Quadratic Programming

The general QP is

$$\min \; q(x) = \frac{1}{2} x^T G x + x^T c$$
$$\text{s.t.} \quad A_e x = b_e$$
$$Ax \geq b$$
$$x \in \mathbb{R}^n$$

almost a linear program except for the quadratic term in the objective

or equivalently:

$$\boxed{\begin{aligned} \min \; q(x) &= \frac{1}{2} x^T G x + x^T c \\ \text{s.t.} \quad a_i^T x &= b_i \quad i \in \mathcal{E} \\ a_i^T x &\geq b_i \quad i \in \mathcal{I} \\ x &\in \mathbb{R}^n \end{aligned}}$$

We will consider convex problems ($G \geq 0$) which serve as subproblems for later methods.

Equality Constrained QP

$$\boxed{\begin{aligned} \min\ & q(x) = \tfrac{1}{2}x^TGx + x^Tc \\ \text{s.t.}\ & Ax = b \\ & x \in \mathbb{R}^n \end{aligned}}$$

The KKT conditions:

$$Gx + c - A^T\lambda = 0$$

$$Ax = b$$

which can be written in matrix form:

$$\begin{bmatrix} G & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

(First order necessary conditions)

Our assumptions:

(1) $G \geq 0$

(2) A is full rank

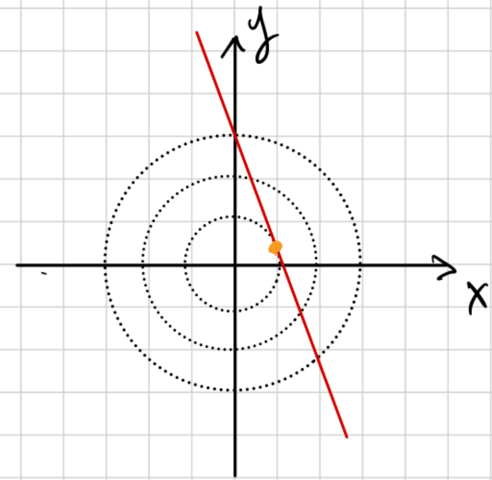
(3) $Z^TGZ > 0$

Example

$$\min \quad x^2+y^2$$
$$\text{s.t.} \quad 3x+y=3$$

$$f(x) = x^2+y^2 = \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix} \overset{G}{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x & y \end{bmatrix} \overset{c}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$3x+y=3 \implies \overset{A}{\begin{bmatrix} 3 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \overset{b}{3}$$



KKT :
$$\begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & -1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$x_1^* = 0.9 \quad x_2^* = 0.3 \quad \lambda^* = 0.6$$

A computational variant

Suppose x not necessarily optimal.
Find a step p so that $x^* = x + p$.

$$\begin{cases} G(x+p) + c - A^T\lambda = 0 \\ A(x+p) = b \end{cases}$$

$$\begin{bmatrix} G & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} p^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -Gx - c \\ b - Ax \end{bmatrix}$$

or

$$\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} -p^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} Gx + c \\ Ax - b \end{bmatrix}$$

Example: $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$Gx + c = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad Ax - b = 0$$

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \Rightarrow p^* = \begin{pmatrix} -0.1 \\ +0.3 \end{pmatrix}$$

$$x^* = x + p^* = \begin{pmatrix} 0.9 \\ 0.3 \end{pmatrix} \quad \checkmark$$

Example: $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Gx + c = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad Ax - b = 1$$

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \Leftrightarrow p^* = \begin{pmatrix} -0.1 \\ -0.7 \end{pmatrix}$$

$$x^* = x + p^* = \begin{pmatrix} 0.9 \\ 0.3 \end{pmatrix} \quad \checkmark$$

An important condition

Let Z be the matrix whose columns form a basis for Null A. $(AZ=0)$.

Lemma. Let A have full row rank and $Z^TGZ > 0$. Then

$$K = \begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix} \text{ is invertible.}$$

Proof: Suppose w, v satisfying

$$\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If we show that $w=0, v=0$ is the unique solution then K is invertible.

First notice that $Aw=0$. So,

$$0 = \begin{bmatrix} w^T & v^T \end{bmatrix}\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix}$$

$$= \begin{bmatrix} w^T & v^T \end{bmatrix}\begin{bmatrix} Gw + A^Tv \\ Aw \end{bmatrix}$$

$$= w^TGw + w^TA^Tv + v^TAw$$

$$= w^TGw$$

Now $w \in$ Null A so $\exists\, u \ni w = Zu$. So,

$$0 = w^TGw = (Zu)^TG(Zu) = u^TZ^TGZu$$

Because $Z^TGZ > 0$, $u=0$ and also $w = Zu = 0$.

Finally $Gw + A^Tv = 0 \Rightarrow v = 0$. $\blacksquare$

Theorem. Let A have full row rank and assume $Z^TGZ > 0$. Then x^* satisfying

$$\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} -x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} c \\ -b \end{bmatrix}$$

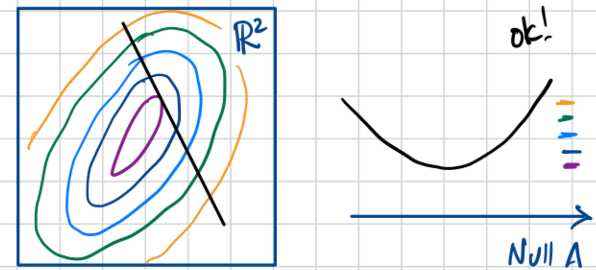
is the unique global minimizer of the ECQP.

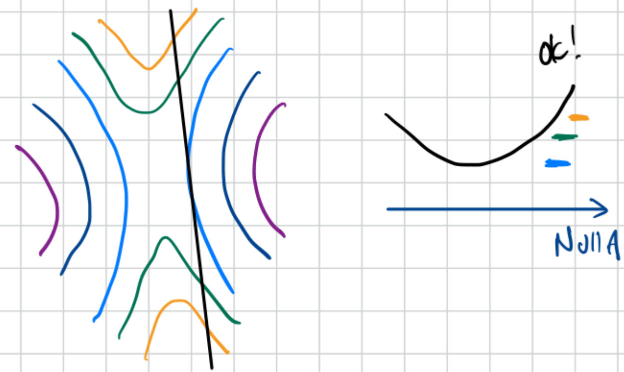
Proof: Let $x \neq x^*$ be a feasible point and $p = x^* - x$. Using the facts that $Ap = 0$ and $p^TGx^* = p^T(-c + A^T\lambda^*) = -p^Tc$, we have

$$\begin{aligned} q(x) &= \tfrac{1}{2}(x^*-p)^TG(x^*-p) + c^T(x^*-p) \\ &= q(x^*) + \tfrac{1}{2}p^TGp - p^TGx^* - c^Tp \\ &= q(x^*) + \tfrac{1}{2}p^TGp \\ &> q(x^*). \end{aligned}$$

Thus x^* is the unique global minimizer. $\blacksquare$

Geometric interpretation





Solving the KKT system

let $x^* = Yx_y + Zx_z$ (c)

Then $A(Zx_z) = (AZ)x_z = 0$

$A(Yx_y) = b - AZx_z = b$

$$\left.\begin{aligned} \min_x \; & q(x) = \tfrac{1}{2}x^TGx + x^Tc \\ \text{s.t.} \; & Ax = b \end{aligned}\right\}$$

$$\min_{x_z} q(x) = \tfrac{1}{2}x_z^TZ^TGZx_z + x_z^T(Z^TGYx_y + Z^Tc)$$

$$\Longrightarrow \boxed{\begin{aligned} & Z^TGZx_z^* = -(Z^TGYx_y^* + Z^Tc) && \text{(a)} \\ & \& \; AYx_y^* = b && \text{(b)} \end{aligned}}$$

Use (b) to solve for x_y^*.

Then use (a) to solve for x_z^*.

Then use (c) to find x^*.

The Projected Gradient CG method

Basic Algorithm

Given: x satisfying $Ax=b$, $P = Z(Z^THZ)^{-1}Z^T$

Compute: $r = Gx + c$, $g = Pr$, $d = -g$

Repeat:

$$\alpha \leftarrow r^Tg \,/\, d^TGd$$

$$x \leftarrow x + \alpha d$$

$$r^+ \leftarrow r + \alpha G d$$

$$g^+ \leftarrow P r^+$$

$$\beta \leftarrow (r^+)^T g \,/\, r^T g$$

$$d \leftarrow -g^+ + \beta d$$

$$g \leftarrow g^+, \quad r \leftarrow r^+$$

Until $r^Tg < tol$

☆ Does not use A or b!
The algorithm naturally operates in the affine subspace of the constraint set.

! Requires Null space basis matrix Z and preconditioner matrix H. These are, in general, not sparse.

A helpful modification

Choose $H = I$. Then

$$P = Z(Z^T Z)^{-1} Z^T = I - A^T (AA^T)^{-1} A$$

and

$$g = Pr = r - A^T (AA^T)^{-1} A r$$

$$g = Pr = r - A^T v \, , \quad AA^T v = Ar$$

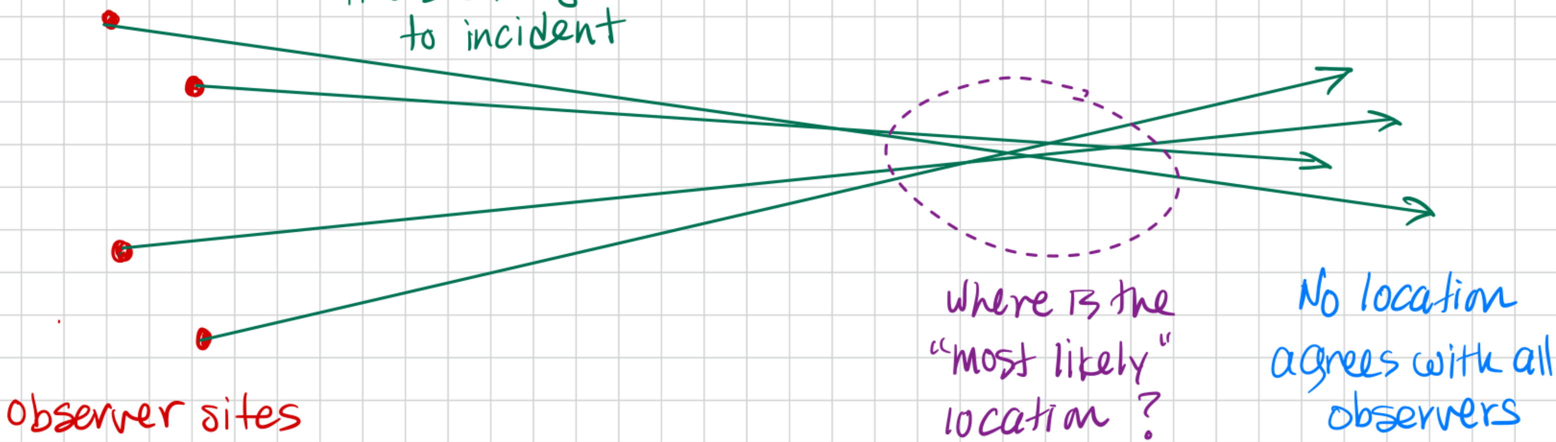
— Many variations exist —

$\Rightarrow$ Procedure for computing g^+

(a) solve $AA^T v = Ar^+$ for v

(b) $g^+ = r^+ - A^T v$

Project 8



we will formulate the problem in three different ways and generalize to $\mathbb{R}^n$.

Method #1

A "solution" can be posed as the task of solving a system of linear equations, one for each line of sight.

$$Ax = b$$

which we can solve approximately using a pseudo-inverse via singular value decomposition (SVD).

$$A = U \Sigma V^T = \tilde{U} \tilde{\Sigma} \tilde{V}^T$$

$$P = \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}^T$$

$$x^* = Pb$$

linear algebra

Method #2

Find the point that minimizes the sum of squared residuals:

For a point in $\mathbb{R}^n$ with m lines of sight:

$$\begin{aligned} \min \quad & \tfrac{1}{2} \|r\|^2 \\ \text{s.t.} \quad & r = Ax - b \\ & x \in \mathbb{R}^n \\ & r \in \mathbb{R}^m \end{aligned}$$

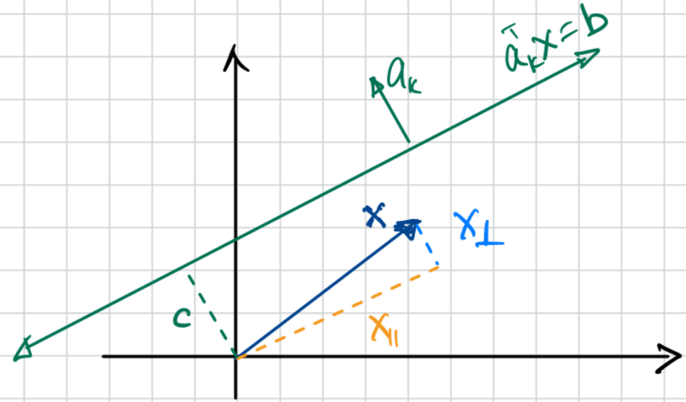
quadratic program

Method #3

minimize the sum of squared distances to each line of sight.

The signed distance from a hyperplane defined by $a_k^T x = b_k$ to a point x is

$$d = \frac{b_k - a_k^T x}{\|a\|} = \frac{b_k}{\|a_k\|} - \frac{a_k^T x}{\|a_k\|}$$



$$x = x_\parallel + x_\perp$$
$$a^T x = a^T x_\parallel + a^T x_\perp$$
$$a^T x = 0 + \|a\| \|x_\perp\|$$
$$\|x_\perp\| = a^T x / \|a\|$$

when $a^T x = b$

then $\|x_\perp\| = c$

$$c = b/\|a\|$$

$$d = c - \|x_\perp\| = \frac{b}{\|a\|} - \frac{a^T x}{\|a\|}$$

So, one simple possibility is to normalize A and b and solve according to method #2.

$$\begin{aligned} \min \quad & \tfrac{1}{2}\|r\|^2 \\ \text{s.t.} \quad & \|a_k\| r_k = a_k^T x - b_k \\ & x \in \mathbb{R}^n \\ & r \in \mathbb{R}^m \end{aligned}$$

quadratic Program

Method #4

We can generalize the previous idea to find the point that minimizes total squared distances to a collection of polyhedral sets.

$$\begin{aligned} \min \quad & \sum_{k=1}^{m} \|x - y_k\|^2 \quad + \alpha\|x\|^2 \\ \text{s.t.} \quad & y_k \in P_k \\ & x \in \mathbb{R}^n \end{aligned}$$

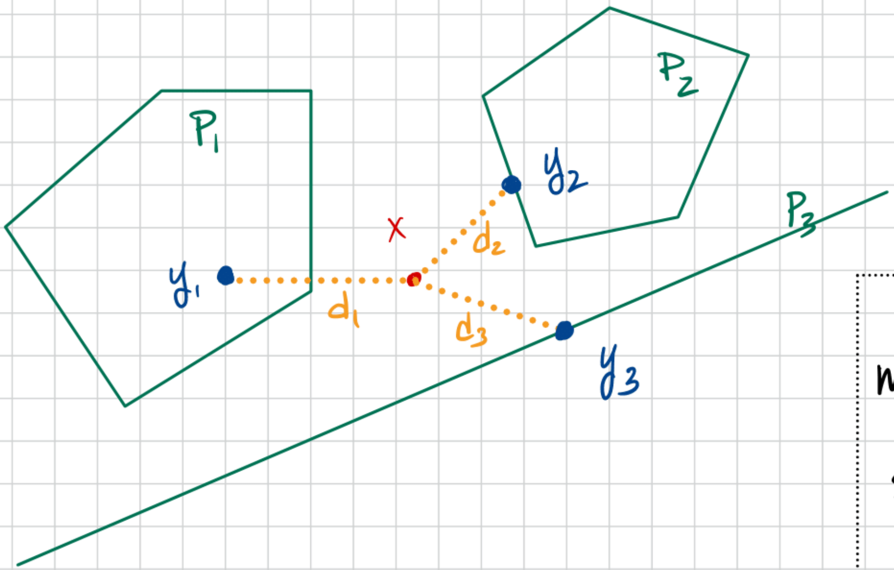
where $P_k = \{x \in \mathbb{R}^n \mid A_k x = b_k,\ C_k x \geq d_k\}$

quadratic program

$P_1 = \{x \in \mathbb{R}^n \mid A_1 x \le b_1\}$

$P_2 = \{x \in \mathbb{R}^n \mid A_2 x \le b_2\}$

$P_3 = \{x \in \mathbb{R}^n \mid A_3 x = b_3\}$



$$\min \ \frac{1}{2}\sum_{k=1}^{3} \|x - y_k\|^2 + \frac{1}{2}\alpha\|x\|^2$$

$$\text{s.t.} \quad A_1 y_1 \le b_1$$

$$A_2 y_2 \le b_2$$

$$A_3 y_3 = b_3$$

$$x, y_k \in \mathbb{R}^n \quad k = 1, 2, 3$$

quadratic program with both inequality and equality constraints.

General Convex QP

Consider some of the first order necessary conditions:

$$\left.\begin{aligned}
&Gx^* + c - \sum_{i \in A(x^*)} \lambda_i^* a_i = 0 \\
&a_i^T x^* = b_i \qquad i \in A(x^*) \\
&a_i^T x^* \geq b_i \qquad i \in \mathcal{I} \setminus A(x^*) \\
&\lambda_i^* \geq 0 \qquad i \in \mathcal{I} \cap A(x^*)
\end{aligned}\right\} \circledast$$

Theorem. If x^* satisfies $\circledast$ for some λ_i^*, $i \in A(x^*)$, and $G \geq 0$, then x^* is a global minimizer of QP.

Proof: Suppose x is any other feasible point.

$$\begin{aligned}
q(x) &= \tfrac{1}{2} x^T G x + x^T c \\
&= \tfrac{1}{2} x^{*T} G x^* + x^{*T} c + (x - x^*)^T (Gx^* + c) + \tfrac{1}{2} (x - x^*)^T G (x - x^*) \\
&\geq q(x^*) + (x - x^*)^T (Gx^* + c) \\
&= q(x^*) + (x - x^*)^T \sum_{i \in A} \lambda_i^* a_i \\
&= q(x^*) + \sum_{i \in A} \lambda_i^* a_i^T (x - x^*) \\
&= q(x^*) + \sum_{i \in A \cap \mathcal{I}} \lambda_i^* a_i^T (x - x^*) \\
&= q(x^*) + \sum_{i \in A \cap \mathcal{I}} \underset{\geq 0}{\lambda_i^*} (\underset{\geq b_i}{a_i^T x} - \underset{= b_i}{a_i^T x^*}) \\
&\geq q(x^*)
\end{aligned}$$

$\blacksquare$

Active Set Method

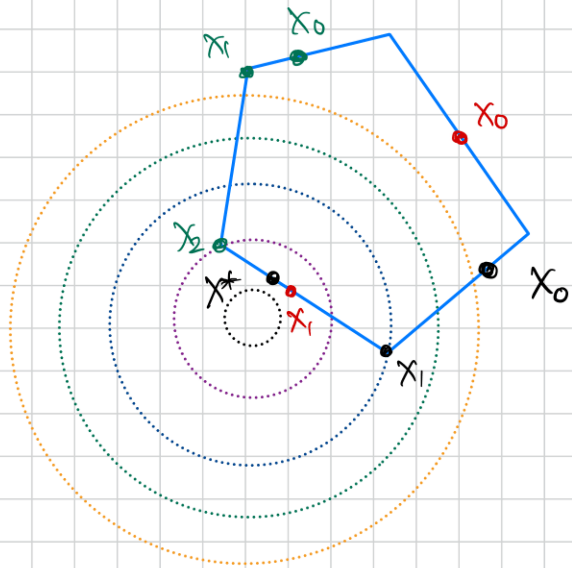
(p 467–476)

If we know the active constraints at optimal point x^* the we could simply solve the equality constrained problem:

$$\min \quad q(x) = \frac{1}{2}x^T G x + x^T c$$
$$\text{s.t.} \quad a_i^T x = b_i \quad i \in A(x^*)$$

But we don't actually know $A(x^*)$.

Concept: Intelligently search combinations of active constraints (working set), improving $q(x)$ with a feasible sequence $x_k \to x^*$.



First, reformulate the problem to get step direction

(i) Use a working set $W \subseteq A(x)$.

(ii) Use a minimizing KKT step p_k on the working set constraints ($x_{k+1} = x_k + \alpha p_k$)

let $p = x - x_k$

$g_k = G x_k + c$

Then

$$
\begin{aligned}
q(x) &= q(p + x_k) \\
&= \tfrac{1}{2}(p+x_k)^T G (p+x_k) + (p+x_k)^T c \\
&= \tfrac{1}{2}p^T G p + p^T G x_k + p^T c + \tfrac{1}{2}x_k^T G x_k + x_k^T c \\
&= \tfrac{1}{2}p^T G p + g_k^T p + \left(\tfrac{1}{2}x_k^T G x_k + x_k^T c\right)
\end{aligned}
$$

$$a_i^T p = a_i^T x - a_i^T x_k = b_i - b_i = 0 \quad \forall i \in W$$

So, solve the following to find p_k:

$$
\min_{p} \ \tfrac{1}{2}p^T G p + g_k^T p
$$
$$
\text{s.t. } a_i^T p = 0, \ i \in W
$$

The equality constrained subproblem!

Constrained Newton step

Step Size

$x_{k+1} = x_k + p_k$ may step to an infeasible point.

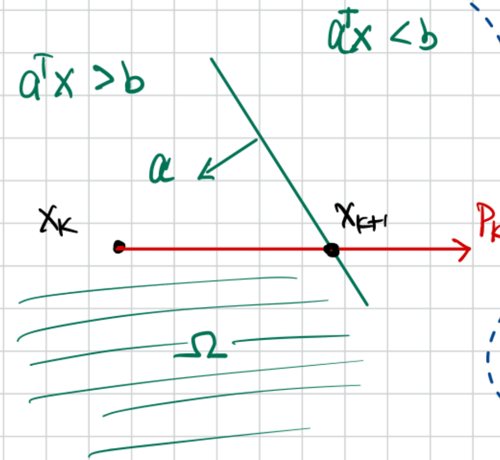
To guarantee feasibility, step only as far as possible in the direction of p_k.

$$x_{k+1} = x_k + \alpha_k p_k$$

$$\alpha_k = \min\left\{ 1,\ \min_{\substack{a_i^T p_k < 0 \\ i \in W}} \frac{b_i - a_i^T x_k}{a_i^T p_k} \right\}$$

full step

Step to a "blocking constraint"



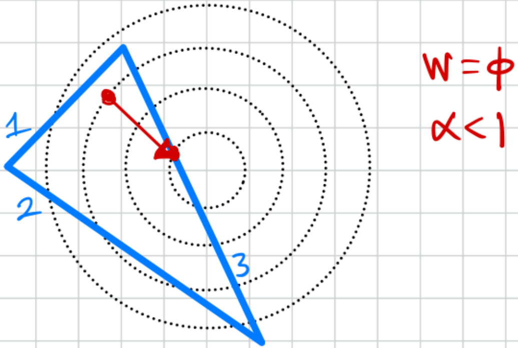
$a^T x_k > b$, If $a^T p_k < 0$

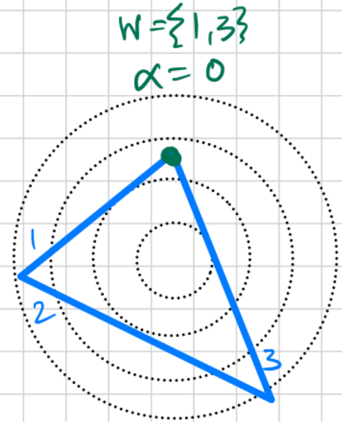
then we are stepping toward the constraint boundary until

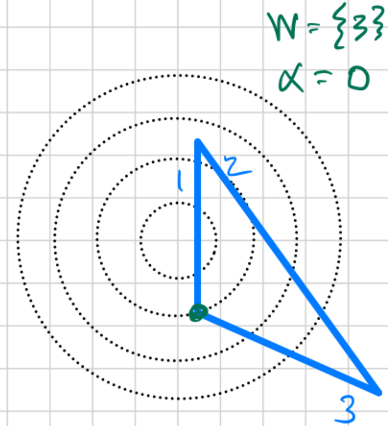
$$a^T(x_k + \alpha p_k) = b$$

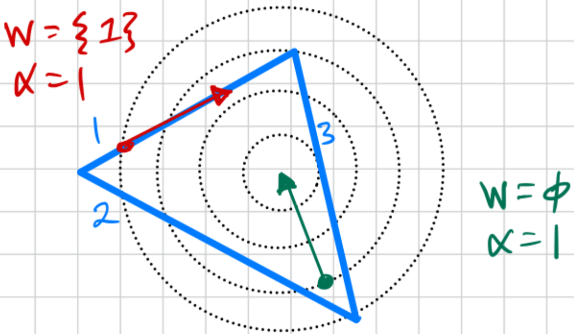
$$\alpha = \frac{b - a^T x_k}{a^T p_k}$$

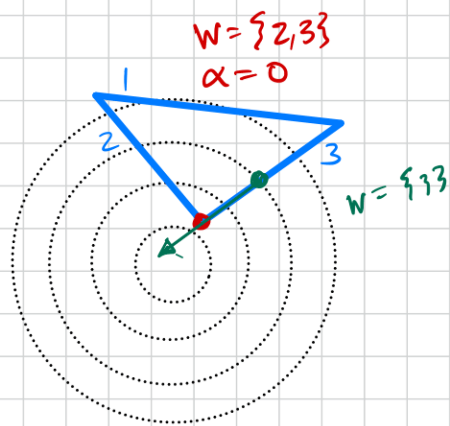
Some Example Steps

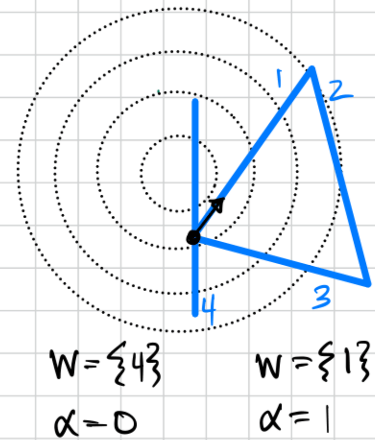












Algorithm Ideas

- If $\|p_k\| > 0$ and $x_k + p_k$ is feasible (full Newton Step is feasible on W)

  $x_{k+1} = x_k + p_k$

- If $\|p_k\| > 0$ and $x_k + p_k$ is infeasible (Newton Step infeasible on W)

  $x_{k+1} = x_k + \alpha_k p_k$ and add the blocking constraint to W

- If $\|p_k\| = 0$ and some $\lambda_\ell < 0$, $\ell \in W$ (a working set constraint prevents a step)

  remove constraint ℓ from W

$$\sum_{i \in W} a_i \lambda_i = g = Gx + c$$

$$A_W \lambda_W = Gx + c$$

- If $\|p_k\| = 0$ and all $\lambda_\ell \geq 0$, $\ell \in W$ ($\vec{0}$ is the only feasible descent direction)

  x_k is KKT optimal.

Active Set Algorithm

Algorithm 16.3 (Active-Set Method for Convex QP).

Compute a feasible starting point x_0;
Set $\mathcal{W}_0$ to be a subset of the active constraints at x_0;
for $k = 0, 1, 2, \ldots$
  Solve (16.39) to find p_k;
  if $p_k = 0$
    Compute Lagrange multipliers $\hat{\lambda}_i$ that satisfy (16.42),
      with $\hat{\mathcal{W}} = \mathcal{W}_k$;
    if $\hat{\lambda}_i \geq 0$ for all $i \in \mathcal{W}_k \cap \mathcal{I}$
      stop with solution $x^* = x_k$;
    else
      $j \leftarrow \arg\min_{j \in \mathcal{W}_k \cap \mathcal{I}} \hat{\lambda}_j$;
      $x_{k+1} \leftarrow x_k$; $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k \backslash \{j\}$;
  else (* $p_k \neq 0$ *)
    Compute α_k from (16.41);
    $x_{k+1} \leftarrow x_k + \alpha_k p_k$;
    if there are blocking constraints
      Obtain $\mathcal{W}_{k+1}$ by adding one of the blocking
        constraints to $\mathcal{W}_k$;
    else
      $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k$;
end (for)

Let W_0 be the set of equality constraints, which are assumed to be linearly independent.