HW5, due 11/3/2022 [redacted]

Problem 1

a. Consider three random variables X, Y, and Z. Let's say that $f_{X,Y,Z}(x,y,z) = f_X(x)f_Y(y)f_Z(z)$, i.e. that their joint PDF is the product of their individual PDFs. Is it necessarily true that X and Y are independent?

b. Is your answer to part *a* different from the answer to the analogous problem for three events? If so, conceptually, why?

Problem 2

Consider a pair of jointly Gaussian random variables X and Y with parameters $m_x = m_y = 0$, $\sigma_x^2 = 1$, $sigma_y^2 = 4$, and $r = 0.5$.

a. Please sketch lines of equal probability for the joint PDF of X and Y.

b. What are the marginal PDFs of X and Y?

c. Please find $f_{Y|X}(y \mid X = x)$.

d. For what value of x is $E[Y \mid X = x] = -2$.

e. What is the PDF of $Z = X + Y - 1$?

f. Let $Z = 2X + 3Y$, and let $W = X - Y$. Please find $E[Z]$, $var(Z)$, $E[W]$, $var(W)$, and $cov(Z, W)$. Also, please find the joint PDF of Z and W.

g. Consider a random variable $R = aX + bY$. For what constants a and b are R and Y independent?

h. Let $Q = aX$, where a equals 1 with probability 0.5 and equals 2 with probability 0.5. Please find the PDF of Q.

Problem 3

a. Consider two random variables X and Y that are independent, and consider $Z = X + Y$. Show that the PDF of Z is given by $f_Z(z) = \int_{-\infty}^{\infty} f_Y(z-x)f_X(x)\,dx$.

b. For independent X and Y, show that $Z = X+Y$ has PDF $f_Z(z) = \int_{-\infty}^{\infty} f_Y(y)f_X(z-y)\,dy$.

c. Say that X and Y are independently and identically distributed exponential random variables with mean 0.5. What is the PDF of $Z = X + Y$?

d. Say that X and Y are independent random variables that are both uniform on $[0, 1]$. Please find the PDF of $Z = X + Y$.

e. Consider the sum of 7 independent random variables that are each uniform on $[0, 1]$. Numerically find the PDF of this sum. What common density does this PDF remind you of?

Problem 4

Say that X and Y are independent random variables that are each uniform on $[0, 1]$. Let $Z = \frac{Y}{X}$.

a. Please find the PDF of Z.

b. Please find $E[X^2 + Y^2]$.

c. Please find the joint PDF of X and Z, where $Z = \frac{Y}{X}$.

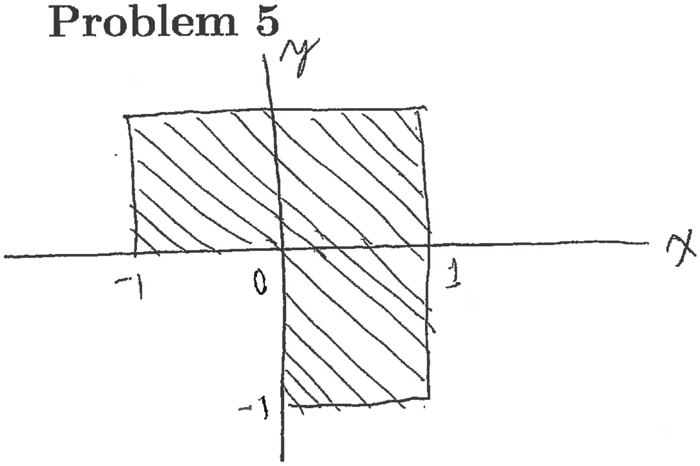
d. Please find $E[XZ]$, $cov(X, Z)$, and $\rho_{X,Z}$.

e. Please find the PDF of X given Z, and $E[X \mid Z]$.

f. Verify that, in this example. $E[E[X \mid Z]] = E[X]$.

g. Please find $E[XZ \mid Z = z]$.

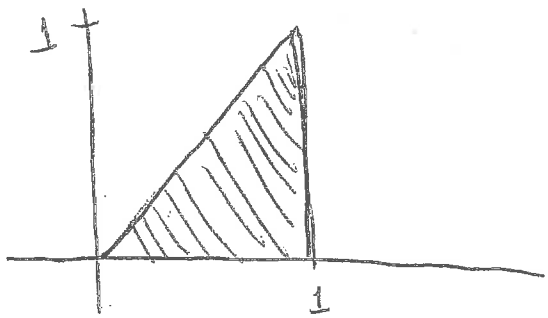
Problem 5



Consider a pair of random variables X and Y uniformly distributed in the region shown above. Please find $E[X]$, $E[Y]$, $E[XY]$, $var(X)$, $var(Y)$, $cov(X, Y)$, and $\rho_{X,Y}$.

Problem 6

Two random variables X and Y are uniformly distributed in the region shown below:



Please find: $f_{X,Y}(x,y)$, $F_{X,Y}(x,y)$, $f_X(x)$, $f_Y(y)$, and $f_{X|Y}(x|Y=y)$.

Problem 7

Is it possible for two random variables X and Y to be each uniform on $[0,1]$, uncorrelated, and not independent?