EE 507 HW 1

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Problem 1

There are two different routes that I typically take between Richland, WA and Pullman, WA. The quickest way is to travel through Kahlotus WA, which is between Connell and Washtucna. However, if I am driving at night, I avoid this route due to its remoteness and abundance of deer. Notice how in Figure 2 the route Via Kahlotus takes less time, which appears as a shift to the left.

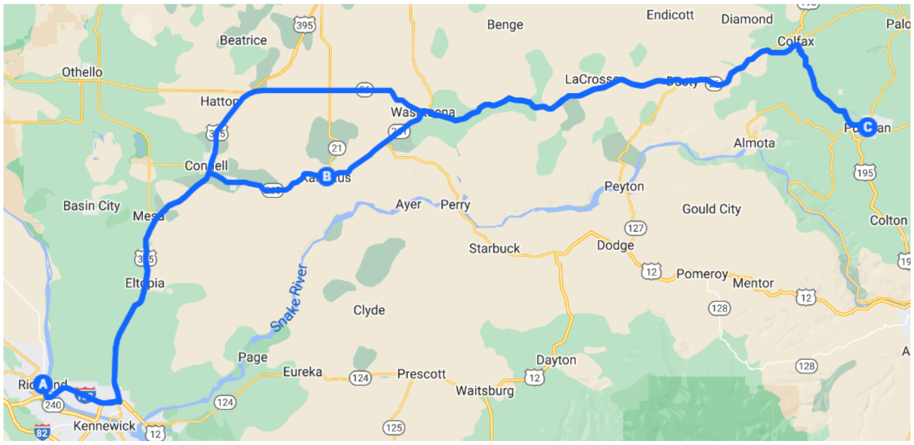

Figure 1: Routes plotted on Google Maps

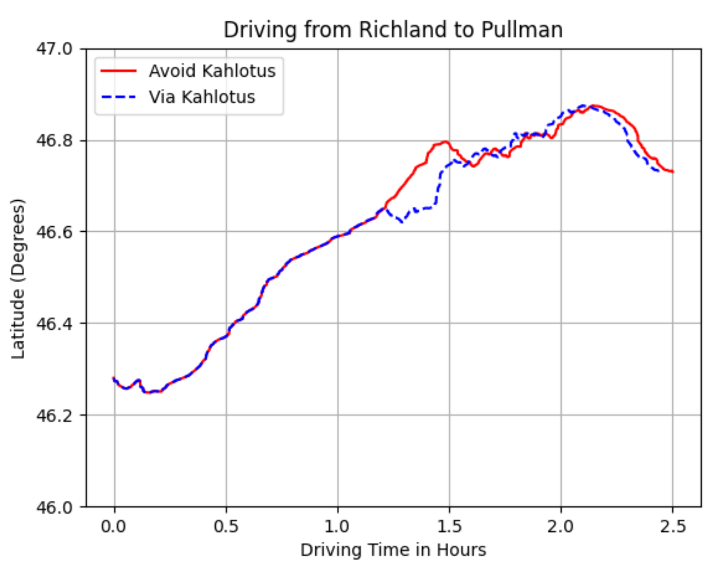

Figure 2: Plot of Latitude vs Time

Problem 2

(2a) Definitions

Probability: a numerical value between 0 and 1 which describes the likelihood or chance that an event occurs.

Outcome: the result of an experiment. For example, when we roll a die, we can assign a outcome to each possible roll, 1-6.

Event: a collection of outcomes. For example, rolling an even number on a die.

Sample Space: the collection of all outcomes, typically denoted Ω.

Axiom: a rule which is known to be true, by definition. Axioms are typically the smallest set of possible rules needed to prove other relationships. In probability theory, we have three axioms.

(2b) The Three Axioms

$P(A) \geq 0$. All event probabilities are associated with a non-negative real number greater than zero. In other words, the probability of a particular event occurring is non-negative.

$P(\Omega) = 1$. The probability that an event will occur is 1. For example, when we roll a die, the outcome will be a number from 1-6. By the design of our experiment, it is impossible for some other event to occur.

$P(AB) = \phi \rightarrow P(A+B) = P(A) + P(B)$. When two events A and B are disjoint (mutually exclusive), the probability that either event occurs is the sum of the individual probabilities. This is very intuitive with an example. When we roll a die, the probability that a 1 or a 2 is rolled is $P(1)+P(2)$.

(2c) Experiment Granularity

Too much granularity: Suppose we are trying to determine the probability that a basketball player sinks his next free throw. Our experiment has too much granularity if we consider unnecessary information, such as his height, date of birth, marital status, etc.

Too little granularity: Suppose we are trying to determine the probability that a bowler converts a spare. However, we are only given the number of pins remaining, as opposed to which pins are left standing. Obviously, the probability a bowler will convert an easy 1-2 pickup is much higher than converting a 7-10 split. We simply don't have enough information to create an accurate probability model.

Problem 3

(3a)

Show that $P(A\overline{B}) = P(A) - P(AB)$.

$$\begin{aligned}
\Omega &= \overline{A}B + A\overline{B} + AB + \overline{A+B} \\
\Omega &= A + \overline{A}B + \overline{A+B} \\
\overline{A}B + A\overline{B} + AB + \overline{A+B} &= A + \overline{A}B + \overline{A+B}
\end{aligned}$$

Because all of these regions are disjoint on both sides of the equality sign, by Axiom 3,

$$\begin{aligned}
P(\overline{A}B) + P(A\overline{B}) + P(AB) + P(\overline{A+B}) &= P(A) + P(\overline{A}B) + P(\overline{A+B}) \\
P(A\overline{B}) + P(AB) &= P(A) \\
\Rightarrow P(A\overline{B}) &= P(A) - P(AB)
\end{aligned}$$

(3b)

Show that if A is contained in B, then $P(A) \leq P(B)$. By the law of total probability,

$$\begin{aligned}
P(B) &= P(B|A)P(A) + P(B|\overline{A})P(\overline{A}) \\
&= P(A) + P(B\overline{A})
\end{aligned}$$

By Axiom 1, $P(B\overline{A}) \geq 0$, so $P(A) \leq P(B)$.

(3c)

Show that $P(A+B) + P(\overline{A}) + P(\overline{B}) - P(\overline{A} + \overline{B}) = 1$. We use the fact that for any two events A and B, we have $P(A+B) = P(A) + P(B) - P(AB)$. Also, we have $\bar{A}\bar{B} = \overline{A+B}$, which is clear from a simple drawing.

$$\begin{aligned}
(A+B) + \overline{A+B} &= \Omega \\
P(A+B) + P(\overline{A+B}) &= P(\Omega) = 1 \\
P(A+B) + P(\bar{A}\bar{B}) &= 1 \\
P(A+B) + \Big(P(\bar{A}) + P(\bar{B}) - P(\bar{A}) - P(\bar{B}) + P(\bar{A}\bar{B})\Big) &= 1 \\
P(A+B) + P(\overline{A}) + P(\overline{B}) - P(\overline{A} + \overline{B}) &= 1
\end{aligned}$$

(3d)

Notice that AB, $\bar{A}C$, $\bar{B}\bar{C}$ are disjoint. Therefore,

$$P(AB) + P(\bar{A}C) + P(\bar{B}\bar{C}) = P(AB + \bar{A}C + \bar{B}\bar{C}) \leq 1$$

From the Venn Diagram on the following page, the bound can be achieved when

$$P(\bar{A}B\bar{C}) = 0 \qquad \text{and} \qquad P(A\bar{B}C) = 0$$

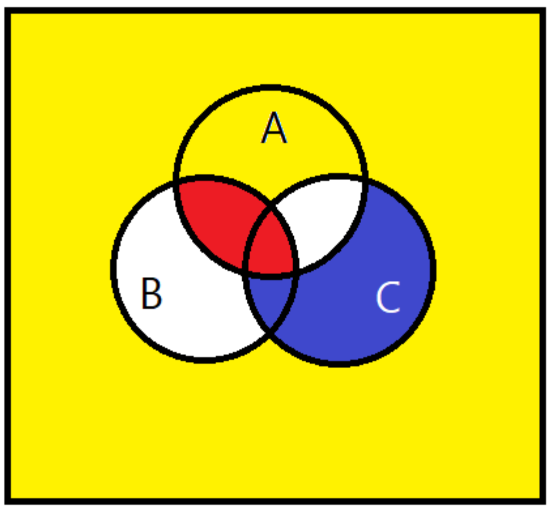

Figure 3: Venn Diagram for (3d) red=AB, blue=$\bar{A}C$, yellow=$\bar{B}\bar{C}$

Problem 4

Experiment: has outcomes A, B, C, and $\square$.
(Note: I will use D rather than $\square$ because it is easier to type).

(4a)

There are $2^4 = 16$ total events.

```
{}, {A}, {B}, {C}, {D}, {A,B}, {A,C}, {A,D}, {B,C}, {B,D}, {C,D}
{A,B,C}, {A,B,D}, {A,C,D}, {B,C,D}, {A,B,C,D}
```

(4b)

The sample space is the set of all outcomes.

$$\Omega = \{A, B, C, D\}$$

(4c)

Find all event probabilities, given

$$P(\{A, D\}) = 0.4 \qquad P(\{A, B, D\}) = 0.8 \qquad P(\{A, C\}) = 0.4$$

By Axiom 2, $P(\{A, B, C, D\}) = 1$. Also, only one outcome can occur at a time, leading to

$$\begin{aligned} P(\{C\}) &= P(\{A, B, C, D\}) - P(\{A, B, D\}) = 0.2 \\ P(\{B\}) &= P(\{A, B, D\}) - P(\{A, D\}) = 0.4 \\ P(\{A\}) &= P(\{A, C\}) - P(\{C\}) = 0.2 \\ P(\{D\}) &= P(\{A, D\}) - P(\{A\}) = 0.2 \end{aligned}$$

From here, the probabilities of each event are just sums of the individual outcome probabilities.

$$\begin{array}{cccc} P(\{\}) = 0 & P(\{A\}) = 0.2 & P(\{B\}) = 0.4 & P(\{C\}) = 0.2 \\ P(\{D\}) = 0.2 & P(\{A, B\}) = 0.6 & P(\{A, C\}) = 0.4 & P(\{A, D\}) = 0.4 \\ P(\{B, C\}) = 0.6 & P(\{B, D\}) = 0.6 & P(\{C, D\}) = 0.4 & P(\{A, B, C\}) = 0.8 \\ P(\{A, B, D\}) = 0.8 & P(\{A, C, D\}) = 0.6 & P(\{B, C, D\}) = 0.8 & P(\{A, B, C, D\}) = 1 \end{array}$$

(4d)

Given $P(\{A, B, D\})$ and $P(\{A, C\})$, find all other events where knowledge of its probability will allow us to compute all 16 probabilities as shown in (4c).

Solution:

By Axiom 2, we know that $P(\{A, B, C, D\}) = 1$. Because outcomes are disjoint, we can represent the probability equations as a linear system:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ a & b & c & d \end{bmatrix} \begin{bmatrix} P(\{A\}) \\ P(\{B\}) \\ P(\{C\}) \\ P(\{D\}) \end{bmatrix} = \begin{bmatrix} P(\{A, B, D\}) \\ P(\{A, C\}) \\ P(\{A, B, C, D\}) \\ P(\{?, ?, ?, ?\}) \end{bmatrix}$$

where $a, b, c, d \in \{0, 1\}$ are used to describe the unknown event. For example, $a = 1, b = 0, c = 1, d = 0$ corresponds to $P(\{A, C\})$. From linear algebra, we know that we can uniquely solve for the outcome probabilities when the left matrix is invertible (determinant is non-zero). Therefore,

$$\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ a & b & c & d \end{vmatrix} = b - d \neq 0$$

So, events where $b \neq d$ allow us to find all of the probabilities. This means that knowledge of the following events will let us find all of the event probabilities. Notice these are the events in which exactly one B or D is present.

$$P(\{D\}), P(\{C, D\}), P(\{B\}), P(\{BC\})$$
$$P(\{A, D\}), P(\{A, C, D\}), P(\{A, B\}), P(\{A, B, C\})$$

Problem 5

(5a)

Flip a fair coin three times. What is the probability of showing at least two heads? There are 8 possible flips of three coins, all with equal probability, which gives

$$P(\#H \geq 2) = P(HHH) + P(HHT) + P(HTH) + P(THH) = \frac{4}{8} = \frac{1}{2}$$

(5b)

What is the probability of showing at least two heads given an even number of heads occurred? There are only four ways to flip three coins and see an even number of heads, which gives

$$P(\#H \geq 2 | \#H \text{ even}) = \frac{P(\#H \geq 2)}{P(\#H \text{ even})} = \frac{3/8}{4/8} = \frac{3}{4}$$

Problem 6

(6a)

outcomes: every point on the board, and every point around the board. There are an infinite number of outcomes.

(6b)

events: dart lands in the 0, 2, 4, 6, or 8 region for a total of 5 events.

(6c)

Let S (score) denote a random variable corresponding to the points acquired from a single dart throw.

$$P(S \geq 3) = P(S = 4) + P(S = 6) + P(S = 8) = \frac{21}{40}$$

(6d)

What is the probability the first score S_1 is greater than three, and the second score S_2 is less than three? Since the experiments are independent,

$$P(S_1 > 3, S_2 \leq 3) = P(S_1 > 3) \cdot P(S_2 \leq 3) = \frac{21}{40} \cdot \frac{19}{40} = 0.249$$

What is the probability that the scores are different?

$$P(S_1 \neq S_2) = 1 - P(S_1 = S_2)$$
$$P(S_1 = S_2) = P(0)^2 + P(2)^2 + P(4)^2 + P(6)^2 + P(8)^2 = 0.2125$$
$$P(S_1 \neq S_2) = 1 - 0.2125 = 0.7875$$

Problem 7

Experiment 1: has outcomes A, B, C
Experiment 2: has outcomes 1, 2, 3

(7a)

How many events are defined for experiment 1? There are 3 outcomes, so there are a total of $2^3 = 8$ events.

```
{}, {A}, {B}, {C}, {A,B}, {A,C}, {B,C}, {A,B,C}
```

(7b)

How many outcomes does the combined experiment have? There a 3 outcomes in each experiment for a total of $3 \cdot 3 = 9$ outcomes in the combined experiment.

```
A1, A2, A3, B1, B2, B3, C1, C2, C3
```

(7c)

How many events does the combined experiment have? There are 9 outcomes, so there are a total of $2^9 = 512$ events in the combined experiment.

(7d)

How many combined-experiment events are not Cartesian products of individual experiments?

```
example of Cartesian product: {A1,A2,B1,B2} = {A,B} x {1,2}
example of not Cartesian product: {A1,A2,B3}
```

If we exclude `{}` as an event from both experiments, there are 7 events from each independent experiment, for a total of $7^2 = 49$ unique Cartesian products. Therefore, the number of combined-experiment events are not Cartesian products of individual experiments is

$$512 - 49 = 463$$

This is equivalent to the method we discussed in office hours, where we considered all 8 events from each experiment and then subtracted out the correct number of empty-set events (64 - 16 + 1 = 49).