1.1)

1. Let $f_X(u) = 0.5 e^{-|u|} \quad \forall u \in R.$

$\Rightarrow f_X(u) = \begin{cases} 0.5 e^{u} & u < 0 \\ 0.5 e^{-u} & u \geq 0 \end{cases}$

$MGF_X(u) = E[e^{sX}]$

$\Rightarrow MGF_X(u) = \int\limits_{u=-\infty}^{u=0} e^{sx} \, 0.5 e^{u} \, du + \int\limits_{u=0}^{u=\infty} e^{sx} \, 0.5 e^{-u} \, du$

$\Rightarrow MGF_X(u) = 0.5 \int\limits_{u=-\infty}^{u=0} e^{(s+1)u} \, du + 0.5 \int\limits_{u=0}^{u=\infty} e^{(s-1)u} \, du$

$\qquad\qquad\qquad\qquad\qquad \text{if } s > -1 \qquad\qquad\qquad \text{if } s < 1$

$\Rightarrow MGF_X(u) = \dfrac{0.5}{s+1} - \dfrac{0.5}{s-1} \qquad\qquad s \in (-1, 1)$

$\Rightarrow MGF_X(u) = 0.5 \times \dfrac{2}{1-s^2} \qquad\qquad\qquad s \in (-1, 1)$

1 (a)

$\boxed{MGF_X(u) = \dfrac{1}{1-s^2}} \qquad\qquad\qquad s \in (-1, 1). \quad \underline{Ans}.$

$E[X^2] = \dfrac{d^2}{ds^2} MGF_X(u) \Big|_{s \to 0} = \dfrac{d}{ds}\left(\dfrac{d}{ds} (1-s^2)^{-1} \right) \Big|_{s=0}$

$\Rightarrow \quad E[X^2] = \quad \dfrac{d}{ds}\left(-(1-s^2)^{-2} \cdot -2s \right)\Big|_{s=0}$

$\Rightarrow \quad E[X^2] = \quad \dfrac{d}{ds}\left(2s(1-s^2)^{-2} \right)\Big|_{s=0}$

$\Rightarrow \quad E[X^2] = \quad 2(1-s^2)^{-2} + 2s(1-s^2)^{-3} \cdot -2 \cdot -2s \Big|_{s=0}$

1(b)

$\Rightarrow \quad E[X^2] = \quad 2$ Ans

—— ✗ —— ✗ —— ✗ ——

2.

X_i	Y_i	$Z_i = X_i + Y_i$	$P(Z_i)$
0	0	0	
0	1	1	
0	2	2	
1	0	1	
1	1	2	
2	0	2	

Z_i	Possible (X_i, Y_i)	$P(Z_i) = \sum P(x_i, y_i) = \sum P(x_i) \cdot P(y_i)$
0	(0,0)	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 \cdot \frac{1}{3}$
1	(0,1), (1,0)	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 \cdot \frac{1}{3} + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^1 \frac{1}{3}$
2	(0,2), (1,1), (2,0)	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 \cdot \frac{1}{3} + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^1 \frac{1}{3}$
		$+ \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 \cdot \frac{1}{3}$

Z_i	$P(Z_i) = \sum P(x_i) P(y_i)$
0	$\frac{1}{9} \cdot 1 = \frac{1}{9} \cdot \frac{9}{9}$
1	$\frac{1}{9} \cdot \frac{5}{3} = \frac{1}{9} \cdot \frac{15}{9}$
2	$\frac{1}{9} \cdot \frac{19}{9}$

$$P(Z \leq 2) = P(X+Y \leq 2) = \frac{1}{9}\left\{ \frac{9}{9} + \frac{15}{9} + \frac{19}{9} \right\} = \frac{43}{81}$$
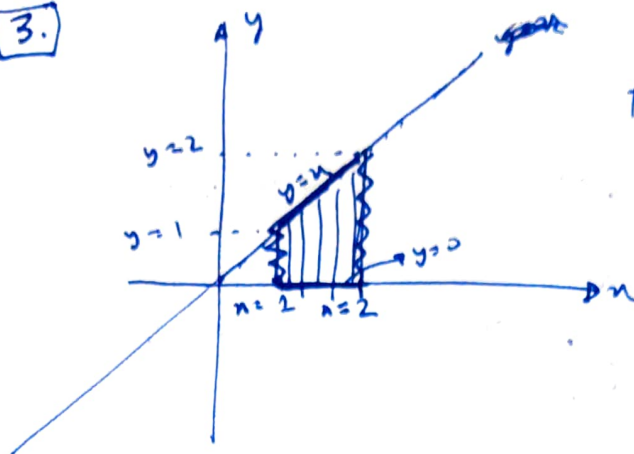
2.

$$P(Z \leq P(X+Y \leq 2) = \frac{43}{81}$$

$$P(X+Y \leq 2) \approx 0.5309$$

Ans

———×—————————×———————×———

3.

$f_{X,Y}(x,y)$ is uniform in the shaded region

$$f_{X,Y}(x,y) = \frac{1}{\text{Area of shaded reg.}}$$

$$\forall\ x \in [1,2]$$
$$y \in [0,x]$$

or $f_{X,Y}(x,y) = \dfrac{1}{\frac{(2+3)(2+1)}{2} \cdot 1}$

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{2}{3} & x \in [1,2] \\ & y \in (0,x] \\ \\ 0 & \text{else} \end{cases}$$

PTO

$$\int_{y=0}^{y=2} 2 - \frac{2y}{3}\, dy$$

$$4y = \frac{2}{3} \cdot \frac{1}{2} y$$

$$\frac{2}{3}(2-y)$$
$$\frac{2}{3}(2y - \frac{y^2}{2}) \quad 2\frac{y}{3}\frac{2}{2}\frac{y^2}{6}$$
$$\frac{2}{3}(4-2) \quad \frac{2-y}{3}$$
$$\frac{4}{3} \quad \frac{2}{9}y - \frac{y^2}{6}\Big|_0^2$$
$$\frac{4}{3} - \frac{4}{6} = \frac{2}{3}$$
$$\frac{2}{3}(2-y)$$
$$\frac{2}{3}(2y - \frac{y^2}{2})\Big|_{y=1}^{y=2}$$

PTO!

$$f_Y(y) = \int_{x=0\,y}^{x=2} f_{X,Y}(x,y)\, dx \quad + \int_{x=1}^{x=2} f_{X,Y}(x,y)\, dx$$

or $f_Y(y) = \int_{x=y}^{x=2} \dfrac{2}{3}\, dx \quad + \int_{x=1}^{x=2} \dfrac{2}{3}\, dx$

$$f_Y(y) = \frac{2}{3} \quad \forall\ y \in [0,2]$$
$$f_Y(y) = 2 - \frac{2y}{3} \quad \forall\ y \in (0,2]$$

$$f_{X|Y=y}(X, Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{X|Y=y}(X, Y=y) = \frac{\frac{2}{3}}{\frac{1}{3}} = 1$$

$$f_Y(y) = \begin{cases} \displaystyle\int_{u=1}^{u=2} f_{X,Y}(u,y)\,du & y \in (0,1) \\[2em] \displaystyle\int_{u=y}^{u=2} f_{X,Y}(u,y)\,du & y \in (1,2) \end{cases}$$

or $f_Y(y) = \begin{cases} \displaystyle\int_{u=1}^{u=2} \frac{2}{3}\,du & y \in (0,1) \\[2em] \displaystyle\int_{u=y}^{u=2} \frac{2}{3}\,du & y \in (1,2) \end{cases}$

3(a)$^{(i)}$

$\leftarrow f_Y(y) = \begin{cases} \dfrac{2}{3} & y \in (0,1) \\[2em] \dfrac{2}{3}(2-y) & y \in (1,2) \end{cases}$

$\underline{\underline{Ans}}$

~~$f_{XY}(u,y)$~~

$$f_{(X|Y=y)} = \frac{f_{XY}(u,y)}{f_Y(y)}$$

• $f_{X|Y}(x|Y=y) = \begin{cases} \dfrac{\frac{2}{3}}{\frac{2}{3}} & y \in (0,1) \\[4mm] \dfrac{\frac{2}{3}}{\frac{2}{3}(2-y)} & y \in (1,2) \end{cases}$

3(a)(ii)

$f_{X|Y}(x|Y=y) = \begin{cases} 1 & y \in [0,1] \\[4mm] \dfrac{1}{1-y} & y \in (1,2) \end{cases}$ Ans

3(a)(iii)

$\therefore$ $f_{X|Y}$ depends ons value of y $\therefore$ X and Y are

NOT ~~todo~~ independent.   Ans

4 $\hat{X}_{X|Y} = \mu_{X|X}$ $\equiv y$

MMSE

$\hat{X}_{X|Y}$
MM

But $\mu_{X|Y=y} = \int\limits_{n=01}^{n=2} n \cdot f_{X|X}(x|Y=y)\, dn$

$\therefore$ $\mu_{X|Y=y} = \begin{cases} \displaystyle\int\limits_{n=1}^{n=2} n \cdot 1 \cdot dn & y \in [0,1] \\[6mm] \displaystyle\int\limits_{n=1}^{n=2} \dfrac{1}{1-y} \cdot n \cdot dn & y \in (1,2] \end{cases}$

on $\mu_{X|Y=y} = \begin{cases} \dfrac{x^2}{2}\Big|_2^2 & y\notin(0,1) \\[4mm] \dfrac{1}{1-y}\cdot\dfrac{x^2}{2}\Big|_1^2 & y\in(1,2) \end{cases}$

o $\mu_{X|Y=y} = \begin{cases} \dfrac{3}{2} & y\in(0,1) \\[4mm] \dfrac{3}{2}\cdot\dfrac{1}{1-y} & y\in(1,2) \end{cases}$

3b)

o $\widehat{X}_{X|Y=y}_{MMSE} = \begin{cases} \dfrac{3}{2} & y\in(0,1] \\[4mm] \dfrac{3}{2}\cdot\dfrac{1}{1-y} & y\in(1,2] \end{cases}$ Ans

— X — X — X —

4. $f_X(x) = N(x, \mu_X = 0, \sigma_X^2 = 4)$

or $f_X(x) = \dfrac{1}{2\sqrt{2\pi}} \, e^{-\frac{1}{2}\left\{\left(\frac{x}{2}\right)^2\right\}}$

$f_Y(y) = N(y, \mu_Y = 0, \sigma_Y^2 = 9)$

or $f_Y(y) = \dfrac{1}{3\sqrt{2\pi}} \, e^{-\frac{1}{2}\left\{\left(\frac{y}{3}\right)^2\right\}}$

$f_{X,Y}(x,y) = N\left(x, y, \mu_X = 0, \mu_Y = 0, \sigma_X^2 = 4, \sigma_Y^2 = 9, \rho = -\frac{1}{2}\right)$

or $f_{X,Y}(x,y) = \dfrac{1}{2\sqrt{2\pi} \cdot 3\sqrt{2\pi}\sqrt{\left(1 - \left(-\frac{1}{2}\right)^2\right)}} \, e^{-\frac{1}{2\left(1-\left(-\frac{1}{2}\right)^2\right)}\left\{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 - 2 \cdot \left(-\frac{1}{2}\right) \cdot \left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}$

4(a)

or $f_{X,Y}(x,y) = \dfrac{1}{6\sqrt{3}\,\pi} \, e^{-\frac{1}{\sqrt{3/4}}\left\{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}$

Again.
(for clarity)

4(a)

$f_{X,Y}(x,y) = \dfrac{1}{6\sqrt{3}\,\pi} \, e^{-\frac{1}{2} \cdot \frac{4}{3}\left\{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}$ Ans

4b)

$Z = X + Y$

$E(Z) = \mu_X + \mu_Y$

or $\mu_Z = 0 + 0$

$\boxed{\mu_Z = 0}$

$W = bX$

$E[W] = E[bX]$

or $\mu_W = b\,E[X]$

or $\mu_W = b\,\mu_X$

or $\boxed{\mu_W = 0}$ Ans

$$\sigma^2_Z = \sigma^2_x + \sigma^2_x + 2\rho\sigma_x\sigma_y \qquad \sigma^2_w = b^2\sigma^2_x$$

$$\text{or } \sigma^2_Z = 4 + 9 + 2 \times -\frac{1}{2} \times 2 \times 3 \qquad \text{or } \boxed{\sigma^2_w = 4b^2}$$

$$\text{or } \sigma^2_Z = 13 \quad - 6$$

$$\text{or } \boxed{\sigma^2_Z = 7}$$

$$Con(Z, W) = Con(X+Y, bX)$$

$$\text{or } Con(Z, W) = E\left[\{X+Y - E[X+Y]\}\{bX - E[bX]\}\right]$$

$$\text{or } Con(Z, W) = E\left[\{X+Y\}\{bX\}\right]$$

$$\text{or } Con(Z, W) = E\left[bX^2 + bXY\right]$$

$$\text{or } Con(Z, W) = bM_{x^2} + b\,Con(XY)$$

$$\longrightarrow \text{Here, } Con(X,Y) = E(XY)$$
$$\text{as } M_x M_y = 0.$$

$$\text{or } Con(Z, W) = b\{\sigma^2_x + M^2_x\} + b\{\rho_{xy}\sigma_x\sigma_x\}$$

$$\text{or } Con(Z, W) = b\left[4 + 0^2 + -\frac{1}{2}\cdot 2\cdot 3\right]$$

$$\text{or } Con(Z, W) = b[1] \quad \boxed{4(b)^{ii)}}$$

$$\text{or } \boxed{Con(Z, W) = b} \Rightarrow \boxed{Con(Z, W) > 0 \ \forall \ b > 0} \ \underline{Ans}$$

$$\rho_{z,w} = \frac{Cov(z,w)}{\sigma_z \cdot \sigma_w}$$

$$\text{or } \rho_{z,w} = \frac{b}{\sqrt{7} \cdot 2b}$$

$$\boxed{\rho_{z,w} = \frac{1}{2\sqrt{7}}}$$

$$\boxed{4(b)^{ii)}}$$

$$\therefore \boxed{f_{z,w}(z,w) \sim \mathcal{N}\left(z,w, \mu_z=0, \mu_w=0, \sigma_z^2=7, \sigma_w^2=4b^2, \rho_{zw}=\frac{1}{2\sqrt{7}}\right)} \underline{Ans}$$

$$4(c) \quad \hat{Y}_{Y|X=n} \underset{MMSE}{=} E[Y|X=n]$$

Let's try to find $f_{Y|X}(Y|X=n)$ first!

$$f_{Y|X}(Y|X=n) = \frac{f_{X,Y}(n,y)}{f_X(n)}$$

$$\text{or } f_{Y|X}(Y|X=n) = \frac{\frac{1}{2\sqrt{2\pi} \cdot 3\sqrt{2\pi} \cdot \frac{\sqrt{3}}{24}} \cdot e^{-\frac{1}{2 \cdot \frac{4}{9}\frac{3}{4}}\left\{\left(\frac{n}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{ny}{2\cdot 3}\right)\right\}}}{\frac{1}{2\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left\{\left(\frac{n}{2}\right)^2\right\}}}$$

$$\text{or } f_{Y|X}(Y|X=n) = \frac{1}{3\sqrt{2\pi} \cdot \frac{3}{4}} e^{-\frac{1}{2}\{\,\,\,\}}$$

$$\text{or } f_{Y|X}(Y|X=n) = \frac{1}{\frac{3\sqrt{3}}{2} \cdot \sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{1}{3}\left(\frac{n}{2}\right)^2 + \frac{4}{3}\left(\frac{y}{3}\right)^2 + \frac{4}{3}\left(\frac{n}{2}\right)\left(\frac{y}{3}\right)\right\}}$$

$$\text{or } f_{Y|X}(Y|X=n) = \frac{1}{\frac{3\sqrt{3}}{2} \cdot \sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{n}{2\sqrt{3}}\right)^2 + \left(\frac{y}{\frac{3\sqrt{3}}{2}}\right)^2 + 2 \cdot 1\left(\frac{n}{2\sqrt{3}}\right)\left(\frac{y}{\frac{3\sqrt{3}}{2}}\right)\right\}}$$

$$\text{or } f_{Y|X}(Y|X=n) = \frac{1}{\left(\frac{3\sqrt{3}}{2}\right)\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y}{\frac{3\sqrt{3}}{2}} + \frac{n}{2\sqrt{3}}\right)^2\right\}}$$

$$\text{or } f_{Y|X}(Y|X=n) = \frac{1}{\left(\frac{3\sqrt{3}}{2}\right)\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y + \frac{3}{4}n}{\left(\frac{3\sqrt{3}}{2}\right)}\right)^2\right\}}$$

$$\therefore f_{Y|X}(Y|X=n) \Rightarrow$$

$$\text{or } Y|X=n \sim N\left(y,\ \mu_Y = -\frac{3}{4}n,\ \sigma_Y^2 = \frac{27}{4}\right)$$

4(C)

$$\Rightarrow \boxed{\hat{Y}_{Y|X=n} = \mu_{Y|X=n} = -\frac{3}{4}n} \quad \text{Ans}$$

MMSE

5. $\quad X \sim \text{unif}[0,1)$

$$Y|X=n \sim \text{unif}[0,n]$$

$$Z|Y=y, X=n \sim \text{unif}[0,y]$$

$$E[X^2 Y^2 Z^2] = \iiint n^2 y^2 z^2 \cdot f_{X,Y,Z}(n,y,z)\, dz\, dy\, dn$$

$\sim$ ~~$E[X^2 Y^2 Z^2] = \iint$~~

But $f_{X,Y,Z}(n,y,z) = $ ~~$f_{z|n}$~~ ~~$f_z|z$~~

$$= f_Z(z|X=n, Y=y) \cdot f_{X,Y}(X=n, Y=y)$$

$\sim \quad f_{X,Y,Z}(n,y,z) = f_Z(z|X=n, Y=y) \cdot f_{Y|X}(y|X=n) \cdot f_X(n)$

$\sim \quad f_{X,Y,Z}(n,y,z) = \dfrac{1}{y} \cdot \dfrac{1}{n} \cdot \dfrac{1}{1} \qquad \forall \quad \begin{array}{l} z \in (0,y) \\ y \in (0,n) \\ n \in (0,1) \end{array}$

$\therefore \quad E[X^2 Y^2 Z^2] = \displaystyle\int_{n=0}^{n=1}\int_{y=0}^{y=n}\int_{z=0}^{z=y} n^2 \cdot y^2 \cdot z^2 \cdot \frac{1}{ny}\, dz\, dy\, dn$

$\sim \quad E[X^2 Y^2 Z^2] = \displaystyle\int_{n=0}^{n=1}\int_{y=0}^{y=n}\int_{z=0}^{z=y} ny\, z^2\, dz\, dy\, dn$

$\sim \quad E[X^2 Y^2 Z^2] = \displaystyle\int_{n=0}^{n=1} n \int_{y=0}^{y=n} y \cdot \left.\frac{z^3}{3}\right|_{z=0}^{z=y}\, dy\, dn$

or $E[X^2Y^2Z^2] = \int\limits_{x=0}^{x=1} x\int\limits_{y=0}^{y=x} \frac{y^4}{3} \, dy \quad dx$

or $E[X^2Y^2Z^2] = \int\limits_{x=0}^{x=1} x \cdot \frac{y^5}{15} \Big|_{y=0}^{y=x} \, dx$

or $E[X^2Y^2Z^2] = \int\limits_{x=0}^{x=1} \frac{x^6}{15} \, dx$

or $E[X^2Y^2Z^2] = \frac{x^7}{105} \Big|_{x=0}^{x=1}$

or $\boxed{E[X^2Y^2Z^2] = \frac{1}{105}}$ Ans

———— X ———— X ———— X ————

$\int\int\int xy0^2$

$x y \frac{y^3}{3}$

$\frac{xy^4}{3}$

$\frac{xy^5}{3 \cdot 5} \frac{xy^4}{3}$

$\frac{xy^5}{3 \cdot 5} \Big|$

$x \cdot \frac{x^5}{3 \cdot 5} \quad \frac{x^6}{3 \cdot 5} \Big| \quad \frac{x^7}{3 \cdot 5 \cdot 7} \Big|_0$

6. Random Processes are mappings of outcomes from a probabilistic experiment to signals.

$$X(A,t) = \cos(t)$$

$$X(C,t) = \begin{cases} 7 & \lfloor t \rfloor \text{ is even} \\ 0 & \lfloor t \rfloor \text{ is odd} \end{cases}$$

$$X(B,t) = 2e^{-2t}$$
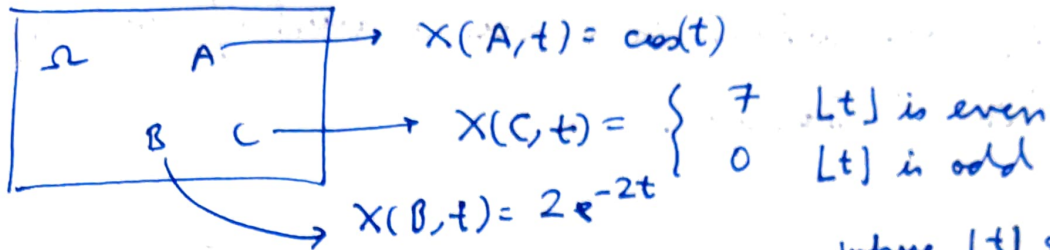
where $\lfloor t \rfloor$ refers to the greatest integer lesser than or equal to t.

Virtually any physical process or simulated phenomenon can be represented as a random process.

Eg. Bus Voltages in a power grid are time-varying signals, and there are multiple buses (outcomes) in a power grid or we can measure a single bus voltage in multiple ways. (sample functions).

$$i=1 \rightarrow V_1(t)$$
$$\vdots$$
$$i=14 \rightarrow V_{14}(t)$$
$$\rightarrow V_i(t)$$
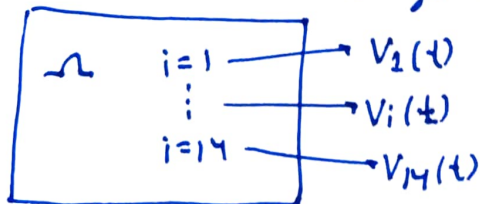
$i \in 1 \text{ to } 14$
$i \in N$

$\Omega \equiv$ IEEE 14 Bus System.

Study of Random Processes is important as it can be used to analyze, estimate, predict, forecast signals which can be very beneficial to us.

Eg. If tomorrow's load demand for a regional substation could be predicted accurately, we can optimize the scheduling of power generation, spot market rates, etc., which can save a grid operator millions of Dollars.

Ans

7.

$$X(t) = \frac{1}{t+1+C} \quad t \in R^+$$
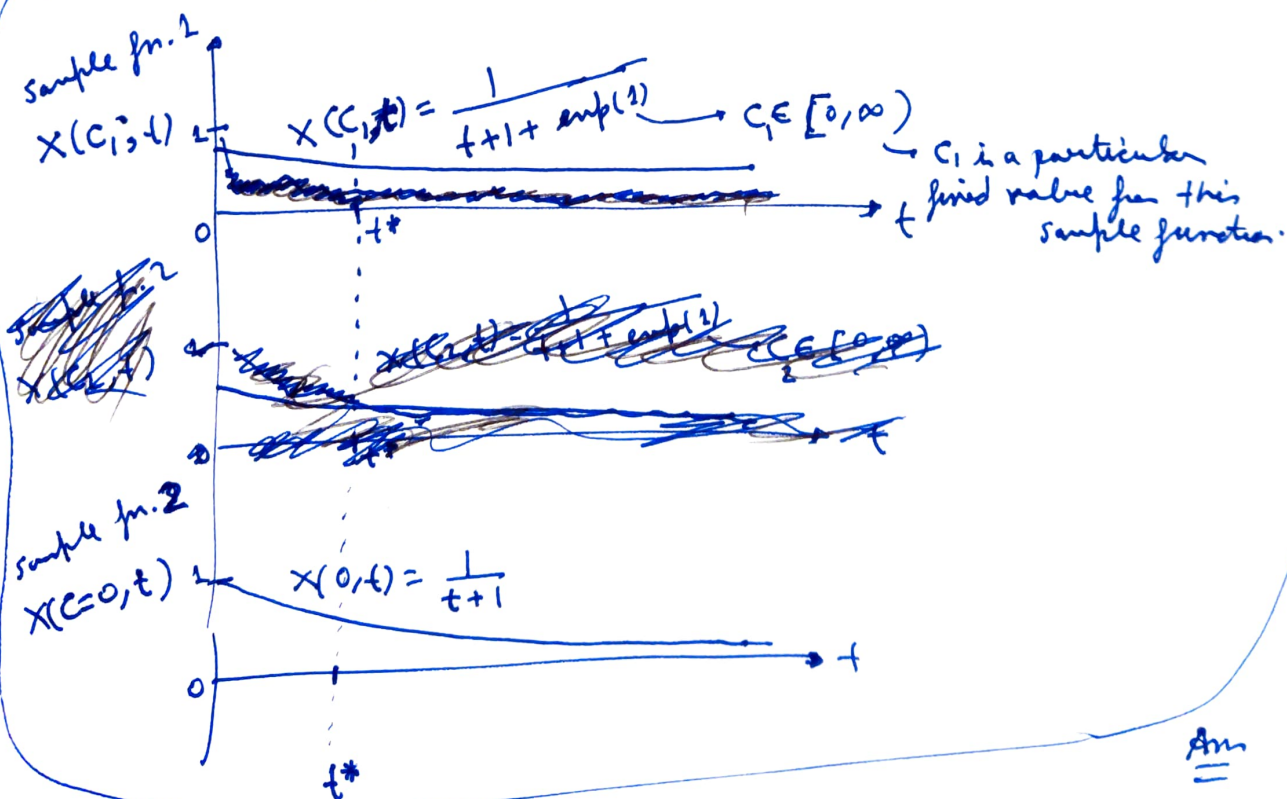
$$C \sim \text{exp}(1)$$

~~$f_C(c) =$~~ or $f_C(c) = e^{-c} \quad c \geq 0$

7(a)

$X(t)$ is a Random Process as

$X(t) = X(C,t)$ is a set of time varying signals, which

are all mapped to the outcome C of a

~~pro~~ probabilistic experiment, where

$$C \sim \text{exp}(1).$$

So $X(C,t)$ can ~~be~~ have a sample function $X(C=2,t)$ or it can

be sampled ~~at~~ time t^* to get a ~~unique~~

set of ~~$X(C,t^*)$~~ $X(C, t=t^*) \ \forall C \sim \text{exp}(1)$.

sample fn. 2
$X(C_i, t)$

$$X(C_i, t) = \frac{1}{t+1+ \text{exp}(1)} \longrightarrow C_i \in [0,\infty)$$

C_i is a particular fixed value from this sample function.

~~sample fn. 2~~

~~$X(C_i, t)$~~ or ~~$X(C,t) = \frac{1}{t+1+\text{exp}(1)} \quad C \in [0,\infty)$~~

sample fn. 2
$X(C=0, t)$

$$X(0,t) = \frac{1}{t+1}$$

t^*

<u>Ans</u>

$X(t) = \dfrac{1}{t+1+c}$ $\qquad t \in R^+ \qquad c \sim exp(1)$

$\qquad\qquad\qquad\qquad\qquad c \in [0, \infty)$

~~$X(t) \in \frac{1}{t+2t} \left[0, \frac{1}{2t+1} \right)$~~

$X(t) \in \left(0, \dfrac{1}{t+1} \right]$

~~$F_{X(t)}$~~

Approach: Find CDF then pdf:

$$F_{X(t)}(n) = P(X(t) \leq n) = P\left(\dfrac{1}{t+1+c} \leq n \right)$$

$F_{X(t)}(n) = P\left(c + t + 1 \geq \dfrac{1}{n} \right)$

$F_{X(t)}(n) = P\left(c \geq \dfrac{1}{n} - (t+1) \right)$

$F_{X(t)}(n) = 1 - P\left(c \leq \dfrac{1}{n} - (t+1) \right)$

$~~F_{X(t)}~~ F_{X(t)}(n) = \begin{cases} 0 & n \leq 0 \\ 1 - \left(1 - e^{-\left(\frac{1}{n} - (t+1) \right)} \right) & n \in \left(0, \frac{1}{t+1} \right] \\ 1 & n \geq \frac{1}{t+1} \end{cases}$

$1 - e^{(t+1)} \cdot e^{-\frac{1}{n}}$

7(b)

$f_{X(t)}(n) = \dfrac{d}{dn} F_{X(t)}(n) = \begin{cases} 0 & n \leq 0 \\ \dfrac{1}{n^2} \cdot e^{-\frac{1}{n}} \cdot e^{(t+1)} & n \in \left(0, \frac{1}{t+1} \right] \\ 0 & n \geq \frac{1}{t+1} \end{cases}$