$\boxed{1.}$ $f_X(x) = Cx^{-3}, \quad x \geq 1$

1(a) $\because$ ~~$(f_X(x))$~~ We know that $\int_{x=-\infty}^{\infty} f_X(x)\,dx = 1$

or $\int_{x=1}^{\infty} Cx^{-3}\,dx = \left.\frac{Cx^{-2}}{-2}\right|_1^{\infty} = 1$

or $\left.\frac{Cx^{-2}}{2}\right|_{\infty}^{1} = 1$

or $\frac{C}{2} = 1$

$\boxed{1a}$ or $\boxed{C = 2}$ Ans

1(b) $E(X) = \int_{x=-1}^{\infty} x f_X(x)\,dx = \int_{x=1}^{\infty} x \cdot 2x^{-3}\,dx =$

or $E(X) = \int_{x=1}^{\infty} 2x^{-2}\,dx = \left.-2x^{-1}\right]_1^{\infty}$

or $E(X) = \left.2x^{-1}\right|_{\infty}^{1}$

$\boxed{1(b)}$ (i) $\boxed{E(X) = 2}$ Ans

$E(X^2) = \int_{x=1}^{\infty} x^2 f_X(x)\,dx = \int_{x=1}^{\infty} x^2 \cdot 2x^{-3}\,dx$

or $E(X^2) = \int_{x=1}^{\infty} 2x^{-1}\,dx$

or $E(X^2) = \left.2\ln(x)\right|_1^{\infty} \to \infty$

$\boxed{1(b)}$ (ii) $\Rightarrow \boxed{V(X) = E(X^2) - \{E(X)\}^2 \to \infty}$ Ans

1 (c) $$F_X(x=\alpha) = \int_{x=1}^{x=\alpha} f_X(x)\,dx \quad \text{for } x \in [1,\infty)$$

or $$F_X(x=\alpha) = \int_{x=1}^{x=\alpha} 2x^{-3}\,dx \quad \text{for } x \in [1,\infty)$$

1(c) or $$\boxed{F_X(x=\alpha) = \begin{cases} 1-\alpha^{-2} & \text{for } x \in [1,\infty) \\ 0 & \text{else} \end{cases}}$$ Ans

1 (d) $$f_X(x \mid x \geq 2) = \frac{f_X(\cancel{\alpha}\, X = \cancel{x} \;\&\&\; X = x \geq 2)}{f_X(\cancel{x\&\&}\, X = x \geq 2)}$$

or $$f_X(x \mid x \geq 2) = \frac{f_X(X=x)}{f_X(X=x \geq 2)} \quad \text{where } x \geq 2$$

or $$f_X(x \mid x \geq 2) = \frac{f_X(X=x)}{1-F_X(x=2)} \quad \text{where } x \geq 2$$

or $$f_X(x \mid x \geq 2) = \frac{2x^{-3}}{1-\{1-\alpha^{-2}\}\big|_{\alpha=2}} \quad \text{where } x \geq 2$$

1(d) or (i) $$\boxed{f_X(x \mid x \geq 2) = \begin{cases} 8x^{-3} & x \geq 2 \\ 0 & \text{else} \end{cases}}$$ Ans

$$E[X \mid X \geq 2] = \int_{x=-\infty}^{x=\infty} x f_{X|X\geq 2}(x \mid x \geq 2)\,dx$$

or $$E[X \mid X \geq 2] = \int_{x=2}^{\infty} x \cdot 8x^{-3}\,dx$$

~~or $E[X \mid X \geq 2] =$~~

or $E[X \mid X \geq 2] = \int_{x=2}^{\infty} 8x^{-2}\,dx$

or $E[X \mid X \geq 2] = -8x^{-1}\Big|_{2}^{\infty}$

$\boxed{1(d)}$-(ii)

or $\boxed{E[X \mid X \geq 2] = \cancel{8}\ 4}$ Ans

2. a. $X \sim B(p)$

2a(i)

$$\text{pmf} \rightarrow P(X) = \begin{cases} 0 & x < 0 \\ 1-p & x = 0 \\ 0 & x = (0,1) \\ p & x = 1 \\ 0 & x > 1 \end{cases}$$

Ans

$$f_X(x) = (1-p)\,\delta(x) + p\,\delta(x-1)$$

2a(ii)

$$\text{CDF} \quad F_X(x) = \begin{cases} 0 & x < 0 \\ 1-p & x \in [0, 1) \\ 1 & x \geq 1 \end{cases}$$

X_i	$P(X_i)$	$X_i\,P(X_i)$	$X_i^2\,P(X_i)$	.....	$X_i^5\,P(X_i)$
0	$1-p$	0	0	...	0
1	p	p	p	...	p
Sum		p	p	...	p

$\Rightarrow$ All first five moments of X are

2a(iii)

$$E(X^i) = p \quad \forall\, i \in N$$

Ans

—— xxx —— xxx —— xx ——

2b. $X \sim BB(p)$

2b(i)

$$\text{pmf} = P(X) = \begin{cases} 0 & x < -1 \\ 1-p & x = -1 \\ 0 & x \in (-1, 1) \\ p & x = 1 \\ 0 & x > 1 \end{cases}$$

Ans

$f_X(x) = (1-p)\,\delta(x+1) + p\,\delta(x-1)$

2b(ii)

$$\boxed{F_X(x) = \begin{cases} 0 & x < -1 \\ 1-p & x \in [-1, 1) \\ 1 & x \geq 1 \end{cases}}$$

Ans

x_i	$P(x_i)$	$x_i^2 P(x_i)$	$x_i^3 P(x_i)$	$x_i^4 P(x_i)$	$x_i^5 P(x_i)$
-1	$1-p$	$1-p$	$p-1$	$1-p$	$p-1$
1	p	p	p	p	p
sum		1	$2p-1$	1	$2p-1$

2b(iii)

$\Rightarrow$ $\cancel{E(X_i}$

$$\boxed{E(X^n) = \begin{cases} 1 & \text{if } n = 2, 4, \ldots \\ 2p-1 & \text{if } n = 1, 3, 5, \ldots \end{cases}}$$

Ans

——— X ——— X ——— X ———

3. $Y \sim \mathcal{N}(\mu = 1, \sigma^2 = 4)$

3a. $\boxed{\text{mean} = \mu = 1}$ Ans

$\boxed{\text{std. dev.} = \sigma = 2}$ Ans

$E(X^2) = \cancel{E(X)}\ V(X) + (E(X))^2$

or $\boxed{E(X^2) = \cancel{\text{Ex}}\ \sigma^2 + \mu^2 = 5}$ Ans

3b. $P(X \in [2, 3])$

or $P\left(Z \in \left[\frac{2-\mu}{\sigma}, \frac{3-\mu}{\sigma}\right]\right)$

or $P(Z \in [0.5, 1]) = G(1) - G(0.5)$

3b

or $\boxed{P(X \in [2, 3]) = 0.1499}$ Ans

3c. $P(Z \geq 1) = 0.3$

or $\cancel{P(}\ 1 - P(Z \leq 1) = 0.3$

or $P(Z \leq 1) = 0.7$

or $G\,\cancel{\Phi}\left(\frac{1-0}{\sigma}\right) = 0.7$

or $\frac{1}{\sigma} = G^{-1}(0.7)$

or $\sigma = \frac{1}{G^{-1}(0.7)} = \frac{1}{0.5244}$

or $\sigma = 1.9069$

3c

So $\boxed{Z \sim \mathcal{N}(0, 1.9069)}$ Ans

— X — X — X —

$\boxed{4.}$

$$\text{Exp.}\ \begin{cases} \xrightarrow{0.2} A & \boxed{f_X(x|A) = 2e^{-2x}} \quad x \geq 0 \quad \text{else} = 0 \\ \xrightarrow{0.3} B & \boxed{f_X(x|B) = e^{-x}} \quad x \geq 0 \quad \text{else} = 0. \\ \xrightarrow{0.5} C & f_X(x|C) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{else.} \end{cases} \end{cases}$$

or $\boxed{f_X(x|C) = \delta(x)}$

$$P(A|X=x) = \frac{f_{X|A}(X=x|A) \cdot P(A)}{f_X(x)}$$

But $f_X(x) = f_X(X=x|A) \cdot P(A) + f_X(X=x|B) \cdot P(B) + f_X(X=x|C) \cdot P(C)$ (By Law of Total probabilities).

So $\boxed{4^{(i)}}$

$$\boxed{P(A|X=x) = \frac{(2e^{-2x})(0.2)}{(2e^{-2x})(0.2) + (1e^{-1x})(0.3) + (\delta(x)) \cdot (0.5)}} \quad x \geq 0$$

Ans

$$\boxed{P(A|X=x) = \begin{cases} 0 & x \to 0 \\ \dfrac{0.4e^{-2x}}{0.4e^{-2x} + 0.3e^{-x}} & x > 0 \end{cases}}$$

Ans

$\boxed{4^{(ii)}}$

$$\boxed{P(B|X=x) = \frac{(1e^{-x})(0.3)}{(2e^{-2x})(0.2) + (1e^{-x})(0.3) + (\delta(x))(0.5)}} \quad x \geq 0$$

Ans

$$\boxed{P(B|X=x) = \begin{cases} 0 & x \to 0 \\ \dfrac{0.3e^{-x}}{0.4e^{-2x} + 0.3e^{-x}} & x > 0 \end{cases}}$$

Ans

4(iii)

$$P(C \mid X = x) = \frac{(\delta(x))(0.5)}{(0.4)e^{-2x} + (0.3)e^{-x} + 0.5\,\delta(x)} \quad x \geq 0$$

Ans

or $P(C \mid X = x) = \begin{cases} 1 & x \to 0 \\ 0 & x > 0 \end{cases}$

Ans

$\boxed{5.}$ $X \sim unif[0,2) \Rightarrow f_X(x) = \begin{cases} \frac{1}{2} & x \in [0,2] \\ 0 & else \end{cases}$

$P(A \mid X = x) = 1 - \frac{x}{2}$ $\qquad P(B \mid X = x) = \frac{x}{2}$

5.a $P(A) = \int_{x=-\infty}^{\infty} P(A \mid X = x) \cdot f_X(x)\, dx$

or $P(A) = \int_{x=0}^{2} \left(1 - \frac{x}{2}\right) \cdot \left(\frac{1}{2}\right) dx$

or $P(A) = \frac{1}{2}x - \frac{x^2}{8} \Big|_0^2$

$\boxed{5a^{(i)}}$

or $\boxed{P(A) = \frac{1}{2}}$ Ans

$P(B) = \int_{x=0}^{2} \left(\frac{x}{2}\right)\left(\frac{1}{2}\right) dx = \frac{x^2}{8}\Big|_0^2$

$\boxed{5a^{(ii)}}$

or $\boxed{P(B) = \frac{1}{2}}$ Ans

5b. $f_{X|A}(X = x \mid A) = \frac{P(A \mid X = x) \cdot f_X(x)}{P(A)}$

or $f_{X|A}(X = x \mid A) = \begin{cases} \dfrac{\left(1 - \frac{x}{2}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} & \text{if } x \in [0,2] \\ 0 & else \end{cases}$

$\boxed{5b}$

or $\boxed{f_{X|A}(X = x \mid A) = \begin{cases} 1 - \frac{x}{2} & x \in [0,2] \\ 0 & else \end{cases}}$ Ans

$\cong$ eq $$\boxed{f_{X|B}(X=x|B) = \begin{cases} \frac{x}{2} & x \in [0,2] \\ 0 & \text{else} \end{cases}}$$

5c. $$P(A|B) = \frac{P(AB)}{P(B)}$$

or $$P(A|B) = \frac{1}{P(B)} \cdot \int_{x=0}^{2} P(AB|X=x) \cdot f_X(x)\, dx$$

or $$P(A|B) = \frac{1}{P(B)} \cdot \int_{x=0}^{2} P(A|X=x) \cdot P(B|X=x) \cdot f_X(x)\, dx$$

[given that $X=x$, A and B are independent events].

or $$P(A|B) = \frac{1}{\cancel{(1/2)}} \cdot \int_{x=0}^{2} \left(1-\frac{x}{2}\right)\left(\frac{x}{2}\right)\cancel{\left(\frac{1}{2}\right)}\, dx$$

or $$P(A|B) = \int_{x=0}^{2} \frac{x^2}{4} - \frac{x^3}{12}$$

or $$\boxed{P(A|B) = \frac{1}{3}}$$ Ans

$\Rightarrow P(A|B) \neq P(A)$

$\Rightarrow$ 5c: **A and B are NOT independent.** Ans

Key Takeaway: Two events may be independent for a given set of conditions/events but that does NOT mean that they are independent in general! Here $P(A|X=x)$ and $P(B|X=x)$ are independent but $P(A)$ and $P(B)$ are NOT!

6. $Q \sim p(1-p)^{q-1}$ $\quad q = 1, 2, 3, \ldots$

$p = 0.7$ = Probability of success ~~after~~ for a trial.

q = # repeated Bernoulli's trials required to obtain a success.

~~$F_Q(q)$=~~

$$F_Q(q=\alpha) = \sum_{q=1}^{q=\alpha} p(1-p)^{q-1}$$

$$\text{or } F_Q(q=\alpha) = \frac{p}{1-p} \cdot \sum_{q=2}^{\alpha} (1-p)^q$$

$$\text{or } F_Q(q=\alpha) = \frac{p}{1-p} \cdot \frac{(1-p)(1-(1-p)^\alpha)}{1-(1-p)}$$

$$\text{or } \boxed{F_Q(q=\alpha) = 1 - (1-p)^\alpha \quad \alpha = 1, 2, 3, \ldots}$$

α consecutive failures.

$$F_Q(q=4) = 1 - (1-p)^4$$

$$\text{or } F_Q(q=4) = 1 - 0.3^4$$

$$\text{or } \boxed{F_Q(q=4) = 0.9919} \boxed{= P(Q \leq 4)}$$

$$P_Q(Q=q \mid Q \leq 4) = \frac{P(Q=q \,\&\, Q \leq 4)}{P(Q \leq 4)}$$

$$\text{or } P_Q(Q=q \mid Q \leq 4) = \begin{cases} \dfrac{P(Q=q)}{P(Q \leq 4)} & q = 1, 2, 3 \text{ or } 4. \\ 0 & \text{else} \end{cases}$$

6a (i)

or $$\boxed{P_Q(Q=q \mid Q\le 4) = \begin{cases} \dfrac{(0.7)(0.3)^{q-1}}{0.9919} & q = 1,2,3 \text{ or } 4 \\ 0 & \text{else.} \end{cases}}$$ Ans

$$E[Q \mid Q\le 4) = \sum_{\forall q} Q \cdot P(Q \mid Q \le 4)$$

or $$E[Q \mid Q\le 4) = \sum_{q=1}^{4} q \cdot \left[\frac{(0.7)(0.3)^{q-1}}{0.9919} \right]$$

6a(ii)

or $$\boxed{E[Q \mid Q\le 4) \approx 1.3959}$$ Ans

6b. $R = Q^2$

Q	R	$P_Q(q) = P_R(q^2)$
1	1	p
2	4	$p(1-p)^1$
3	9	$p(1-p)^2$
4	16	$p(1-p)^3$
$\vdots$		
q	r	$p(1-p)^q$

$$P_R(r) = P_Q(\sqrt{r}) \quad r = 1, 4, 9, 16, 25 \ldots$$

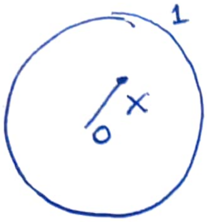
or $$P_R(r) = (p)(1-p)^{\sqrt{r}-1} \quad r = 1, 4, 9, 16, 25 \ldots$$

6b

or $$\boxed{P_R(r) = (0.7)(0.3)^{\sqrt{r}-1} \quad r = 1, 4, 9, 16, 25 \ldots}$$ Ans

———— x ———— x ———— x ————

$\boxed{7.}$



$Z = X^a \quad a > 0$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{\pi x^2}{\pi 1^2} = x^2 & x \in [0,1) \\ 1 & x \geq 1 \end{cases}$$

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 2x & x \in [0,1) \\ 0 & x \geq 1 \end{cases}$$

$$F_Z(\alpha) = P(Z \leq \alpha) = P(X^a \leq \alpha) \qquad a \geq 0 \quad Z \in [0,1]$$

or
$$F_Z(\alpha) = \begin{cases} 0 & \alpha^{\frac{1}{a}} < 0 \\ \left(\alpha^{\frac{1}{a}}\right)^2 & \alpha^{\frac{1}{a}} \in [0,1) \\ 1 & \alpha^{\frac{1}{a}} \geq 1 \end{cases}$$

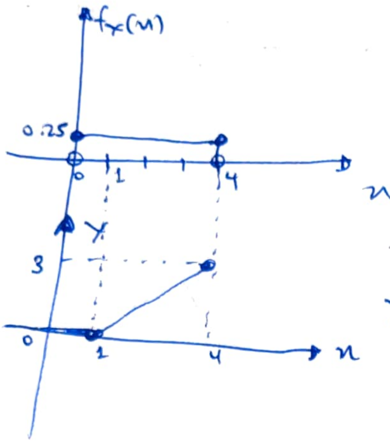
or
$$F_Z(\alpha) = \begin{cases} 0 & \alpha < 0 \\ \alpha^{\frac{2}{a}} & \alpha \in [0,1) \\ 1 & \alpha \geq 1 \end{cases}$$

$$\Rightarrow f_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{2}{a} z^{\left(\frac{2}{a} - 1\right)} & z \in [0,1] \\ 0 & z > 1 \end{cases}$$

Yes, for $a = 2$, $\frac{2}{a} - 1 = 0$ and $f_Z(z) = 1$ for $z \in [0,1]$.

i.e. z has a uniform distribution.

$\boxed{8.}$ $X \sim unif(0,4)$ $\qquad Y = \begin{cases} 0 & X < 1 \\ X-1 & X \geq 1 \end{cases}$



$Y \in [0,3]$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 0.25x & x \in [0,4] \\ 1 & x \geq 4 \end{cases}$$

$$F_Y(y) = P(Y \leq y) = \begin{cases} 0 & y < 0 \\ 0.25(y+1) & y \in [0,3) \\ 1 & y \geq 3 \end{cases}$$

$P(Y \leq y) = P(0 \leq X \leq 1) + P(1 \leq X < y+1)$

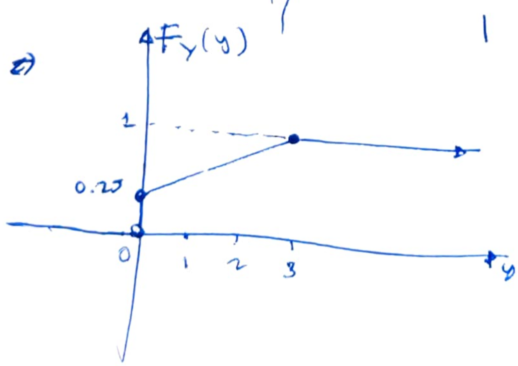
$\Rightarrow P(Y \leq y) = P(X \in (0, y+1))$

$\Rightarrow P(Y \leq y) = \int_{x=0}^{x=y+1} f_X(x)\, dx$

$\Rightarrow P(Y \leq y) = \int_{x=0}^{x=y+1} 0.25\, dx$

$\Rightarrow P(Y \leq y) = \begin{cases} 0 & y < 0 \\ 0.25(y+1) & y \in [0,3) \\ 1 & y \geq 3 \end{cases}$

$\Rightarrow$



$\boxed{8}$

$$\boxed{f_Y(y) = \begin{cases} 0 & y < 0 \\ 0.25\delta(y) + 0.25 & y \in [0,3) \\ 0 & y \geq 3 \end{cases}}$$

Ans

—— X —— X —— X ——