2. 2(a) $X, Y \sim \mathcal{N}(\mu_X = 0, \mu_Y = 0, \sigma_X^2 = 1, \sigma_Y^2 = 4, \rho = 0.5)$

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{1-\rho^2}\,\sigma_X\sqrt{2\pi}\,\sigma_Y\sqrt{2\pi}}\, e^{\underbrace{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right\}}_{\text{Only component (say } g(x,y)) \text{ dependent on } x \text{ and } y.}}$$

Taking out the component $g(x,y)$ and putting the given values of $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ_{XY}:

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - \frac{2\,(x)(y)}{(1)(2)}$$

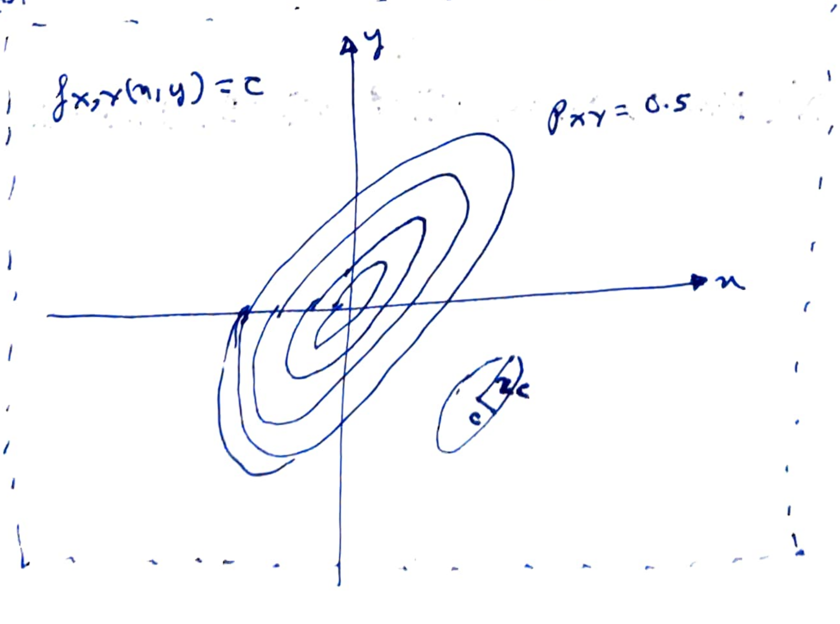
$\Rightarrow f_{X,Y}(x,y) \underset{\substack{\text{i.e.}\\ \text{is}\\ \text{constant}}}{\rightarrow} c$ for $g(x,y) \underset{\substack{\text{i.e.}\\ \text{is}\\ \text{constant}}}{\rightarrow} c$

$\Rightarrow$ Contour of $f_{X,Y}(x,y)$ is

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - xy = c$$

which is the equation for an ellipse, tilted and with positive slope.

$\boxed{2(a)}$



Ans

2(b)

$$\boxed{\begin{aligned} f_X(x) &= \mathcal{N}(x; 0, 1^2) = \frac{1}{1.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}} \quad \forall x \\ f_Y(y) &= \mathcal{N}(y, 0, 2^2) = \frac{1}{2.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y}{2}\right)^2\right\}} \quad \forall y \end{aligned}}$$

Ans

2(c)

$$f_{Y|X}(y|X=x) = \frac{f_{X,Y}(X=x, Y)}{f_X(X=x)} \quad \forall x, y$$

or
$$f_{Y|X}(y|X=x) = \frac{\dfrac{1}{\sqrt{\left(1-\frac{1}{2^2}\right)}\,1.\sqrt{2\pi}\,.\,2.\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2^2}}\cdot\left\{\left(\frac{x}{1}\right)^2+\left(\frac{y}{2}\right)^2 - 2\times\frac{1}{2}\times\frac{x}{1}\times\frac{y}{2}\right\}} \quad \cancel{\forall x, y}}{\dfrac{1}{1.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}}}$$

$\forall (x, y)$

or
$$f_{Y|X}(y|X=x) = \frac{1}{\sqrt{3}.\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\left\{\left(\frac{x}{\sqrt{3}}\right)^2+\left(\frac{y}{\sqrt{3}}\right)^2-2.1.\left(\frac{x}{\sqrt{3}}\right)\left(\frac{y}{\sqrt{3}}\right)\right\}}$$

$\forall (x, y)$

2(c)

or $\boxed{f_{Y|X}(y|X=x) = \mathcal{N}(\mu_X=0, \mu_Y=0, \sigma^2_X=3, \sigma^2_Y=3, \rho_{XY}=1)} \quad \boxed{\forall (x,y)}$

Ans

PTO.

2(d) Find x s.t. $E[Y|X=x] = -2$.

From 2(c), we know that $Y|X=x$ is fully correlated to $X=x$. ($\rho_{Y|X, X} = 1$).

$\because$ Y and X ~~share~~ have the same support/domain, of $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

$\therefore$ $E_Y[Y|X=x] = -2 \equiv E_X[X=x] = -2$

But $E_X[X=x] = x$

~~$\Rightarrow x =$~~ $\Rightarrow \boxed{x = -2}$ 2(d) Ans

2(e) $Z = X + Y - 1$

Z is also a gaussian, we need to only compute μ_Z and σ_Z^2 to find $f_Z(z)$:

$E(Z) = E(X) + E(Y) - E(1)$

or $\mu_Z = \mu_X + \mu_Y - 1$

or $\mu_Z = 0 + 0 - 2$

or $\boxed{\mu_Z = -1}$

$\sigma_Z^2 = E\left[\cancel{X-\mu_Z}\{(X+Y-1) - E[X+Y-1]\}^2\right]$

or $\sigma_Z^2 = E\left[\{(X-\mu_X) + (Y-\mu_Y) + (-1 - E(-2))\}^2\right]$

or $\sigma_Z^2 = E\left[\{(X-\mu_X) + (Y-\mu_Y)\}^2\right]$

or $\sigma_Z^2 = E[(X-\mu_X)^2] + E[(Y-\mu_Y)^2] + \cancel{2\rho_{XY}}\ 2E[(X-\mu_X)(Y-\mu_Y)]$

or $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 1^2 + 2^2 + 2 \times 0.5 \times 1 \times 2$

or $\boxed{\sigma_Z^2 = 7}$

2(e)

$\therefore \boxed{Z \sim \mathcal{N}(z, \mu_Z = -1, \sigma_Z^2 = 7)}$ Ans

2(f) $\boxed{2.5}$

$Z = 2X + 3Y$ $\qquad$ $W = X - Y$

$E(Z) = 2\mu_X + 3\mu_Y$ $\qquad$ $E(W) = \mu_X - \mu_Y$

2(f)(i)

or $\boxed{\mu_Z = 0}$ Ans

2(f)(iii)

$\boxed{\mu_W = 0}$ Ans

$\sigma_Z^2 = E\left[\{(2X+3Y) - (2\mu_X + 3\mu_Y)\}^2\right]$ $\qquad$ $\sigma_W^2 = E\left[\{(X-Y) - (\mu_X - \mu_Y)\}^2\right]$

or $\sigma_Z^2 = E\left[\{2(X-\mu_X)\}^2 + \{3(Y-\mu_Y)\}^2 + 2.2.3.(X-\mu_X)(Y-\mu_Y)\right]$

or $\sigma_W^2 = E\left[\{X-\mu_X\}^2 + \{Y-\mu_Y\}^2 - \{2.1.1.(X-\mu_X)(Y-\mu_Y)\}\right]$

or $\sigma_Z^2 = 4\sigma_X^2 + 9\sigma_Y^2 + 12\rho_{XY}\sigma_X\sigma_Y$ $\qquad$ or $\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 4.1^2 + 9.2^2 + 12.(0.5).1.2$ $\qquad$ or $\sigma_W^2 = 1^2 + 2^2 - 2.(0.5).1.2$

or $\sigma_Z^2 = 4 + 36 + 12$ $\qquad$ or $\sigma_W^2 = 1 + 4 - 2$

2(f)(ii)

or $\boxed{\sigma_Z^2 = 52}$ Ans

2(f)(iv)

$\boxed{\sigma_W^2 = 3}$ Ans

2(g)

$Cov(Z,W) = E\left[(Z-\mu_Z)(W-\mu_W)\right]$

or $Cov(Z,W) = E\left[(2X+3Y-2\mu_X-3\mu_Y)(X-Y-\mu_X-(-\mu_Y))\right]$

or $Cov(Z,W) = E\left[\{2(X-\mu_X)+3(Y-\mu_Y)\}\{1(X-\mu_X)-1(Y-\mu_Y)\}\right]$

or $Cov(Z,W) = E\left[2(X-\mu_X)^2 - 3(Y-\mu_Y)^2 + 1(X-\mu_X)(Y-\mu_Y)\right]$

or $Cov(Z,W) = 2\sigma_X^2 - 3\sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y$

or $Cov(Z,W) = 2.1^2 - 3.2^2 + (0.5).1.2$

or $Cov(Z,W) = 2 - 12 + 1$

2(f)(v)

or $\boxed{Cov(Z, W) = -9}$ Ans

$$\rho_{Z,W} = \frac{Cov(Z, W)}{\sigma_Z \sigma_W} = \frac{-9}{\sqrt{52}\cdot\sqrt{3}} \approx -0.7206$$

2(f)(vi)

$$f_{Z,W}(z, w) = \mathcal{N}(\mu_Z = 0, \mu_W = 0, \sigma_Z^2 = 52, \sigma_W^2 = 3, \rho_{Z,W} = -0.7206)$$ Ans

2(g) $R = aX + bY$ $\Rightarrow$ $\sigma_R^2 = a^2 \cdot \sigma_X^2 + b^2 \sigma_Y^2$

or $\sigma_R^2 = a^2 \cdot 1^2 + b^2 \cdot 2^2$

or $\sigma_R^2 = a^2 + 4b^2$

$Cov(R, Y) = 0$ is a sufficient condition for R and Y to be independent, as both are gaussian.

$$Cov(R, Y) = E\left[(aX + bY - a\mu_X - b\mu_Y)(Y - \mu_Y)\right]$$

or $Cov(R, Y) = E\left[a(X - \mu_X)(Y - \mu_Y) + b(Y - \mu_Y)^2\right]$

or $Cov(R, Y) = a\,\rho_{X,Y}\sigma_X\sigma_Y + b\sigma_Y^2$

or $Cov(R, Y) = a(0.5) \cdot 1 \cdot 2 + b \cdot 2^2$

or $Cov(R, Y) = a + 4b$

2(g)

For R, Y to be independent distributions, $a + 4b = 0$

we can use any set of Real numbers to do so,

say $a = -4, b = 1$

Ans

2(b) $Q = aX$

given $P(a=2) = 0.5 \qquad f(Q \mid a=2) = f_X(X = 2x)$

$P(a=1) = 0.5 \qquad f(Q \mid a=1) = f_X(X = x)$

$$Q \xrightarrow{0.5} Q = X$$

$$Q \xrightarrow{0.5} Q = 2X$$

$f_Q(q) = f_{Q|a}(Q \mid a=1) \cdot P(a=1) + f_{Q|a}(Q \mid a=2) \cdot P(a=2)$ (LoTP)

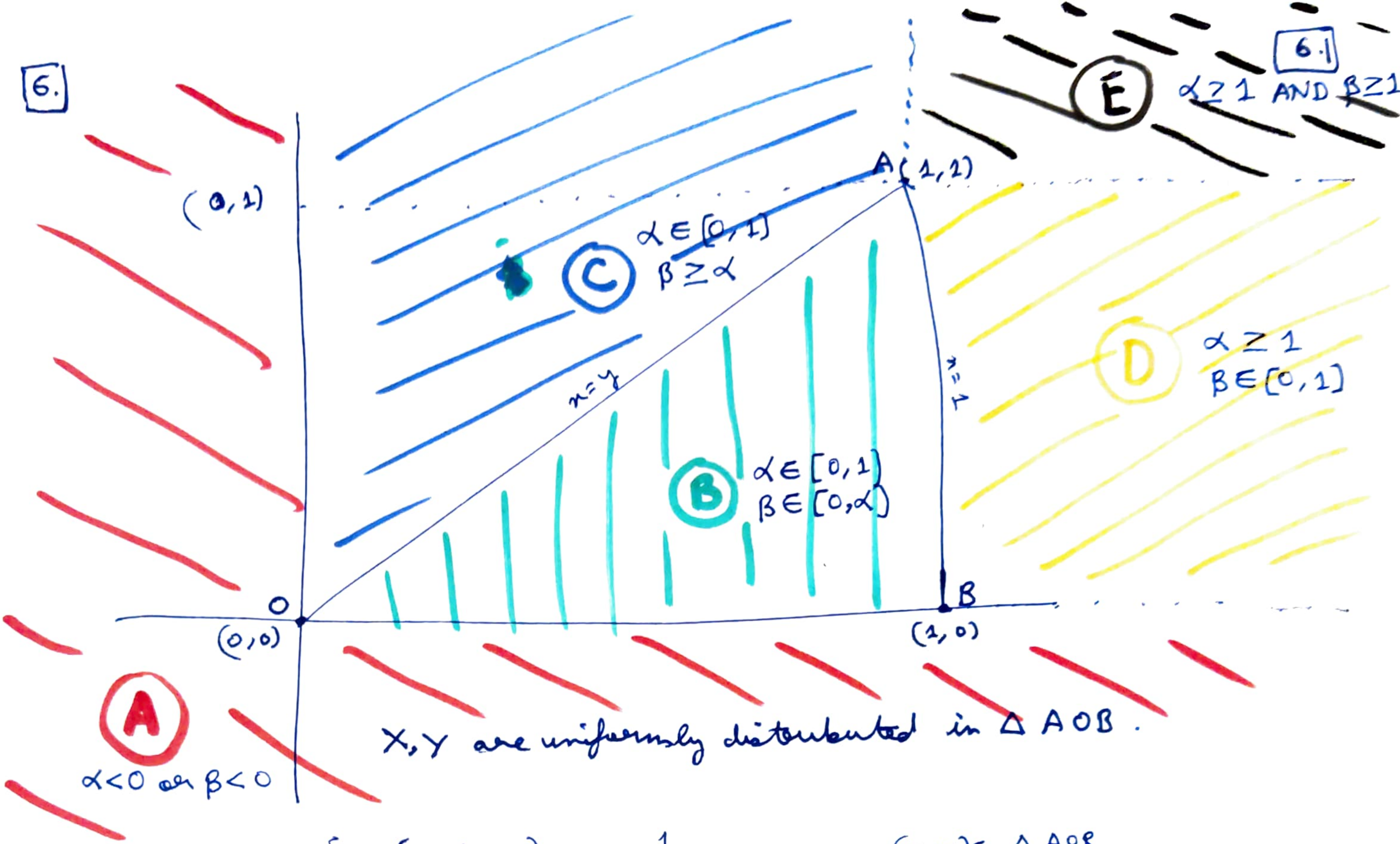
or $f_Q(q) = 0.5\,\mathcal{N}(q, \mu_q = \mu_x, \sigma_q^2 = \sigma_x^2) + 0.5\,\mathcal{N}(q, \mu_q = 2\mu_x, \sigma_q^2 = 2^2\sigma_x^2)$

$\boxed{2(b)}$

or $\boxed{f_Q(q) = 0.5\,\mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 1) + 0.5\,\mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 4)}$ Ans

———— × ———— × ———— × ———— × ————

6.



X, Y are uniformly distributed in Δ AOB.

So $f_{X,Y}(x,y) = \dfrac{1}{\text{Area of } \Delta AOB}$ $\quad (x,y) \in \Delta AOB$

or $f_{X,Y}(x,y) = \dfrac{1}{\frac{1}{2} \times 1 \times 1}$ $\quad y \in [0,1]$ and $x \in [y,1]$

6(a)

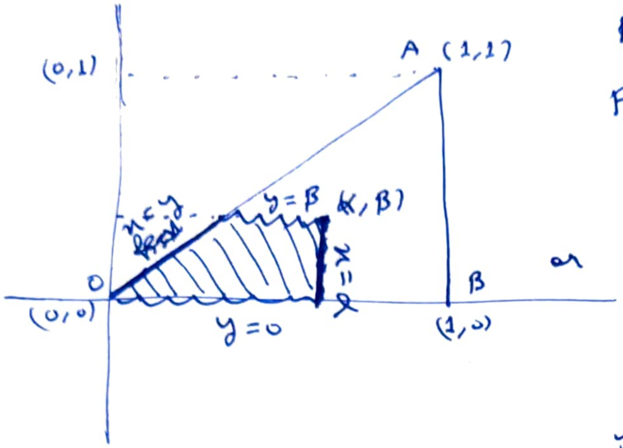
or $\boxed{f_{X,Y}(x,y) = \begin{cases} 2 & y \in [0,1],\ x \in (y,1) \\ 0 & \text{else} \end{cases}}$ Ans

6(b) (Calculations on next pages.)

$$\boxed{F_{X,Y}(x=\alpha, y=\beta) = \begin{cases} 0 & \alpha < 0 \text{ or } \beta < 0 \quad \text{A} \\ 2\alpha\beta - \beta^2 & \alpha \in [0,1],\ \beta \in [0,\alpha] \quad \text{B} \\ \alpha^2 & \alpha \in [0,1],\ \beta \geq \alpha \quad \text{C} \\ 2\beta - \beta^2 & \alpha \geq 1,\ \beta \in [0,\alpha] \quad \text{D} \\ 1 & \alpha \geq 1,\ \beta \geq 1 \quad \text{E} \end{cases}}$$ Ans

For (α, β) in B:

i.e. $\alpha \in [0,1]$, $\beta \in [0, \alpha]$



~~$F_{X,Y}(x,y)$~~

$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\,dx\,dy \quad (\alpha,\beta) \in B$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\beta}\int_{x=y}^{x=\alpha} 2\,dx\,dy \quad (\alpha,\beta) \in B$$

or $$F_{X,Y}(\alpha, \beta) = \int_{y=0}^{y=\beta} 2x\Big|_{y}^{\alpha} dy \quad (\alpha,\beta) \in B$$

or $$F_{X,Y}(\alpha, \beta) = \int_{y=0}^{y=\beta} 2(\alpha - y)\,dy \quad (\alpha,\beta) \in B$$

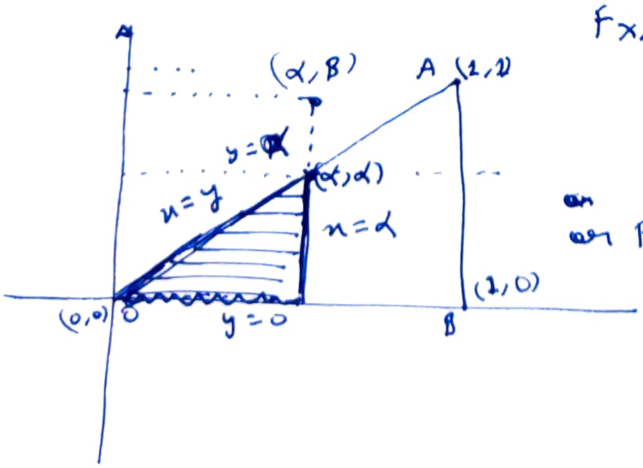
or $$F_{X,Y}(\alpha, \beta) = 2\alpha y - y^2 \Big|_0^{\beta} \quad (\alpha,\beta) \in B$$

or $$\boxed{F_{X,Y}(\alpha, \beta) = 2\alpha\beta - \beta^2 \quad (\alpha,\beta) \in B}$$

$\boxed{\text{or } \alpha \in [0,1),\ \beta \in [0,\alpha]}$

For (α, β) in C:

i.e. $\alpha \in [0,1]$, $\beta \geq \alpha$



$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\,dx\,dy \quad (\alpha,\beta) \in C$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha}\int_{x=y}^{x=\alpha} 2\,dx\,dy \quad (\alpha,\beta) \in C$$

or $F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha} 2(\alpha - y)\, dy \qquad (\alpha,\beta) \in$ C

or $F_{X,Y}(x=\alpha, y=\beta) = 2\alpha^2 - \alpha^2 \qquad (\alpha,\beta) \in$ C

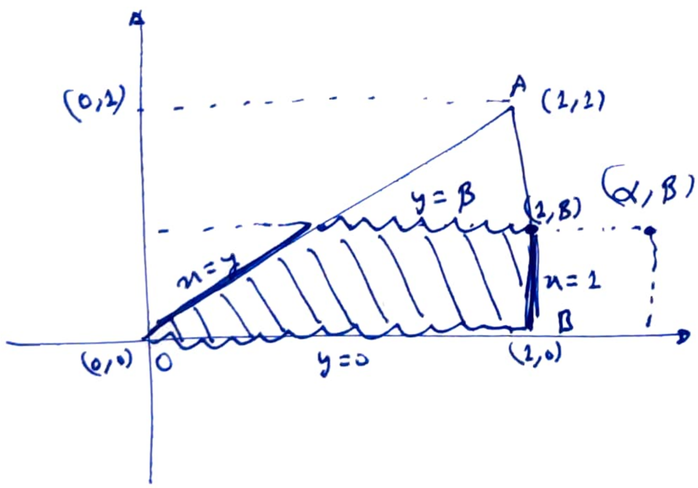
or $\boxed{F_{X,Y}(x=\alpha, y=\beta) = \alpha^2 \qquad (\alpha,\beta) \in \text{C} \quad \text{or} \quad \alpha \in [0,1],\ \beta \geq \alpha}$

Area of the shaded Δ.
$\times f_{X,Y}(x,y)$ in the region.

For (α,β) in D

i.e. $\alpha \geq 1$, $\beta \in [0,1]$

$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\, dx\, dy \qquad (\alpha,\beta) \in$ D



or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} \int_{x=y}^{x=1} 2\, dx\, dy \qquad (\alpha,\beta) \in$ D

or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2(1-y)\, dy \qquad (\alpha,\beta) \in$ D

or $F_{X,Y}(\alpha,\beta) = 2y - y^2 \Big|_0^{\beta} \qquad (\alpha,\beta) \in$ D

or $\boxed{F_{X,Y}(\alpha,\beta) = 2\beta - \beta^2 \qquad (\alpha,\beta) \in \text{D} \quad \text{or} \quad \alpha \geq 1,\ \beta \in [0,1]}$

$$f_X(x) = \int_{\forall y} f_{X,Y}(x,y)\,dy$$

or
$$f_X(x) = \begin{cases} 0 & x < 0 \\ \int_{y=0}^{y=x} 2\,dy & x \in [0,1] \\ 0 & x > 1 \end{cases}$$

6(c)

or
$$\boxed{f_X(x) = \begin{cases} 0 & x < 0 \\ 2x & x \in [0,1] \\ 0 & x > 1 \end{cases}}$$ Ans

$$f_Y(y) = \int_{\forall x} f_{X,Y}(x,y)\,dx$$

or
$$f_Y(y) = \begin{cases} 0 & y < 0 \\ \int_{x=y}^{x=1} 2\,dx & y \in [0,1] \\ 0 & y > 1 \end{cases}$$

6(d)

or
$$\boxed{f_Y(y) = \begin{cases} 0 & y < 0 \\ 2(1-y) & y \in [0,1] \\ 0 & y > 1 \end{cases}}$$ Ans

$$f_{X|Y}(x|Y=y) = \frac{f_{X,Y}(x, y=Y)}{f_Y(y)}$$

6(e)

or

$$f_{X|Y}(x|Y=y) = \begin{cases} \text{not defined} & y < 0 \\ \dfrac{1}{1-y} & y \in (0,1) \\ \text{not defined.} & y \geq 1 \end{cases}$$

Ans