Distributed Computing for Scalable Optimal Power Flow in Large Radial Electric Power Distribution Systems with Distributed Energy Resources

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Abstract

The increasing penetrations of distributed energy resources (DERs) at the power distribution level augments the complexity of optimally operating the grid edge assets, primarily because of the nonlinearity and scale of the system. An alternative is to solve the relaxed convex or linear-approximated problem, but these methods lead to sub-optimal or power-flow infeasible solutions. This paper proposes a scalable and fast approach to solve the large nonlinear optimal power flow (OPF) problem using a developed distributed method. The full network-level OPF problem is decomposed into multiple smaller sub-problems that are easy to solve - the distributed method attains network-level optimal solutions upon consensus. This effective decomposition technique reduces the number of iterations required for a consensus by order of magnitude compared to traditional distributed algorithms. We demonstrate the proposed approach by solving different nonlinear OPF problems (different problem objectives) for a distribution system with more than fifty-thousands (50,000) problem variables.

Keywords: Distributed Control, Optimal Power Flow, Distributed Optimization, Power Distribution Systems.

1. Introduction

The nature and the requirements of the power systems, especially at the distribution level are changing rapidly with the large-scale integration of controllable distributed energy resources (DERs). The continued proliferation of DERs, which include Photovoltaic (PV) systems, battery energy storage units (BESS), and controllable loads such as Electric Vehicles (EVs) is leading to a drastic increase in the number of active nodes at the distribution level that need to be controlled/managed optimally for efficient and resilient grid operations. Traditionally, grid operations are centrally managed upon solving an optimal power flow (OPF) problem where centralized optimization techniques are used to solve the resulting difficult non-linear non-convex OPF problem [1], [2]. Unfortunately, the computational challenges, primarily posed by the non-convex power flow constraints in OPF formulation, increase drastically with the size of the distribution systems motivating computationally efficient approaches [3].

Existing methods manage the computational challenges using convex relaxation or linear approximation techniques [4], [5]. The primary drawbacks of (i) the relaxed models are the possibilities of inexact and/or infeasible power flow solutions [6], and (ii) the approximated models may lead to NLP infeasible solutions and high optimality gap depending upon the problem type [7]. Moreover, methods based on both approximation and relaxation techniques use a centralized paradigm that may lead to scalability challenges as the problem size increases. With a majority of DER integration happening at the secondary feeder level, the OPF problem will need to be solved for even larger feeders with thousands of secondaries. For example, the largest IEEE test feeder is an 8500 node test system that terminates at the secondary transformer level and does not include secondary feeders. If each service transformer is expanded to a 20 node secondary feeder, it will lead to a total of 22000 secondary nodes added to the problem formulation. Such problem complexities motivate the move towards a distributed computing and/or control paradigm. Fortunately, the radial operational topology of power distribution systems makes them highly conducive for parallelization and distributed computing. This paper develops a distributed computing approach for distribution-level OPF problems that can scale for very large distribution feeders and converge using fewer iterations among distributed computing nodes, thus significantly reducing the overall compute time.

Within this context, existing literature includes numerous approaches on the application of distributed

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optimization algorithms for power distribution systems [8], [9]. In general, these methods adopt the traditional distributed optimization techniques to model a distributed optimal power flow (D-OPF) problem [8]–[11]. A D-OPF formulation decomposes the OPF into several smaller subproblems that require multiple micro- and macro-iterations for convergence. Within each micro-iteration, the distributed sub-problems are solved in parallel. While, during macro-iterations, the solutions or more specifically the updated boundary variables obtained from the distributed subproblems are exchanged to enforce network-level consensus. Both micro and macro-iterations together decide the time-of-convergence for the algorithm. Unfortunately, the existing distributed optimization algorithms require a very large number of macro-iterations to converge for medium-scale distribution feeders [9], [12]–[14]. A practical implementation of such algorithms requires to solve distributed sub-problems a large number of times as well to reach a converged solution within a reasonable time. A large number of communication rounds/message-passing events among distributed computing agents is not preferred since this leads to significant delays in decision-making. Lately, to address some of these challenges, real-time feedback-based online distributed algorithms have been explored in the related literature for network optimization [15]–[20]. Generally, these algorithms do not wait to optimize for a time-step but asymptotically arrive at an optimal decision over several steps of real-time decision-making. However, these algorithms also take hundreds of iterations to track the optimal solution for a mid-size feeder. This raises further challenges to the performance of the algorithm for larger feeders, especially during the fast-varying phenomenon.

To address these challenges, we have developed a distributed OPF formulation for the radial distribution systems based on the equivalence of networks principle [21], [22]. In this paper, we test the performance of the distributed algorithm for scaled systems and various network-level objectives. Specifically, we have (i) proposed distributed computation method to solve OPF for very large notional distribution test feeders *(with 10,000 nodes & 50,000 problem variables)* where the centralized computation can solve for at max 20% of the feeders, and (ii) solved OPF for several different operational problem objectives with different control variables, that face different levels of challenges mathematically. In brief, we have tested the proposed D-OPF method for scalability while subjected to severe non-linearities in terms of problem objectives. The proposed approach solves the original non-convex OPF problem for power distribution systems using a novel decomposition technique combined with distributed computing approach. First, the low-compute distributed OPF sub-problems are locally solved. The consensus of the boundary variables is achieved using a Fixed-Point Iteration (FPI) algorithm. Upon consensus, the solutions converge to network-level OPF solutions. We demonstrate the proposed approach for three problem objectives *(1) loss minimization, (2) DER generation maximization, and (3) voltage deviation minimization* using a balanced synthetic 10,000 node distribution feeder and single-phase equivalent of 8500-node test feeder (with 2500 nodes). The proposed approach is shown to scale for all problem objectives while most centralized formulations can't be solved for more than 2000 nodes using off-the-shelf optimization solvers such as *Artelys Knitro*. To our knowledge, this is the first paper to demonstrate an approach that solves such a large-scale D-OPF on a regular CPU without the use of any high-performance computing (HPC) machines.

2. Centralized OPF Model

In this paper, $(\cdot)^*$ represents the complex-conjugate; $(\cdot)^T$ represents matrix transpose; $(\cdot)^n$ represents the n^{th} iteration; $\overline{(.)}$ and $\underline{(.)}$ denotes the maximum and minimum limit of a given quantity. Also for complex numbers, we denote $j = \sqrt{-1}$.

2.1. System Variable Definitions

Let us represent a balanced radial power distribution network by the directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ be the set of all nodes in the system and $\mathcal{E}$ denotes the set of all distribution lines connecting the pair of buses (i, j) i.e., from node i to node j. Also, $r_{ij} + jx_{ij}$ is the series impedance $\forall\{ij\} \in \mathcal{E}$. Let, for node j, k be the set of all children nodes. Next we denote $v_j = |V_j|^2 = V_j V_j^*$, as the squared magnitude of voltage at node j. Let l_{ij} be the squared magnitude of current flow in branch $\{ij\}$. We denote P_{ij}, Q_{ij} as the sending-end active and reactive power flows for branch ij, and complex power $p_{L_j} + jq_{L_j}$ is the load connected and $p_{Dj} + jq_{Dj}$ is the power output of DER connected at node j.

2.2. System Models

The network is modeled using the branch flow equations [23] defined for each line $\{ij\} \in \mathcal{E}$ and $\forall j \in \mathcal{N}$ in (1).

$$P_{ij} - r_{ij}l_{ij} - p_{L_j} + p_{Dj} = \sum_{k:j\to k} P_{jk} \tag{1a}$$

$$Q_{ij} - x_{ij}l_{ij} - q_{L_j} + q_{Dj} = \sum_{k:j\to k} Q_{jk} \tag{1b}$$

$$v_j = v_i - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) + (r_{ij}^2 + x_{ij}^2)l_{ij} \tag{1c}$$

$$v_i l_{ij} = P_{ij}^2 + Q_{ij}^2 \tag{1d}$$

The DERs are modeled as Photovoltaic modules (PVs) interfaced using smart inverters, capable of 2-quadrant operation. If the reactive power generation, q_{Dj}, is controllable and modeled as the decision variable for the optimal operation, then the real power generation by the DER, p_{Dj}, is assumed to be known(measured). Let the rating of the DER connected at node j be S_{DRj}, then the limits on q_{Dj} are given by (2).

$$-\sqrt{S_{DRj}^2 - p_{Dj}^2} \leq q_{Dj} \leq \sqrt{S_{DRj}^2 - p_{Dj}^2} \tag{2}$$

On the contrary, if the active power generation, p_{Dj}, is modeled as the decision variable, then q_{Dj} is set to 0, and p_{Dj} can vary between 0 and S_{DRj}, see (3).

$$0 \leq p_{Dj} \leq S_{DRj} \tag{3}$$

2.3. Centralized OPF problems

To optimize the network for some cost function, we define a centralized OPF (C-OPF) problem by (i) a network-level problem objective, (ii) the power flow models in equation (1), and (iii) the operating constraints on the power flow variables. In this paper, we formulate three different optimal power flow problems for the power distribution grids, *(1) active power loss minimization, (2) DER generation maximization, and (3) Voltage deviation (ΔV) minimization*. The corresponding OPF problems are detailed below.

2.3.1. Loss Minimization: The problem objective is to reduce the network losses by controlling the reactive power output from DERs (q_{Dj}). Let $X_{lm} = [P_{ij},\ Q_{ij},\ l_{ij},\ v_j,\ q_{Dj}]^T$ be the problem variables $\forall j \in \mathcal{N}$, and $\forall \{ij\} \in \mathcal{E}$. Note that, if node j doesn't have any DER, then $q_{D_j} = 0$. Also, let $F_{lm}(X_{lm})$ denote the objective function representing the total power loss in the given distribution system. Note that $F_{lm}(X_{lm})$ is a function of both the power flow variables and decision variables. Then, the OPF problem is defined as the following in **(C1)**.

$$\textbf{(C1)} \quad \min \quad F_{lm}(X_{lm}) = \sum_{\{ij\} \in \mathcal{E}} l_{ij} r_{ij} \tag{4a}$$

$$\text{s.t.} \quad (1) \text{ and } (2) \tag{4b}$$

$$\underline{V}^2 \leq v_j \leq \overline{V}^2 \qquad ;\forall j \in \mathcal{N} \tag{4c}$$

$$l_{ij} \leq \left(I_{ij}^{rated}\right)^2 \qquad ;\forall \{ij\} \in \mathcal{E} \tag{4d}$$

where, $\overline{V} = 1.05$ pu and $\underline{V} = 0.95$ pu are the limits on bus voltages, and $(I_{ij}^{rated})^2$ is the thermal limit for the branch $\{ij\}$.

2.3.2. DER Maximization: In the DER maximization problem objective, the DER active power generation is maximized without violating the operational limits of the distribution system. This is achieved by maximizing the active power output from DERs (p_{Dj}). Let $X_{dm} = [P_{ij},\ Q_{ij},\ l_{ij},\ v_j,\ p_{Dj}]^T$ be the problem variables. Here, the objective function is denoted by $F_{dm}(X_{dm})$, representing the total active power generation by DERs. Then, this DER maximization OPF problem is defined as the following in **(C2)**. Similar to the previous formulation, if any node j doesn't have any DER, then we set $p_{D_j} = 0$.

$$\textbf{(C2)} \quad \max \quad F_{dm}(X_{dm}) = \sum_{j \in \mathcal{N}} p_{Dj} \tag{5a}$$

$$\text{s.t.} \quad (1) \text{ and } (3) \tag{5b}$$

$$\underline{V}^2 \leq v_j \leq \overline{V}^2 \qquad ;\forall j \in \mathcal{N} \tag{5c}$$

$$l_{ij} \leq \left(I_{ij}^{rated}\right)^2 \qquad ;\forall \{ij\} \in \mathcal{E} \tag{5d}$$

2.3.3. Voltage Deviation Minimization: In this specific network-level optimization problem, we try to keep the nodal voltages as close as possible to a pre-specified reference, V_{ref}. The problem objective is to minimize the nodal voltage deviations from the reference value by controlling the reactive power output from DERs (q_{Dj}). The problem variables are denoted by $X_{dv} = [P_{ij},\ Q_{ij},\ l_{ij},\ v_j,\ q_{Dj}]^T, \forall j \in \mathcal{N}$ and $\forall \{ij\} \in \mathcal{E}$. Also, the cost function, $F_{dv}(X_{dv})$, represents the total two-norm distances of nodal voltages, v_j, from reference voltage v_{ref}. Mathematically $F_{dv}(X_{dv}) = \sqrt{\sum (v_j - v_{ref})^2},\ \forall j \in \mathcal{N}$. The OPF problem is defined as the following in **(C3)**. Here in this paper, we used $V_{ref} = 1.00$ pu as the bus reference voltage.

$$\textbf{(C3)} \quad \min \quad F_{dv}(X_{dv}) = \sqrt{\sum_{\forall j \in \mathcal{N}} (v_j - v_{ref})^2} \tag{6a}$$

$$\text{s.t.} \quad (1) \text{ and } (2) \tag{6b}$$

$$\underline{V}^2 \leq v_j \leq \overline{V}^2 \qquad ;\forall j \in \mathcal{N} \tag{6c}$$

$$l_{ij} \leq \left(I_{ij}^{rated}\right)^2 \qquad ;\forall \{ij\} \in \mathcal{E} \tag{6d}$$

Assumption 1: The loads in the network for all three OPFs are modeled as constant power loads; i.e., in *ZIP* load model, $(Z, I, P) = (0, 0, 1)$.

In the next section, we detail the method of how to decompose the optimization problems for large-scale distribution grids into several sub-problems, solve in parallel, and converge into the final solution.

3. Decomposition of the OPF Problem

The OPF problems described in the previous section are formulated as a centralized optimization problem for the radial power distribution systems. For a large scale distribution system with thousands of nodes and decision variables, solving the NLP OPF is computationally expensive and difficult to converge

for very large-scale distribution systems. Since the power distribution system is operated radially, the OPF problems defined in **(C1)-(C3)** are naturally decomposable into multiple sub-problems defined for the connected areas. The details of the proposed problem decomposition technique and the resulting distributed OPF problem are discussed next.

3.1. Decomposition Method

First, we decompose the whole distribution grid into N smaller areas. Let $\mathcal{A}_R = \{A_1,\ A_2, \ldots,\ A_N\}$, be the set of all decomposed areas. Also, let each area, $A_m \in \mathcal{A}_R$, be defined as a directed graph $A_m = \mathcal{G}(\mathcal{N}_m, \mathcal{E}_m)$. Here, each area A_m has a maximum number of nodes/variables, so that the respective OPF sub-problems for that area can easily be solved using off-the-shelf NLP solvers. The coupling/complicating variables among these smaller sub-problems, associated with respective areas, are defined based on the network topology. Since the grid was radial to begin with, the decomposed areas or the sub-trees of the networks are also connected radially with each other. This specific structure of the network helps to identify the unique parent area and the child areas for any area A_m, which in turn associates the complicating/shared variables that are exchanged among sub-problems to solve the overall master problem. For this decomposition method, the complicating variables are the shared bus voltages and power flows in the shared bus. Computationally, sub-problems associated with each area are solved in parallel by assuming a fixed voltage at the shared bus with the unique parent area, and constant loads at the shared buses with child areas. After solving the sub-problems, the respective complicating variables, i.e., the total power requirements in that area are shared with the sub-problem for the parent area and the shared bus voltages are shared with sub-problems associated with child areas; then the sub-problems are solved again with updated values. The step of exchanging the complicating variables is termed macro-iteration. This process is repeated until all the complicating/shared variables have converged.

The proposed decomposition approach is elaborated using a two-area system as an example. Let us assume the network is decomposed into 2 areas – area A_1 and A_2. Here area A_1 is the parent area of area A_2; each with their purely own local variables defined by x_1 and x_2. Let $\mathbf{Y} = [y_1, y_2]^T$ be the complicating variable that couples the sub-problems for the two areas. Here, y_1 and y_2 are the bus voltage magnitude (v) and the complex power flow through the bus ($S = P + jQ$) shared between A_1 and A_2, respectively; i.e., $[y_1, y_2]^T = [v, S]^T$. If the set of all local variables for A_1 and A_2 is denoted by X_1 and X_2, respectively, then $X_1 = \{x_1, y_1\}$ & $X_2 = \{x_2, y_2\}$. Let $X = X_1 \cup X_2$ be the set of all problem variables and $\mathcal{S}$ is the set of constraints for the overall centralized optimization problem. If F is a decomposable cost function, then the problem can be decomposed and written as (7), where, $\mathcal{S}_1$ and $\mathcal{S}_2$ are the set of constraints on local variables for decomposed area A_1 and A_2, respectively. Also, f_1, f_2 are the cost functions for the respective local sub-problems.

$$\min_{X \in \mathcal{S}} \ F(X) = \min_{X_1 \in \mathcal{S}_1, X_2 \in \mathcal{S}_2} \ f_1(X_1, y_2) + f_2(X_2, y_1) \quad (7)$$

The original problem defined in (7) can be readily decomposable into the following two sub-problems (see (8)), associated for respective decomposed areas; i.e., equation (8a) and (8b) for area A_1 and A_2, respectively.

$$\text{For } A_1 \quad : \min_{X_1 \in S_1} \quad f_1(X_1, y_2) \quad (8a)$$

$$\text{For } A_2 \quad : \min_{X_2 \in S_2} \quad f_2(X_2, y_1) \quad (8b)$$

Remark 1: Please note that the decomposition of the OPFs also works for any maximization problem, such as **(C2)**.

Remark 2: The decomposition method described here can easily be extended for a network, where a multiple area decomposition is required to make the individual sub-problems small enough to be solved efficiently. Similar to the 2-area distributed OPF, the optimization problem can be decomposed into several smaller sub-problems, each representing one decomposed area.

Remark 3: The decomposition approach can be further extended to nodal decompositions, where each node represents one area.

3.2. Consensus for the Decomposed Sub-problems

After decomposing the optimization problem into several smaller sub-problems, the proposed distributed algorithm solves the sub-problems individually to obtain respective local and complicating variables. Here, at each boundary among decomposed areas, the complicating variable y_2 and y_1 are kept fixed to solve sub-problem (8a) and sub-problem (8b), respectively. Then the solved y_1 by sub-problem (8a) and solved y_2 by sub-problem (8b) are exchanged again between areas. After each macro-iteration, the update step of complicated variable, $\mathbf{Y}$, is performed using Fixed Point Iteration (FPI) method, described by (9) for n^{th} macro-iteration, where $\alpha \in [0,\ \infty)$.

$$\mathbf{Y}^{(n)} := \frac{\mathbf{Y}^{(n)} + \alpha \mathbf{Y}^{(n-1)}}{1 + \alpha} \quad (9)$$

While $\alpha = 0$ signifies directly using the respective computed shared variable's value to approximate respective parent or children areas, $\alpha > 0$ represents a weighted approximation that helps to reduce oscillations

of the complicated variables, $\mathbf{Y}$, at the shared boundaries. Here, instead of a constant value, alpha can be made adaptive for better convergence in case of oscillation. The macro-iterations continue until the change in all complicating variables for all decomposed boundaries are within tolerance ϵ_{tol}, see (10).

$$\left|\mathbf{Y}^{(n)} - \mathbf{Y}^{(n-1)}\right| \leq \epsilon_{tol} \tag{10}$$

4. Distributed OPF for Scalability

We detail the proposed distributed approach to solve large-scale OPF problems for distribution networks using the aforementioned decomposition technique. First, we discuss the formulation of the sub-problems, and then we describe the proposed algorithm.

4.1. Distributed Sub-Problems

For a system decomposed into multiple areas, the sub-problems are defined for each $A_m \in \mathcal{A}_R$. While decomposing the network, it is ensured that the number of variables in each decomposed area does not exceed a certain number so that the local problems can be solved fast. Here, the power flow model is defined in (11b), and are used by the corresponding sub-problem for area A_m, defined $\forall j \in \mathcal{N}_m$ and $\forall\{ij\} \in \mathcal{E}_m$. Let Area A_m shares bus 'o' with its parent area, and C_h be the set of buses shared with its child areas. The sub-problems for *(1) loss minimization, (2) DER maximization, and (3) ΔV minimization* are detailed next. For these OPF objectives, we use the same OPF formulation as the central problem, but only define it for the respective area, A_m. Also, at $(n)^{th}$ iteration, the complicating variables are updated using (9) and approximates the parent area as a constant voltage source at bus 'o' (eq. (11c)) and child areas as constant power loads at the shared buses $k \in C_h$ (eq. (11d)-(11e)). The sub-problem for loss minimization is described below.

$$\textbf{(D1)} \quad \min \;\; f_m = \sum_{\{ij\}\in\mathcal{E}_m} l_{ij} r_{ij} \tag{11a}$$

$$\text{s.t. equation (1)} \tag{11b}$$

$$v_o = v'_o; \tag{11c}$$

$$p_{L_k} = P'_{kl} \;\; ; \forall k \in C_h, \;\; \{kl\} \in \mathcal{E}m_{Ch} \tag{11d}$$

$$q_{L_k} = Q'_{kl} \;\; ; \forall k \in C_h, \;\; \{kl\} \in \mathcal{E}m_{Ch} \tag{11e}$$

$$-\sqrt{S_{DRj}^2 - p_{Dj}^2} \leq q_{Dj} \leq \sqrt{S_{DRj}^2 - p_{Dj}^2} \tag{11f}$$

$$\underline{V}^2 \leq v_j \leq \overline{V}^2 \qquad ; \forall j \in \mathcal{N}_m \tag{11g}$$

$$l_{ij} \leq \left(I_{ij}^{rated}\right)^2 \qquad ; \forall\{ij\} \in \mathcal{E}_m \tag{11h}$$

Here, $\mathcal{E}_{m_{Ch}}$ represents the set of lines in the child areas of Area A_m. In this example, the bus voltage, v'_o, is obtained from the parent area based on its converged solutions for the previous $(n-1)^{th}$ micro-iteration. Similarly, $\forall k \in C_h$, & $\{kl\} \in \mathcal{E}_{m_{Ch}}$, P'_{kl},& Q'_{kl} is the solved branch flows obtained from the converged solutions of $(n-1)^{th}$ micro-iteration, executed by child areas of A_m. Note that, the symbol $(.)'$ depicts that the variable is solved by other neighboring areas. The sub-problems for DER maximization and ΔV minimization are also formulated ad defined in (12) and (13), respectively.

$$\textbf{(D2)} \quad \max \;\; f_m = \sum_{j\in\mathcal{N}_m} p_{D_j} \tag{12a}$$

$$\text{s.t. (11b) - (11e), (3), (11g) - (11h)} \tag{12b}$$

$$\textbf{(D3)} \quad \min \;\; f_m = \sqrt{\sum_{\forall j\in\mathcal{N}_m} (v_j - v_{ref})^2} \tag{13a}$$

$$\text{s.t. (11b) - (11e), (2), (11g) - (11h)} \tag{13b}$$

Algorithm 1: Distributed Algorithm for Scaled OPFs

1. Decompose the network into N areas, so that each area has a maximum specified node numbers
2. Initialize complicating variables, $\mathbf{Y}^0 \in \mathcal{S}$; error, $e = 1$; and macro-iteration count $n = 0$
3. If $|e| \leq \epsilon_{tol}$, stop iteration count, and go to step 10
4. Else, increase iteration count n: $n \leftarrow n + 1$
5. Solve Φ_m in parallel using **Algorithm 2**, for all decomposed areas A_m, where, Φ_m depicts the sub-problem –
   $\Phi_m : \; X_m^{(n)} := \underset{X_m \in S_m}{\text{argmin/argmax}} \;\; f_m\left(X_m, y_{m'}^{(n-1)}\right)$
6. Update all the complicating variables, $\mathbf{Y}$, using (9), where α can be constant or adaptive
7. Check residual vector $\mathcal{R}^{(n)} = \left[\;\; \mathbf{Y}^{(n)} - \mathbf{Y}^{(n-1)} \;\; \right]$
8. $e = \max |\mathcal{R}^{(n)}|$
9. Go to step 2
10. Return Global Minimizer:
    $X^* = \{X_m^{(n)} \mid m = 1, 2, ..., N\}$

4.2. Algorithm

For completeness, now we discuss the full distributed algorithm that decomposes the OPFs for large-scale power distribution systems and solves iteratively to reach the global solution. Here, we use the decomposition technique that we developed in Section 3, and solve sub-problems for different network level objectives, i.e., **(D1)**-**(D3)** until convergence. We use tolerance of $\epsilon_{tol} = 0.001$ to meet the convergence criterion. The algorithm is detailed in Algorithm 1. To better understand the distributed computing of the OPFs, the sub-routine in step 5 of Algorithm 1 is described in Algorithm 2.

Algorithm 2: Sub-routine to Solve Sub-Problems Φ_m at Step-5 of **Algorithm 1**

Sub-Problem : For decomposed area $A_m \in \mathcal{A}_R$

Macro-Iteration step : (n)

Complicated Variables : $y_{m'}^{(n-1)}$, variables that are used for coupling sub-problems

Optimization Variable : X_m

Steps :

1 Approximate the neighboring areas of A_m using complicating variables, shared by the neighbors; i.e., $y_{m'}^{(n-1)}$ is set to either constant voltage (if it has parent area), or constant loads (if it has child areas), or both – depending on the position of the area A_m (See equation (11c)-(11e))

2 Solve distributed sub-problems of minimizing or maximizing the decomposed cost function f_m, e.g., **(D1)**, **(D3)**, etc. using off-the-shelf NLP solvers

3 Store the local minimizer in the variable $X_m^{(n)}$

5. Result

The proposed approach is evaluated using a large, balanced synthetic 10,000 node distribution system and medium-size balanced IEEE-8500 node test system with 2500 nodes. All experiments are simulated in Matlab 2018b on a machine with 8GB memory and Core i7-8700 CPU @3.19 GHz. The NLP subproblems for the proposed distributed method are solved using Matlab's *fmincon* using *'sqp'* algorithm. However, given the NP-hard nature of the centralized OPF problems, to bench-mark against centralized OPF, we also use a commercial NLP solver *Artelys Knitro* with *'active-set'* algorithm that scales relatively well with the problem size [24]. Note that, the solution time in this paper includes (i) writing the problem (creating the matrices for OPF solvers), (ii) calling the solvers, and (iii) getting the solution from the solver (solver time).

5.1. Simulated System

The simulations are conducted using the following two test systems: (i) Synthetic 10,000 node distribution system with different DER penetration levels, and (ii) Balanced IEEE-8500 node test system with 100% DER penetration for nodal decomposition. The % penetration here implies the percentage of DER nodes relative to the total load nodes in the system. The synthetic 10,000 node system is shown in Fig. 1. This system is comprised of 1 main feeder, and 20 laterals, where each lateral supplies 20 neighborhoods. It is assumed that each neighborhood is comprised of 20 households. Thus, each lateral supplies a total of 400 houses. Also, in between 2 laterals, we assume 4 nodes in the main feeder that represent the distributed loads. Every load in this distribution network is set to consume a total of

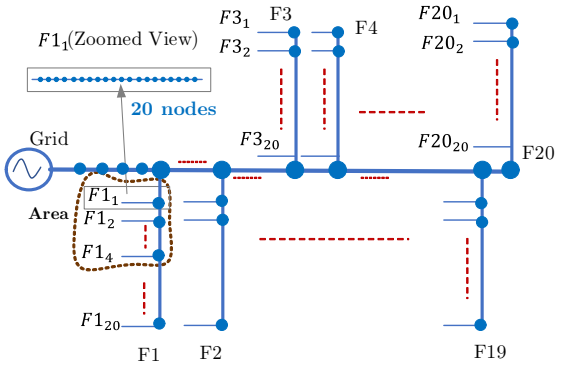


Figure 1: Synthetic 10,000 Node System

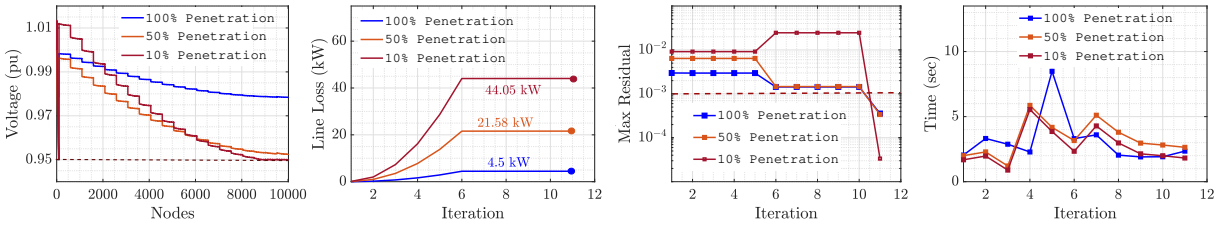
$S_L = 0.1 + 0.01j$ pu, and the line impedance of all the branches is assumed to be $z = 0.07 + 0.01j$ pu. The base voltage for the network is 12.47 kV (V_{LL}) and the base kVA is 1000. For loss minimization and ΔV minimization objectives, each DER in the system can generate 7 kW of real power, with a nominal rating of 8.4 kVA. For the DER maximization problem, the rating of the DERs is increased to at-most 10 times to stress the system. We decompose the distribution system in multiple areas where each area is composed of 100 nodes (see Fig. 2).

For the IEEE 8500 node test system, the DER sizes are chosen randomly with a rating ranging from 1.3 to 5.8 kVA. We use this medium-scale distribution system to further decompose the problem into the nodal level, i.e., each node is considered as an area. The simulated system is a balanced, single-phase equivalent distribution system of the original 8500-node test system with 2522 nodes. The base values are assumed to be the same with 12.47 kV and 1000 kVA. The proposed decomposition technique is then simulated for various DER penetration with different network objectives.

5.2. Loss Minimization Objective: (D1)

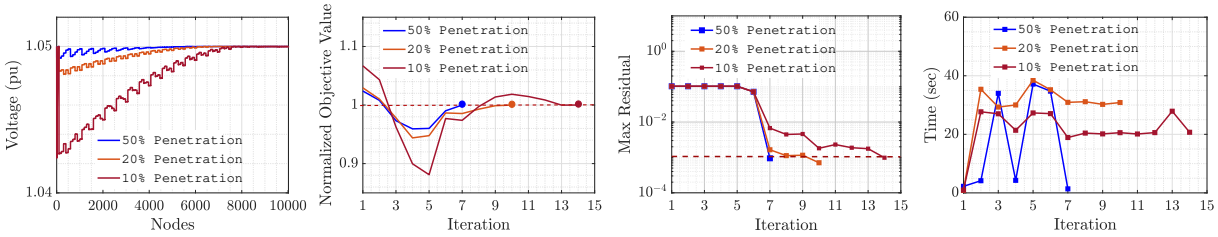
We solve the loss minimization problem **(D1)** for a 10,000 node test system with varying DER penetration levels. The reactive power from DERs is optimized to reduce the system power losses. All the loads are assumed at a nominal value (S_L). The kVA rating of the DERs is assumed to be 120% of their nominal active power generation rating. We have simulated (i) 100%, (ii) 50%, and (iii) 10% DER penetration levels for loss minimization objective. The results are detailed next. We have used $\alpha = 0$ for FPI update in (9).

The converged solution of the decomposed central OPF and the convergence properties of the proposed method for the loss minimization problem are shown in Fig. 2. We can see that the converged voltage does not violate any voltage constraints, i.e., the voltage is not outside of the specified limit of 0.95-1.05 pu bound



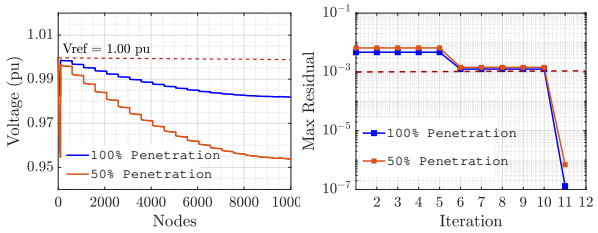
(a) Nodal voltages (b) Objective value (c) Convergence (d) Simulation time

Figure 2: Numerical Results for Loss Minimization Objective for Synthetic 10,000 Node System



(a) Nodal voltages (b) Objective value (c) Convergence (d) Simulation time

Figure 3: Numerical Results for DER Maximization Objective for Synthetic 10,000 Node System



(a) Nodal voltages (b) Convergence

Figure 4: Numerical Results for ΔV Minimization Objective for Synthetic 10,000 Node System

(Fig. 2a). Fig. 2b shows the change in the objective function value with macro-iterations. 100% penetration can reduce the line losses to 4.5 kW. The convergence properties for this case are shown in Fig. 2c. For all the cases, it takes around 11 macro-iterations to satisfy the convergence criterion. The time taken at each iteration for this case is also plotted in Fig. 2d. This time represents the highest time it takes to solve any sub-problem at each iteration. It only takes $\sim$ 30 seconds to solve the OPF by decomposing the problems into several sub-problems for all the DER penetration levels (see Table 1).

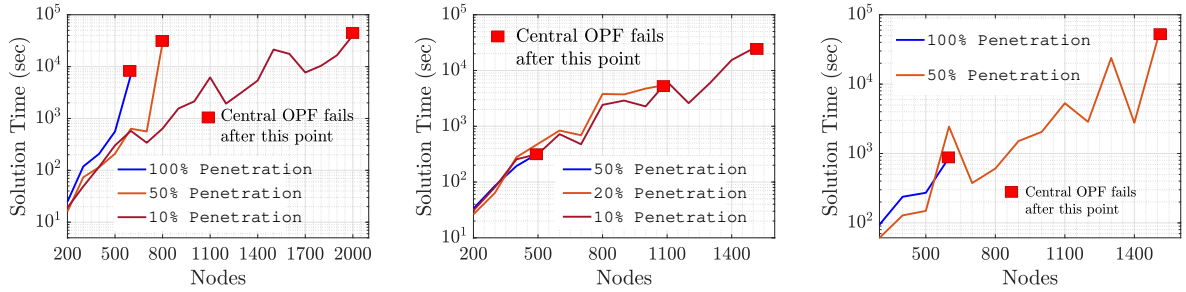
Table 1: Results Summary

Problem	DER%	Objective Value	Time (s)
Loss Min	100	4.5 kW	34
	50	21.58 kW	35
	10	44.05 kW	30
DER Max	50	1.03 MW (cap. 1.05 MW)	120
	20	0.55 MW (cap. 0.56 MW)	240
	10	0.66 MW (cap. 0.70 MW)	300
ΔV Min	100	2.65 pu	30
	50	6.68 pu	15

5.3. DER Maximization Objective: (D2)

In this section, we present the result for DER maximization OPF problem for the power distribution networks. Here, we solve the decomposed problem **(D2)** for a 10,000 node test system with different DER penetration levels. The active power generation of the DERs is maximized while maintaining the operation limits of the network, such as voltage limits. For this optimization problem, we have used various load and generation multipliers to stress the system. We simulate 3 different cases – (i) 50% DER penetrations where the active power generation capacities of DERs are 21 kW and loads are set to their nominal values, (ii) 20% DER penetrations with 28 kW of active power generation capacities for each DERs and load multiplier is set to 0.5, and (iii) 10% DER penetration levels with max of 70 kW generation capabilities and load multiplier set to 0.5. The result of this OPF is discussed next. We have used $\alpha = 2.33$ for FPI update in (9).

Fig. 3 shows the results for the DER maximization OPF. Similar to the previous objective, we can see that the nodal voltages do not violate the pre-specified voltage limits (Fig. 3a). The voltages are near their upper bound which implies that the systems were highly stressed for different simulated cases. With increased DER penetrations, more nodes have voltages that are closer to the upper limits of 1.05 pu. Fig. 3b shows the normalized objective value of the OPF problem w.r.t. macro-iterations. Here, the objective value is scaled w.r.t. the converged/final cost as the orders of the final costs are different. The actual values of the objective function upon solving OPFs using distributed approach



(a) Loss minimization objective (b) DER maximization objective (c) ΔV minimization objective

Figure 5: Solution Time for Central OPF (C-OPF) Problems for Different Sizes of Networks

is shown in Table 1. Note that, even though the number of DERs in 10% penetration case is lower than 20% case, individual DERs have higher capacity in 10% penetration case than later, and thus total generation is higher in 10% penetration case than 20% penetration case. It takes 7, 10 and 14 iterations to converge for 50, 20 and 10% DER penetrations (Fig. 3b, 3c). The simulation time per macro-iteration is shown in Fig. 3d. The solution time to for the OPF using the proposed decomposition approach is reported in Table 1.

5.4. ΔV Minimization Objective: (D3)

Next we detail the results for the ΔV minimization problem. Here, we solve the decomposed problem **(D3)** for a 10,000 node test system with different DER penetration levels, but with nominal values of DER generation and loads. The reactive power generation of the DERs are optimized for the given problem objective with a reference value of $V_{ref} = 1.00$ pu. For this optimization problem, we have simulated two different cases: (i) 100% DER penetrations and (ii) 50% DER penetrations. We have set $\alpha = 0$ for the FPI updates.

The results of the OPF are shown in Fig. 4, where Fig. 4a shows the nodal voltages after optimization. The higher penetrations of DERs result in a better network-level reduction in voltage deviation. Also, for both cases, it only takes 11 macro-iterations to reach convergence (Fig. 4b). The objective value and the solution time are shown in Table 1. As can be seen, the OPF problem converges within a reasonable time.

5.5. Failure of Central Solution

In this section, we solve the centralized version of the same OPF problem for increasing system size and demonstrate the scalability challenges. We also highlight the system size for which central OPF is unable to converge. It can be observed from Fig. 5 that with increasing DER penetration, it takes a higher time to solve the NLP OPFs. Also, for the same system size (node number), an OPF with higher DER penetration fails to converge sooner. For example, in the case of loss minimization, the NLP solver can solve the OPF problems for 800 nodes for a 50% DER penetration case (Fig. 5a). However, it can only solve for 600 nodes with 100% DER penetration. Similarly, for the DER maximization objective, with 20% DER penetration, the central problem can be solved for no more than 1100 nodes. On the contrary, for 50% DER penetration, central OPF fails to solve for more than 500 nodes. None of the OPFs can be solved using a centralized optimization technique for a distribution feeder with more than 2000 nodes. Kindly note that all the centralized optimization problems have been solved using *KNITRO* with the active-set algorithm that has shown better optimization performances than other solvers, such as, *fmincon* with different algorithms - *sqp*, *active-set*, and *interior-point-method*. Using different commercial solvers may lead to different convergence performances, however, all of them will still be challenged by the scale of the nonlinear optimization problem for modern power distribution systems with high penetrations of renewable energy resources.

Further, we have compared the objective values from converged centralized cases with proposed distributed solutions. For all the cases, the maximum difference of objective values is not more than 0.01 pu. For better demonstration, in this paper, we have shown objective function values for 500 and 1000 node systems with 50% and 10% PV penetration cases, respectively, in Table 2. From the table, it is clear that the D-OPF results match the centralized computation, validating the proposed method as well.

Table 2: Comparison with converged C-OPF

Problem	Node / DER %	C-OPF	D-OPF
Loss Min	500 / 50%	0.188 kW	0.184 kW
	1000 / 10%	0.840 kW	0.845 kW
DER Max	500 / 50%	50 kW	50.01 kW
	1000 / 10%	18.5 kW	18.2 kW
ΔV Min	500 / 50%	0.218 pu	0.210 pu
	1000 / 10%	0.614 pu	0.619 pu

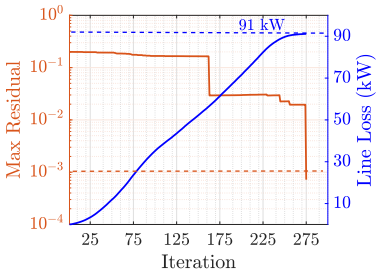


Figure 6: Result for Nodal Decomposition for IEEE-8500 Node Test System

5.6. Nodal Decomposition

To show the efficacy of the proposed decomposition and distributed computing approach, we further decompose the problem at the individual node level and solve the resulting OPF for the loss minimization objective for a balanced IEEE-8500 bus test system. While the commercial NLP solver, *KNITRO*, failed to optimize C-OPF, the nodal decomposition is able to solve the system with significant speed up. This method decomposes the OPF into 2522 sub-problems; each sub-problem needs to solve a nodal OPF with 5 variables. Overall network-level convergence took 275 macro-iterations, and each sub-problem took 480 microseconds (avg) to solve. Thus, the overall solution time for network-level optimization is relatively short, ~ 5 seconds. The convergence is shown in Fig. 6.

5.7. Three-Phase Unbalanced Network

Now we extend our proposed decomposition-based method for 3-phase unbalanced distribution systems. The IEEE-123 bus system has been used (decomposed into 4 sections) as a test system with randomly placed 50 DERs in the network with (i) 40 kW active power generation (Rated 50 kVar) for Loss minimization & ΔV minimization OPF, and (ii) up to 100 kW capacity For DER maximization OPF. As a power flow model, we have used the equations developed in [3]. The result is shown in Table 3. It only takes 6-9 macro-iteration for the proposed method and the results match with the centralized computation. For example, the line loss is 25.1 kW using the D-OPF method, and the C-OPF solution is 23.2 kW. Implementing the method for a larger unbalanced system with binary variables is a part of our future work.

Table 3: Result for Unbalanced Network

Problem	C-OPF	D-OPF	Iteration
Loss Min	23.2 kW	25.1 kW	6
DER Max	4.56 MW	4.58 MW	6
ΔV Min	7.30 pu	7.63 pu	9

6. Conclusion

This paper presents a scalable distributed computing algorithm to solve non-linear OPF problems for power distribution systems that scale well for all general classes of distributed OPF problems. The proposed distributed approach converges within a short time for large feeders even when the centralized OPF takes a significant amount of time or fails to converge. We have demonstrated the successful application of the proposed approach for a synthetic 10,000-node distribution test system with a total of $\sim 50,000$ variables on a regular CPU. All OPF problems are shown to converge within a reasonable time. To the best of our knowledge, this is the first work to demonstrate the application of distributed algorithms to solve the OPF problem for large distribution feeders without requiring HPC machines. It should be noted that the proposed decomposition is amenable to implementation on many-core machines. Moreover, the fast convergence and fewer communication requirements among decomposed problems demonstrated using several case studies further make the algorithm appealing for a distributed implementation. Although the convergence of the proposed method is not guaranteed theoretically, it is observed in all our simulation cases, even for a 3-phase unbalanced system. The convergence properties of the proposed algorithms are being thoroughly examined as part of ongoing research. Additionally, as part of our ongoing research, we will adapt the proposed algorithms to meshed systems, OPFs with binary variables, and large-scale real-world feeders.

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