beta) \in \textcircled{B}$$

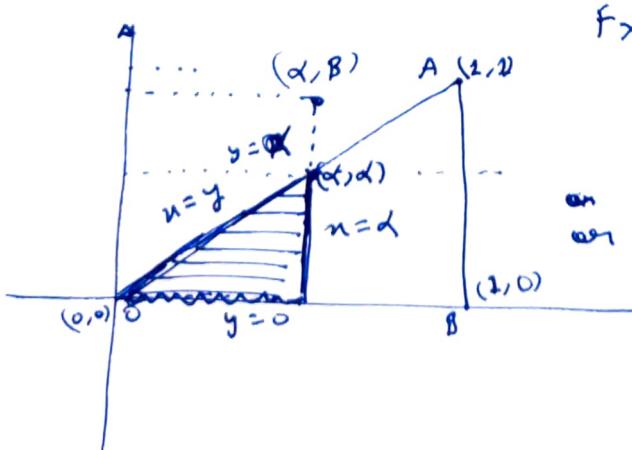
$$\text{or } F_{X,Y}(\alpha, \beta) = 2\alpha y - y^2 \Big|_0^{\beta} \quad (\alpha, \beta) \in \textcircled{B}$$

$$\text{or } \boxed{F_{X,Y}(\alpha, \beta) = 2\alpha\beta - \beta^2 \quad (\alpha, \beta) \in \textcircled{B}}$$

or $\alpha \in [0, 1]$
 $\beta \in [0, \alpha]$

For (α, β) in **C**:

i.e. $\alpha \in [0, 1], \beta \geq \alpha$



$$f_{X,Y}(n=\alpha, y=\beta) = \iint_{\text{shaded region}} 2 dndy \quad (\alpha, \beta) \in \textcircled{C}$$

$$\text{or } F_{X,Y}(n=\alpha, y=\beta) = \iint_{y=0}^{y=\alpha} \iint_{n=0}^{n=\beta} 2 dndy \quad (\alpha, \beta) \in \textcircled{C}$$

or $F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha} 2(\alpha-y) dy \quad (\alpha, \beta) \in \textcircled{C}$

or $F_{X,Y}(x=\alpha, y=\beta) = 2\alpha^2 - \alpha^2 \quad (\alpha, \beta) \in \textcircled{C}$

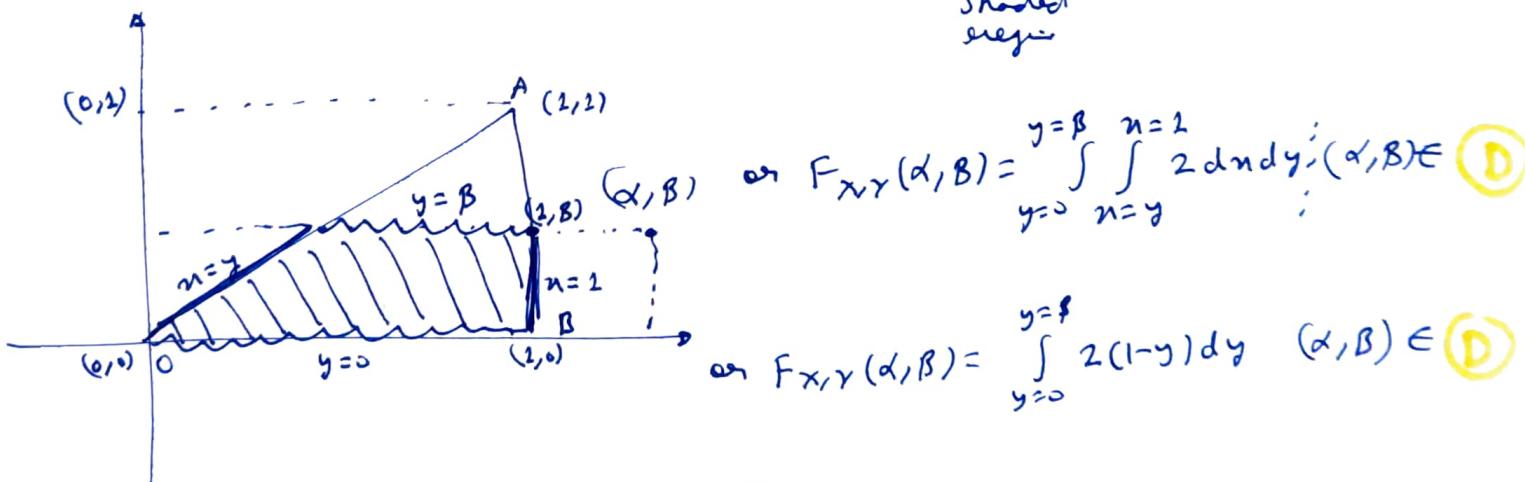
or $F_{X,Y}(x=\alpha, y=\beta) = \alpha^2 \quad (\alpha, \beta) \in \textcircled{C}$
 or $\alpha \in [0, 1], \beta \leq \alpha$

Area of the shaded Δ .
 * $f_{X,Y}(x,y)$ in the region.

For $(\alpha, \beta) \in \textcircled{D}$

i.e. $\alpha \geq 1, \beta \in [0, 1]$

$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded reg}} 2 dx dy \quad (\alpha, \beta) \in \textcircled{D}$



or $F_{X,Y}(\alpha, \beta) = 2y - y^2 \Big|_0^\beta \quad (\alpha, \beta) \in \textcircled{D}$

or $F_{X,Y}(\alpha, \beta) = 2\beta - \beta^2 \quad (\alpha, \beta) \in \textcircled{D}$
 or $\alpha \geq 1, \beta \in [0, 1]$

$$f_{X,Y}(n) = \int_{-\infty}^{\infty} f_{X,Y}(n,y) dy$$

$$\text{or } f_{X,Y}(n) = \begin{cases} 0 & n < 0 \\ \int_{y=0}^{y=n} 2 dy & n \in [0, 2] \\ 0 & n > 2 \end{cases}$$

6(c)

$$\text{or } f_X(n) = \begin{cases} 0 & n < 0 \\ 2n & n \in [0, 2] \\ 0 & n > 2 \end{cases} \quad \underline{\text{Ans}}$$

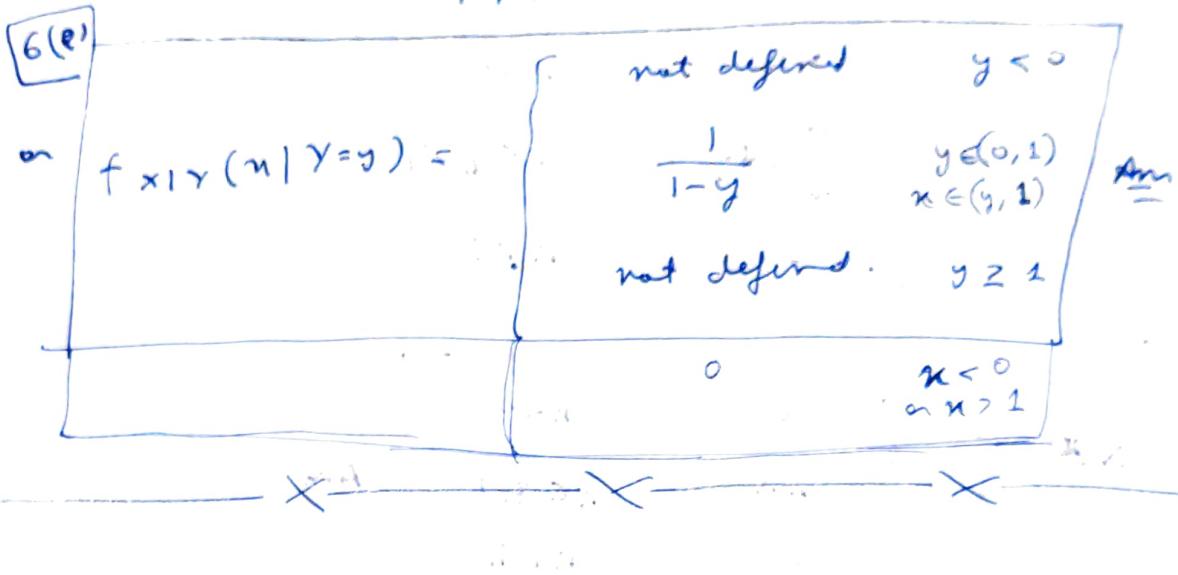
$$f_{Y|X}(y) = \int_{-\infty}^{\infty} f_{X,Y}(n,y) dn$$

$$\text{or } f_Y(y) = \begin{cases} 0 & y < 0 \\ \int_{n=y}^{n=1} 2 dn & y \in [0, 2] \\ 0 & y > 2 \end{cases}$$

6(d)

$$\text{or } f_Y(y) = \begin{cases} 0 & y < 0 \\ 2(1-y) & y \in [0, 2] \\ 0 & y > 2 \end{cases} \quad \underline{\text{Ans}}$$

$$f_{X|Y}(u|Y=y) = \frac{f_{XY}(x,y=y)}{f_Y(y)}$$

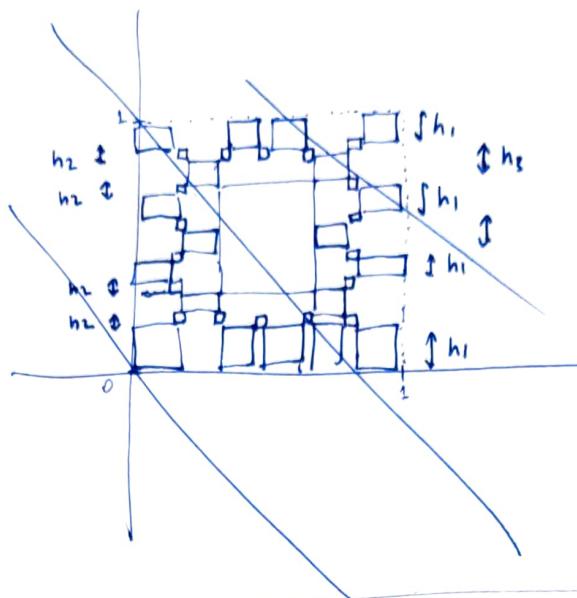
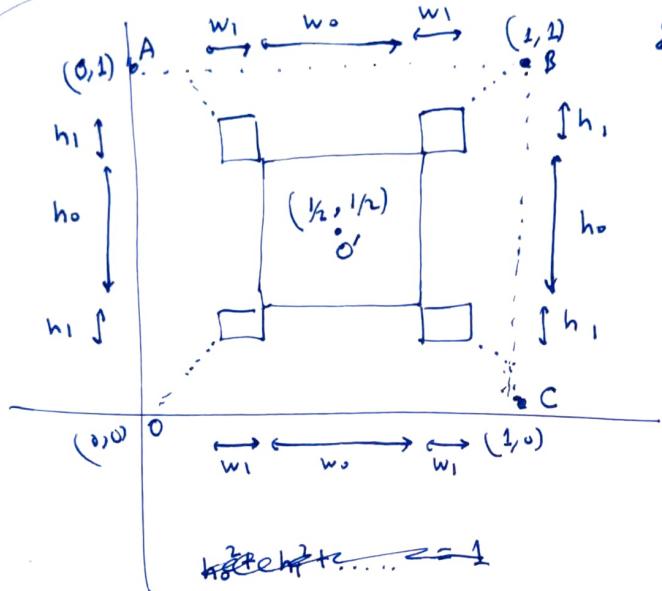


Explain the following statement:

For x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n we have

$$\Pr(X_1=x_1, X_2=x_2, \dots, X_n=x_n | Y_1=y_1, Y_2=y_2, \dots, Y_n=y_n) = \Pr(X_1=x_1 | Y_1=y_1) \Pr(X_2=x_2 | Y_2=y_2) \dots \Pr(X_n=x_n | Y_n=y_n)$$

7.

Ans~~but $h_0^2 + h_1^2 + h_2^2 + \dots = 1$~~

$$\text{where } h_0 + h_1 + h_2 + \dots = 1$$

$$w_0 + w_1 + w_2 + \dots = 1$$

$$(\text{uniform pdf}) \times \{h_0 w_0 + h_1 w_1 + h_2 w_2 + \dots\} = 1$$

In each square OABC of size 1×1 unit squares, any pdf distribution which is

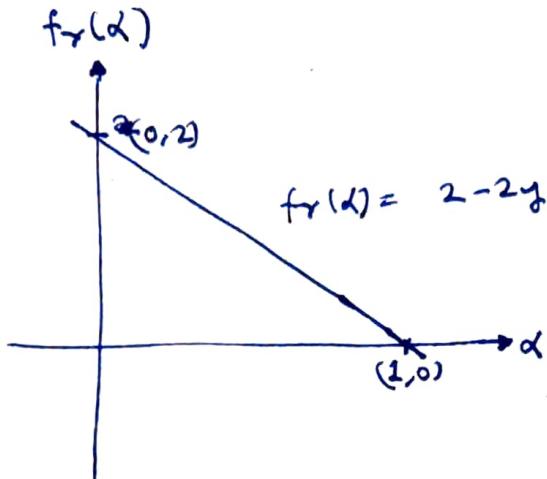
- 1) symmetric about $(\frac{1}{2}, \frac{1}{2})$.
- 2) NOT a single rectangle.
- 3) ~~filled with-~~ a set of more than one (possibly infinite) rectangles which are touching together (to maintain continuity in both x and y) as well as are symmetrically placed around $(\frac{1}{2}, \frac{1}{2})$ (to maintain zero covariance).

can qualify for a distribution rev in x, y s.t.

- 1) X and Y are uncorrelated.
- 2) X and Y are NOT independent.
- 3) $X \sim \text{unif}(0,1)$ and $Y \sim \text{unif}(0,1)$



8.



$$8(a) \hat{Y}_{MP} = \underset{x}{\operatorname{arg\,min}} (f_Y(x)) \Big|_{x \in [0, 1]}$$

$$\text{or } \hat{Y}_{MP} = \underset{x}{\operatorname{arg\,min}} (2 - 2x) \Big|_{x \in [0, 1]}$$

$$\boxed{\begin{array}{l} 8(a) \\ \text{or } \hat{Y}_{MP} = 0 \end{array}} \quad \text{Ans for which } f_Y(x=0) = 2$$

Ans

$$8(b) \hat{Y}_{MMSE} = \underset{x}{\operatorname{arg\,min}} \cdot [E[(Y-x)^2]]$$

$$\text{We know that } \hat{Y}_{MMSE} = \mu_Y = E(Y)$$

$$\text{or } \hat{Y}_{MMSE} = \int_{y=0}^{y=1} y \cdot f_Y(y) dy$$

$$\text{or } \hat{Y}_{MMSE} = \int_{y=0}^{y=1} y \cdot (2 - 2y) dy$$

$$\text{or } \hat{Y}_{MMSE} = \int_{y=0}^{y=1} 2y - 2y^2 dy$$

$$\text{or } \hat{Y}_{MMSE} = \left[y^2 - \frac{2}{3}y^3 \right]_{y=0}^{y=1}$$

$$\boxed{\begin{array}{l} 8(b) \\ \text{or } \hat{Y}_{MMSE} = \frac{1}{3} \end{array}} \quad \text{Ans}$$

None the less, let us derive \hat{y}_{MMSE} using only first principles.

$$\hat{y}_{MMSE} = \underset{\alpha}{\operatorname{argmin}} \left[E[(y-\alpha)^2] \right]$$

$$\text{or } \hat{y}_{MMSE} = \underset{\alpha}{\operatorname{argmin}} \left[\int_{y=0}^{y=1} (y-\alpha)^2 \cdot (2-2y) dy \right]$$

$$\text{or } \hat{y}_{MMSE} = \underset{\alpha}{\operatorname{argmin}} \left[2 \int_{y=0}^{y=1} \left\{ y^2 - 2\alpha y + \alpha^2 \right\} \{1-y\} dy \right]$$

$$\text{or } \hat{y}_{MMSE} = \underset{\alpha}{\operatorname{argmin}} \left[2 \int_{y=0}^{y=1} \left\{ -y^3 + (1+2\alpha)y^2 - (\alpha^2 + 2\alpha)y + \alpha^2 \right\} dy \right]$$

$$\text{or } \hat{y}_{MMSE} = \underset{\alpha}{\operatorname{argmin}} \left[2 \left[-\frac{y^4}{4} + \frac{(1+2\alpha)}{3} y^3 - \frac{(\alpha^2+2\alpha)}{2} y^2 + \alpha^2 y \right] \Big|_{y=0}^{y=1} \right]$$

$$\text{or } \hat{y}_{MMSE} = \underset{\alpha}{\operatorname{argmin}} \left[2 \left[-\frac{1}{4} + \frac{(1+2\alpha)}{3} - \frac{(\alpha^2+2\alpha)}{2} + \alpha^2 \right] \right]$$

$$\text{or } \hat{y}_{MMSE} = \underset{\alpha}{\operatorname{argmin}} \left[2 \left[\frac{\alpha^2}{2} - \frac{1}{3}\alpha + \frac{1}{12} \right] \right]$$

$$\text{or } \hat{y}_{MMSE} = \underset{\alpha}{\operatorname{argmin}} \left[\left(\alpha - \frac{1}{3} \right)^2 + \frac{1}{18} \right]$$

8(b)

$\hat{y}_{MMSE} = \frac{1}{3}$	Ans
with error = $\frac{1}{18}$	

$$\hat{y}_{\text{MMAE}} = \underset{\alpha}{\operatorname{arg\,min}}(8(c)) = \underset{\alpha}{\operatorname{arg\,min}} \left(\int_{y=0}^{y=\alpha} (\alpha - y)(2 - 2y) dy + \int_{y=\alpha}^{y=1} (y - \alpha)(2 - 2y) dy \right)$$

$$\text{or } \hat{y}_{\text{MMAE}} = \underset{\alpha}{\operatorname{arg\,min}} \left(2 \int_{y=0}^{y=\alpha} (\alpha - y)(y - 1) dy + 2 \int_{y=1}^{y=\alpha} (y - \alpha)(y - 1) dy \right)$$

$$\text{or } \hat{y}_{\text{MMAE}} = \underset{\alpha}{\operatorname{arg\,min}} \left(2 \int_{y=0}^{y=\alpha} \{y^2 - (\alpha+1)y + \alpha\} dy + 2 \int_{y=1}^{y=\alpha} \{y^2 - (\alpha+1)y + \alpha\} dy \right)$$

$$\text{or } \hat{y}_{\text{MMAE}} = \underset{\alpha}{\operatorname{arg\,min}} \left(4 \left[\frac{y^3}{3} - \frac{(\alpha+1)y^2 + \alpha y}{2} \right] \Big|_{y=\alpha} - 2 \left[\frac{y^3}{3} - \frac{(\alpha+1)y^2 + \alpha y}{2} \right] \Big|_{y=1} - 2 \left[\frac{y^3}{3} - \frac{(\alpha+1)y^2 + \alpha y}{2} \right] \Big|_{y=0} \right)$$

$$\text{or } \hat{y}_{\text{MMAE}} = \underset{\alpha}{\operatorname{arg\,min}} \left(4 \left[\frac{\alpha^3}{3} - \frac{(\alpha+1)\alpha^2 + \alpha^2}{2} \right] - 2 \left[\frac{1}{3} - \frac{(\alpha+1)}{2} + \alpha \right] \right)$$

$$\text{or } \hat{y}_{\text{MMAE}} = \underset{\alpha}{\operatorname{arg\,min}} \left(4 \left[-\frac{\alpha^3}{6} + \frac{\alpha^2}{2} \right] - 2 \left[\frac{\alpha}{2} - \frac{1}{6} \right] \right)$$

$$\text{or } \hat{y}_{\text{MMAE}} = \underset{\alpha}{\operatorname{arg\,min}} \left(-\frac{4}{6} \alpha^3 + 2\alpha^2 - \alpha + \frac{1}{6} \right)$$

$$\text{or } \hat{y}_{\text{MMAE}} = \underset{\alpha}{\operatorname{arg}} \left(\frac{d}{d\alpha} \left[-\frac{2}{3} \alpha^3 + 2\alpha^2 - \alpha + \frac{1}{3} \right] = 0 \right)$$

$$\text{or } \hat{y}_{\text{MMAE}} = \underset{\alpha}{\operatorname{arg}} \left(-2\alpha^2 + 4\alpha - 1 = 0 \right)$$

$$\text{or } \hat{y}_{\text{MMAE}} = \underset{\alpha}{\operatorname{arg}} \left(-2(\alpha - 1)^2 + 1 = 0 \right)$$

$$\text{or } \hat{y}_{\text{MMAE}} = \underset{\alpha}{\operatorname{arg}} \left(\alpha(\alpha - 1)^2 = \frac{1}{2} \right)$$

$\therefore \hat{y}_{\text{MMAE}} = 1 - \frac{1}{\sqrt{2}}$

$\therefore \hat{y}_{\text{MMAE}} \approx 0.293$ Ans



From the former class, we require :

9.1

$$\boxed{9.1} \quad Y|X \sim \text{exp}\left(\frac{1}{n^2}\right) \Rightarrow f_{Y|X}(y|x=n) = \frac{1}{n^2} e^{-\frac{1}{n^2}y} \quad y \geq 0$$

$$X \sim \text{unif}(0,1) \Rightarrow f_X(n) = 1 \quad n \in [0,1]$$

$\hat{Y}|X=n$ MMSE = $E[Y|X] = n^2$ Ans

$\hat{Y}|X=n$ $= \sigma^2_{Y|X} = n^4$ Ans

or std. error $\hat{Y}|X = \sigma_{Y|X} = n^2$

overall error = $\int_{n=0}^{\infty} \sigma^2_{Y|X} dm =$

or overall error = $\int_{n=0}^{\infty} n^4 \cdot 1 \cdot dm$

$\hat{Y}|X=n$ $= \frac{1}{5}$ Ans

$\hat{Y}^2|X$ MMSE = $E(Y^2|X=n)$

or $\hat{Y}^2|X$ MMSE = $\int_{y=0}^{y=\infty} \frac{1}{n^2} e^{-\frac{1}{n^2}y} \cdot y^2 dy$

or $\hat{Y}^2|X$ MMSE = $(y^3 + 2n^2y + 2n^4) e^{-\frac{1}{n^2}y} \Big|_{y=0}^{y=\infty}$

$\hat{Y}^2|X$ MMSE = $2n^4$ Ans

Mean of squares \neq Square of mean Ans

Part b

$$\begin{aligned} & \hat{Y} A \hat{Y} e^{-AY} \\ &= \frac{d}{dy} [(C \hat{Y}^2 + D \hat{Y} + E) e^{-AY}] \end{aligned}$$

$$\Rightarrow -CA = A$$

$$2C - AD = 0$$

$$D - AE = 0$$

$$\Rightarrow C = -1$$

$$D = -2n^2$$

$$E = 2n^4$$

$$\begin{aligned} & \Rightarrow \int A \hat{Y}^2 e^{-AY} \\ &= \int (y^3 + 2n^2y + 2n^4) e^{-\frac{1}{n^2}y} dy \end{aligned}$$

g(c)

$$f_{X|Y=y}(x|Y=y) = \frac{f_{Y|X}(y|X=x) f_X(x)}{f_Y(y)}$$

or $f_{X|Y}(x|Y=y) = \left(\frac{1}{n^2} e^{-\frac{1}{n^2} y} \right) \cdot (1)$

$\int_{y=0}^{y=\infty} \frac{1}{n^2} e^{-\frac{1}{n^2} y} dy$

 $y \in [0, \infty)$
 $n \in (0, \infty)$

g(c) $\mu_{X|Y} = \sum_{n=0}^{n=2} n \cdot f_{X|Y}(x|Y=y) dn$ $\hat{\quad}$

g(d) $\hat{y}|x_m$ LMMSE = $a^* n + b^*$ where,

g(d) $a^* = \frac{\text{Cov}(X, Y)}{\sigma_x^2}$

$$\sigma_a^* = \frac{\mu_{XY} - \mu_X \cdot \mu_Y}{\sigma_x^2}$$

or $a^* = \frac{\sum_{n=0}^{n=1} \int_{y=0}^{y=\infty} ny \cdot \frac{1}{n^2} e^{-\frac{1}{n^2} y} dy dn - \frac{1}{2} \sum_{n=0}^{n=1} \int_{y=0}^{y=\infty} y \cdot \frac{1}{n^2} e^{-\frac{1}{n^2} y} dy dn}{(\frac{1}{12})}$

or ~~$a^* = \frac{1}{n^2} \int_{y=0}^{y=\infty} y \cdot \frac{1}{n^2} e^{-\frac{1}{n^2} y} dy$~~

or $a^* = \frac{\sum_{n=0}^{n=1} \frac{1}{n} \left(\int_{y=0}^{y=\infty} y \cdot e^{-\frac{1}{n^2} y} dy \right) dn - \frac{1}{2} \sum_{n=0}^{n=1} \frac{1}{n^2} \left(\int_{y=0}^{y=\infty} e^{-\frac{1}{n^2} y} dy \right) dn}{1/12}$

or $a^* = \frac{\sum_{n=0}^{n=1} \frac{1}{n} \cdot \left(\left. \left(ny + \frac{n^2}{2} y^2 \right) \right|_{y=0}^{y=\infty} \right) dn - \frac{1}{2} \sum_{n=0}^{n=1} \frac{1}{n^2} \left(\left. \left(ny + \frac{n^2}{2} y^2 \right) e^{-\frac{1}{n^2} y} \right|_{y=0}^{y=\infty} \right) dn}{1/12}$

Hilf Rechenregel

$$\text{or } a^* = \frac{\int_{n=2}^{n=1} \frac{1}{n} \cdot (n^4) dn - \frac{1}{2} \int_{n=0}^{n=1} \frac{1}{n^2} \cdot (n^4) dn}{1/n}$$

$$\text{or } a^* = \frac{\frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{n}}$$

$$\text{or } \boxed{a^* = 1}$$

$$b^* = \mu_x - a^* m_x$$

$$\text{or } b^* = \frac{1}{3} - 1 \cdot \frac{1}{2}$$

$$\text{or } \boxed{b^* = -\frac{1}{6}}$$

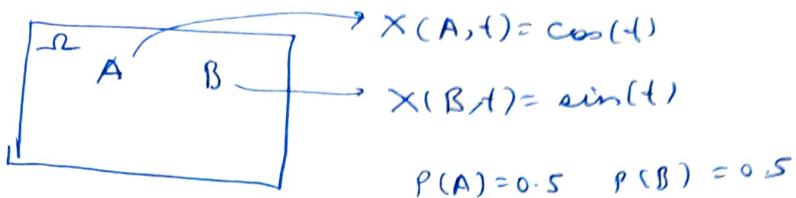
$g(d)$

$$\hat{x}|x_{\text{LMMSE}} = 1 \cdot n - \frac{1}{6} \quad \text{Ans}$$

— x — x — x — x —

[11]

[11.2]

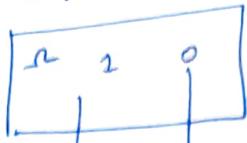


$$11.2) \quad f_{X(t)}^{(n)} = \underset{\cancel{X \in A}}{\delta(n - X(A,t))} P(A) + P(B) \delta(n - X(B,t))$$

or $f_{X(t)}^{(n)} = 0.5 \delta(n - \cos(t)) + 0.5 \delta(n - \sin(t))$ $\xrightarrow{\text{Ans}}$

— x — x — x —

12.

 $x(k) =$ 

$$P(x(k)=0) = 0.6$$

$$P(x(k)=1) = 0.4$$

$$\text{Def } f_{x(k)}(n, k) = \begin{cases} 0.6 \delta(n) & n=0 \\ 0.4 \delta(n-1) & n=1, 2, \dots \end{cases}$$

$$12(c) \quad Y(k) = \sum_{i=0}^{k-1} x[i]$$

$$Y(k) = \begin{matrix} 0 & \text{w.p.} \\ 1 & \text{w.p.} \\ \vdots & \\ k & \text{w.p.} \end{matrix}$$

$$\begin{aligned} & \text{w.p. } k \in \{0, 1\} \\ & \text{w.p. } k \in \{0, 1\} \\ & \text{w.p. } k \in \{0, 1\} \end{aligned}$$

$$\text{z.B. } Y(k) = i \quad \text{w.p. } \begin{matrix} k \in \{0, 1\} \\ k \in \{0, 1\} \\ \vdots \\ k \in \{0, 1\} \end{matrix} \quad i = 0, 1, \dots, k$$

$$\boxed{12(c) \Rightarrow f_{Y(k)}(y, k) = \sum_{i=0}^{k-1} k c_i (0.6)^{k-i} (0.4)^i \delta(y-i)}$$

—X—X—X—X—