WASHINGTON STATE UNIVERSITY

EE 521/ECE 582 – Analysis of Power systems

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Newton-Raphson and Power Problem

To apply the Newton-Raphson method to the solution of the power-flow equations, we express bus voltages and line admittances in polar form. When n is set equal to i in Eqs. (9.6) and (9.7) and the corresponding terms are separated from the summations, we obtain

$$P_i = |V_i|^2 G_{ii} + \sum_{\substack{n=1 \\ n \neq i}}^{N} |V_i V_n Y_{in}| \cos(\theta_{in} + \delta_n - \delta_i) \tag{9.38}$$

$$Q_i = -|V_i|^2 B_{ii} - \sum_{\substack{n=1 \\ n \neq i}}^{N} |V_i V_n Y_{in}| \sin(\theta_{in} + \delta_n - \delta_i) \tag{9.39}$$

These equations can be readily differentiated with respect to voltage angles and magnitudes. The terms involving G_{ii} and B_{ii} come from the definition of Y_{ij} in Eq. (9.1) and the fact that the angle $(\delta_n - \delta_i)$ is zero when $n = i$.

$$\Delta P_i = \boxed{P_{i,\text{sch}}} - P_{i,\text{calc}} \tag{9.40}$$

$$\Delta Q_i = \boxed{Q_{i,\text{sch}}} - Q_{i,\text{calc}} \tag{9.41}$$

mismatch

Calculating the Jacobian

Each nonslack bus of the system has two equations like those for ΔP_i and ΔQ_i. Collecting all the mismatch equations into vector-matrix form yields

$$\underbrace{\left[\begin{array}{ccc|ccc} \frac{\partial P_2}{\partial \delta_2} & \cdots & \frac{\partial P_2}{\partial \delta_4} & |V_2|\frac{\partial P_2}{\partial |V_2|} & \cdots & |V_4|\frac{\partial P_2}{\partial |V_4|} \\ \vdots & \mathbf{J}_{11} & \vdots & \vdots & \mathbf{J}_{12} & \vdots \\ \frac{\partial P_4}{\partial \delta_2} & \cdots & \frac{\partial P_4}{\partial \delta_4} & |V_2|\frac{\partial P_4}{\partial |V_2|} & \cdots & |V_4|\frac{\partial P_4}{\partial |V_4|} \\ \hline \frac{\partial Q_2}{\partial \delta_2} & \cdots & \frac{\partial Q_2}{\partial \delta_4} & |V_2|\frac{\partial Q_2}{\partial |V_2|} & \cdots & |V_4|\frac{\partial Q_2}{\partial |V_4|} \\ \vdots & \mathbf{J}_{21} & \vdots & \vdots & \mathbf{J}_{22} & \vdots \\ \frac{\partial Q_4}{\partial \delta_2} & \cdots & \frac{\partial Q_4}{\partial \delta_4} & |V_2|\frac{\partial Q_4}{\partial |V_2|} & \cdots & |V_4|\frac{\partial Q_4}{\partial |V_4|} \end{array}\right]}_{\text{Jacobian}} \underbrace{\left[\begin{array}{c} \Delta\delta_2 \\ \vdots \\ \Delta\delta_4 \\ \hline \frac{\Delta|V_2|}{|V_2|} \\ \vdots \\ \frac{\Delta|V_4|}{|V_4|} \end{array}\right]}_{\text{Corrections}} = \underbrace{\left[\begin{array}{c} \Delta P_2 \\ \vdots \\ \Delta P_4 \\ \hline \Delta Q_2 \\ \vdots \\ \Delta Q_4 \end{array}\right]}_{\text{Mismatches}} \tag{9.45}$$

The partitioned form of Eq. (9.45) emphasizes the four different types of partial derivatives which enter into the jacobian $\mathbf{J}$. The elements of $\mathbf{J}_{12}$ and $\mathbf{J}_{22}$ have voltage-magnitude multipliers because a simpler and more symmetrical jacobian results. In choosing this format, we have used the identity

$$\underbrace{|V_j|\frac{\partial P_i}{\partial |V_j|}}_{\text{Element of } \mathbf{J}_{12}} \times \underbrace{\frac{\Delta|V_j|}{|V_j|}}_{\text{Correction}} = \frac{\partial P_i}{\partial |V_j|} \times \Delta|V_j| \tag{9.46}$$

and the corrections become $\Delta|V_j|/|V_j|$ as shown rather than $\Delta|V_j|$.

$\Delta\delta$

Radians

Flat Start $1.0\angle 0^\circ$

The solution of Eq. (9.45) is found by iteration as follows:

- Estimate values $\delta_i^{(0)}$ and $|V_i|^{(0)}$ for the state variables.
- Use the estimates to calculate:
  $P_{i,\text{calc}}^{(0)}$ and $Q_{i,\text{calc}}^{(0)}$ from Eqs. (9.38) and (9.39),
  mismatches $\Delta P_i^{(0)}$ and $\Delta Q_i^{(0)}$ from Eqs. (9.40) and (9.41), and
  the partial derivative elements of the jacobian $\mathbf{J}$.
- Solve Eq. (9.45) for the initial corrections $\Delta\delta_i^{(0)}$ and $\Delta|V_i|^{(0)}/|V_i|^{(0)}$.
- Add the solved corrections to the initial estimates to obtain

$$\delta_i^{(1)} = \delta_i^{(0)} + \Delta\delta_i^{(0)} \tag{9.47}$$

$$|V_i|^{(1)} = |V_i|^{(0)} + \Delta|V_i|^{(0)} = |V_i|^{(0)}\left(1 + \frac{\Delta|V_i|^{(0)}}{|V_i|^{(0)}}\right) \tag{9.48}$$

- Use the new values $\delta_i^{(1)}$ and $|V_i|^{(1)}$ as starting values for iteration 2 and continue.

In more general terms, the update formulas for the starting values of the state variables are

$$\delta_i^{(k+1)} = \delta_i^{(k)} + \Delta\delta_i^{(k)} \tag{9.49}$$

$$|V_i|^{(k+1)} = |V_i|^{(k)} + \Delta|V_i|^{(k)} = |V_i|^{(k)}\left(1 + \frac{\Delta|V_i|^{(k)}}{|V_i|^{(k)}}\right) \tag{9.50}$$

$\delta_1^{(0)}, \delta_1^{(1)}, \delta_1^{(2)}, \ldots$

Δ 0.1 0.005

Newton-Raphson and Jacobian

$$P_i = |V_i|^2 G_{ii} + \sum_{\substack{n=1\\n\neq i}}^{N} |V_i V_n Y_{in}| \cos(\theta_{in} + \delta_n - \delta_i) \qquad (9.38)$$

$\frac{\partial}{\partial \delta}$

$$Q_i = -|V_i|^2 B_{ii} - \sum_{\substack{n=1\\n\neq i}}^{N} |V_i V_n Y_{in}| \sin(\theta_{in} + \delta_n - \delta_i) \qquad (9.39)$$

Each nonslack bus of the system has two equations like those for ΔP_i and ΔQ_i. Collecting all the mismatch equations into vector-matrix form yields

$$\underbrace{\left[\begin{array}{ccc|ccc} \frac{\partial P_2}{\partial \delta_2} & \cdots & \frac{\partial P_2}{\partial \delta_4} & |V_2|\frac{\partial P_2}{\partial |V_2|} & \cdots & |V_4|\frac{\partial P_2}{\partial |V_4|} \\ \vdots & \mathbf{J}_{11} & \vdots & \vdots & \mathbf{J}_{12} & \vdots \\ \frac{\partial P_4}{\partial \delta_2} & \cdots & \frac{\partial P_4}{\partial \delta_4} & |V_2|\frac{\partial P_4}{\partial |V_2|} & \cdots & |V_4|\frac{\partial P_4}{\partial |V_4|} \\ \hline \frac{\partial Q_2}{\partial \delta_2} & \cdots & \frac{\partial Q_2}{\partial \delta_4} & |V_2|\frac{\partial Q_2}{\partial |V_2|} & \cdots & |V_4|\frac{\partial Q_2}{\partial |V_4|} \\ \vdots & \mathbf{J}_{21} & \vdots & \vdots & \mathbf{J}_{22} & \vdots \\ \frac{\partial Q_4}{\partial \delta_2} & \cdots & \frac{\partial Q_4}{\partial \delta_4} & |V_2|\frac{\partial Q_4}{\partial |V_2|} & \cdots & |V_4|\frac{\partial Q_4}{\partial |V_4|} \end{array}\right]}_{\text{Jacobian}} \underbrace{\left[\begin{array}{c} \Delta\delta_2 \\ \vdots \\ \Delta\delta_4 \\ \hline \frac{\Delta|V_2|}{|V_2|} \\ \vdots \\ \frac{\Delta|V_4|}{|V_4|} \end{array}\right]}_{\text{Corrections}} = \underbrace{\left[\begin{array}{c} \Delta P_2 \\ \vdots \\ \Delta P_4 \\ \hline \Delta Q_2 \\ \vdots \\ \Delta Q_4 \end{array}\right]}_{\text{Mismatches}} \qquad (9.45)$$

$$\frac{\partial P_i}{\partial \delta_i} = \sum_{n=2}^{4} |V_i V_n Y_{in}| \sin (\theta_{in} + \delta_n - \delta_i)$$

Expressions for the elements of this equation are easily found by differentiating the appropriate number of terms in Eq. (9.38). When the variable n equals the particular value j, only one of the cosine terms in the summation of Eq. (9.38) contains δ_j, and by partial differentiating that single term with respect to δ_j, we obtain the typical off-diagonal element of $\mathbf{J}_{11}$,

$\mathbf{J}_{11}$

$$\frac{\partial P_i}{\partial \delta_j} = -|V_i V_j Y_{ij}| \sin(\theta_{ij} + \delta_j - \delta_i) \qquad (9.52)$$

On the other hand, *every* term in the summation of Eq. (9.38) contains δ_i, and so the typical diagonal element of $\mathbf{J}_{11}$ is

$$\frac{\partial P_i}{\partial \delta_i} = \sum_{\substack{n=1\\n\neq i}}^{N} |V_i V_n Y_{in}| \sin(\theta_{in} + \delta_n - \delta_i) = -\sum_{\substack{n=1\\n\neq i}}^{N} \frac{\partial P_i}{\partial \delta_n} \qquad (9.53)$$

By comparing this expression and that for Q_i in Eq. (9.39), we obtain

$$\frac{\partial P_i}{\partial \delta_i} = -Q_i - |V_i|^2 B_{ii} \qquad (9.54)$$

$$\frac{\partial P_i}{\partial \delta_i} = +\cancel{|V_i|^2 B_{ii}} + \boxed{\Sigma} \cancel{- V_i^2 B_{ii}}$$

In a quite similar manner, we can derive formulas for the elements of submatrix $\mathbf{J}_{21}$ as follows:

$$\frac{\partial Q_i}{\partial \delta_j} = -|V_i V_j Y_{ij}| \cos(\theta_{ij} + \delta_j - \delta_i) \qquad (9.55)$$

$$\frac{\partial Q_i}{\partial \delta_i} = \sum_{\substack{n=1\\n\neq i}}^{N} |V_i V_n Y_{in}| \cos(\theta_{in} + \delta_n - \delta_i) = -\sum_{\substack{n=1\\n\neq i}}^{N} \frac{\partial Q_i}{\partial \delta_n} \qquad (9.56)$$

Comparing this equation for $\partial Q_i/\partial \delta_i$ with Eq. (9.38) for P_i, we can show that

$$\frac{\partial Q_i}{\partial \delta_i} = P_i - |V_i|^2 G_{ii} \qquad (9.57)$$

Newton-Raphson and Power Problem

$$P_i = |V_i|^2 G_{ii} + \sum_{\substack{n=1 \\ n \neq i}}^{N} |V_i V_n Y_{in}| \cos(\theta_{in} + \delta_n - \delta_i) \tag{9.38}$$

$$Q_i = -|V_i|^2 B_{ii} - \sum_{\substack{n=1 \\ n \neq i}}^{N} |V_i V_n Y_{in}| \sin(\theta_{in} + \delta_n - \delta_i) \tag{9.39}$$

Each nonslack bus of the system has two equations like those for ΔP_i and ΔQ_i. Collecting all the mismatch equations into vector-matrix form yields

$$\underbrace{\left[\begin{array}{ccc|ccc} \frac{\partial P_2}{\partial \delta_2} & \cdots & \frac{\partial P_2}{\partial \delta_4} & |V_2|\frac{\partial P_2}{\partial |V_2|} & \cdots & |V_4|\frac{\partial P_2}{\partial |V_4|} \\ \vdots & \mathbf{J}_{11} & \vdots & \vdots & \mathbf{J}_{12} & \vdots \\ \frac{\partial P_4}{\partial \delta_2} & \cdots & \frac{\partial P_4}{\partial \delta_4} & |V_2|\frac{\partial P_4}{\partial |V_2|} & \cdots & |V_4|\frac{\partial P_4}{\partial |V_4|} \\ \hline \frac{\partial Q_2}{\partial \delta_2} & \cdots & \frac{\partial Q_2}{\partial \delta_4} & |V_2|\frac{\partial Q_2}{\partial |V_2|} & \cdots & |V_4|\frac{\partial Q_2}{\partial |V_4|} \\ \vdots & \mathbf{J}_{21} & \vdots & \vdots & \mathbf{J}_{22} & \vdots \\ \frac{\partial Q_4}{\partial \delta_2} & \cdots & \frac{\partial Q_4}{\partial \delta_4} & |V_2|\frac{\partial Q_4}{\partial |V_2|} & \cdots & |V_4|\frac{\partial Q_4}{\partial |V_4|} \end{array}\right]}_{\text{Jacobian}} \underbrace{\left[\begin{array}{c} \Delta\delta_2 \\ \vdots \\ \Delta\delta_4 \\ \hline \frac{\Delta|V_2|}{|V_2|} \\ \vdots \\ \frac{\Delta|V_4|}{|V_4|} \end{array}\right]}_{\text{Corrections}} = \underbrace{\left[\begin{array}{c} \Delta P_2 \\ \vdots \\ \Delta P_4 \\ \hline \Delta Q_2 \\ \vdots \\ \Delta Q_4 \end{array}\right]}_{\text{Mismatches}} \tag{9.45}$$

Slack Bus --- $V\angle\delta$ ← set PQ Float

Voltage Controlled Bus ---

PV set P, Q, δ Float

Jacobian Matrix is

2 Nbus − #PV − 2 ← slack

Let us now bring together the results developed above in the following definitions:

Off-diagonal elements, $i \neq j$

$$M_{ij} \triangleq \frac{\partial P_i}{\partial \delta_j} = |V_j| \frac{\partial Q_i}{\partial |V_j|} \tag{9.64}$$

$$N_{ij} \triangleq \frac{\partial Q_i}{\partial \delta_j} = -|V_j| \frac{\partial P_i}{\partial |V_j|} \tag{9.65}$$

Diagonal elements, $i = j$

$$M_{ii} \triangleq \frac{\partial P_i}{\partial \delta_i} \qquad |V_i| \frac{\partial Q_i}{\partial |V_i|} = -M_{ii} - 2|V_i|^2 B_{ii} \tag{9.66}$$

$$N_{ii} \triangleq \frac{\partial Q_i}{\partial \delta_i} \qquad |V_i| \frac{\partial P_i}{\partial |V_i|} = N_{ii} + 2|V_i|^2 G_{ii} \tag{9.67}$$

Interrelationships among the elements in the four submatrices of the jacobian are more clearly seen if we use the definitions to rewrite Eq. (9.45) in the following form:

$$\left[\begin{array}{ccc|ccc} M_{22} & M_{23} & M_{24} & N_{22} + 2|V_2|^2 G_{22} & -N_{23} & -N_{24} \\ M_{32} & M_{33} & M_{34} & -N_{32} & N_{33} + 2|V_3|^2 G_{33} & -N_{34} \\ M_{42} & M_{43} & M_{44} & -N_{42} & -N_{43} & N_{44} + 2|V_4|^2 G_{44} \\ \hline N_{22} & N_{23} & N_{24} & -M_{22} - 2|V_2|^2 B_{22} & M_{23} & M_{24} \\ N_{32} & N_{33} & N_{34} & M_{32} & -M_{33} - 2|V_3|^2 B_{33} & M_{34} \\ N_{42} & N_{43} & N_{44} & M_{42} & M_{43} & -M_{44} - 2|V_4|^2 B_{44} \end{array}\right] \times \left[\begin{array}{c} \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \\ \hline \Delta|V_2|/|V_2| \\ \Delta|V_3|/|V_3| \\ \Delta|V_4|/|V_4| \end{array}\right] = \left[\begin{array}{c} \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \\ \hline \Delta Q_2 \\ \Delta Q_3 \\ \Delta Q_4 \end{array}\right] \tag{9.68}$$

Newton-Raphson and Power Problem

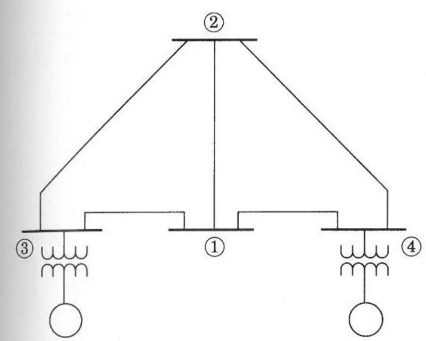


FIGURE 7.3
Single-line diagram of the four-bus system of Example 7.1. Reference node is not shown.

Slack Bus ---

Voltage Controlled Bus ---

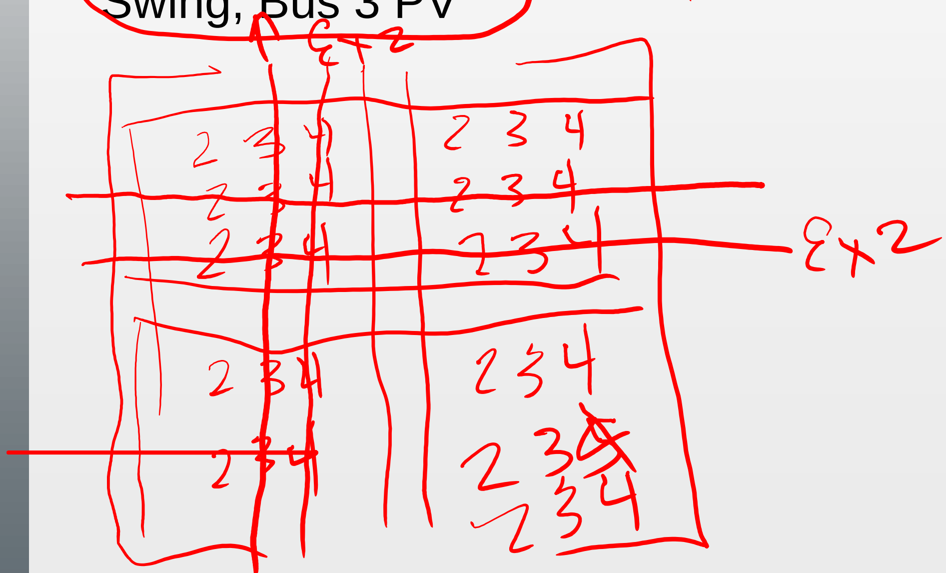
Jacobian Matrix is

8-2-2=4

8-1-2

5x5

Example – Bus 1 is Swing, Bus 3 PV



Example – Bus 1 is Swing, Buses 3&4 PV

4x4

100 10PV

200-10-2=

188 x 188

Newton-Raphson and Power Problem

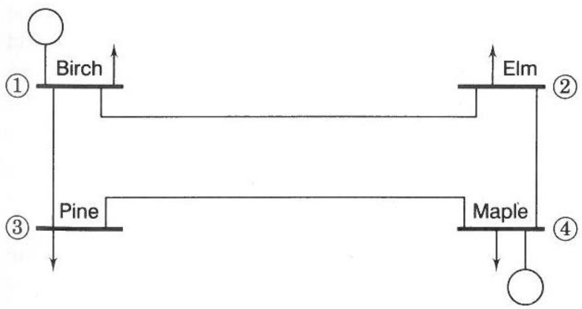


FIGURE 9.2 One-line diagram for Exampl 9.2 showing the bus names an numbers.

TABLE 9.3
Bus data for Example 9.2

Bus	Generation P, MW	Generation Q, Mvar	Load P, MW	Load Q, Mvar†	V, per unit	Remarks
1	—	—	50	30.99	$1.00\angle 0°$	Slack bus
2	0	0	170	105.35	$1.00\angle 0°$	Load bus (inductive)
3	0	0	200	123.94	$1.00\angle 0°$	Load bus (inductive)
4	318	—	80	49.58	$1.02\angle 0°$	Voltage controlled

†The Q values of load are calculated from the corresponding P values assuming a power factor of 0.85.

TABLE 9.2
Line data for Example 9.2†

Line, bus to bus	Series Z: R per unit	Series Z: X per unit	Series $Y = Z^{-1}$: G per unit	Series $Y = Z^{-1}$: B per unit	Shunt Y: Total charging Mvar‡	Shunt Y: Y / 2 per unit
1–2	0.01008	0.05040	3.815629	−19.078144	10.25	0.05125
1–3	0.00744	0.03720	5.169561	−25.847809	7.75	0.03875
2–4	0.00744	0.03720	5.169561	−25.847809	7.75	0.03875
3–4	0.01272	0.06360	3.023705	−15.118528	12.75	0.06375

†Base 100MVA, 230 kV.
‡At 230 kV.

TABLE 9.4
Bus admittance matrix for Example 9.2†

Bus no.	①	②	③	④
①	8.985190 −j44.835953	−3.815629 +j19.078144	−5.169561 +j25.847809	0
②	−3.815629 +j19.078144	8.985190 −j44.835953	0	−5.169561 +j25.847809
③	−5.169561 +j25.847809	0	8.193267 −j40.863838	−3.023705 +j15.118528
④	0	−5.169561 +j25.847809	−3.023705 +j15.118528	8.193267 −j40.863838

†Per-unit values rounded to six decimal places

Newton-Raphson and Power Problem

Example 9.5. The small power system of Example 9.2 has the line data and bus data given in Tables 9.2 and 9.3. A power-flow study of the system is to be made by the Newton-Raphson method using the polar form of the equations for P and Q. Determine the number of rows and columns in the jacobian. Calculate the initial mismatch $\Delta P_3^{(0)}$ and the initial values of the jacobian elements of the (second row, third column); of the (second row, second column); and of the (fifth row, fifth column). Use the specified values and initial voltage estimates shown in Table 9.3.

Solution. Since the slack bus has no rows or columns in the jacobian, a 6×6 matrix would be necessary if P and Q were specified for the remaining three buses. In fact, however, the voltage magnitude is specified (held constant) at bus ④, and thus the jacobian will be a 5×5 matrix. In order to calculate $P_{3,\text{calc}}$ based on the estimated and the specified voltages of Table 9.3, we need the polar form of the off-diagonal entries of Table 9.4,

$$Y_{31} = 26.359695\angle 101.30993^\circ\ ; \qquad Y_{34} = 15.417934\angle 101.30993^\circ$$

and the diagonal element $Y_{33} = 8.193267 - j40.863838$. Since Y_{32} and the initial values $\delta_3^{(0)}$ and $\delta_4^{(0)}$ are all zero, from Eq. (9.38) we obtain

$$P_{3,\text{calc}}^{(0)} = |V_3|^2 G_{33} + |V_3 V_1 Y_{31}| \cos\theta_{31} + |V_3 V_4 Y_{34}| \cos\theta_{34}$$

$$= (1.0)^2 8.193267 + (1.0 \times 1.0 \times 26.359695)\cos(101.30993^\circ)$$

$$+ (1.0 \times 1.02 \times 15.417934)\cos(101.30993^\circ)$$

$$= -0.06047 \text{ per unit}$$

Scheduled real power into the network at bus ③ is -2.00 per unit, and so the initial mismatch which we are asked to calculate has the value

$$\Delta P_{3,\text{calc}}^{(0)} = -2.00 - (-0.06047) = \boxed{-1.93953 \text{ per unit}}$$

From Eq. (9.52) the (second row, third column) jacobian element is

$$\frac{\partial P_3}{\partial \delta_4} = -|V_3 V_4 Y_{34}| \sin(\theta_{34} + \delta_4 - \delta_3)$$

$$= -(1.0 \times 1.02 \times 15.417934)\sin(101.30993^\circ)$$

$$= -15.420898 \text{ per unit}$$

and from Eq. (9.53) the element of the (second row, second column) is

$$\boxed{\frac{\partial P_3}{\partial \delta_3} = -\frac{\partial P_3}{\partial \delta_1} - \frac{\partial P_3}{\partial \delta_2} - \frac{\partial P_3}{\partial \delta_4}}$$

$$= |V_3 V_1 Y_{31}| \sin(\theta_{31} + \delta_1 - \delta_3) - 0 - (-15.420898)$$

$$= (1.0 \times 1.0 \times 26.359695)\sin(101.30993^\circ) + 15.420898$$

$$\boxed{= 41.268707 \text{ per unit}}$$

For the element of the (fifth row, fifth column) Eq. (9.63) yields

$$|V_3| \frac{\partial Q_3}{\partial |V_3|} = \boxed{-\frac{\partial P_3}{\partial \delta_3} - 2|V_3|^2 B_{33}}$$

$$= -41.268707 - 2(1.0)^2(-40.863838) = 40.458969 \text{ per unit}$$

Newton-Raphson and Power Problem

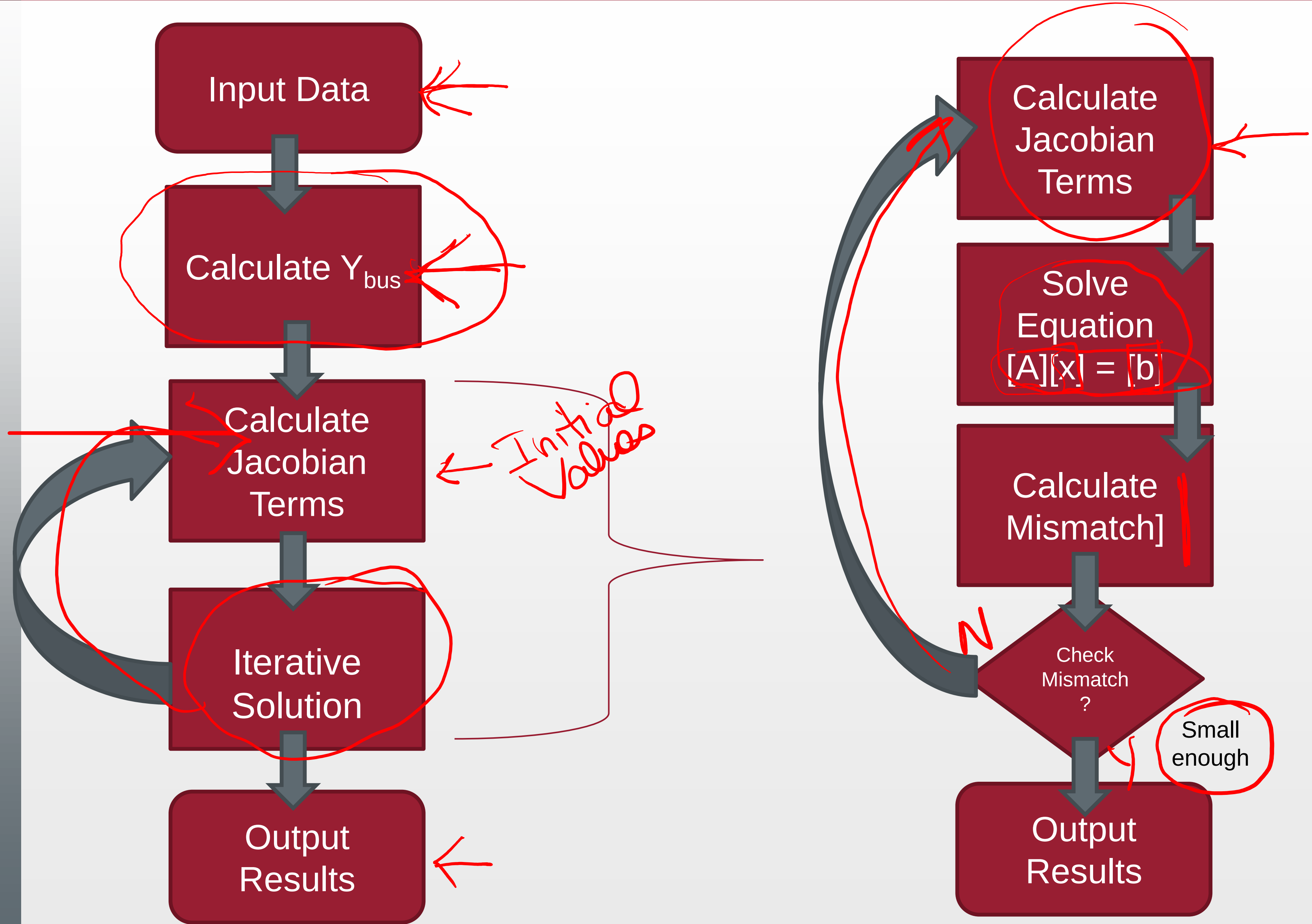
$$
\begin{array}{c|c}
 & \begin{array}{ccccc} ② & ③ & ④ & ② & ③ \end{array} \\
\begin{array}{c} ② \\ ③ \\ ④ \\ ② \\ ③ \end{array} &
\left[\begin{array}{ccc|cc}
45.443 & 0 & -26.365 & 8.882 & 0 \\
0 & 41.269 & -15.421 & 0 & 8.133 \\
-26.365 & -15.421 & 41.786 & -5.273 & -3.084 \\
\hline
-9.089 & 0 & 5.273 & 44.229 & 0 \\
0 & -8.254 & 3.084 & 0 & 40.459
\end{array}\right]
\end{array}
\left[\begin{array}{c}
\Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \\ \hline \dfrac{\Delta|V_2|}{|V_2|} \\ \dfrac{\Delta|V_3|}{|V_3|}
\end{array}\right]
$$

$$
= \left[\begin{array}{c} -1.597 \\ -1.940 \\ 2.213 \\ \hline -0.447 \\ -0.835 \end{array}\right]
$$

This system of equations yields values for the voltage corrections of the first iteration which are needed to update the state variables according to Eqs. (9.49) and (9.50). At the end of the first iteration the set of updated voltages at the buses is:

Bus no. i =	①	②	③	④
δ_i (deg.)	0	−0.93094	−1.78790	−1.54383
$\lvert V_i \rvert$ (per unit)	1.00	0.98335	0.97095	1.02

Newton-Raphson and Parts



Solving the Matrix once you calculate it

$$
\begin{array}{cc}
 & \begin{array}{ccccc} ② & ③ & ④ & ② & ③ \end{array} \\
\begin{array}{c} ② \\ ③ \\ ④ \\ ② \\ ③ \end{array} &
\left[\begin{array}{ccc|cc}
45.443 & 0 & -26.365 & 8.882 & 0 \\
0 & 41.269 & -15.421 & 0 & 8.133 \\
-26.365 & -15.421 & 41.786 & -5.273 & -3.084 \\
\hline
-9.089 & 0 & 5.273 & 44.229 & 0 \\
0 & -8.254 & 3.084 & 0 & 40.459
\end{array}\right]
\end{array}
\left[\begin{array}{c}
\Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \\ \hline \dfrac{\Delta|V_2|}{|V_2|} \\ \dfrac{\Delta|V_3|}{|V_3|}
\end{array}\right]
=
\left[\begin{array}{c}
-1.597 \\ -1.940 \\ 2.213 \\ \hline -0.447 \\ -0.835
\end{array}\right]
$$

Solve $\boxed{A^{-1}}$

$$-3 \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -32 \end{bmatrix}$$

$$-2x_2 = -32 \quad x_2 = 16$$

$$x_1 + 2(16) = 10$$

$$x_1 = -22$$

A linear system may be generically modeled as:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{22}x_3 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n2}x_3 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

Or in matrix form as: $Ax = b$

Matrix of coefficients

Vector of unknown quantities

Vector of known quantities

$$A^{-1}Ax = A^{-1}b$$

$$\boxed{I}\,x = A^{-1}b$$

If A^{-1} exists, then $x^* = A^{-1}b$ is the unique solution

Old method you probably learned is Cramer's Rule:

$$A^{-1}(i,j) = \frac{1}{\det(A)} \left(A_{ij}\right)^T$$

determinant

Cofactor of ij^{th} entry

This method requires $2(n+1)!$ multiplications to find A^{-1}

Not computationally efficient!

Considerations for solving $Ax=b$:

Do we really need to find A^{-1} if we can find the solution x^* in another, more computationally efficient, way?

- Direct methods
- Iterative methods

Direct methods for solving $Ax=b$:

Direct methods (also known as elimination methods) find the exact solution through a finite number of arithmetic operations.

These methods are accurate to within the roundoff error of the computer on which they are implemented

J

Matrix

Examples include:

- Gaussian elimination
- LU factorization

Gaussian Elimination for solving $Ax=b$
is the process of using a series of elementary row operations to convert:

ERO

Identity matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & x_1^* \\ 0 & 1 & \cdots & 0 & x_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & x_n^* \end{array}\right]$$

I

solution

An elementary row operation is one of the following:

1. Interchange any two rows of the matrix
2. Multiply any row by a constant
3. Take a linear combination of rows and add it to another row

Basic Approach to Gaussian Elimination

Upper Triangularize (i.e. zero out all elements below the diagonal) using elementary row operations (EROs)

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \xrightarrow{\text{(EROs)}} \left[\begin{array}{cccc|c} 1 & a'_{12} & \cdots & a'_{1n} & b_1^* \\ 0 & 1 & \cdots & a'_{2n} & b_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n^* \end{array}\right]$$

All zeros below diagonal

$b_n^*, \; b_{n-1}^*, \; b_{n-2}^*$

Then back substitute to find solutions

Example: Use Gaussian Elimination to solve:

$-9 \quad -4 \quad -2$

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 2 & 1 & 2 & 3 \\ 4 & 3 & 5 & 8 \\ 9 & 2 & 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Use elementary row operations to eliminate elements below diagonal in first column.

Since there are 3x2x1 lower diagonal elements and 4 diagonals, there will be 10 EROs

Column 1: We want each ERO to multiply row *j* (*j*=2, …, 4) by $-a_{j1}/a_{11}$ and add it to row 1 to zero out the elements.

$$-\frac{a_{j1}}{a_{11}}$$

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 2 & 1 & 2 & 3 \\ 4 & 3 & 5 & 8 \\ 9 & 2 & 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$-\frac{a_{21}}{a_{11}} \to ERO(2,1)$

$-\frac{a_{31}}{a_{11}} \to ERO(3,1)$

$-\frac{a_{41}}{a_{11}} \to ERO(4,1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -9 & 0 & 0 & 1 \end{bmatrix}$$

ERO(1) ERO(2) ERO(3)

ERO(4)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scale diagonal a_{11} to 1
(in this case ERO(4) = identity since a_{11} is already 1)

Update A by EROs 1 through 4

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{ERO(4)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -9 & 0 & 0 & 1 \end{bmatrix}}_{\text{ERO(3)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{ERO(2)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{ERO(1)}} \underbrace{\begin{bmatrix} 1 & 3 & 4 & 8 \\ 2 & 1 & 2 & 3 \\ 4 & 3 & 5 & 8 \\ 9 & 2 & 7 & 4 \end{bmatrix}}_{\text{A}} = \underbrace{\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -5 & -6 & -13 \\ 0 & -9 & -11 & -24 \\ 0 & -25 & -29 & -68 \end{bmatrix}}_{\text{A'}}$$

All zeros

Repeat process on A’ to zero out elements below diagonal 2

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -5 & -6 & -13 \\ 0 & -9 & -11 & -24 \\ 0 & -25 & -29 & -68 \end{bmatrix}$$

$$-\frac{-9}{-5}$$

$$-\frac{-25}{-5} = -5$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{9}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ERO(5)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{25}{5} & 0 & 1 \end{bmatrix}$$

ERO(6)

diagonalize

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ERO(7)

Update A' by EROs 5 through 7

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{ERO(7)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{25}{5} & 0 & 1 \end{bmatrix}}_{\text{ERO(6)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{9}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{ERO(5)}} \underbrace{\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -5 & -6 & -13 \\ 0 & -9 & -11 & -24 \\ 0 & -25 & -29 & -68 \end{bmatrix}}_{\text{A'}} = \underbrace{\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & 1 & 1.2 & 2.6 \\ 0 & 0 & -0.2 & -0.6 \\ 0 & 0 & 1 & -3 \end{bmatrix}}_{\text{A''}}$$

All zeros

Now eliminate third column

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}}_{\text{ERO(8)}}$$

And diagonalize

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{ERO(9)}} \Rightarrow \underbrace{\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & 1 & 1.2 & 2.6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix}}_{\text{A'''}}$$

$$\overset{A'''}{\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & 1 & 1.2 & 2.6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix}} \quad \text{Normalize to 1} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{6} \end{bmatrix}}_{\text{ERO(10)}} \Rightarrow \begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & 1 & 1.2 & 2.6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[A][x] = [b]$$

Multiply b by EROs 1 through 10:

$$\boxed{\text{ERO(10)ERO(9)ERO(8)...ERO(1)}}b = \boxed{b''''} = \begin{bmatrix} 1 \\ 0.2 \\ 6 \\ 1.5 \end{bmatrix}$$

$$[A \,\vdots\, b]$$

Now we can use back substitution to find x_1, x_2, x_3, and x_4

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & 1 & 1.2 & 2.6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.2 \\ 6 \\ 1.5 \end{bmatrix}$$

$$x_4 = 1.5$$

$$x_3 = 6 - 3(1.5) = 1.5$$

$$x_2 = 0.2 - 2.6(1.5) - 1.2(1.5) = -5.5$$

$$x_1 = 1 - 8(1.5) - 4(1.5) - 3(-5.5) = -0.5$$

$$\Rightarrow \underbrace{\begin{bmatrix} -0.5 \\ -5.5 \\ 1.5 \\ 1.5 \end{bmatrix}}_{x^*}$$

Check: $Ax^{*}=b$?

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 2 & 1 & 2 & 3 \\ 4 & 3 & 5 & 8 \\ 9 & 2 & 7 & 4 \end{bmatrix} \begin{bmatrix} -0.5 \\ -5.5 \\ 1.5 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

Announcements

- Remember two discussions
- Read Chapter 2.5-2.7
- Review Chapter 2 examples
- Working on MATLAB Program
  - Input the CDF file
  - Start working on building the Ybus
  - Think about developing Jacobian values