EE 507 HW05

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1.1

1. (a) Check if $\boxed{f_X(x) \cdot f_Y(y) \cdot f_Z(z) = f_{X,Y,Z}(x,y,z)}$ (0)
implies that ~~the three~~ X and Y are independent,
i.e. check if (0) implies $\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$ (?)

~~Given~~ Integrating both sides of (0) w.r.t. z:

$$\int_{\forall z} f_X(x) \cdot f_Y(y) \cdot f_Z(z)\, dz = \int_{\forall z} f_{X,Y,Z}(x,y,z)\, dz$$

or $$f_X(x) f_Y(y) \int_{\forall z} f_Z(z) dz = f_{X,Y}(x,y)$$ (marginal pdf of X, Y)

or $$f_X(x) \cdot f_Y(y) \cdot 1 = f_{X,Y}(x,y)$$

or $\boxed{f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)}$ which is (?) which is what we set to check for!

$\therefore$ 1(a) $\boxed{f_X(x) \cdot f_Y(y) f_Z(z) = f_{X,Y,Z}(x,y,z) \Rightarrow X \text{ and } Y \text{ are independent}}$ Ans

1(b) check if $\boxed{P(X)P(Y)P(Z) = P(X,Y,Z)}$ (0) implies that X and Y are independent i.e. $\boxed{P(X) \cdot P(Y) = P(X,Y)}$ (?)
where X, Y, Z are events of a probabilistic experiment.

No.
Counterexample:
~~Let experiment~~
In an experiment, ~~let~~ we are picking a number from 1 to 8 randomly: Each number has equal probability of being picked. This experiment is like throwing a fair 8-sided die.

Now let us define three events:

$X = \{1, 2, 3, 4\} \Rightarrow P(X) = \frac{1}{2}$ ~~$P(X, Y)$~~

$Y = \{1, 3, 4, 5\} \Rightarrow P(Y) = \frac{1}{2}$

$Z = \{1, 6, 7, 8\} \Rightarrow P(Z) = \frac{1}{2}$

$X \cap Y \cap Z = \{1\} \Rightarrow P(X \cap Y \cap Z) = \frac{1}{8}$

~~So $P(X) \cdot P(Y)$~~

So equation ⓪ is followed by events X, Y, Z, as

$$P(X) \cdot P(Y) \cdot P(Z) = \left(\frac{1}{2}\right)^3 = \frac{1}{8} = P(X, Y, Z)$$

But $P(X) \cdot P(Y) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$P(X \cap Y) = P(\{1, 3, 4\}) = \frac{3}{8}$ — X and Y are NOT independent.

In fact, $P(Y) \cdot P(Z) = \frac{1}{4}$

$P(Y \cap Z) = P(\{1\}) = \frac{1}{8}$ — Y and Z are NOT independent either

And $P(X) \cdot P(Z) = \frac{1}{4}$

$P(X \cap Z) = P(\{1\}) = \frac{1}{8}$ — X and Z are NOT independent either.

1(b)

∴ The analogous event $P(X) \cdot P(Y) \cdot P(Z) = P(XYZ)$ does NOT imply independence of X and Y. Ans

$\boxed{2.1}$

$\boxed{2.}$ 2(a) $X, Y \sim \mathcal{N}(\mu_X = 0, \mu_Y = 0, \sigma_X^2 = 1, \sigma_Y^2 = 4, \rho = 0.5)$

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{1-\rho^2}\,\sigma_X\sqrt{2\pi}\,\sigma_Y\sqrt{2\pi}}\, e^{\underbrace{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right\}}_{\text{Only component (say } g(x,y)) \text{ dependent on } x \text{ and } y.}}$$

Taking out the component $g(x,y)$ and putting the given values of $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ_{XY}:

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - \frac{2(x)(y)}{(1)(2)}$$

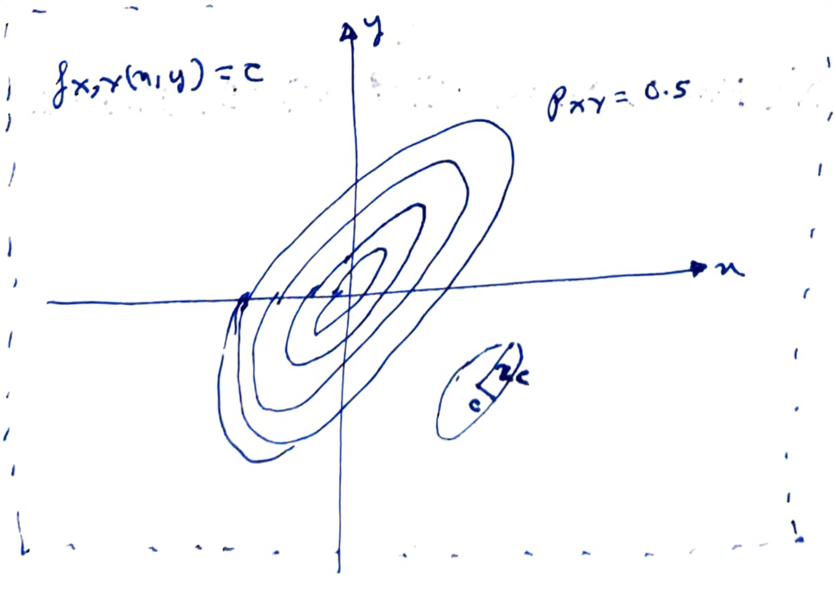
$\Rightarrow$ $f_{X,Y}(x,y) \underset{\substack{\text{i.e.}\\ \text{is}\\ \text{constant}}}{\rightarrow} c$ for $g(x,y) \underset{\substack{\text{i.e.}\\ \text{is}\\ \text{constant}}}{\rightarrow} c$

$\Rightarrow$ Contour of $f_{X,Y}(x,y)$ is

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - \cancel{2}\, xy = c$$

which is the equation for an ellipse, tilted and with positive slope.

$\boxed{2(a)}$



Ans

2(b)

$$\boxed{f_X(x) = \mathcal{N}(x, 0, 1^2) = \frac{1}{1 \cdot \sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}} \quad \forall x}$$

$$\boxed{f_Y(y) = \mathcal{N}(y, 0, 2^2) = \frac{1}{2 \cdot \sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y}{2}\right)^2\right\}} \quad \forall y}$$

Ans

2(c)

$$f_{Y|X}(y|X=x) = \frac{f_{X,Y}(X=x, Y)}{f_X(X=x)} \quad \forall x, y$$

or
$$f_{Y|X}(y|X=x) = \frac{\dfrac{1}{\sqrt{(1-\frac{1}{2^2})}\, 1 \cdot \sqrt{2\pi} \cdot 2 \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \cdot \frac{1}{1-\frac{1}{2^2}} \cdot \left\{ \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - 2 \times \frac{1}{2} \times \frac{x}{1} \times \frac{y}{2} \right\}} \quad \cancel{\forall x, y}}{\dfrac{1}{1 \cdot \sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}}}$$

$\forall (x, y)$

or
$$f_{Y|X}(y|X=x) = \frac{1}{\sqrt{3} \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \cdot \left\{ \left(\frac{x}{\sqrt{3}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 - 2 \cdot 1 \cdot \left(\frac{x}{\sqrt{3}}\right)\left(\frac{y}{\sqrt{3}}\right) \right\}}$$

$\forall (x, y)$

~~2(c)~~

or
$$\cancel{\boxed{f_{Y|X}(y|X=x) = \sqrt{2\pi}\, \mathcal{N}(\mu_X = 0, \mu_Y = 0, \sigma_X^2 = 3, \sigma_Y^2 = 3, \rho_{XY} = 1) \quad \forall (x, y)}}$$

~~Ans~~

or
$$\cancel{f_{Y|X}(y|X=x) = \mathcal{N}_y(\mu_Y = \ldots, \sigma_Y^2 = 3) \quad \forall y}$$

or
$$f_{Y|X}(y|X=x) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y-x}{\sqrt{3}}\right)^2\right\}} \quad \forall y$$

2(c)

or
$$\boxed{f_{Y|X}(y|X=x) = \mathcal{N}(y, \mu_Y = x, \sigma_Y^2 = 3) \quad \forall y}$$

Ans

PTO.

~~2(d) Find x s.t. $E[Y|X=x] = -2$.~~

~~From 2(c), we know that $Y|X=x$ is fully correlated to $X=x$. ($\rho_{Y|X,X} = 1$).~~

~~$\because$ Y and X have the same support/domain of $x \in \mathbb{R}$ and $y \in \mathbb{R}$,~~

~~$\therefore$ $E_Y[Y|X=x] = -2 \equiv E_X[X=x] = -2$~~

~~But $E_X[X=x] = x$~~

~~$\Rightarrow \boxed{x = -2}$ Ans~~

2(d) Find x s.t. $E[Y|X=x] = -2$

From 2(c), we know that $Y|X=x$ is a gaussian with mean x.

So $E[Y|X=x] = -2 \Rightarrow \boxed{x=-2}$ Ans

2(e) $Z = X + Y - 1$

Z is also a gaussian, we need to only compute μ_Z and σ_Z^2 to find $f_Z(z)$:

$E(Z) = E(X) + E(Y) - E(1)$

or $\mu_Z = \mu_X + \mu_Y - 1$

or $\mu_Z = 0 + 0 - 2$

or $\boxed{\mu_Z = -1}$

$\sigma_Z^2 = E[\cancel{X - \mu_Z}\{(X + Y - 1) - E[X + Y - 1]\}^2]$

or $\sigma_Z^2 = E[\{(X - \mu_X) + (Y - \mu_Y) + (-1 - E(-1))\}^2]$

or $\sigma_Z^2 = E[\{(X - \mu_X) + (Y - \mu_Y)\}^2]$

or $\sigma_Z^2 = E[(X - \mu_X)^2] + E[(Y - \mu_Y)^2] + \cancel{2\rho_{XY}}$ $2E[(X - \mu_X)(Y - \mu_Y)]$

or $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 1^2 + 2^2 + 2 \times 0.5 \times 1 \times 2$

or $\boxed{\sigma_Z^2 = 7}$

2(e)

$\therefore \boxed{Z \sim \mathcal{N}(z, \mu_Z = -1, \sigma_Z^2 = 7)}$ Ans

2(f) $\boxed{2.5}$

$Z = 2X + 3Y$

$E(Z) = 2\mu_X + 3\mu_Y$

2(f)(i)

or $\boxed{\mu_Z = 0}$ Ans

$W = X - Y$

$E(W) = \mu_X - \mu_Y$

2(f)(iii)

$\boxed{\mu_W = 0}$ Ans

$\sigma_Z^2 = E\left[\{(2X+3Y) - (2\mu_X + 3\mu_Y)\}^2\right]$

or $\sigma_Z^2 = E\left[\{2(X-\mu_X)\}^2 + \{3(Y-\mu_Y)\}^2 + 2.2.3.(X-\mu_X)(Y-\mu_Y)\right]$

or $\sigma_Z^2 = 4\sigma_X^2 + 9\sigma_Y^2 + 12\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 4.1^2 + 9.2^2 + 12.(0.5).1.2$

or $\sigma_Z^2 = 4 + 36 + 12$

2(f)(ii)

or $\boxed{\sigma_Z^2 = 52}$ Ans

$\sigma_W^2 = E\left[\{(X-Y) - (\mu_X - \mu_Y)\}^2\right]$

or $\sigma_W^2 = E\left[\{X-\mu_X\}^2 + \{Y-\mu_Y\}^2 - \{2.1.1.(X-\mu_X)(Y-\mu_Y)\}\right]$

or $\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_W^2 = 1^2 + 2^2 - 2.(0.5).1.2$

or $\sigma_W^2 = 1 + 4 - 2$

2(f)(iv)

$\boxed{\sigma_W^2 = 3}$ Ans

2(g)

$Cov(Z, W) = E\left[(Z - \mu_Z)(W - \mu_W)\right]$

or $Cov(Z, W) = E\left[(2X + 3Y - 2\mu_X - 3\mu_Y)(X - Y - \mu_X - (-\mu_Y))\right]$

or $Cov(Z, W) = E\left[\{2(X-\mu_X) + 3(Y-\mu_Y)\}\{1(X-\mu_X) - 1(Y-\mu_Y)\}\right]$

or $Cov(Z, W) = E\left[2(X-\mu_X)^2 - 3(Y-\mu_Y)^2 + 1(X-\mu_X)(Y-\mu_Y)\right]$

or $Cov(Z, W) = 2\sigma_X^2 - 3\sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y$

or $Cov(Z, W) = 2.1^2 - 3.2^2 + (0.5).1.2$

or $Cov(Z, W) = 2 - 12 + 1$

2(f)(v)

or $\boxed{Cov(Z,W) = -9}$ Ans

$$\rho_{Z,W} = \frac{Cov(Z,W)}{\sigma_Z \sigma_W} = \frac{-9}{\sqrt{52}.\sqrt{3}} \approx -0.7206$$

2(f)(vi)

$$\boxed{f_{Z,W}(z,w) = \mathcal{N}(\mu_Z = 0, \mu_W = 0, \sigma_Z^2 = 52, \sigma_W^2 = 3, \rho_{Z,W} = -0.7206)}$$ Ans

2(g) $R = aX + bY \Rightarrow \sigma_R^2 = a^2.\sigma_X^2 + b^2\sigma_Y^2$

or $\sigma_R^2 = a^2.1^2 + b^2.2^2$

or $\sigma_R^2 = a^2 + 4b^2$

$Cov(R,Y) = 0$ is a sufficient condition for R and Y to be independent, as both are gaussian.

$$Cov(R,Y) = E[(aX + bY - a\mu_X - b\mu_Y)(Y - \mu_Y)]$$

or $Cov(R,Y) = E[a(X-\mu_X)(Y-\mu_Y) + b(Y-\mu_Y)^2]$

or $Cov(R,Y) = a\rho_{X,Y}\sigma_X\sigma_Y + b\sigma_Y^2$

or $Cov(R,Y) = a(0.5).1.2 + b.2^2$

or $Cov(R,Y) = a + 4b$

2(g)

For R, Y to be independent distributions, $a + 4b = 0$

we can use any set of Real numbers to do so,

say $a = -4, b = 1$

Ans

2(b) $\quad Q = aX$

given $\quad p(a=2) = 0.5 \qquad f(Q \mid a=2) = f_X(\cancel{X} = 2x)$

$\qquad\qquad p(a=1) = 0.5 \qquad f(Q \mid a=1) = f_X(\cancel{X} = x)$

$$
\begin{tikzcd}
 & Q = X \\
Q \arrow[ur, "0.5"] \arrow[dr, "0.5"'] & \\
 & Q = 2X
\end{tikzcd}
$$

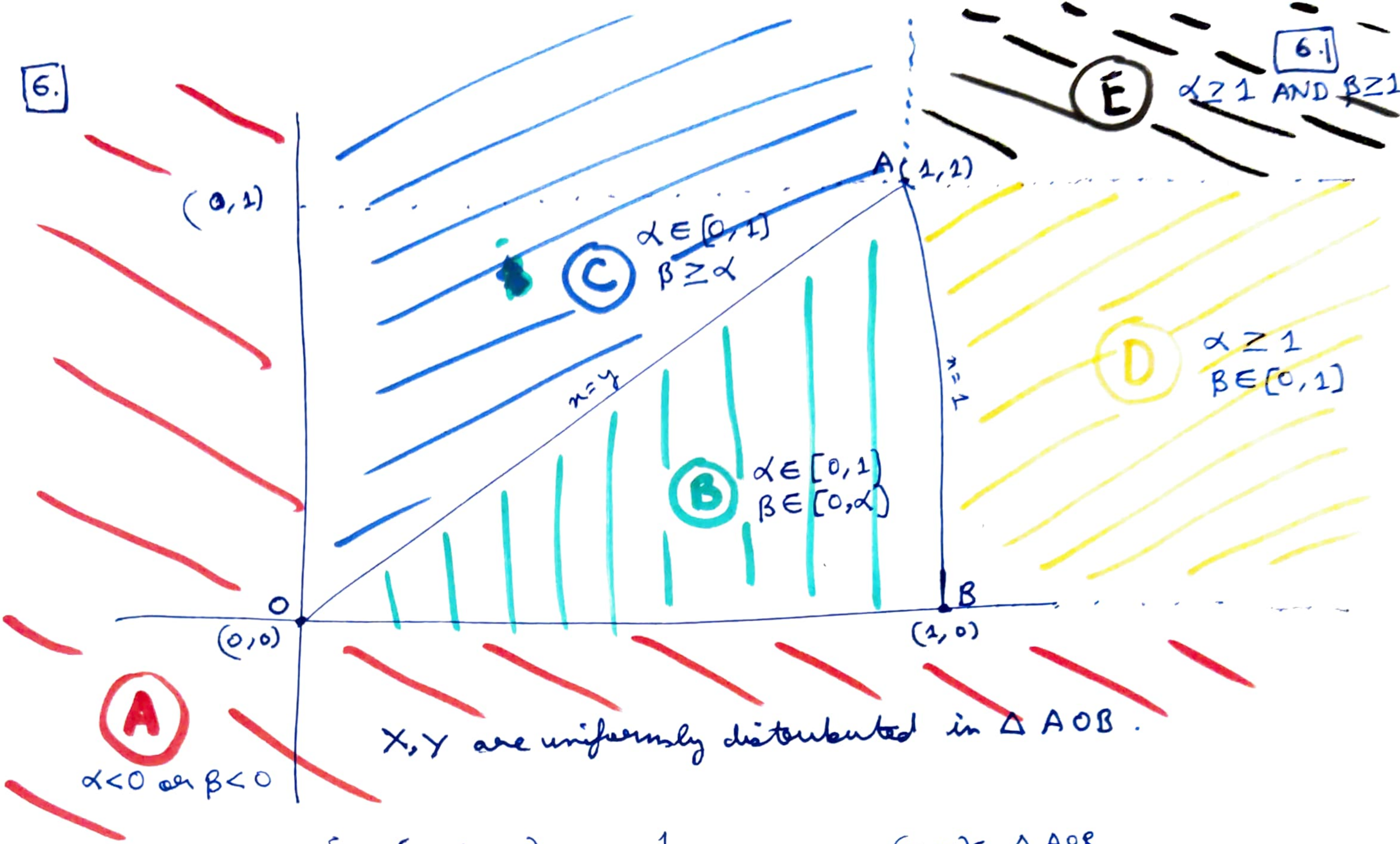
$$f_Q(q) = f_{Q|a}(Q \mid a=1) \cdot p(a=1) + f_{Q|a}(Q \mid a=2) \cdot p(a=2) \quad \text{(LoTP)}$$

or $$f_Q(q) = 0.5\,\mathcal{N}(q, \mu_q = \mu_x, \sigma_q^2 = \sigma_x^2) + 0.5\,\mathcal{N}(q, \mu_q = 2\mu_x, \sigma_q^2 = 2^2\sigma_x^2)$$

$\boxed{2(b)}$

or $$\boxed{f_Q(q) = 0.5\,\mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 1) + 0.5\,\mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 4)}$$ Ans

6.



X, Y are uniformly distributed in Δ AOB.

So $f_{X,Y}(x,y) = \dfrac{1}{\text{Area of } \Delta AOB}$ $\quad (x,y) \in \Delta AOB$

or $f_{X,Y}(x,y) = \dfrac{1}{\frac{1}{2}\times 1 \times 1}$ $\quad y \in [0,1]$, $x \in [y,1]$ or

6(a)

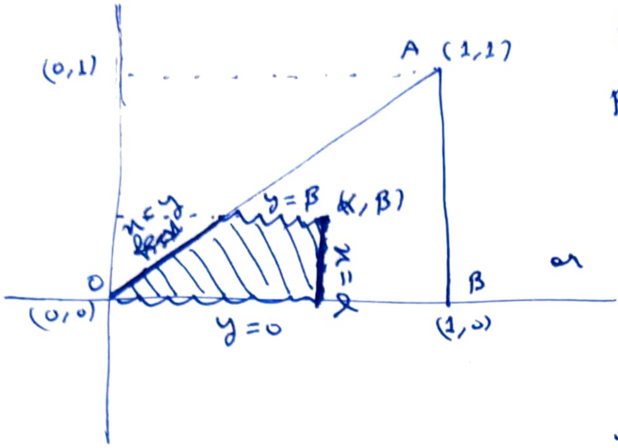
or $$\boxed{f_{X,Y}(x,y) = \begin{cases} 2 & y \in [0,1],\ x \in [y,1] \\ 0 & \text{else} \end{cases}}$$ Ans

6(b) (Calculations on next pages.)

$$\boxed{F_{X,Y}(x=\alpha, y=\beta) = \begin{cases} 0 & \alpha < 0 \text{ or } \beta < 0 \quad \text{A} \\ 2\alpha\beta - \beta^2 & \alpha \in [0,1],\ \beta \in [0,\alpha] \quad \text{B} \\ \alpha^2 & \alpha \in [0,1],\ \beta \geq \alpha \quad \text{C} \\ 2\beta - \beta^2 & \alpha \geq 1,\ \beta \in [0,\alpha] \quad \text{D} \\ 1 & \alpha \geq 1,\ \beta \geq 1 \quad \text{E} \end{cases}}$$ Ans

For (α, β) in $\textcircled{B}$:

i.e. $\alpha \in [0,1]$, $\beta \in [0, \alpha]$



~~$F_{X,Y}(x,y)$~~

$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\,dx\,dy \quad (\alpha,\beta) \in \textcircled{B}$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\beta} \int_{x=y}^{x=\alpha} 2\,dx\,dy \quad (\alpha,\beta) \in \textcircled{B}$$

or $$F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2x\Big|_{y}^{\alpha} dy \quad (\alpha,\beta) \in \textcircled{B}$$

or $$F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2(\alpha - y)\,dy \quad (\alpha,\beta) \in \textcircled{B}$$

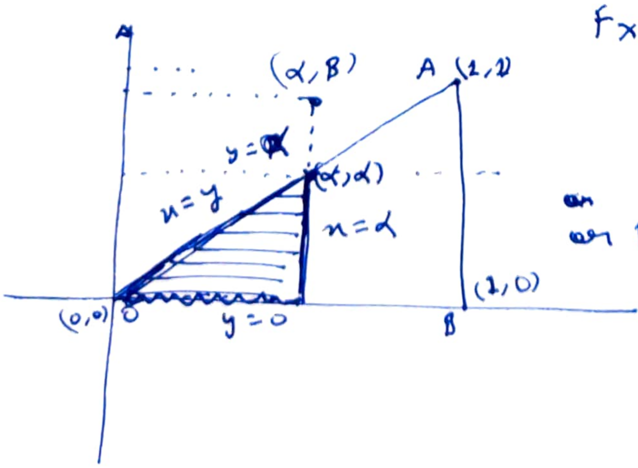
or $$F_{X,Y}(\alpha,\beta) = 2\alpha y - y^2\Big|_0^{\beta} \quad (\alpha,\beta) \in \textcircled{B}$$

or $$\boxed{F_{X,Y}(\alpha,\beta) = 2\alpha\beta - \beta^2 \quad (\alpha,\beta) \in \textcircled{B}}$$

$\boxed{\text{or } \alpha \in [0,1],\ \beta \in [0,\alpha]}$

For (α, β) in $\textcircled{C}$:

i.e. $\alpha \in [0,1]$, $\beta \geq \alpha$



$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\,dx\,dy \quad (\alpha,\beta) \in \textcircled{C}$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha} \int_{x=y}^{x=\alpha} 2\,dx\,dy \quad (\alpha,\beta) \in \textcircled{C}$$

or $F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha} 2(\alpha - y)\, dy \qquad (\alpha,\beta) \in$ C

or $F_{X,Y}(x=\alpha, y=\beta) = 2\alpha^2 - \alpha^2 \qquad (\alpha,\beta) \in$ C

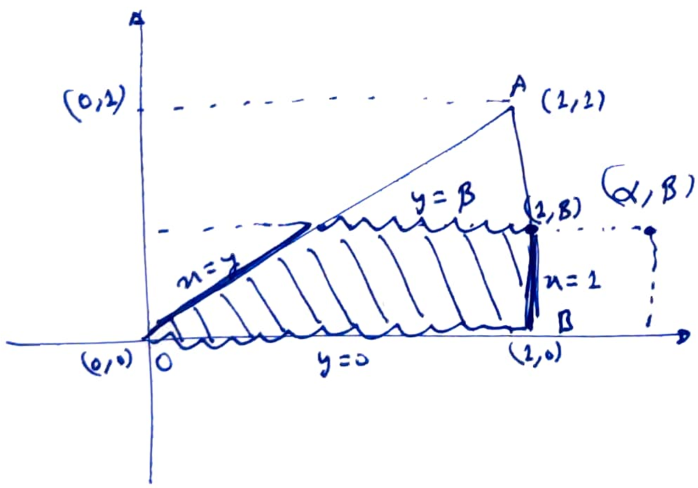
or $\boxed{F_{X,Y}(x=\alpha, y=\beta) = \alpha^2 \qquad (\alpha,\beta) \in \text{C} \quad \text{or } \alpha \in [0,1],\ \beta \geq \alpha}$

Area of the shaded Δ.
$\times f_{X,Y}(x,y)$ in the region.

For (α,β) in D

i.e. $\alpha \geq 1,\ \beta \in [0,1]$

$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\, dx\, dy \qquad (\alpha,\beta) \in$ D



or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} \int_{x=y}^{x=1} 2\, dx\, dy \qquad (\alpha,\beta) \in$ D

or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2(1-y)\, dy \qquad (\alpha,\beta) \in$ D

or $F_{X,Y}(\alpha,\beta) = 2y - y^2 \Big|_0^\beta \qquad (\alpha,\beta) \in$ D

or $\boxed{F_{X,Y}(\alpha,\beta) = 2\beta - \beta^2 \qquad (\alpha,\beta) \in \text{D} \quad \text{or } \alpha \geq 1,\ \beta \in [0,1)}$

$$f_X(x) = \int_{\forall y} f_{X,Y}(x,y)\,dy$$

or
$$f_X(x) = \begin{cases} 0 & x < 0 \\ \int_{y=0}^{y=x} 2\,dy & x \in [0,1] \\ 0 & x > 1 \end{cases}$$

6(c)

or
$$\boxed{f_X(x) = \begin{cases} 0 & x < 0 \\ 2x & x \in [0,1] \\ 0 & x > 1 \end{cases}}$$
Ans

$$f_Y(y) = \int_{\forall x} f_{X,Y}(x,y)\,dx$$

or
$$f_Y(y) = \begin{cases} 0 & y < 0 \\ \int_{x=y}^{x=1} 2\,dx & y \in [0,1] \\ 0 & y > 1 \end{cases}$$

6(d)

or
$$\boxed{f_Y(y) = \begin{cases} 0 & y < 0 \\ 2(1-y) & y \in [0,1] \\ 0 & y > 1 \end{cases}}$$
Ans

$$f_{X|Y}(x \mid Y=y) = \frac{f_{X,Y}(x, y=y)}{f_Y(y)}$$

6(e)

or

$$\boxed{f_{X|Y}(x \mid Y=y) = \begin{cases} \text{not defined} & y < 0 \\ \dfrac{1}{1-y} & y \in (0,1) \\ \text{not defined.} & y \geq 1 \end{cases}}$$

Ans