EE 507 HW05

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1. (a) Check if $\boxed{f_X(x) \cdot f_Y(y) \cdot f_Z(z) = f_{X,Y,Z}(x,y,z)}$ ⓪

implies that ~~the three~~ X and Y are independent,

i.e. check if ⓪ implies $\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$ ⓘ

~~Given:~~ Integrating both sides of ⓪ w.r.t. z:

$$\int_{\forall z} f_X(x) \cdot f_Y(y) \cdot f_Z(z)\, dz = \int_{\forall z} f_{X,Y,Z}(x,y,z)\, dz$$

or $$f_X(x) f_Y(y) \int_{\forall z} f_Z(z) dz = f_{X,Y}(x,y)$$ (marginal pdf of X, Y)

or $$f_X(x) \cdot f_Y(y) \cdot 1 = f_{X,Y}(x,y)$$

or $$\boxed{f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)}$$ which is ⓘ which is what we set to check for!

$\therefore$ 1(a) $\boxed{f_X(x) \cdot f_Y(y)\, f_Z(z) = f_{X,Y,Z}(x,y,z) \Rightarrow X \text{ and } Y \text{ are independent}}$ Ans

1(b) check if $\boxed{P(X)P(Y)P(Z) = P(X,Y,Z)}$ ⓪

implies that X and Y are independent i.e. $\boxed{P(X) \cdot P(Y) = P(X,Y)}$ ⓘ

where X, Y, Z are events of a probabilistic experiment.

No.

Counterexample:

~~Let experiment~~

In an experiment, ~~let~~ we are picking a number from 1 to 8 randomly: Each number has equal probability of being picked. This experiment is like throwing a fair 8-sided die.

Now let us define three events:

$X = \{1, 2, 3, 4\} \Rightarrow P(X) = \frac{1}{2}$ ~~$P(X,Y)$~~

$Y = \{1, 3, 4, 5\} \Rightarrow P(Y) = \frac{1}{2}$

$Z = \{1, 6, 7, 8\} \Rightarrow P(Z) = \frac{1}{2}$

$X \cap Y \cap Z = \{1\} \Rightarrow P(X \cap Y \cap Z) = \frac{1}{8}$

~~So $P(X)+P(Y)$~~

So equation ⓪ is followed by events X, Y, Z, as

$$P(X).P(Y).P(Z) = \left(\frac{1}{2}\right)^3 = \frac{1}{8} = P(X, Y, Z)$$

But $P(X).P(Y) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$P(X \cap Y) = P(\{1, 3, 4\}) = \frac{3}{8}$

X and Y are NOT independent.

In fact, $P(Y).P(Z) = \frac{1}{4}$

$P(Y \cap Z) = P(\{1\}) = \frac{1}{8}$

Y and Z are NOT independent either

And $P(X).P(Z) = \frac{1}{4}$

$P(X \cap Z) = P(\{1\}) = \frac{1}{8}$

X and Z are NOT independent either.

1(b)

$\therefore$ The analogous event $P(X).P(Y).P(Z) = P(XYZ)$ does NOT imply independence of X and Y.

Ans

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2. 2(a) $X, Y \sim \mathcal{N}(\mu_X = 0, \mu_Y = 0, \sigma_X^2 = 1, \sigma_Y^2 = 4, \rho = 0.5)$

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{1-\rho^2}\,\sigma_X\sqrt{2\pi}\,\sigma_Y\sqrt{2\pi}}\, e^{-\frac{1}{2(1-\rho^2)}\underbrace{\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right\}}_{\text{Only component (say } g(x,y)) \text{ dependent on } x \text{ and } y.}}$$

Taking out the component $g(x,y)$ and putting the given values of $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ_{XY}:

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - \frac{2(x)(y)}{(1)(2)}$$

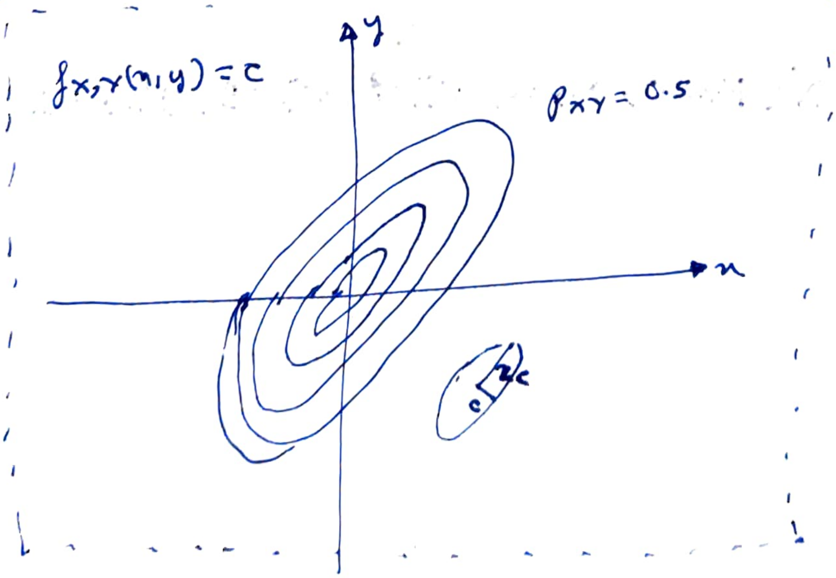
$\Rightarrow f_{X,Y}(x,y) \underset{\text{i.e. is constant}}{\rightarrow} c$ for $g(x,y) \underset{\text{i.e. is constant}}{\rightarrow} c$

$\Rightarrow$ Contour of $f_{X,Y}(x,y)$ is

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - xy = c$$

which is the equation for an ellipse, tilted and with positive slope.

2(a)



Ans

2(b)   2.2

$$\boxed{\begin{aligned} f_X(x) &= \mathcal{N}(x, 0, 1^2) = \frac{1}{1.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}} \quad \forall x \\ f_Y(y) &= \mathcal{N}(y, 0, 2^2) = \frac{1}{2.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y}{2}\right)^2\right\}} \quad \forall y \end{aligned}}$$

Ans

2(c)

$$f_{Y|X}(y|X=x) = \frac{f_{X,Y}(X=x, Y)}{f_X(X=x)} \quad \forall x, y$$

or
$$f_{Y|X}(y|X=x) = \frac{\dfrac{1}{\sqrt{\left(1-\frac{1}{2^2}\right)}\, 1.\sqrt{2\pi}\,.\, 2.\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2^2}}\cdot\left\{\left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - 2\times\frac{1}{2}\times\frac{x}{1}\times\frac{y}{2}\right\}} \quad \cancel{\forall x, y}}{\dfrac{1}{1.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}}} \quad \forall (x, y)$$

or
$$f_{Y|X}(y|X=x) = \frac{1}{\sqrt{3}.\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\left\{\left(\frac{x}{\sqrt{3}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 - 2.1.\left(\frac{x}{\sqrt{3}}\right)\left(\frac{y}{\sqrt{3}}\right)\right\}} \quad \forall (x, y)$$

2(c)

or $\cancel{f_{Y|X}(y|X=x) = \sqrt{2\pi}\,\mathcal{N}(\mu_X=0, \mu_Y=0, \sigma_X^2=3, \sigma_Y^2=3, \rho_{XY}=1) \quad \forall (x,y)}$ Ans

or $\cancel{f_{Y|X}(y|X=x) = \mathcal{N}_y(\mu_Y = \ldots, \sigma_Y^2 = 3) \quad \forall y}$

or
$$f_{Y|X}(y|X=x) = \frac{1}{\sqrt{3}}\,.\,\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y-x}{\sqrt{3}}\right)^2\right\}} \quad \forall y$$

2(c)

or
$$\boxed{f_{Y|X}(y|X=x) = \mathcal{N}(y, \mu_Y = x, \sigma_Y^2 = 3) \quad \forall y}$$
Ans

PTO.

~~2(d) Find x s.t. $E[Y|X=x] = -2$.~~

~~From 2(c), we know that $Y|X=x$ is fully correlated to $X=x$. ($\rho_{Y|X, X} = 1$).~~

~~$\because$ Y and X have the same support/domain of $x \in \mathbb{R}$ and $y \in \mathbb{R}$,~~

~~$\therefore$ $E_Y[Y|X=x] = -2 \equiv E_X[X=x] = -2$~~

~~But $E_X[X=x] = x$~~

~~$\boxed{2(d)}$~~

~~$\Rightarrow \boxed{x = -2}$ Ans~~

2(d) Find x s.t. $E[Y|X=x] = -2$

From 2(c), we know that $Y|X=x$ is a gaussian with mean x.

So $E[Y|X=x] = -2 \Rightarrow \boxed{x = -2}$ Ans

2(e) $Z = X + Y - 1$

Z is also a Gaussian, we need to only compute μ_Z and σ_Z^2 to find $f_Z(z)$:

$$E(Z) = E(X) + E(Y) - E(1)$$

or $\mu_Z = \mu_X + \mu_Y - 1$

or $\mu_Z = 0 + 0 - 1$

or $\boxed{\mu_Z = -1}$

$$\sigma_Z^2 = E\left[\{(X+Y-1) - E[X+Y-1]\}^2\right]$$

or $\sigma_Z^2 = E\left[\{(X-\mu_X) + (Y-\mu_Y) + (-1 - E(-1))\}^2\right]$

or $\sigma_Z^2 = E\left[\{(X-\mu_X) + (Y-\mu_Y)\}^2\right]$

or $\sigma_Z^2 = E[(X-\mu_X)^2] + E[(Y-\mu_Y)^2] + 2E[(X-\mu_X)(Y-\mu_Y)]$

or $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 1^2 + 2^2 + 2 \times 0.5 \times 1 \times 2$

or $\boxed{\sigma_Z^2 = 7}$

2(e)

$\therefore \boxed{Z \sim \mathcal{N}(z, \mu_Z = -1, \sigma_Z^2 = 7)}$ Ans

2(f) $\boxed{2.5}$

$Z = 2X + 3Y$ $\qquad$ $W = X - Y$

$E(Z) = 2\mu_X + 3\mu_Y$ $\qquad$ $E(W) = \mu_X - \mu_Y$

2(f)(i)

or $\boxed{\mu_Z = 0}$ Ans

2(f)(iii)

$\boxed{\mu_W = 0}$ Ans

$\sigma_Z^2 = E\left[\{(2X+3Y) - (2\mu_X + 3\mu_Y)\}^2\right]$ $\qquad$ $\sigma_W^2 = E\left[\{(X-Y) - (\mu_X - \mu_Y)\}^2\right]$

or $\sigma_Z^2 = E\left[\{2(X-\mu_X)\}^2 + \{3(Y-\mu_Y)\}^2 + 2.2.3.(X-\mu_X)(Y-\mu_Y)\right]$

or $\sigma_W^2 = E\left[\{X-\mu_X\}^2 + \{Y-\mu_Y\}^2 - \{2.1.1.(X-\mu_X)(Y-\mu_Y)\}\right]$

or $\sigma_Z^2 = 4\sigma_X^2 + 9\sigma_Y^2 + 12\rho_{XY}\sigma_X\sigma_Y$ $\qquad$ or $\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 4.1^2 + 9.2^2 + 12.(0.5).1.2$ $\qquad$ or $\sigma_W^2 = 1^2 + 2^2 - 2.(0.5).1.2$

or $\sigma_Z^2 = 4 + 36 + 12$ $\qquad$ or $\sigma_W^2 = 1 + 4 - 2$

2(f)(ii)

or $\boxed{\sigma_Z^2 = 52}$ Ans

2(f)(iv)

$\boxed{\sigma_W^2 = 3}$ Ans

2(g)

$Cov(Z, W) = E\left[(Z - \mu_Z)(W - \mu_W)\right]$

or $Cov(Z, W) = E\left[(2X + 3Y - 2\mu_X - 3\mu_Y)(X - Y - \mu_X - (-\mu_Y))\right]$

or $Cov(Z, W) = E\left[\{2(X-\mu_X) + 3(Y-\mu_Y)\}\{1(X-\mu_X) - 1(Y-\mu_Y)\}\right]$

or $Cov(Z, W) = E\left[2(X-\mu_X)^2 - 3(Y-\mu_Y)^2 + 1(X-\mu_X)(Y-\mu_Y)\right]$

or $Cov(Z, W) = 2\sigma_X^2 - 3\sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y$

or $Cov(Z, W) = 2.1^2 - 3.2^2 + (0.5).1.2$

or $Cov(Z, W) = 2 - 12 + 1$

2(f)$^{(v)}$

or $\boxed{Cov(Z, W) = -9}$ Ans

$$\boxed{\rho_{Z,W} = \frac{Cov(Z,W)}{\sigma_Z \sigma_W} = \frac{-9}{\sqrt{52}\cdot\sqrt{3}} \approx -0.7206}$$

2(f)$^{(vi)}$

$$\boxed{f_{Z,W}(z,w) = \mathcal{N}(\mu_Z = 0, \mu_W = 0, \sigma_Z^2 = 52, \sigma_W^2 = 3, \rho_{Z,W} = -0.7206)}$$

Ans

2(g) $R = aX + bY$ $\Rightarrow$ $\sigma_R^2 = a^2 \cdot \sigma_X^2 + b^2 \sigma_Y^2$

or $\sigma_R^2 = a^2 \cdot 1^2 + b^2 \cdot 2^2$

or $\sigma_R^2 = a^2 + 4b^2$

$Cov(R, Y) = 0$ is a sufficient condition for R and Y to be independent, as both are gaussian.

$$Cov(R,Y) = E\left[(aX + bY - a\mu_X - b\mu_Y)(Y - \mu_Y)\right]$$

or $Cov(R,Y) = E\left[a(X-\mu_X)(Y-\mu_Y) + b(Y-\mu_Y)^2\right]$

or $Cov(R,Y) = a\,\rho_{X,Y}\sigma_X\sigma_Y + b\sigma_Y^2$

or $Cov(R,Y) = a(0.5)\cdot 1\cdot 2 + b\cdot 2^2$

or $Cov(R,Y) = a + 4b$

2(g)

For R, Y to be independent distributions, $a + 4b = 0$

we can use any set of Real numbers to do so,

say $a = -4, b = 1$

Ans

2(b) $\quad Q = aX$

given $\quad p(a=2) = 0.5 \qquad f(Q \mid a=2) = \cancel{2} \, f_X(X = 2x)$

$\qquad\qquad p(a=1) = 0.5 \qquad f(Q \mid a=1) = f_X(X = x)$

$$\begin{tikzcd}[row sep=small] & Q = X \\ Q \arrow[ur, "0.5"] \arrow[dr, "0.5"'] & \\ & Q = 2X \end{tikzcd}$$

$$f_Q(q) = f_{Q|a}(Q \mid a=1) \cdot p(a=1) + f_{Q|a}(Q \mid a=2) \cdot p(a=2) \quad \text{(LoTP)}$$

or $f_Q(q) = 0.5\,\mathcal{N}(q, \mu_q = \mu_x, \sigma_q^2 = \sigma_x^2) + 0.5\,\mathcal{N}(q, \mu_q = 2\mu_x, \sigma_q^2 = 2^2\sigma_x^2)$

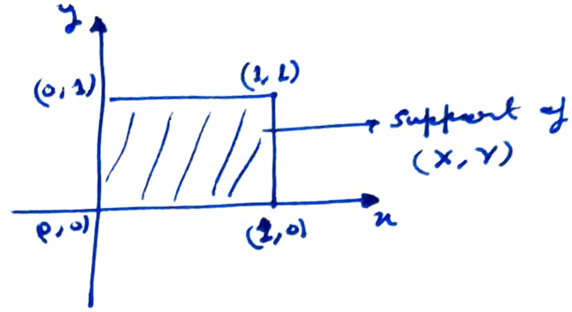
$\boxed{2(h)}$

or $\boxed{f_Q(q) = 0.5\,\mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 1) + 0.5\,\mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 4)}$ Ans

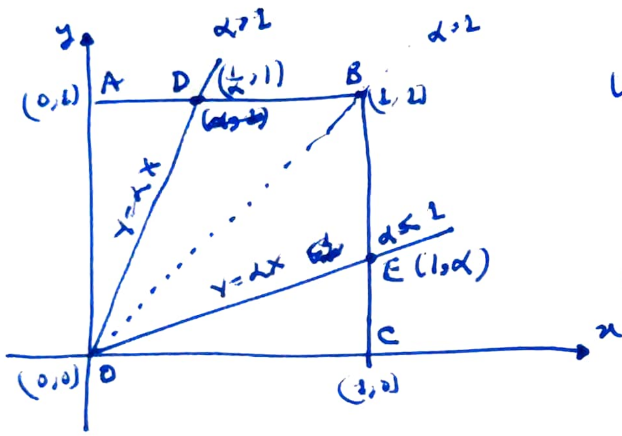
4. $X \sim \text{unif}[0,1]$ X, Y are independent

$Y \sim \text{unif}[0,1]$

$Z = \dfrac{Y}{X}$



4(a) Find $f_Z(z) = ?$



Using CDF method:

$F_{Z}(\underset{(=\alpha)}{z}) = P(Z \leq \alpha)$ $\alpha \in (0,\infty)$

or $F_{Z}(\underset{(=\alpha)}{z}) = P\left(\frac{Y}{X} \leq \alpha\right)$

or $F_{Z}(\underset{(=\alpha)}{z}) = P(Y \leq \alpha X)$

$$\text{or} \quad F_Z(z=\alpha) = \begin{cases} \text{Area of } \triangle\, OEC & \alpha \leq 1 \\ \text{Area of } ODBC = 1 - \text{Area}(\triangle ADO) & \alpha \geq 1 \end{cases}$$

Note: $\because \underbrace{f_{X,Y}(x,y) = 1}_{\text{uniform}}$ $\forall (x,y) \in OABC$, simply computing the area under the curves is equivalent to computing the corresponding CDFs.

$$\text{or} \quad F_Z(z=\alpha) = \begin{cases} \frac{1}{2} & 0 < \alpha \leq 1 \\ \frac{1}{2\alpha^2} & \alpha \geq 1 \\ 0 & \alpha < 0 \end{cases}$$

4.2

4(a)

$$\Rightarrow \boxed{f_Z(z) = \frac{d}{d\alpha} F_Z(z=\alpha)\Big|_{\alpha\to z} = \begin{cases} 0 & z<0 \\ \frac{1}{2} & z \in (0,1] \\ \frac{1}{2z^2} & z \geq 1 \end{cases}}$$

Ans

4 b) $E[X^2+Y^2] = E[X^2] + E[Y^2]$

or $E[X^2+Y^2] = \int_{x=0}^{x=1} x^2 \cdot 1 \cdot dx + \int_{y=0}^{y=1} y^2 \cdot 1 \cdot dy$

or $E[X^2+Y^2] = \frac{1}{3} + \frac{1}{3}$

4(b)

or $\boxed{E[X^2+Y^2] = \frac{2}{3}}$ Ans

4 c) $f_{X,Z}(x,z) = \begin{cases} f_{Z|X}(z|X=x) \cdot f_X(X=x) & x \in (0,1],\ z \in (0,\frac{1}{x}] \\ 0 & \text{else} \end{cases}$

or $f_{X,Z}(x,z) = \begin{cases} \text{pdf of } \left(\frac{1}{x} \cdot Y\right) \cdot 1 & x \in (0,1],\ z \in (0,\frac{1}{x}] \\ 0 & \text{else} \end{cases}$

But $Y \sim unif(0,1)$

$Z|X=x \sim \frac{1}{x} \cdot unif(0,1)$

$\equiv Z|X=x \sim unif\left(0, \frac{1}{x}\right)$

$$\Rightarrow \boxed{pdf(Z|X=x) = \begin{cases} x & z \in (0, \frac{1}{x}) \\ 0 & \text{else} \end{cases}}$$

4.3

4(c)

So ~~f~~ $\boxed{f_{X,Z}(x,z) = \begin{cases} x & x \in (0,1] \\ & z \in (0, \frac{1}{x}] \\ 0 & \text{else} \end{cases}}$ Ans

4(d) $E[XZ] = E[Y] = \frac{1}{2}$

~~But anyway using~~ $\cancel{E[XZ] = \iint f_{X,Z}(x,z) \cdot x \cdot z \cdot dy\, dx}$

4(d)(i) $\boxed{E[XZ] = \frac{1}{2}}$ Ans

$$\mathrm{Cov}(X,Z) = E[XZ] - \mu_X \mu_Z$$

or $\mathrm{Cov}(X,Z) = E[Y] - \frac{1}{2}\mu_Z$

or $\mathrm{Cov}(X,Z) = \cancel{E}\ \frac{1}{2} - \frac{1}{2}\mu_Z$

$$\mu_Z = \int f_Z(z) \cdot z \cdot dz$$

or $\mu_Z = \int_{z=0}^{z=1} \left(\frac{1}{2}\right) \cdot z \cdot dz + \int_{z=1}^{z=\infty} \left(\frac{1}{2z^2}\right) z \, dz$

or $\mu_Z = \frac{1}{4} z^2 \Big|_{z=0}^{z=1} + \frac{1}{2} \ln z \Big|_{z=1}^{z=\infty}$

or $\boxed{\mu_Z \to \infty}$

4(d)(ii) $\boxed{\mathrm{Cov}(X,Z) \to -\infty}$ Ans

$\boxed{4.4}$

$$\rho_{X,Z}(x,z) = \frac{Cov(X,Z)}{\sigma_X \sigma_Z}$$

~~σ_Z~~ $\sigma_X = \sqrt{\frac{1}{12}}$

$$\sigma_Z = \int (z-\mu_Z)^2 \cdot f_Z(z)\, dz \to \infty$$

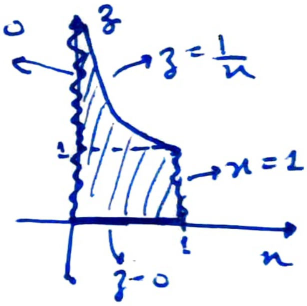
$$\mu_Z \to \infty$$

So $\rho_{X,Z}(x,z) \sim \dfrac{-\infty}{\sqrt{\frac{1}{12}} \cdot \infty} \approx -1$

$\boxed{4(d)^{(iii)}}$

or $\boxed{\rho_{X,Z}(x,z) = -1}$ Ans

support of (X,Z) :



4(e) $f_{X|Z}(x|Z=z) = \dfrac{f_{Z|X}(z|X=x) \cdot f_X(X=x)}{f_Z(Z=z)}$

or $f_{X|Z}(x|Z=z) = \begin{cases} \text{n.d.} & z<0 \text{ or } \cancel{z < \frac{1}{x}} \\ \dfrac{x \cdot 1}{\frac{1}{2}} & z \in (0,1],\ x \in (0,1] \\ \dfrac{x \cdot 1}{\frac{1}{2} z^{-2}} & z \in [1,\infty),\ x \in (0, \frac{1}{z}] \\ 0 & \cancel{x < \frac{1}{z}} \text{ else} \end{cases}$

4.5

4(e)(i)

or $f_{X|Z}(x|Z=z) = \begin{cases} \text{n.d.} & z<0 \\ 2x & z\in(0,1] \\ & x\in(0,1] \\ 2z^2x & z\in[1,\infty) \\ & x\in(0,\frac{1}{z}] \\ 0 & \text{else} \end{cases}$ Ans

$$E[X|Z=z] = \int_{x=0}^{x=1} x\cdot f_{X|Z}(x|Z=z)\,dx$$

$$\text{or } E[X|Z=z] = \begin{cases} \int_{x=0}^{x=1} x\cdot 2x\,dx & z\in(0,1] \\ \int_{x=0}^{x=\frac{1}{z}} x\cdot 2z^2x\,dx & z\in[1,\infty) \end{cases}$$

4(e)(ii)

or $E[X|Z=z] = \begin{cases} \frac{2}{3} & z\in(0,1] \\ \frac{2}{3z} & z\in[1,\infty) \end{cases}$ Ans

4 (f) $E_Z[E_X[X|Z=z]]$

$$= \int_{z} E_X[X|Z=z] \cdot f_Z(z)\, dz$$

or $$E_Z[E_X(X|Z=z)) = \int_{z=0}^{z=1} \frac{2}{3} \cdot \frac{1}{2}\, dz + \int_{z=1}^{z=\infty} \frac{2}{3z} \cdot \frac{1}{2z^2}\, dz$$

or $$E_Z[E_X(X|Z=z)) = \int_{z=0}^{z=1} \frac{1}{3}\, dz + \int_{z=1}^{z=\infty} \frac{1}{3} z^{-3}\, dz$$

or $$E_Z[E_X[X|Z=z]] = \frac{1}{3} + \Big[\frac{1}{6} z^{-2}\Big]_{z=\infty}^{z=1}$$

or $$E_Z[E_X[X|Z=z)) = \frac{1}{3} + \frac{1}{6}$$

4(f)

or $$\boxed{E_Z[E_X[X|Z=z]) = \frac{1}{2} = E[X]}$$ Ans

Hence Verified ! ☺

4(g) $$E[XZ|Z=z] = E[zX|Z=z]$$

or $$E[XZ|Z=z] = z\,E[X|Z=z]$$

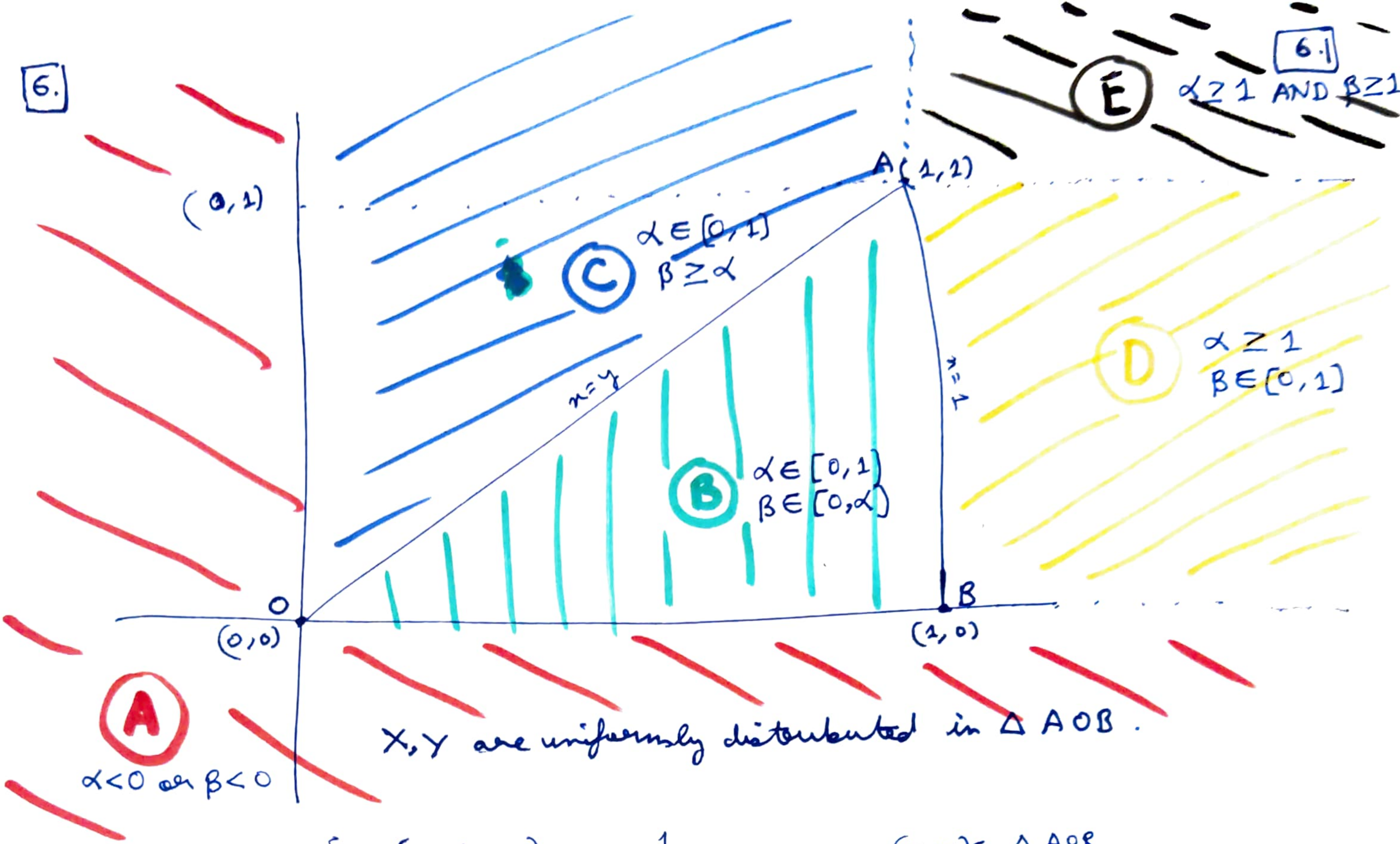
or $$\boxed{\underset{\text{or } E[Y|Z=z]}{E[XZ|Z=z]} = \begin{cases} \frac{2}{3} z & z \in [0,1],\ x \in [0,z] \\ \frac{2}{3} & z \in [1,\infty),\ x \in [0,1] \end{cases}}$$ Ans

Verify (optional) if $E_Z[E_Y[Y|Z=z]] = E_Y[Y] \rightarrow \frac{1}{2}$

$$E_Z[E_Y[Y|Z=z]] = \int_{z=0}^{z=1} \frac{2}{3} z \cdot \frac{1}{2}\, dz + \int_{z=1}^{z=\infty} \frac{2}{3} \cdot \frac{1}{2z^2}\, dz$$

or $$E_Z[E_Y[Y|Z=z]) = \frac{1}{3} \frac{z^2}{2}\Big|_0^1 + \frac{1}{3z}\Big|_\infty^1$$

or $$E_Z[E_Y[Y|Z=z]) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} = E[Y]$$ Hence verified ! ☺



X, Y are uniformly distributed in Δ AOB.

So $f_{X,Y}(x,y) = \dfrac{1}{\text{Area of } \Delta AOB}$ $\quad (x,y) \in \Delta AOB$

or $f_{X,Y}(x,y) = \dfrac{1}{\frac{1}{2} \times 1 \times 1}$ $\quad y \in [0,1]$ and $x \in [y, 1]$

6(a)

or
$$f_{X,Y}(x,y) = \begin{cases} 2 & y \in [0,1],\ x \in [y,1] \\ 0 & \text{else} \end{cases}$$

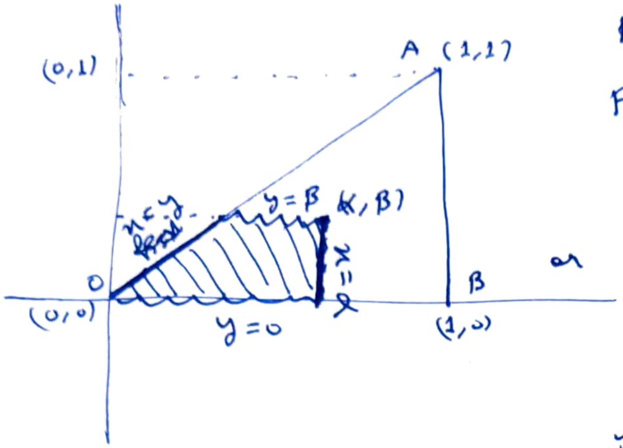
Ans

6(b) (Calculations on next pages.)

$$F_{X,Y}(x=\alpha, y=\beta) = \begin{cases} 0 & \alpha < 0 \text{ or } \beta < 0 \quad \text{A} \\ 2\alpha\beta - \beta^2 & \alpha \in [0,1],\ \beta \in [0,\alpha] \quad \text{B} \\ \alpha^2 & \alpha \in [0,1],\ \beta \geq \alpha \quad \text{C} \\ 2\beta - \beta^2 & \alpha \geq 1,\ \beta \in [0,\alpha] \quad \text{D} \\ 1 & \alpha \geq 1,\ \beta \geq 1 \quad \text{E} \end{cases}$$

Ans

For (α,β) in (B):
i.e. $\alpha \in (0,1)$, $\beta \in [0,\alpha]$



~~$F_{X,Y}(x,y)$~~

$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{Shaded region}} 2\,dx\,dy \quad (\alpha,\beta)\in (B)$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\beta}\int_{x=y}^{x=\alpha} 2\,dx\,dy \quad (\alpha,\beta)\in (B)$$

or $$F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2x\Big|_{y}^{\alpha} dy \quad (\alpha,\beta)\in (B)$$

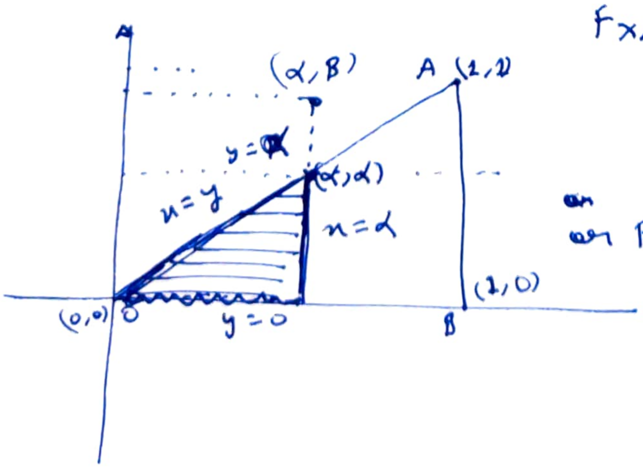
or $$F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2(\alpha-y)\,dy \quad (\alpha,\beta)\in (B)$$

or $$F_{X,Y}(\alpha,\beta) = 2\alpha y - y^2\Big|_0^{\beta} \quad (\alpha,\beta)\in (B)$$

or $$\boxed{F_{X,Y}(\alpha,\beta) = 2\alpha\beta - \beta^2 \quad (\alpha,\beta)\in (B)}$$

$\boxed{\text{or } \alpha\in(0,1),\ \beta\in[0,\alpha]}$

For (α,β) in (C):
i.e. $\alpha \in [0,1]$, $\beta \geq \alpha$



$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{Shaded region}} 2\,dx\,dy \quad (\alpha,\beta)\in (C)$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha}\int_{x=y}^{x=\alpha} 2\,dx\,dy \quad (\alpha,\beta)\in (C)$$

6.3

or $F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha} 2(\alpha-y)\,dy \quad (\alpha,\beta) \in C$

or $F_{X,Y}(x=\alpha, y=\beta) = 2\alpha^2 - \alpha^2 \quad (\alpha,\beta) \in C$

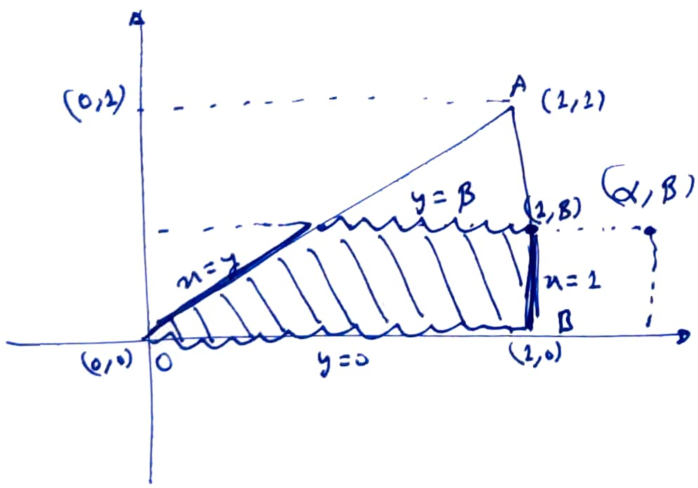
or $\boxed{F_{X,Y}(x=\alpha, y=\beta) = \alpha^2 \quad (\alpha,\beta) \in C \quad \text{or } \alpha \in [0,1], \beta \geq \alpha}$

Area of the shaded Δ.
* $f_{X,Y}(x,y)$ in the region.

For (α,β) in D

i.e. $\alpha \geq 1$, $\beta \in [0,1]$

$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\,dx\,dy \quad (\alpha,\beta) \in D$



or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} \int_{x=y}^{x=1} 2\,dx\,dy \quad (\alpha,\beta) \in D$

or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2(1-y)\,dy \quad (\alpha,\beta) \in D$

or $F_{X,Y}(\alpha,\beta) = 2y - y^2 \Big|_0^{\beta} \quad (\alpha,\beta) \in D$

or $\boxed{F_{X,Y}(\alpha,\beta) = 2\beta - \beta^2 \quad (\alpha,\beta) \in D \quad \text{or } \alpha \geq 1, \beta \in [0,1)}$

$$f_X(x) = \int_{\forall y} f_{X,Y}(x,y)\,dy$$

or $$f_X(x) = \begin{cases} 0 & x < 0 \\ \int_{y=0}^{y=x} 2\,dy & x \in [0,1] \\ 0 & x > 1 \end{cases}$$

6(c)

or $$\boxed{f_X(x) = \begin{cases} 0 & x < 0 \\ 2x & x \in [0,1] \\ 0 & x > 1 \end{cases}}$$ Ans

$$f_Y(y) = \int_{\forall x} f_{X,Y}(x,y)\,dx$$

or $$f_Y(y) = \begin{cases} 0 & y < 0 \\ \int_{x=y}^{x=1} 2\,dx & y \in [0,1] \\ 0 & y > 1 \end{cases}$$

6(d)

or $$\boxed{f_Y(y) = \begin{cases} 0 & y < 0 \\ 2(1-y) & y \in [0,1] \\ 0 & y > 1 \end{cases}}$$ Ans

$$f_{X|Y}(x|Y=y) = \frac{f_{X,Y}(x,y=y)}{f_Y(y)}$$

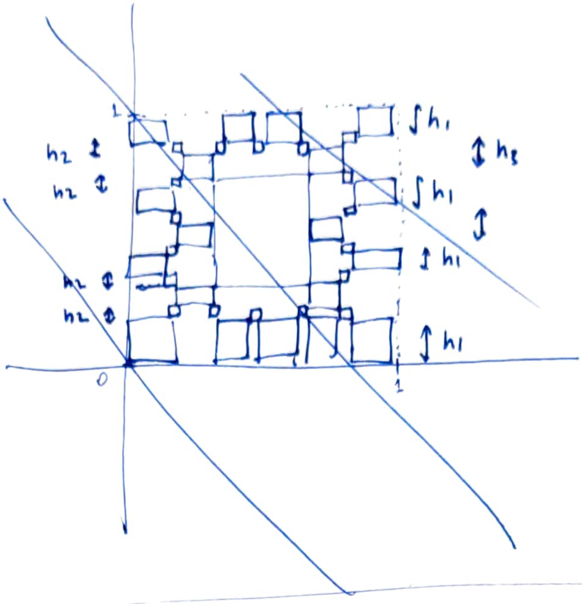
6(e)

or

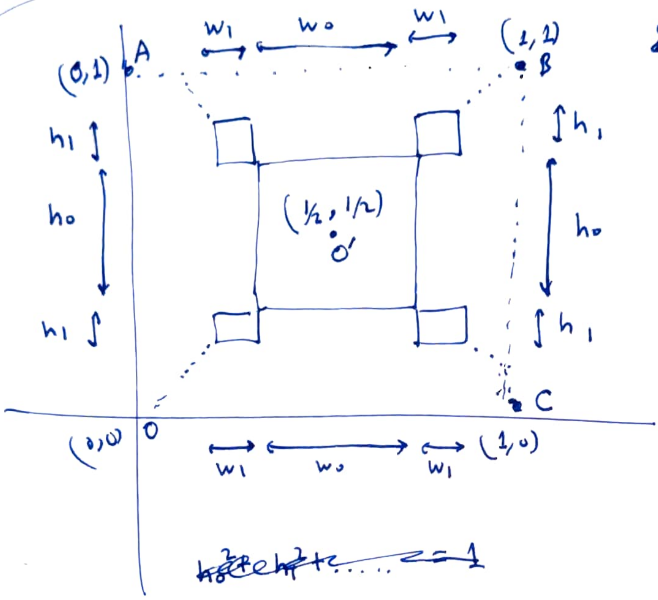
$$f_{X|Y}(x|Y=y) = \begin{cases} \text{not defined} & y < 0 \\ \frac{1}{1-y} & y \in (0,1),\ x \in (y,1) \\ \text{not defined} & y \geq 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Ans

$\boxed{7.}$



Ans



In ~~rectangle~~ square OABC of size 1×1 unit squares, any pdf distribution which is

1) symmetric about $\left(\frac{1}{2}, \frac{1}{2}\right)$.
2) NOT a single rectangle.
3) ~~But~~ Filled with a set of more than one (possibly infinite) rectangles which are touching together (to maintain continuity in both x and y) as well as are symmetrically placed around $\left(\frac{1}{2}, \frac{1}{2}\right)$ (to maintain zero covariance), ~~and~~

can qualify for a distribution ~~rv~~ in x, y s.t.

1) X and Y are uncorrelated.
2) X and Y are NOT independent.
3) $X \sim \text{unif}[0,1]$ and $Y \sim \text{unif}[0,1]$.

~~$h_0 + h_1 + h_2 + \ldots = 1$~~

~~$h_0 + h_1 + h_2 + \ldots = 1$~~

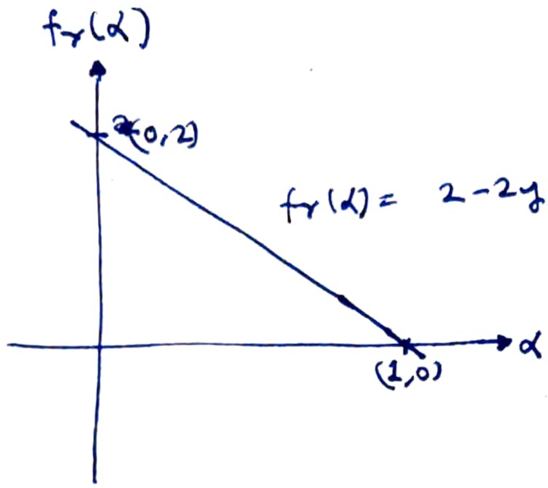
where $h_0 + h_1 + h_2 + \ldots = 1$

$w_0 + w_1 + w_2 + \ldots = 1$

$(\text{uniform pdf}) \times \{h_0 w_0 + h_1 w_1 + h_2 w_2 + \ldots\} = 1$

— X — X — X — X —

8.



8 (a) $\hat{Y}_{MP} = \arg\max_{\alpha} \left(f_Y(\alpha) \right) \Big|_{\alpha \in [0,1]}$

or $\hat{Y}_{MP} = \arg\max_{\alpha} \left(2-2\alpha \right) \Big|_{\alpha \in [0,1]}$

8(a)

or $\boxed{\hat{Y}_{MP} = 0}$ Ans for which $f_Y(\alpha=0) = 2$

Ans

8(b) $\hat{Y}_{MMSE} = \arg\min_{\alpha} \left[E\left[(Y-\alpha)^2 \right] \right]$

We know that $\hat{Y}_{MMSE} = \mu_Y = E(Y)$

$$\text{or } \hat{Y}_{MMSE} = \int_{y=0}^{y=1} y \cdot f_Y(y)\, dy$$

$$\text{or } \hat{y}_{MMSE} = \int_{y=0}^{y=1} y \cdot (2-2y)\, dy$$

$$\text{or } \hat{y}_{MMSE} = \int_{y=0}^{y=1} 2y - 2y^2 \; dy$$

$$\text{or } \hat{y}_{MMSE} = y^2 - \frac{2}{3}y^3 \Big|_{y=0}^{y=1}$$

8(b)

or $\boxed{\hat{y}_{MMSE} = \frac{1}{3}}$ Ans

PTO

8.2

None the less, let us derive $\hat{Y}_{MMSE}$ using only first principles.

$$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[E\left[(Y-\alpha)^2\right] \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[\int_{y=0}^{y=1} (y-\alpha)^2 \cdot (2-2y)\, dy \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2\int_{y=0}^{y=1} \{ y^2 - 2\alpha y + \alpha^2 \} \{1-y\}\, dy \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2\int_{y=0}^{y=1} \{ -y^3 + (1+2\alpha) y^2 - (\alpha^2 + 2\alpha) y + \alpha^2 \} dy \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2 \left[-\frac{y^4}{4} + \frac{(1+2\alpha)}{3} y^3 - \frac{(\alpha^2+2\alpha)}{2} y^2 + \alpha^2 y \right]_{y=0}^{y=1} \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2 \left[-\frac{1}{4} + \frac{(1+2\alpha)}{3} - \frac{(\alpha^2+2\alpha)}{2} + \alpha^2 \right] \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[2 \left[\frac{\alpha^2}{2} - \frac{1}{3}\alpha + \frac{1}{12} \right] \right]$$

or $$\hat{y}_{MMSE} = \underset{\alpha}{\arg\min} \left[\left(\alpha - \frac{1}{3}\right)^2 + \frac{1}{18} \right]$$

8(b)

or $$\boxed{\hat{y}_{MMSE} = \frac{1}{3} \quad \text{with error} = \frac{1}{18}}$$ Ans

8(c)

$$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(E(|y-\alpha|)\right) = \underset{\alpha}{\arg\min}\left(\int_{y=0}^{y=\alpha} (\alpha - y)(2-2y)\,dy + \int_{y=\alpha}^{y=1} (y-\alpha)(2-2y)\,dy \right)$$

$$\text{or } \hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(2\int_{y=0}^{y=\alpha} (\cancel{\alpha-}(y-\alpha)(y-1)\,dy + 2\int_{y=1}^{y=\alpha} (y-\alpha)(y-1)\,dy \right)$$

$$\text{or } \hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(2\int_{y=0}^{y=\alpha} \{y^2 - (\alpha+1)y + \alpha\}\,dy + 2\int_{y=1}^{y=\alpha} \{y^2 - (\alpha+1)y + \alpha\}\,dy \right)$$

$$\text{or } \hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\times\left[\frac{y^3}{3} - \frac{(\alpha+1)}{2}y^2 + \alpha y\right]\Bigg|_{y=\alpha} - 2\left[\frac{y^3}{3} - \frac{(\alpha+1)}{2}y^2 + \alpha y\right]\Bigg|_{y=1} - 2\left[\frac{y^3}{3} - \cancel{\frac{(\alpha+1)}{2}y^2} + \alpha y\right]\Bigg|_{y=0} \right)$$

$$\text{or } \hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\left[\frac{\alpha^3}{3} - \frac{(\alpha+1)}{2}\alpha^2 + \alpha^2\right] - 2\left[\frac{1}{3} - \frac{(\alpha+1)}{2} + \alpha\right] \right)$$

$$\text{or } \hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\left[-\frac{\alpha^3}{6} + \frac{\alpha^2}{2}\right] - 2\left[\frac{\alpha}{2} - \frac{1}{6}\right] \right)$$

$$\text{or } \hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left[-\frac{4}{6}\alpha^3 + 2\alpha^2 - \alpha + \frac{1}{3} \right]$$

$$\text{or } \hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(\frac{d}{d\alpha}\left[-\frac{2}{3}\alpha^3 + 2\alpha^2 - \alpha + \frac{1}{3}\right] = 0 \right)$$

$$\text{or } \hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(-2\alpha^2 + 4\alpha - 1 = 0 \right)$$

$$\text{or } \hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(-2(\alpha-1)^2 + 1 = 0 \right)$$

$$\text{or } \hat{y}_{MMAE} = \underset{\alpha}{\arg}\left((\alpha-1)^2 = \frac{1}{2} \right)$$

$$\text{or } \hat{y}_{MMAE} = 1 - \frac{1}{\sqrt{2}}$$

8(c) $\text{or } \boxed{\hat{y}_{MMAE} \approx 0.293}$ Ans