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EE 507 MIDTERM 2

17 NOVEMBER 2022

81.5/85

Great!

1.1

1. $f_X(x) = 0.5 e^{-|x|} \quad \forall x \in \mathbb{R}$.

+10/10/10

$$\Rightarrow f_X(x) = \begin{cases} 0.5 e^{x} & x < 0 \\ 0.5 e^{-x} & x \geq 0 \end{cases}$$

$$MGF_X(s) = E[e^{sX}]$$

$$\Rightarrow MGF_X(s) = \int_{x=-\infty}^{x=0} e^{sx} 0.5 e^{x} dx + \int_{x=0}^{x=\infty} e^{sx} 0.5 e^{-x} dx$$

$$\Rightarrow MGF_X(s) = 0.5 \int_{x=-\infty}^{x=0} e^{(s+1)x} dx + 0.5 \int_{x=0}^{x=\infty} e^{(s-1)x} dx$$

if $s > -1$ if $s < 1$

$$\Rightarrow MGF_X(s) = \frac{0.5}{s+1} - \frac{0.5}{s-1} \qquad s \in (-1, 1)$$

$$\Rightarrow MGF_X(s) = 0.5 \times \frac{2}{1-s^2} \qquad s \in (-1,1)$$

1(a)

$$\Rightarrow \boxed{MGF_X(s) = \frac{1}{1-s^2} \qquad s \in (-1,1).}$$ Ans.

$$E[X^2] = \frac{d^2}{ds^2} MGF_X(s)\Big|_{s\to 0} = \frac{d}{ds}\left(\frac{d}{ds}(1-s^2)^{-1}\right)\Big|_{s=0}$$

$\Rightarrow E[X^2] = \frac{d}{ds}\left(-(1-s^2)^{-2} \cdot -2s \right)\Big|_{s=0}$

$\Rightarrow E[X^2) = \frac{d}{ds}\left(2s(1-s^2)^{-2} \right)\Big|_{s=0}$

$\Rightarrow E[X^2] = 2(1-s^2)^{-2} + 2s(1-s^2)^{-3} \cdot -2 \cdot -2s \Big|_{s=0}$

1(b)

$\Rightarrow \boxed{E[X^2) = 2}$ Ans

———— X ———— X ———— X ————

$\boxed{2.}$

x_i	y_i	$z_i = x_i + y_i$	~~$P(z_i$~~
0	0	0	
0	1	1	
0	2	2	
1	0	1	
1	1	2	
2	0	2	

+10/10

z_i	Possible (x_i, y_i)	$p(z_i) = \sum p(x_i, y_i) = \sum p(x_i)\cdot p(y_i)$
0	(0,0) ~~(0,2), (1,1), (2,0)~~	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 \cdot \frac{1}{3}$
1	(0,1), (1,0)	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 \cdot \frac{1}{3} + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^1 \frac{1}{3}$
2	(0,2), (1,1), (2,0)	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 \cdot \frac{1}{3} + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^1 \frac{1}{3} + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 \cdot \frac{1}{3}$

z_i	$p(z_i) = \sum p(x_i) p(y_i)$
0	$\frac{1}{9} \cdot 1 = \frac{1}{9} \cdot \frac{9}{9}$
1	$\frac{1}{9} \cdot \frac{5}{3} = \frac{1}{9} \cdot \frac{15}{9}$
2	$\frac{1}{9} \cdot \frac{19}{9}$

$$p(Z \le 2) = P(X + Y \le 2) = \frac{1}{9}\left\{\frac{9}{9} + \frac{15}{9} + \frac{19}{9}\right\} = \frac{43}{81}$$

$\boxed{2.}$

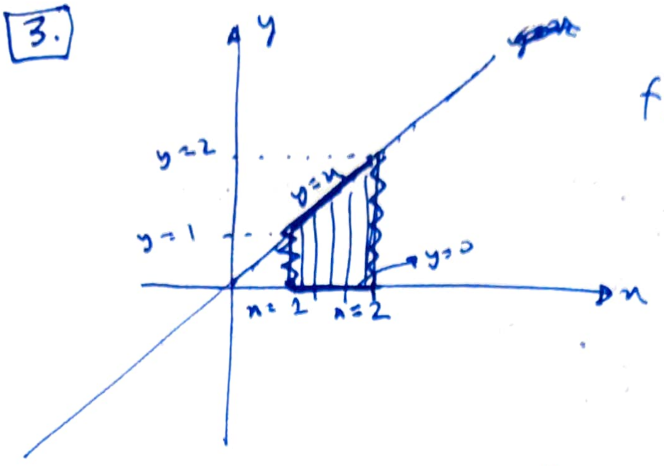
$$\boxed{\begin{gathered} P(X+Y \le 2) = \frac{43}{81} \\ P(X+Y \le 2) \approx 0.5309 \end{gathered}}$$

Ans

+12.5/15

3.1

3.



$f_{X,Y}(x,y)$ is uniform in the shaded region

$$f_{X,Y}(x,y) = \frac{1}{\text{Area of shaded region}} \quad \forall\, x \in (1,2],\ y \in (0,x]$$

$$\Rightarrow f_{X,Y}(x,y) = \frac{1}{\frac{(2+1)}{2}\cdot 1}$$

$$\Rightarrow \boxed{f_{X,Y}(x,y) = \begin{cases} \frac{2}{3} & x \in [1,2],\ y \in (0,x] \\ 0 & \text{else} \end{cases}}$$

PTO

PTO!

$$f_Y(y) = \int_{x=y}^{x=2} f_{X,Y}(x,y)\,dx + \int_{x=1}^{x=2} f_{X,Y}(x,y)\,dx$$

$$\Rightarrow f_Y(y) = \int_{x=y}^{x=2} \frac{2}{3}\,dx + \int_{x=1}^{x=2} \frac{2}{3}\,dx$$

$$\Rightarrow f_Y(y) = \frac{6-2y}{3} \quad \forall\ y \in (0,2]$$

$$\Rightarrow f_Y(y) = 2 - \frac{2y}{3} \quad \forall\ y \in (0,2]$$

$$f_{X|Y=y}(X, Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$\Rightarrow f_{X|Y=y}(X, Y=y) = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$$

$\int_{y=0}^{y=2} 2 - \frac{2y}{3}\,dy$

$\Rightarrow 4 - \frac{2}{3}\cdot\frac{1}{2}\cdot 4$

$\frac{2}{3}(2-y)$

$\frac{2}{3}\left(2y - \frac{y^2}{2}\right)$ $2 - \frac{2}{6}\cdot\frac{y^2}{2}$ $2 - \frac{4}{6} =$

$\frac{2}{3}(4-2)$

$\frac{4}{3}$

$\frac{2-y}{3}$

$\frac{2}{3}y - \frac{y^2}{6}\Big|_0^2$

$\frac{4}{3} - \frac{4}{6} = \frac{2}{3}$

$\frac{2}{3}(2-y)$

$\frac{2}{3}\left(2y - \frac{y^2}{2}\right)\Big|_{y=1}^{y=2}$

$\boxed{3.2}$

$$f_Y(y) = \begin{cases} \int_{x=1}^{x=2} f_{X,Y}(x,y)\,dx & y \in (0,1) \\ \int_{x=y}^{x=2} f_{X,Y}(x,y)\,dx & y \in [1,2) \end{cases}$$

$$\text{or } f_Y(y) = \begin{cases} \int_{x=1}^{x=2} \frac{2}{3}\,dx & y \in (0,1) \\ \int_{x=y}^{x=2} \frac{2}{3}\,dx & y \in [1,2) \end{cases}$$

3(a)(i)

$$\boxed{\text{or } f_Y(y) = \begin{cases} \frac{2}{3} & y \in (0,1) \\ \cancel{\frac{2}{3}y} \\ \frac{2}{3}(2-y) & y \in [1,2) \end{cases}}$$

Ans

$\cancel{f_{X,Y}(x,y}$

$$f(x \mid Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

3.3

$$\therefore f_{X|Y}(X|Y=y) = \begin{cases} \dfrac{\frac{2}{3}}{\frac{2}{3}} & y \in (0,1) \\[2ex] \dfrac{\frac{2}{3}}{\frac{2}{3}(2-y)} & y \in (1,2] \end{cases}$$

3(a)(ii)

$$\boxed{\therefore f_{X|Y}(X|Y=y) = \begin{cases} 1 & y \in [0,1] \\ \dfrac{1}{1-y} & y \in (1,2) \end{cases}}$$

Ans

3(a)(iii)

$\because f_{X|Y}$ depends on value of y $\therefore$ X and Y are NOT ~~taken~~ independent.

Ans

$$\therefore \hat{X}_{X|Y_{MMSE}} = \mu_{X|Y} \quad \cancel{=\{}$$

$$\therefore \cancel{X_q}\ \cancel{X_{X|Y_{MM}}}$$

$$\text{But} \quad \mu_{X|Y=y} = \int_{x=1}^{x=2} x \cdot f_{X|X}(X|Y=y)\, dx$$

$$\therefore \mu_{X|Y=y} = \begin{cases} \displaystyle\int_{x=1}^{x=2} x \cdot 1 \cdot dx & y \in [0,1] \\[2ex] \displaystyle\int_{x=1}^{x=2} \frac{1}{1-y} \cancel{x} \cdot x \cdot dx & y \in (1,2] \end{cases}$$

wrong x range

$$\Rightarrow \quad \mu_{X|Y=y} = \begin{cases} \left.\frac{x^2}{2}\right|_1^2 & y \in (0,1] \\ \frac{1}{1-y} \cdot \left.\frac{x^2}{2}\right|_1^2 & y \in [1,2] \end{cases}$$

$$\Rightarrow \quad \mu_{X|Y=y} = \begin{cases} \frac{3}{2} & y \in (0,1] \\ \frac{3}{2} \cdot \frac{1}{1-y} & y \in [1,2] \end{cases}$$

$\boxed{3(b)}$

$$\Rightarrow \quad \boxed{\hat{X}_{\substack{X|Y=y \\ MMSE}} = \begin{cases} \frac{3}{2} & y \in (0,1] \\ \frac{3}{2} \cdot \frac{1}{1-y} & y \in [1,2] \end{cases}} \quad \text{Ans}$$

———— X ———— X ———— X ————

4.1

4. $f_X(x) = \mathcal{N}(x, \mu_X = 0, \sigma_X^2 = 4)$ +14.5/15

or $f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{2}\right)^2\right\}}$

$f_Y(y) = \mathcal{N}(y, \mu_Y = 0, \sigma_Y^2 = 9)$

or $f_Y(y) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y}{3}\right)^2\right\}}$

$f_{X,Y}(x,y) = \mathcal{N}(x, y, \mu_X = 0, \mu_Y = 0, \sigma_X^2 = 4, \sigma_Y^2 = 9, \rho = -\frac{1}{2})$

or $f_{X,Y}(x,y) = \frac{1}{2\sqrt{2\pi} \cdot 3\sqrt{2\pi} \sqrt{(1-(-\frac{1}{2})^2)}} e^{-\frac{1}{2(1-(-\frac{1}{2})^2)}\left\{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 - 2\cdot\left(-\frac{1}{2}\right)\cdot\left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}$

4(a)

or $\boxed{f_{X,Y}(x,y) = \frac{1}{\cancel{4\sqrt{3}}\ 6\sqrt{3}\pi} e^{-\frac{1}{3/2}\left\{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}}$

4(a)

Again. (for clarity)

$\boxed{f_{X,Y}(x,y) = \frac{1}{6\sqrt{3}\pi} e^{-\frac{1}{2}\cdot\frac{4}{3}\left\{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}}$ Ans

4b)

$Z = X + Y$

$E(Z) = \mu_X + \mu_Y$

or $\mu_Z = 0 + 0$

$\boxed{\mu_Z = 0}$

$W = bX$

$E[W] = E[bX]$

or $\mu_W = b\, E[X]$

or $\mu_W = b\,\mu_X$

or $\boxed{\mu_W = 0}$

$\sigma^2_Z = \sigma^2_X + \sigma^2_X + 2\rho\sigma_X\sigma_Y$ $\qquad$ $\sigma^2_W = b^2\sigma^2_X$

or $\sigma^2_Z = 4 + 9 + 2 \times -\frac{1}{2} \times 2 \times 3$ $\qquad$ or $\boxed{\sigma^2_W = 4b^2}$

or $\sigma^2_Z = 13 - 6$

or $\boxed{\sigma^2_Z = 7}$

$Cov(Z, W) = Cov(X+Y, bX)$

or $Cov(Z, W) = E[\{X + Y - \cancel{\mu} E[X+Y]\}\{bX - E[bX]\}]$

or $Cov(Z, W) = E[\{X+Y\}\{bX\}]$

or $Cov(Z, W) = E[bX^2 + bXY]$

or $Cov(Z, W) = b\mu_{X^2} + b\underline{Cov(XY)}$

Here, $Cov(X,Y) = E[XY]$ as $\mu_X\mu_Y = 0$.

or $Cov(Z, W) = b\{\sigma_X^2 + \mu_X^2\} + b\{\rho_{XY}\sigma_X\sigma_X\}$

or $Cov(Z, W) = b[4 + 0^2 + -\frac{1}{2} \cdot 2 \cdot 3]$

or $Cov(Z, W) = b[1]$

$\boxed{4(b)^{ii)}}$

or $\boxed{Cov(Z, W) = b}$ $\Rightarrow$ $\boxed{Cov(Z, W) > 0 \ \forall \ b > 0}$ Ans

$$\rho_{Z,W} = \frac{Cov(Z,W)}{\sigma_Z \cdot \sigma_W}$$

or $$\rho_{Z,W} = \frac{b}{\sqrt{7} \cdot 2b}$$

or $$\boxed{\rho_{Z,W} = \frac{1}{2\sqrt{7}}}$$

4(b)(i)

$$\therefore \boxed{f_{Z,W}(z,w) \sim \mathcal{N}\left(z, w, \mu_Z = 0, \mu_W = 0, \sigma_Z^2 = 7, \sigma_W^2 = 4b^2, \rho_{ZW} = \frac{1}{2\sqrt{7}}\right)}$$ Ans

4(c) $$\hat{Y}_{Y|X=x}^{\text{MMSE}} = E[Y \mid X = x]$$

Let's try to find $f_{Y|X}(Y|X=x)$ first!

$$f_{Y|X}(Y|X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

or $$f_{Y|X}(Y|X=x) = \frac{\dfrac{1}{2\sqrt{2\pi} \cdot 3\sqrt{2\pi} \cdot \frac{\sqrt{3}}{2}} \cdot e^{-\frac{1}{2 \cdot \frac{3}{4}}\left\{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{x}{2}\frac{y}{3}\right)\right\}}}{\dfrac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{2}\right)^2\right\}}}$$

$$\Rightarrow \cancel{f_{Y|X}(Y|X=x) = \frac{1}{3\sqrt{2\pi}\cdot\frac{3}{4}} e^{-\frac{1}{2}\{}}$$

$$\Rightarrow f_{Y|X}(Y|X=x) = \frac{1}{\frac{3\sqrt{3}}{2}\cdot\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{1}{3}\left(\frac{x}{2}\right)^2 + \frac{4}{3}\left(\frac{y}{3}\right)^2 + \frac{4}{3}\left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}$$

$$\Rightarrow f_{Y|X}(Y|X=x) = \frac{1}{\frac{3\sqrt{3}}{2}\cdot\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{2\sqrt{3}}\right)^2 + \left(\frac{y}{\frac{3\sqrt{3}}{2}}\right)^2 + 2.1\left(\frac{x}{2\sqrt{3}}\right)\left(\frac{y}{\frac{3\sqrt{3}}{2}}\right)\right\}}$$

$$\Rightarrow f_{Y|X}(Y|X=x) = \frac{1}{\left(\frac{3\sqrt{3}}{2}\right)\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y}{\frac{3\sqrt{3}}{2}} + \frac{x}{2\sqrt{3}}\right)^2\right\}}$$

$$\Rightarrow f_{Y|X}(Y|X=x) = \frac{1}{\left(\frac{3\sqrt{3}}{2}\right)\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y + \frac{3}{4}x}{\left(\frac{3\sqrt{3}}{2}\right)}\right)^2\right\}}$$

$$\Rightarrow \cancel{f_{Y|X}(Y|X=x) =}$$

$$\Rightarrow Y|X=x \sim \mathcal{N}\left(y, \mu_Y = -\frac{3}{4}x, \sigma_Y^2 = \frac{27}{4}\right)$$

$\boxed{4(C)}$

$$\Rightarrow \boxed{\hat{Y}_{\substack{Y|X=x \\ MMSE}} = \mu_{Y|X=x} = -\frac{3}{4}x}$$ Ans

———— × ———— × ———— × ————

+10/10

5. $X \sim unif[0,1)$

$Y | X=x \sim unif[0,x]$

$Z | Y=y, X=x \sim unif[0,y)$

$$E[X^2Y^2Z^2] = \iiint x^2y^2z^2 \cdot f_{X,Y,Z}(x,y,z)\, dz\, dy\, dx$$

or ~~$E[X^2Y^2Z^2] = \iiint$~~

But $f_{X,Y,Z}(x,y,z) =$

$$= f_Z(z | X=x, Y=y) \cdot f_{X,Y}(X=x, Y=y)$$

or $f_{X,Y,Z}(x,y,z) = f_Z(z | X=x, Y=y) \cdot f_{Y|X}(y | X=x) \cdot f_X(x)$

or $f_{X,Y,Z}(x,y,z) = \frac{1}{y} \cdot \frac{1}{x} \cdot \frac{1}{1} \quad \forall \; z \in [0,y), \; y \in [0,x], \; x \in [0,1)$

$$\therefore E[X^2Y^2Z^2] = \int_{x=0}^{x=1}\int_{y=0}^{y=x}\int_{z=0}^{z=y} x^2 \cdot y^2 \cdot z^2 \cdot \frac{1}{xy}\, dz\, dy\, dx$$

$$\text{or } E[X^2Y^2Z^2] = \int_{x=0}^{x=1}\int_{y=0}^{y=x}\int_{z=0}^{z=y} xyz^2\, dz\, dy\, dx$$

$$\text{or } E[X^2Y^2Z^2] = \int_{x=0}^{x=1} x \int_{y=0}^{y=x} y \cdot \frac{z^3}{3}\Big|_{z=0}^{z=y} dy\, dx$$

$$\text{or} \quad E[X^2Y^2Z^2] = \int_{x=0}^{x=1} x \int_{y=0}^{y=x} \frac{y^4}{3}\, dy\, dx$$

$$\text{or} \quad E[X^2Y^2Z^2] = \int_{x=0}^{x=1} x \cdot \frac{y^5}{15}\Bigg|_{y=0}^{y=x} dx$$

$$\text{or} \quad E[X^2Y^2Z^2] = \int_{x=0}^{x=1} \cancel{x \cdot x^5}\ \frac{x^6}{15}\, dx$$

$$\text{or} \quad E[X^2Y^2Z^2] = \frac{x^7}{105}\Bigg|_{x=0}^{x=1}$$

$$\text{or} \quad \boxed{E[X^2Y^2Z^2] = \frac{1}{105}} \quad \underline{\text{Ans}}$$

——— × ——— × ——— × ———

$\int\int\int xyz^2$

$xy\frac{y^3}{3}\Big|$

$x\frac{y^4}{3}$

$\cancel{\frac{xy^5}{3 \cdot 5}}$ $x\frac{y^4}{3}$

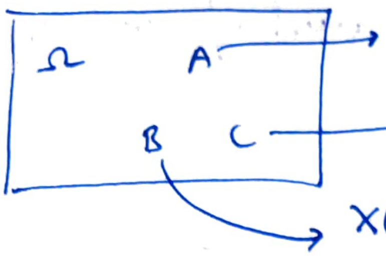
$x\frac{y^5}{3 \cdot 5}\Big|$

$x \cdot \frac{x^5}{3 \cdot 5}$ $\frac{x^6}{3 \cdot 5}\Big|$ $\frac{x^7}{3 \cdot 5 \cdot 7}\Big|_0^1$

$\boxed{6.1}$

+10/10

$\boxed{6.}$ Random Processes are mappings ~~of~~ from outcomes of a probabilistic experiment to signals.



$X(A,t) = \cos(t)$

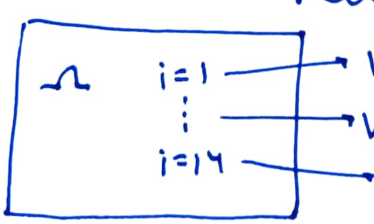
$X(C,t) = \begin{cases} 7 & \lfloor t \rfloor \text{ is even} \\ 0 & \lfloor t \rfloor \text{ is odd} \end{cases}$

$X(B,t) = 2e^{-2t}$

where $\lfloor t \rfloor$ refers to the greatest integer lesser than or equal to t.

Virtually any physical process or simulated phenomenon can be represented as a random process.

Eg. Bus Voltages in a power grid are time-varying signals, and there are multiple buses (outcomes) in a power grid or we can measure a single bus voltage in multiple ways. (sample functions).



$V_1(t)$

$V_i(t)$

$V_{14}(t)$

$i \in 1$ to 14

$i \in N$

$\Omega \equiv$ IEEE 14 Bus System.

Study of Random Processes is important as it can be used to analyze, estimate, predict, forecast signals which can be very beneficial to us.

Eg. If tomorrow's load demand for a regional substation could be predicted accurately, we can optimize the scheduling of power generation, spot market rates, etc., which can save a grid operator millions of Dollars.

Ans

+14.5/15

$\boxed{7.1}$

$\boxed{7.}$ $X(t) = \dfrac{1}{t+1+C}$ $t \in R^+$

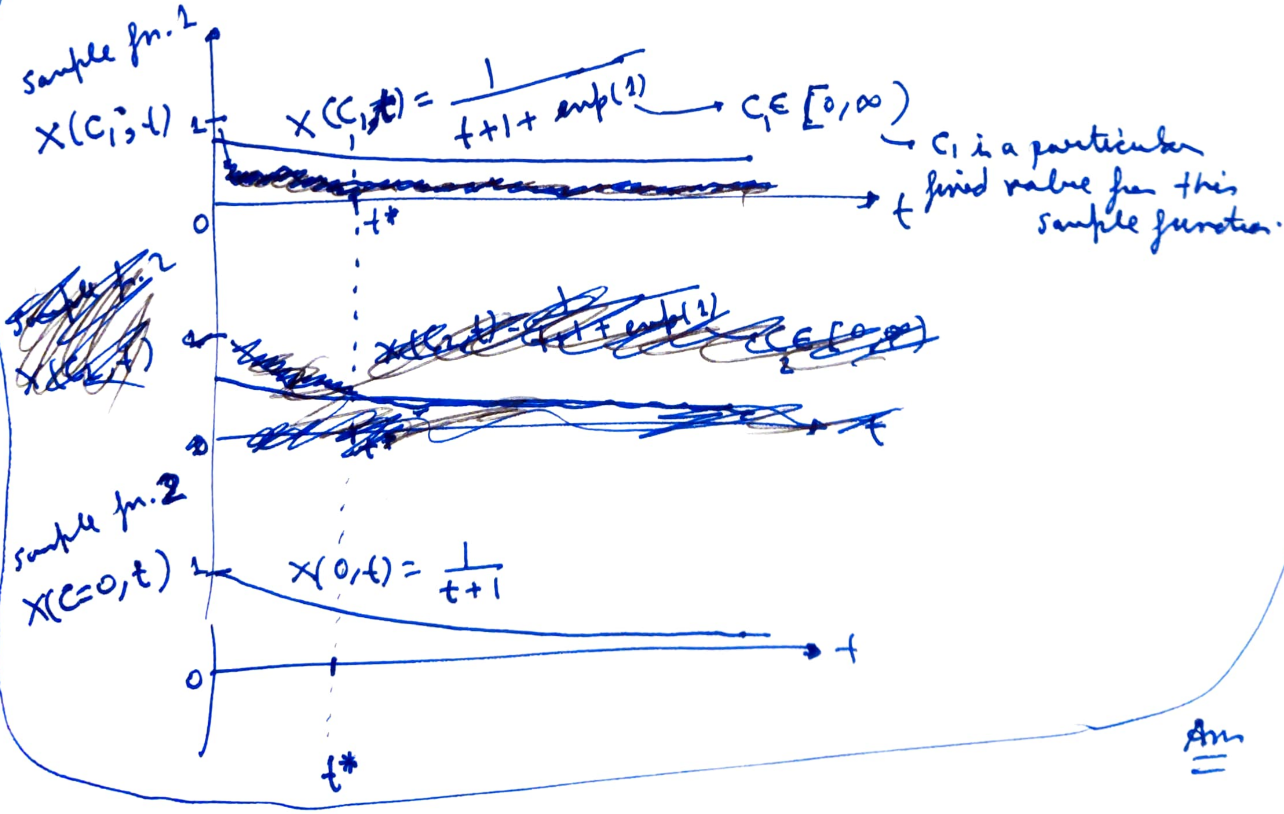
$C \sim exp(1)$

~~or $f_C(c) =$~~ or $f_C(c) = e^{-c}$ $c \geq 0$

$\boxed{7(a)}$ $X(t)$ is a Random Process as $X(t) = X(C,t)$ is a set of time varying signals which are all mapped to the outcome C of a ~~proba~~ probabilistic experiment, where

$C \sim exp(1)$.

So $X(C,t)$ can ~~be~~ have a sample function $X(C=2,t)$ or it can be sampled ~~at~~ time t^* to get a ~~unique~~ set of ~~$X(C,t^*)$~~ $X(C, t=t^*)$ $\forall$ $C \sim exp(1)$.



Am

7.2

$X(t) = \frac{1}{t+1+C} \quad t \in \mathbb{R}^+ \quad C \sim \exp(1)$

$C \in [0, \infty)$

$\Rightarrow$ ~~$X(t) \in \frac{1}{t+1} \left[0, \frac{1}{t+1}\right)$~~

$\Rightarrow X(t) \in \left(0, \frac{1}{t+1}\right]$

Approach: Find CDF then pdf:

$$F_{X(t)}(x) = P(X(t) \le x) = P\left(\frac{1}{t+1+C} \le x\right)$$

$$\Rightarrow F_{X(t)}(x) = P\left(C + t + 1 \ge \frac{1}{x}\right)$$

$$\Rightarrow F_{X(t)}(x) = P\left(C \ge \frac{1}{x} - (t+1)\right)$$

$$\Rightarrow F_{X(t)}(x) = 1 - P\left(C \le \frac{1}{x} - (t+1)\right)$$

$$\Rightarrow F_{X(t)}(x) = \begin{cases} 0 & x \le 0 \\ 1 - \left(1 - e^{-\left(\frac{1}{x} - (t+1)\right)}\right) & x \in \left(0, \frac{1}{t+1}\right] \\ 1 & x \ge \frac{1}{t+1} \end{cases}$$

$1 - e^{(t+1)} \cdot e^{-\frac{1}{x}}$

$\boxed{7(b)}$

$$\Rightarrow \boxed{f_{X(t)}(x) = \frac{d}{dx} F_{X(t)}(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{x^2} \cdot e^{-\frac{1}{x}} \cdot e^{(t+1)} & x \in \left(0, \frac{1}{t+1}\right] \\ 0 & x \ge \frac{1}{t+1} \end{cases}}$$

Ans