13 Oct 2022

T

6PM to 9PM

EE 507 midterm 1

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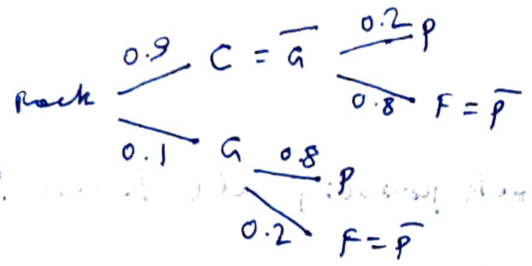
1.2

1.



(a) $P(G|P) = \dfrac{P(P|G) \cdot P(G)}{P(P)}$

or $P(G|P) = \dfrac{(0.8)(0.1)}{P(P|G) \cdot P(G) + P(P|\bar{G}) \cdot P(\bar{G})}$

or $P(G|P) = \dfrac{(0.8)(0.1)}{\underbrace{(0.8)(0.1) + (0.2)(0.9)}_{P(P) = 0.26}}$

1 (a)

or $\boxed{P(G|P) \approx 0.3077}$ Ans

(b) $P(\bar{P}_1 \cdot P_2\ P_3 |_G) = P(P_1|_G) \cdot P(P_2|_G) \cdot P(P_3|_G)$ (Independent Repeated trials). for given rock

$P(P3|G) = [P(P|G)]^3$

or $P(P_1 P_2 P_3) = \{P(P)\}^3$

$P(G|P_1 P_2 P_3) = \dfrac{\overset{P(P3)|G) \cdot P(G)}{\overset{P(P_3|G)}{\cancel{P(P_1P_2P_3|G)}}} \cdot P(G)}{\cancel{P(P_1P_2P_3)}\ P(P_3)}$

$P(G|P_3)$

or $P(G|P_1 P_2 P_3) = \dfrac{\{P(P|G)\}^3 \cdot P(G)}{\cancel{\{P(P)\}^3}\ P(P3|G) \cdot P(G) + P(P3|C) \cdot P(C)}$

~~or $P(G \mid P_1 P_2 P_3) = \dfrac{(0.8)^3 (0.1)}{\{0.8 \times 0.1 + 0.2 \times 0.9\}^3}$~~

~~or $P(G \mid P_1 P_2 P_3) =$~~

(b) Let event P3 denote a work passing all three tests.

given $P(P3 \mid G) = [P(P \mid G)]^3$

and $P(P3 \mid C) = [P(P \mid C)]^3$

$$P(G \mid P3) = \frac{P(P3 \mid G) \cdot P(G)}{P(P3 \mid G) \cdot P(G) + P(P3 \mid C) \cdot P(C)}$$

or $$P(G \mid P3) = \frac{[P(P \mid G)]^3 \cdot P(G)}{[P(P \mid G)]^3 \cdot P(G) + [P(P \mid C)]^3 \cdot P(C)}$$

or $$P(G \mid P3) = \frac{(0.8)^3 (0.1)}{(0.8)^3 (0.1) + (0.2)^3 (0.9)}$$

1(b)

or $\boxed{P(G \mid P3) = 0.8767}$ Ans

$\boxed{2.}$ (a) Given $\boxed{A+B+C=\Omega}$ ①

TPT $\boxed{P(A)+P(B)+P(C)+P(\overline{AB+AC}) \geq 2}$ (P)

Starting with the LHS of (P):

$LHS = P(A)+P(B)+P(C)+P[(\overline{AB}).(\overline{AC})]$ (Using Duality principle).

or $LHS = P(A)+P(B)+P(C)+P[(\bar{A}+\bar{B}).(\bar{A}+\bar{C})]$

or $LHS = P(A)+P(B)+P(C)+P[\bar{A}+\bar{A}\bar{B}+\bar{A}\bar{C}+\bar{B}\bar{C}]$

or $LHS = P(A)+P(B)+P(C)+P[\bar{A}(1+\bar{B}+\bar{C})+\bar{B}\bar{C}]$

or $LHS = P(A)+P(B)+P(C)+P[\bar{A}+\bar{B}\bar{C}]$

or $LHS = P(A)+P(B)+P(C)+P[\bar{A}+A\bar{B}\bar{C}]$

or $LHS = P(A)+P(B)+P(C)+P(\bar{A})+P(A\bar{B}\bar{C})$ (split using Axiom 3)

or $LHS = 1+P(B)+P(C)+P(A\bar{B}\bar{C})$

or $LHS = 1+P(B)+P(C)+P[A(\overline{B+C})]$ (using Duality)

or $LHS = 1+\cancel{P(B)+P(C)}+$ $P(B+C)+P(BC)+P[A(\overline{B+C})]$

or $LHS = 1+P[A(B+C)+\bar{A}(B+C)]+P(BC)+P[A(\overline{B+C})]$

or $LHS = 1+P[A(B+C)]+P(\bar{A}(B+C))+P[BC]+P[A(\overline{B+C})]$ (Split using Axiom 3).

or $LHS = 1+P(A)+P(\bar{A}(B+C))+P(BC)$

or $LHS = 1+P(A+\bar{A}(B+C))+P(BC)$ (Corollary of Axiom 3).

or $LHS = 1+P(A+B+C)+P(BC) \Rightarrow$ $\boxed{LHS = 2+P(BC) \geq 2}$ Ans Hence Proved!

2 (b)

Given $P(D) = 0.6$

$P(E) = 0.7$

$P(D+E) = 0.8$

$$P(D+E) = P(D) + P(E) - P(DE)$$

or ~~P(D+E) =~~

or $0.8 = 0.6 + 0.7 - P(DE)$

or $\boxed{P(DE) = 0.5}$

$\boxed{P(D).P(E) = 0.42}$

$\because P(DE) > P(D).P(E)$

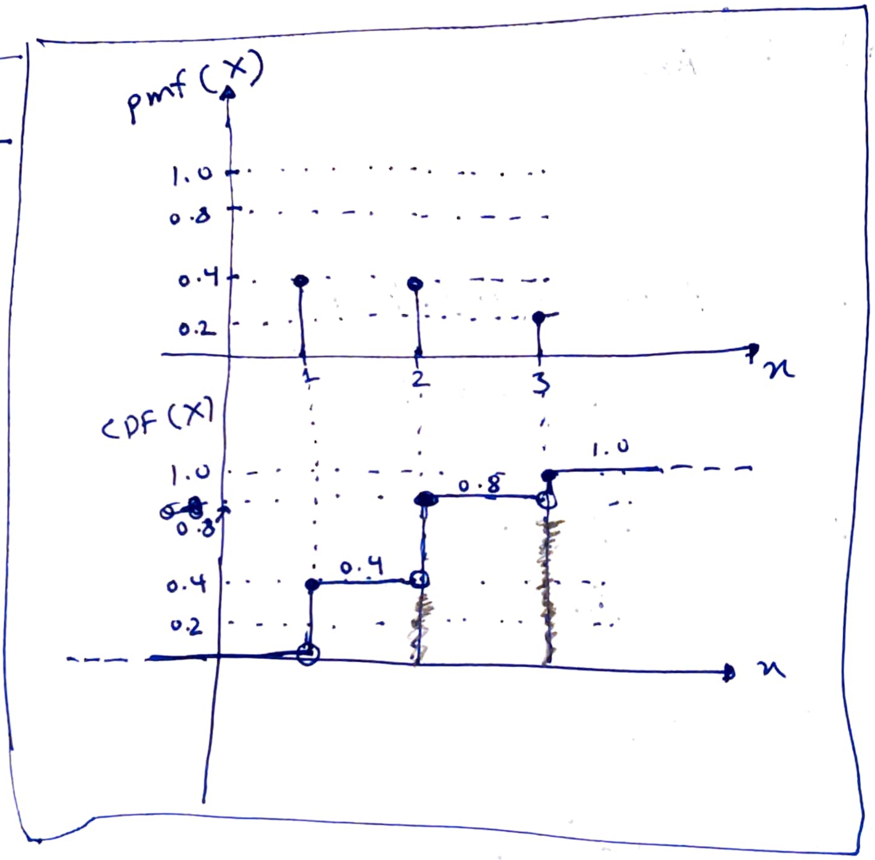
$\therefore$ ~~D and E are positively independent~~

$\boxed{2(b)}$ $\therefore$ **D and E are positively dependent.** Ans

3.

X_i	constituent elements.	~~CDF(Xi)~~ PMF(Xi)	CDF(Xi)
1	a, e	~~P({a,e}) = 0.2~~ P({a,e}) = 0.4	0.4
2	b, d	P({b,d}) = 0.4	0.8
3	c	P({c}) = 0.2	1.0

3(a)



Ans

(b) $P(A \mid X \leq 2)$ But $X \leq 2 \equiv \{a, b, d, e\}$

So $P(\{a,b,d\} \mid \{a,b,d,e\}) = 1$

3(b) $\boxed{P(A \mid X \leq 2) = 1}$ Ans

Note: Solving for 3(c) first, then 3(b):

$$\cancel{P(X \le 2 \mid A) = \frac{P(A \mid X \le 2) \cdot P(X \le 2)}{P(A)}}$$

$\cancel{P(X \le 2 \mid A) =}$

(c) $P(X \le 2 \mid A) = P(\{a,b,d,e\} \mid \{a,b,d\}) = 1$

3(c)

$$\boxed{P(X \le 2 \mid A) = 1}$$ Ans

$$P(A \mid X \le 2) = \frac{P(X \le 2 \mid A) \cdot P(A)}{P(X \le 2)}$$

$$P(A \mid X \le 2) = \frac{(1)(0.6)}{CDF(X=2)}$$

$$P(A \mid X \le 2) = \frac{0.6}{0.8}$$

3(b)

$$\boxed{P(A \mid X \le 2) = 0.75}$$ Ans

—— X —— X —— X ——

4.

$$f_X(x) = \begin{cases} c\,e^{-x} & \text{for } x \in [0, T) \\ 0 & \text{else.} \end{cases}$$

(a) $\displaystyle\int_{x=0}^{x=T} c\,e^{-x}\,dx = 1$ (Area under pdf = 1)

or $\displaystyle c\,e^{-x}\Big|_{x=T}^{x=0} = 1$

or $c(1 - e^{-T}) = 1$

or 4(a) $\boxed{c = \dfrac{1}{1 - e^{-T}}}$ Ans

(b) $\displaystyle F_X(x=\alpha) = \int_{x=0}^{x=\alpha} c\,e^{-x}\,dx \quad , x \in [0, T)$

or $F_X(x=\alpha) = c\,(1 - e^{-\alpha}) \quad x \in [0, T)$

4(b)

$$\boxed{F_X(x=\alpha) = \begin{cases} 0 & \alpha < 0 \\ \dfrac{1-e^{-\alpha}}{1-e^{-T}} & \alpha \in [0, T) \\ 1 & \alpha > T \end{cases}}$$

Ans

$$E(X) = \int_{x=0}^{x=T} f_X(x)\cdot x \, dx$$

$$\Rightarrow E(X) = c\int_{x=0}^{x=T} \underset{\mathrm{I}}{x}\, \underset{\mathrm{II}}{e^{-x}} \, dx$$

$$\Rightarrow E(X) = c\left[-xe^{-x}\Big|_{x=0}^{x=T} - \int_{x=0}^{x=T} -1\cdot e^{-x}\, dx \right]$$

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right)$$

$$f(x)\cdot g(x)$$

$$\Rightarrow E(X) = c\left[xe^{-x}\Big|_{x=T}^{x=0} + \int_{x=0}^{x=T} e^{-x}\, dx \right]$$

$$xe^{-x} - \cancel{e^{-x}}$$

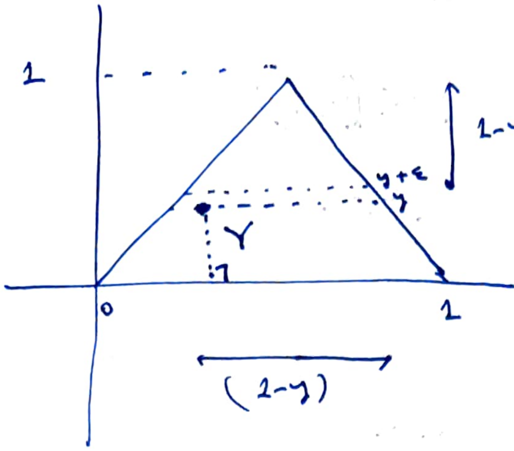
$$\Rightarrow E(X) = c\left[-Te^{-T} + (1-e^{-T}) \right]$$

4(c)

$$\Rightarrow \boxed{E(X) = \frac{1-(1+T)e^{-T}}{1-e^{-T}}}$$

Ans

5.)



$1-y$ $\quad F_Y(y=\alpha) = \begin{cases} 0 & \alpha < 0 \\ 2\left[\frac{1}{2} - \frac{1}{2}(1-\alpha)^2\right] & \alpha \in (0,1). \\ 1 & \alpha \geq 1 \end{cases}$

$x \in [0,1]$

$2 \times$ Area of $\triangle$ from $y=0$ to $y=\alpha$.

$$\Rightarrow F_Y(y=\alpha) = \begin{cases} 0 & \alpha < 0 \\ 1-(1-\alpha)^2 & \alpha \in [0,1) \\ 1 & \alpha \geq 1 \end{cases}$$

5 (a)

$$\boxed{f_Y(y=\alpha) = \begin{cases} 0 & \alpha < 0 \\ 2(1-\alpha) & \alpha \in [0,1) \\ 0 & \alpha \geq 1 \end{cases}}$$

Ans

(b) $\quad Z = \sqrt{Y} \qquad Z \in [0,1]$

$F_Z(z) = P(Z \leq \alpha)$

$\Rightarrow F_Z(z) = P(\sqrt{Y} \leq \alpha)$

$\Rightarrow F_Z(z) = P(Y \leq \alpha^2) = F_Y(y=\alpha^2)$

5.2

$$F_Z(z \leq \alpha) = \begin{cases} 0 & \alpha^2 < 0 \\ 1-(1-\alpha^2)^2 & \alpha^2 \in [0,1) \\ 1 & \alpha^2 > 1 \end{cases}$$

$\Rightarrow$ ~~$F_Z(z=\alpha)$~~

$$F_Z(z=\alpha) = \begin{cases} 0 & \alpha < 0 \\ 1-(1-\alpha^2)^2 & \alpha \in [0,1) \\ 1 & \alpha > 1 \end{cases}$$

5(b)

$$\Rightarrow \boxed{f_Z(z=\alpha) = \begin{cases} 0 & \alpha < 0 \\ 2\alpha(1-\alpha^2) & \alpha \in [0,1) \\ 0 & \alpha > 1 \end{cases}}$$

Ans

— X — X —

6.1

6.

Coin toss

0.5 H — $f_{X|H}(x|H) = \mathcal{N}(\mu=0, \sigma^2=4)$ $\forall x \in R$

0.5 T — $f_{X|T}(x|T) = \text{unif}[0,2)$. $x \in [0,2)$

(a) $f_X(x) = f_{X|H}(x|H)\cdot P(H) + f_{X|T}(x|T).P(T)$ [LoTP]

or $f_X(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} \cdot \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) & x \in [0,2) \\ \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2}\left(\frac{1}{2}\right) & x \text{ else} \end{cases}$

6(a)

$$\boxed{\text{or } f_X(x) = \begin{cases} \frac{1}{4\sqrt{2\pi}} e^{-\left(\frac{x}{2}\right)^2} + \frac{1}{4} & x \in [0,2) \\ \frac{1}{4\sqrt{2\pi}} e^{-\left(\frac{x}{2}\right)^2} & \text{else} \end{cases}}$$

Ans

(b) $P(1 \le X \le 3) = P(1 \le X \le 2) + P(2 \le X \le 3)$

~~or $P(1 \le X \le 3) = \{F_X(x=2) - F_X(x \ge 1)\} + \{F_X(x=3) - F_X(x=2)\}$~~

or $P(1 \le X \le 3) = \left[\frac{1}{2}\left\{G\left(\frac{2-0}{2}\right) - G\left(\frac{1-0}{2}\right)\right\} + \frac{1}{2}\left\{\frac{1}{2}(1-0)\right\}\right]$

$+ \left[\frac{1}{2}\left\{G\left(\frac{3-0}{2}\right) - G\left(\frac{2-0}{2}\right)\right\}\right]$

6(b)

$$\Rightarrow \boxed{P(1 \le X \le 3) = \frac{1}{2}\left[G(1.5) - G(0.5)\right] + \frac{1}{4}} \text{ Ans}$$

$\cancel{\Rightarrow P(1 \le X \le 3) =}$

(c)

$$P(H \mid X=1) = \frac{f(X=1 \mid H) \cdot P(H)}{f(X=1)}$$

$$\Rightarrow P(H \mid X=1) = \frac{f_X(x=1 \mid H) \cdot P(H)}{f_X(x=1)}$$

$$\Rightarrow P(H \mid X=1) = \frac{f_X(x=1 \mid H) \cdot P(H)}{f_X(x=1 \mid H) \cdot P(H) + f_X(x=1 \mid T) \cdot P(T)}$$

$$\Rightarrow P(H \mid X=1) = \frac{\left(\frac{1}{2\sqrt{2\pi}} e^{-\left(\frac{1}{2}\right)^2}\right) \cdot \left(\frac{1}{2}\right)}{\left(\frac{1}{2\sqrt{2\pi}} e^{-\left(\frac{1}{2}\right)^2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$

~~6(c)~~

$\Rightarrow \cancel{P(H \mid X=1) = 0.2370}$ ~~Ans~~

6(c) $\boxed{P(H \mid X=1) = 0.2370}$ Ans

$\boxed{7.}$ given $p_X(x) = \frac{x}{21}$ $x = 1, 2, 3, 4, 5, 6.$ else zero.

(a) $P(2 \le X < 5) = P(X=2) + P(X=3) + P(X=4)$

$\therefore P(2 \le X < 5) = \frac{2}{21} + \frac{3}{21} + \frac{4}{21}$

$\boxed{7(a)}$ $\boxed{P(2 \le X < 5) = \frac{9}{21}}$ Ans

(b) $E(X) = \sum x \, p_X(x)$

$\therefore E(X) = \sum_{x=1}^{6} \frac{x^2}{21}$

$\boxed{7(b)}$ $\boxed{E(X) = 4.3333}$ Ans

8.

(a) Given an experiment S with set of outcomes Ω;

i.e. ~~set~~ $\Omega = \{x_1, x_2, x_3, \ldots\}$

such that $x_1, x_2, x_3 \ldots$ are ~~all~~ possible outcomes of S,

we define an event E uniquely via ~~what~~ whether outcome x_i happened as part of the event or not, for every possible outcome x_i in Ω.

Event E is basically a set of outcomes of Ω.

For example, if the experiment S has a finite number of possible outcomes x_1, x_2 and x_3.

we can define 2^3 events depending on whether a particular outcome is part of the event or not.

The complete set of events for this hypothetical experiment would be:

E_0	$\bar{x}_1, \bar{x}_2, \bar{x}_3 \equiv \phi$	E_4	$x_1, \bar{x}_2, \bar{x}_3$
E_1	$\bar{x}_1, \bar{x}_2, x_3$	E_5	$x_1, \bar{x}_2, x_3$
E_2	$\bar{x}_1, x_2, \bar{x}_3$	E_6	$x_1, x_2, \bar{x}_3$
E_3	$\bar{x}_1, x_2, x_3$	E_7	x_1, x_2, x_3

where a dash on x_i outcome indicates that x_i is not included in the event.

Here the events are labelled $E_0, E_1, \ldots E_7$.

In other words, an event E is an element of the power set of Ω: $\mathcal{P}$.

where $\mathcal{P} = \{\{\phi\}, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_2 x_3\}, \{x_1, x_3\}, \{x_1, x_2, x_3\}\}$

as per the previous example of $\Omega = \{x_1, x_2, x_3\}$.

Am

8 (b)

Random Variables are numerical mappings of the outcomes of an experiment.

~~A single~~
Every outcome will be assigned only one Random Variable value, but a particular Random Variable value can ~~have~~ be reverse-mapped to multiple outcomes.

Ans

eg. If in experiment 5 three fair coins are independently tossed, we can define a random variable X as the # Heads obtained.

Ω outcome	X_i
TTT	0
TTH	1
THT	1
THH	2
HTT	1
HTH	2
HHT	2
HHH	3

X X X