Radial Distribution Load Flow Using Conic Programming

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***Abstract*—This paper shows that the load flow problem of a radial distribution system can be modeled as a convex optimization problem, particularly a conic program. The implications of the conic programming formulation are threefold. First, the solution of the distribution load flow problem can be obtained in polynomial time using interior-point methods. Second, numerical ill-conditioning can be automatically alleviated by the use of scaling in the interior-point algorithm. Third, the conic formulation facilitates the inclusion of the distribution power flow equations in radial system optimization problems. A state-of-the-art implementation of an interior-point method for conic programming is used to obtain the solution of nine different distribution systems. Comparisons are carried out with a previously published radial load flow program by R. Cespedes.**

***Index Terms*—Load flow control, nonlinear programming, optimization methods.**

I. Introduction

THE LOAD flow program is an essential tool for the efficient operation and control of power distribution networks. The distribution systems are characterized by their prevailing radial nature and high R/X ratio. This renders the load flow problem ill-conditioned. Early research indicated that standard load flow methods fail to converge for ill-conditioned test systems [1]. Methods for radial distribution have therefore been predominantly based on forward/backward sweeping of the network tree representation [2]. An efficient radial load flow technique that employs in the forward sweep the solution to a biquadratic equation was developed by Cespedes [3]. The biquadratic equation in [3] relates the voltage magnitudes at the sending and receiving ends of each branch to the power flow at the receiving end. Cespedes' approach has been accepted by many power system researchers. In fact, minor variants of this method have been applied in [4] and [5]. Moreover, Cespedes' load flow has been generalized in [6] to cater for exponential load models. It has been also applied in voltage stability analysis of radial distribution networks [7].

This letter presents a radial load flow solution based on conic programming. Section II formulates the load flow problem as a set of linear constraints and active rotated quadratic cones. The formulation as a conic program together with a summary of numerical results is given in Section III. This paper is concluded in Section IV.

II. Distribution Load Flow

Consider the single-line equivalent circuit shown in Fig. 1 (all relevant quantities are in per-unit). The line model without shunt connections is sufficient to describe fully the concepts in this paper.

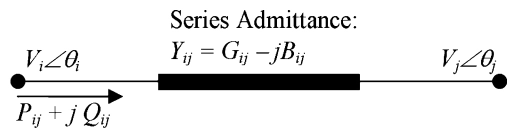

Fig. 1. Distribution line model.

The real/reactive power flows from node i to node j are

$$P_{ij} = G_{ij}V_i^2 - G_{ij}V_iV_j\cos\theta_{ij} + B_{ij}V_iV_j\sin\theta_{ij}, \quad (1)$$
$$Q_{ij} = B_{ij}V_i^2 - B_{ij}V_iV_j\cos\theta_{ij} - G_{ij}V_iV_j\sin\theta_{ij}, \quad (2)$$

where $\theta_{ij} = \theta_i - \theta_j$. By defining $u_i = V_i^2/\sqrt{2}$, $R_{ij} = V_iV_j\cos\theta_{ij}$, and $I_{ij} = V_iV_j\sin\theta_{ij}$, (1) and (2) become

$$P_{ij} = \sqrt{2}G_{ij}u_i - G_{ij}R_{ij} + B_{ij}I_{ij}, \quad (3)$$
$$Q_{ij} = \sqrt{2}B_{ij}u_i - B_{ij}R_{ij} - G_{ij}I_{ij}. \quad (4)$$

In (3) and (4), R_{ij} and I_{ij} are constrained such that

$$2u_iu_j = R_{ij}^2 + I_{ij}^2. \quad (5)$$

Equations (3)–(5) can be used to define the radial load flow problem. Let N be the number of nodes in the distribution system with node 1 being connected to the power substation. It is assumed that the magnitude of the voltage at this node is specified. The power injection constraints at each of the $(N-1)$ remaining nodes $(i = 2, \ldots, N)$ are

$$-\sum_{j\in k(i)} P_{ij} = -\sqrt{2}u_i\sum_{j\in k(i)} G_{ij} + \sum_{j\in k(i)} (G_{ij}R_{ij} - B_{ij}I_{ij}) = P_{Li}, \quad (6)$$
$$-\sum_{j\in k(i)} Q_{ij} = -\sqrt{2}u_i\sum_{j\in k(i)} B_{ij} + \sum_{j\in k(i)} (B_{ij}R_{ij} + G_{ij}I_{ij}) = Q_{Li}. \quad (7)$$

In (6) and (7), $k(i)$ denotes the set of nodes connected to node i, and P_{Li}/Q_{Li} denote the real/reactive power loads at node i. Equations (6) and (7), when evaluated for $i = 2, \ldots, N$, define a linear system with $2\times(N-1)$ equations. Note that the number of lines in a radial network is $(N-1)$. Because $R_{ij} = R_{ji}$ and $I_{ij} = -I_{ji}$, the total number of variables is $3\times(N-1)$. Therefore, $(N-1)$ additional equations are required to solve for the variables. These equations result from (5) enforced for all ij lines.

The system of equations defined by (5)–(7) can be solved to obtain the radial load flow solution. The original variables can be easily deduced once the new adopted variables are computed. In fact, Expósito and Ramos [8] have proposed a radial load flow technique based on solving a very similar system of equations using the Newton approach. In the accompanying discussion of

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[8], the authors pointed out that when using single precision arithmetic, the ill-conditioning resulting from the existence of very short line sections along with longer ones can be alleviated by the use of block arithmetic and symbolic computations. For an increase in the numerical reliability of the method, the authors in [8] strongly recommend the use of double precision arithmetic.

III. Conic Programming Formulation

The radial load flow solution satisfying (5)–(7) can be obtained by solving the following second-order cone program:

$$\text{maximize} \sum_{ij \text{ lines}} R_{ij} \text{ subject to}$$

$$\text{equations (6) and (7) for } i = 2, \ldots N \tag{8}$$

$$2u_i u_j \geq R_{ij}^2 + I_{ij}^2 \text{ for all } ij \text{ lines} \tag{9}$$

$$u_1 = V_1^2/\sqrt{2},\ u_i \geq 0 \text{ for } i = 2, \ldots N \tag{10}$$

$$R_{ij} \geq 0 \text{ for all } ij \text{ lines.} \tag{11}$$

The above optimization problem calls for increasing the values of R_{ij} until all inequality constraints in (9) become active. It should be noted that because the load flow solution of a practical radial distribution system is unique [7], the solution to the optimization problem converges to the true load flow solution. This has been numerically confirmed on nine different distribution systems.

The second-order cone program differs from the standard linear program by the inclusion of rotated quadratic cones (9). Nevertheless, it has been shown that primal-dual interior-point methods developed for linear optimization can be generalized to the conic quadratic case while maintaining their efficiency [9]. In this letter, the conic programs were solved using the MOSEK implementation [10], which is based on one of the best known theoretical algorithms. Note that MOSEK requires introducing additional variables and linear constraints such that no variable belongs to two different cones. On the positive side, the choice of a commercial solver implies that one can benefit from the time and effort that has been put into the development of high-powered software tools. In particular, MOSEK insures accurate results, even in the presence of ill-conditioned problems, by scaling the problem appropriately so that the search direction is well defined [9]. This has been confirmed by comparing the load flow results obtained from solving the conic optimization problem with those computed using an implementation of the Cespedes radial load flow [3]. Table I shows a summary of the results obtained for several ill-conditioned radial networks gathered from the literature. The fourth column of Table I gives the maximum voltage difference between the conic optimization solution and Cespedes' radial load flow solution (the error values designated by an asterisk sign are in p.u.). It is evident that for practical purposes, the solutions are almost identical.

The computational effort in columns 5 and 6 of Table I was obtained with MOSEK running on a Pentium IV, 1.89-GHz PC, with 256 Mbytes of RAM. The termination criteria were set to their default values [10]. In all cases, the CPU time was less than 0.3 s. As the optimizer technology improves, solution time would continue to improve.

TABLE I
Solution Statistics

N	Data from Ref.	Base voltage (kV)	Error (V)	MOSEK time (s.)	MOSEK iterations
12	[4]	11	7.30E-4	0.16	16
15	[5]	11	1.42E-3	0.16	15
28	[4]	11	1.93E-3	0.20	21
30	[3]	-	1.31E-6*	0.21	22
30	[6]	-	1.47E-7*	0.19	18
33	[2]	12.66	4.44E-3	0.20	19
43	[1]	-	4.23E-7*	0.24	29
69	[7]	12.66	2.80E-2	0.26	30
85	[5]	11	6.83E-2	0.29	27

In case the radial load flow problem has no solution, the conic optimization software gives a reliable certificate of infeasibility [9]. This is unlike other radial load flow algorithms ([1], [3], and [8], for example), which would diverge and terminate because of exceeding the maximum allowed number of iterations. In addition, the proposed implementation of the distribution load flow is more amenable to integration within an optimization function, such as optimal capacitor placement for loss reduction.

IV. Conclusion

This letter has shown that the radial distribution load flow can be formulated as a conic quadratic optimization problem. Such problems can be solved efficiently using interior-point methods [9]. Numerical results indicate that even ill-conditioned systems can be solved by a commercial interior-point conic optimizer. The formulation can be extended to unbalanced three-phase networks.

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