Does $f_x f_y f_z = f_{xyz}$ imply $f_x f_y = f_{xy}$? Yes.

1) a)

$$\int f_x f_y f_z \, dz \overset{?}{=} \int f_{xyz} \, dz$$

$$f_x f_y \underbrace{\int f_z \, dz}_{1} \overset{?}{=} f_{xy}(x,y)$$

$$f_x f_y = f_{xy}(x,y) \Rightarrow X \text{ and } Y \text{ are independent!}$$

b)

Does $P(X)P(Y)P(Z) = P(X,Y,Z)$ imply $P(X)P(Y) = P(X,Y)$? No. Counter example:

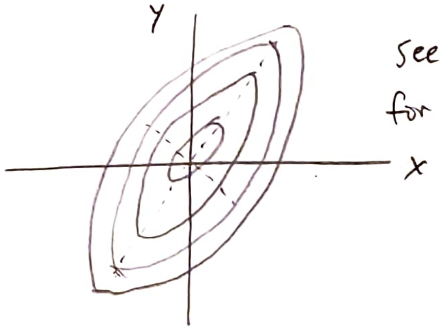
Flip a coin.

$Z = \{\emptyset\}$, $X = Y = \{H\}$

$\Rightarrow P(X)P(Y)P(Z) = 0 = P(X,Y,Z)$, but $P(X)P(Y) = (\frac{1}{2}) \cdot (\frac{1}{2}) \neq P(X,Y) = \frac{1}{2}$.

Z can be independent from X and Y, even if X and Y are not independent from each other.

2a) $(X,Y) \sim N(0,0,1,4,1/2)$. Lines of equal probability are ellipses



See MATLAB plot for exact solution

2b) $X \sim N(0,1) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$

$Y \sim N(0,4) \Rightarrow f_Y(y) = \frac{1}{2\sqrt{2\pi}} \exp\left(\frac{-y^2}{8}\right)$

2c) $$f_{Y|X}(y) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\frac{1}{2\pi\sqrt{3}} \exp\left[\frac{-2}{3}\left(x^2 + y^2/4 - xy/2\right)\right]}{\frac{1}{\sqrt{2\pi}} \exp\left[\frac{-1}{2} x^2\right]}$$

$$= \frac{\sqrt{2\pi}}{2\pi\sqrt{3}} \exp\left[\frac{-1}{6}\left(x^2 - 2xy + y^2\right)\right]$$

$$= \frac{1}{\sqrt{3}\sqrt{2\pi}} \exp\left[\frac{-1}{6}(y-x)^2\right]$$

$$= \frac{1}{\sqrt{3}\sqrt{2\pi}} \exp\left[\frac{-(y-x)^2}{2(\sqrt{3})^2}\right]$$

$Y|X=x \sim N(x, 3)$

2d) choose $x = -2$ so that $E[Y|X=x] = -2$ from the definition of $Y|X=x$.

2e) $E[Z] = E[X+Y-1] = E[X] + E[Y] - 1 = 0 + 0 - 1 = -1$

$E[Z^2] = E[(X+Y-1)^2] = E[\underline{X^2} + \underline{XY} - \underline{X} + \underline{YX} + \underline{Y^2} - \underline{Y} + -\underline{X} - \underline{Y} + 1]$

$= E[X^2 + Y^2 + 2XY - 2X - 2Y + 1] = E[X^2] + E[Y^2] + E[2XY] - E[2X]^{0} - E[2Y]^{0} + 1.$

$E[XY] = E[(X-0)(Y-0)] = \text{cov}(X,Y)$

$r = 0.5 = \dfrac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \dfrac{\text{cov}(X,Y)}{2} \Rightarrow \text{cov}(X,Y) = 1 \Rightarrow E[XY] = 1.$

$E[X^2] = \text{var}(X) + E^2[X] = 1 + 0 = 1$

$E[Y^2] = \text{var}(Y) + E^2[Y] = 4 + 0 = 4$

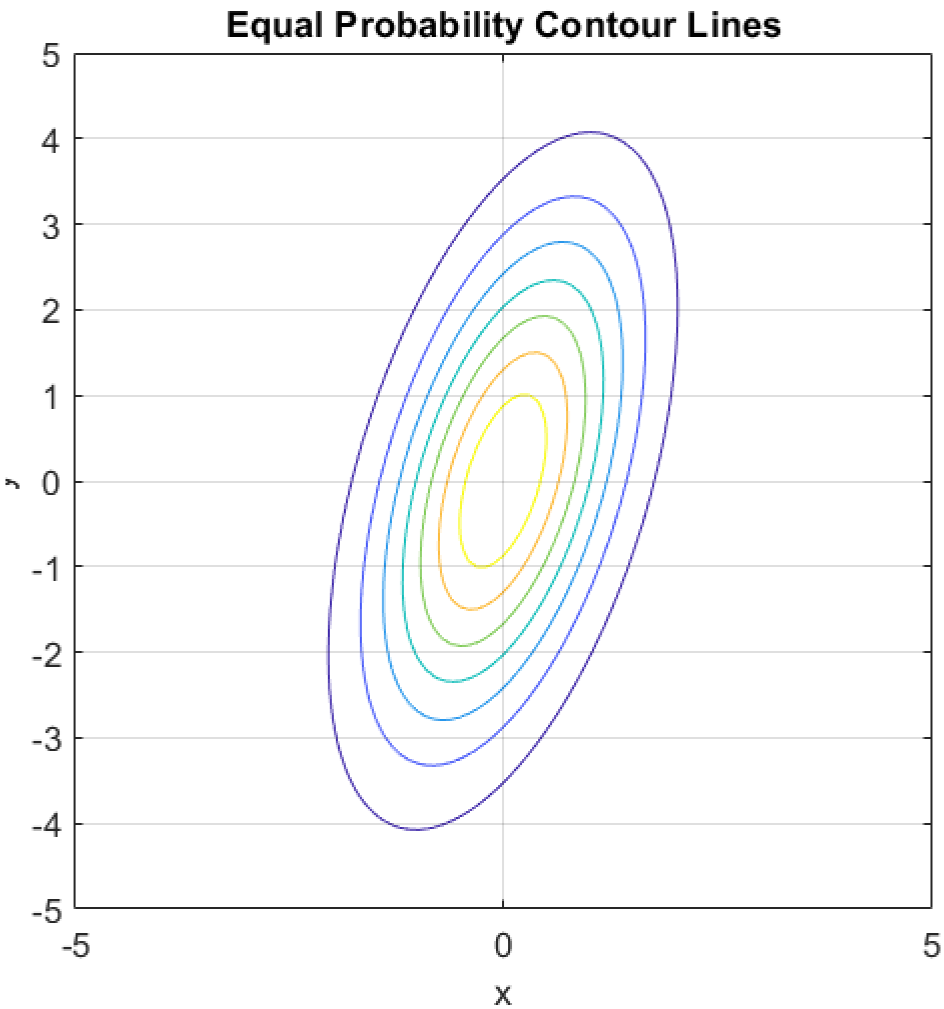
$E[Z^2] = E[X^2] + E[Y^2] + 2E[XY] + 1$

$= 1 + 4 + 2 + 1$

$= 8.$

$\text{Var}(Z) = E[Z^2] - E^2[Z] = 8 - (-1)^2 = 7$

$\Rightarrow Z \sim N(\underset{\mu}{-1}, \underset{\sigma^2}{7}) \Rightarrow f_Z(z) = \dfrac{1}{\sqrt{14\pi}} \exp\left(\dfrac{-(z+1)^2}{14}\right)$


Equal Probability Contour Lines
5
4
3
2
1
0
-1
-2
-3
-4
-5
y
-5
0
5
x

2f) $Z = 2X + 3Y, \quad W = X - Y$

$E[Z] = 2E[X] + 3E[Y] = 0 + 0 = 0$

$Var(Z) = E[Z^2] - E^2[\overset{0}{\cancel{Z}}] = E[(2X+3Y)^2] = E[\underbrace{4X^2}_{E[X^2]=1} + \underbrace{12XY}_{E[XY]=1} + \underbrace{9Y^2}_{E[Y^2]=4}] = 4 + 12 + 36 = 52.$

$E[W] = E[X] - E[Y] = 0$

$var(W) = E[W^2] - E^2[\overset{0}{\cancel{W}}] = E[X^2 - 2XY + Y^2] = 1 - 2 + 4 = 3.$

$cov(Z,W) = E[(Z-0)(W-0)] = E[ZW] = E[(2X+3Y)(X-Y)]$

$= E[\underbrace{2X^2}_{1} \underbrace{- 2XY + 3XY}_{1} - 3\underbrace{Y^2}_{4}] = 2 + 1 - 12 = -9$

$r = \dfrac{cov(Z,W)}{\sigma_Z \sigma_W} = \dfrac{-9}{\sqrt{52 \cdot 3}} = -0.7206$

$(Z, W) \sim N(\mu_Z = 0, \mu_W = 0, \sigma_Z^2 = 52, \sigma_W^2 = 3, r = -0.7206)$

g) $R = aX + bY$. For what (a,b) are R, Y independent? R is gaussian, so $cov(R,Y) = 0 \Rightarrow$ independence. $E[R] = aE[X] + bE[Y] = 0$

$cov(R,Y) = E[(R - \mu_R)(Y - 0)] = E[RY] = E[(aX+bY)Y]$

$= E[a\underbrace{XY}_{1} + b\underbrace{Y^2}_{4}] = a + 4b = 0 \Rightarrow$ R and Y are independent when $\boxed{\forall (a,b) \ni a = -4b}$

h) $Q = aX$. $a = 1$ w/ pr. 1/2, $a = 2$ w/ pr. 1/2

$E[Q] = aE[X] = 0.$ $E[Q^2|a=1] = E[X^2] = \overset{\sigma^2_{Q|a=1}}{1}.$ $E[Q^2|a=2] = E[4X^2] = \overset{\sigma^2_{Q|a=2}}{4}.$

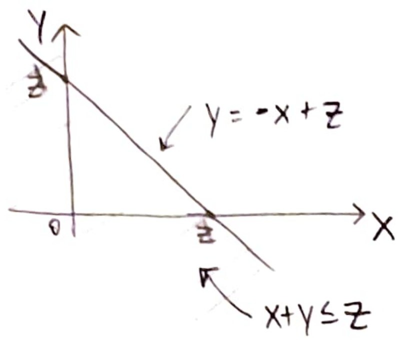
$f_Q(q) = \frac{1}{2} f_{Q|a=1}(q) + \frac{1}{2} f_{Q|a=2}(q) = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-q^2}{2}\right) + \frac{1}{2\sqrt{2\pi}}\exp\left(\frac{-q^2}{8}\right)\right]$

3a) X, Y are independent, with $Z = X + Y$. show $f_Z(z) = \int_{-\infty}^{\infty} f_Y(z-x) f_X(x)\, dx$.

that is, $f_Z(z) = f_X(z) * f_Y(z)$.

$f_{X,Y}(x,y) = f_X(x) f_Y(y)$.

$$F_Z(z) = P(Z \le z) = P(X+Y \le z) = \iint_R f_{X,Y}(x,y)\,dy\,dx = \iint_R f_X(x) f_Y(y)\,dy\,dx$$



$$F_Z(z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_X(x) f_Y(y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{z-x} f_Y(y)\,dy \;\; dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot \left[F_Y(z-x) - \cancel{F_Y(-\infty)}^{\,0} \right] dx$$

$$= \int_{-\infty}^{\infty} f_X(x) F_Y(z-x)\,dx$$

✓

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_X(x) \frac{d}{dz} F_Y(z-x)\,dx = \boxed{\int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx}$$

3b) just change the limits on the original integral:

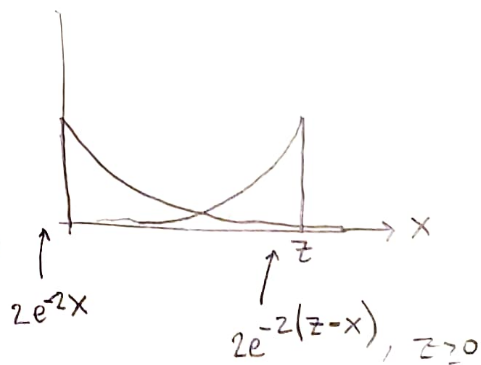
$$F_Z(z) = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f_X(x) f_Y(y)\,dx\,dy = \int_{-\infty} f_Y(y) F_X(z-y)\,dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) \frac{d}{dz} F_X(z-y)\,dy = \boxed{\int_{-\infty}^{\infty} f_Y(y) f_X(z-y)\,dy}$$ ✓

3c) $X \sim \exp(2)$, $Y \sim \exp(2)$, (X,Y) iid.

$f_X(x) = 2e^{-2x}$, $f_Y(y) = 2e^{-2y}$, $x \ge 0, y \ge 0$

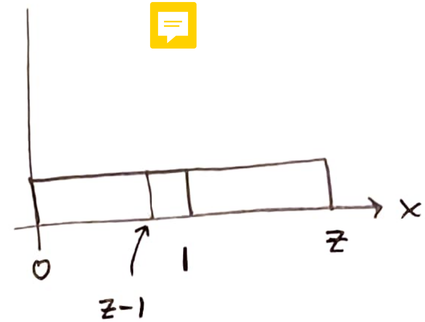
$$f_Z(z) = f_X(z) * f_Y(z) = \int_0^z 4e^{-2(x+z-x)}\,dx = \boxed{4ze^{-2z}}$$



$$\Rightarrow f_Z(z) = \begin{cases} 0 & z \le 0 \\ 4ze^{-2z}, & z > 0 \end{cases}$$

3d) X, Y are iid $\sim unif(0,1)$. $Z = X+Y$

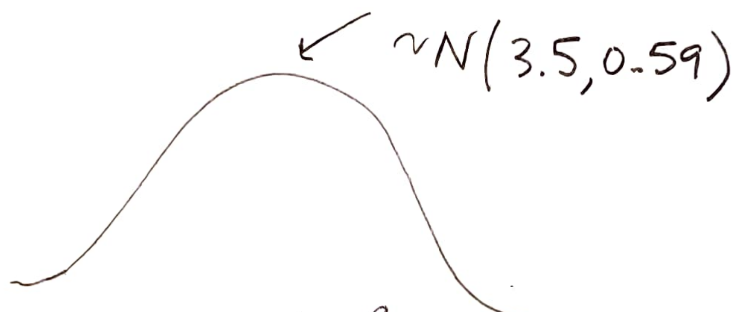
$$f_Z(z) = \int_{z-1}^{1} 1 \cdot 1 \, dx = 1-(z-1) = 2-z$$



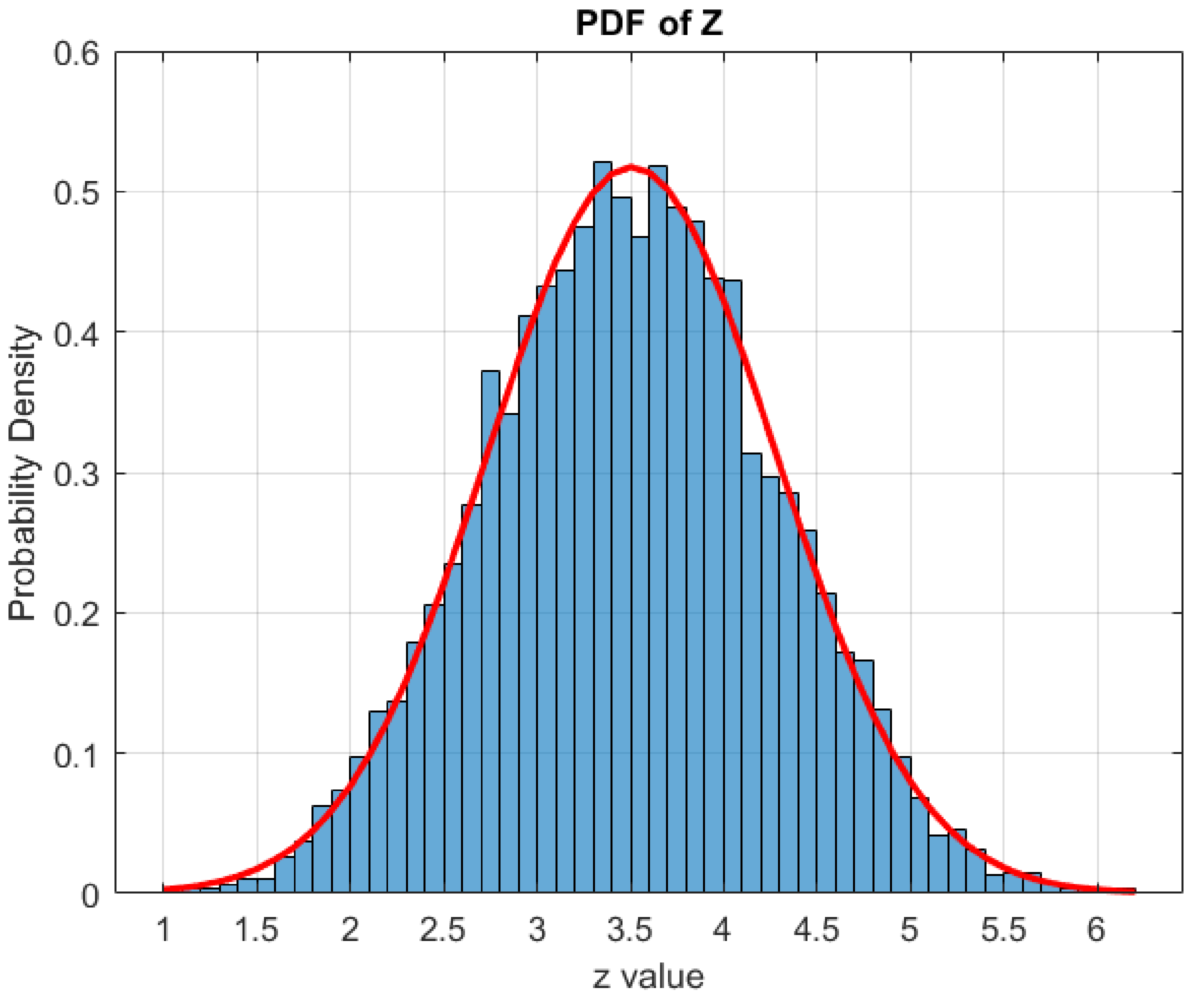
$$f_Z(z) = \begin{cases} 2-z, & 1 \le z \le 2 \\ 0, & o.w. \end{cases}$$

3e) We should expect to see a Gaussian distribution!

$$\left.\begin{array}{l} X_1 \sim unif(0,1) \\ X_2 \sim unif(0,1) \\ \\ X_7 \sim unif(0,1) \end{array}\right\}$$



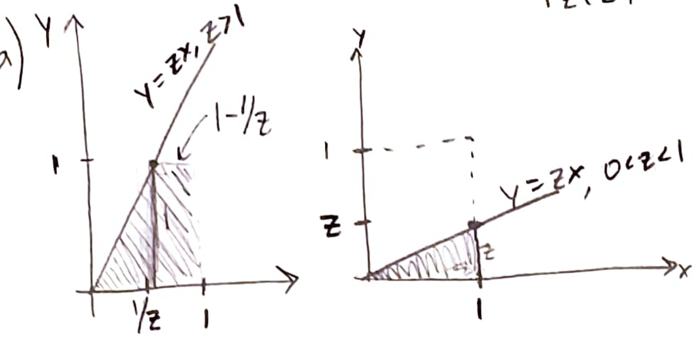
in MATLAB


PDF of Z
Probability Density
0
0.1
0.2
0.3
0.4
0.5
0.6
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
z value

4. X, Y are iid ~unif(0,1). $Z = Y/X$. $f(x,y) = 1$.

$$F_Z(z) = P(Z \le z) = P(Y/X \le z) = P(Y \le zX)$$

a)



$z < 0 \Rightarrow F_Z(z) = 0$

$0 < z < 1 \Rightarrow F_Z(z) = \dfrac{\triangle = z/2}{\square \cdot 1} = z/2$

$1 < z \Rightarrow F_Z(z) = \dfrac{\triangle + \square}{\square \cdot 1} = \frac{1}{2}\cdot\left(\frac{1}{z}\cdot 1\right) + \left(1 - \frac{1}{z}\right)\cdot 1$

$= \dfrac{1}{2z} + 1 - \dfrac{2}{2z}$

$= 1 - \dfrac{1}{2z}$

$$F_Z(z) = \begin{cases} 0 & z < 0 \\ z/2 & 0 < z < 1 \\ 1 - \frac{1}{2z} & z > 1 \end{cases}$$

$$\boxed{f_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{2} & 0 < z < 1 \\ \frac{1}{2z^2}, & z > 1 \end{cases}}$$

$-(2z)^{-1} \Rightarrow 1\cdot(2z)^{-2}\cdot 2 = \dfrac{2}{2^2 z^2}$

$\dfrac{d}{dz}(-z^{-1}) =$

b) $E[X^2 + Y^2] = E[X^2] + E[Y^2] = var(X) + E^2(x) + Var(y) + E^2(Y)$

$= \frac{1}{12} + \frac{1}{4} + \frac{1}{12} + \frac{1}{4} = \boxed{\frac{2}{3}}$

c) $F_{Z,X}(z,x) = P(Z \le z, X \le x) = \begin{cases} 0, & z < 0 \text{ or } x < 0 \text{ or } x > 1 \\ ?, & 0 \le zx \le 1, 0 \le x \le 1 \\ ??, & xz > 1, 0 \le x \le 1 \end{cases}$

$= P(Y \le zX, X \le x) =$

$\left.\begin{array}{l} Z = g(x,y) = y/x \\ X = h(x,y) = x \end{array}\right] \Rightarrow \begin{array}{l} X = X \\ Y = ZX \end{array}$

$f(x,y) = f_X(x) f_Y(y) = 1.$

$$J = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix}\Bigg|_{\substack{x = x \\ y = zx}} = \begin{pmatrix} \frac{-y}{x^2} & 1/x \\ 1 & 0 \end{pmatrix}\Bigg|_{\substack{x = x \\ y = zx}} = \begin{pmatrix} \frac{-zx}{x^2} & 1/x \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{-z}{x} & 1/x \\ 1 & 0 \end{pmatrix}. \quad \det(J) = -\frac{1}{x}$$

$$f_{Z,X}(z,x) = \frac{\sum f_{XY}(x_i, y_i)}{|\det(J)|} = \frac{1}{1/x} = \boxed{\begin{cases} x, & 0 \le x \le 1, \quad 0 \le z \le 1/x \\ 0, & \text{o.w.} \end{cases}}$$

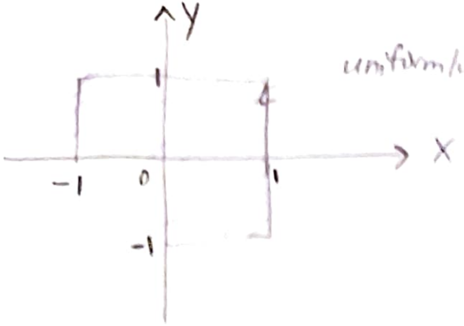
d) $E[\underbrace{XZ}_{g = X \cdot Z}] = \int_0^1 \int_0^{1/x} xz \cdot x \, dz \, dx = \int_0^1 \cancel{x^2} \cdot \frac{1}{2} (1/\cancel{x})^2 \, dx = \boxed{\frac{1}{2}.}$

$\text{cov}(X, Z) = E[(x - \bar{x})(z - \bar{z})] = E[(x - 1/2)(z - \infty)] = \infty$ $\underline{\underline{\text{no}}}$?

$E[z] = \bar{z} = \int_0^1 \frac{1}{2} z \, dz + \int_1^{\infty} \frac{1}{z^2} \cdot z \, dz = \frac{1}{2} \cdot \frac{1}{2}(1) + \ln\cancel{(\infty)}^{\to \infty} - \ln\cancel{(1)}^{\to 0}$

$\rho = \frac{\text{cov}(x, z)}{\sqrt{\text{var}(x) \cdot \text{var}(z)}} = \frac{\infty}{\infty} = \text{undefined}$

5.



uniformly distributed.

$1/4 \cdot 5/4 = 5/16$

$.3125$

$.33$

$$E[X] = \frac{2}{3} E[X \mid Y>0] + \frac{1}{3} E[X \mid Y<0] = 0 + \frac{1}{3} \cdot \frac{1}{2} = \boxed{\frac{1}{6}}$$

$$E[Y] = \frac{2}{3} E[Y \mid X>0] + \frac{1}{3} E[Y \mid X<0] = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{2} = \boxed{\frac{1}{6}} \text{ (or by symmetry)}$$

$$E[XY] = \underbrace{\frac{2}{3} E[XY \mid Y>0]} + \frac{1}{3} E[XY \mid Y<0]$$

$$\Rightarrow \frac{2}{3} \int_{-1}^{1} \int_{0}^{1} \frac{1}{2} \cdot xy \, dy\, dx = \frac{1}{3} \int_{-1}^{1} \int_{0}^{1} xy \, dy\, dx = \frac{1}{3} \int_{-1}^{1} x \cdot \frac{1}{2}(1^2 - 0^2)\, dx = 0.$$

odd function, symmetric limits

$$\Rightarrow \frac{1}{3} \int_{0}^{1} \int_{-1}^{0} 1 \cdot xy \, dy\, dx = \frac{1}{3} \int_{0}^{1} -\frac{1}{2} \cdot x \, dx = \frac{-1}{12} \Rightarrow \boxed{E[XY] = -1/12}$$

$$\text{var}(X) = E[X^2] - E^2[X]. \quad E[X^2] = \frac{2}{3} E[X^2 \mid Y>0] + \frac{1}{3} E[X^2 \mid Y<0]$$

$$= \frac{2}{3} \cdot \int_{-1}^{1} \frac{1}{2} x^2 \, dx + \frac{1}{3} \int_{0}^{1} 1 \cdot x^2 \, dx$$

$$= 2/9 + 1/9 = 1/3.$$

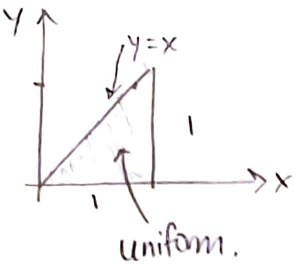
$$\Rightarrow \text{var}(X) = E[X^2] - E^2[X] = 1/3 - 1/36 = \boxed{11/36}$$

By symmetry, $\boxed{\text{var}(Y) = 11/36.}$

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{-1}{12} - \frac{1}{36} = -4/36 = \boxed{-1/9}$$

$$\rho_{XY} = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}} = \frac{-1/9}{\sqrt{(11/36)^2}} = \boxed{\frac{-4}{11}}$$

6.



we know $\iint_R f(x,y)\,dy\,dx = 1$, so obviously we have

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{over the region } 0 \le x \le 1,\ 0 \le y \le x \\ 0 & \text{o.w.} \end{cases}$$

$$f_X(x) = \int_0^x 2\,dy = \begin{cases} 2x, & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

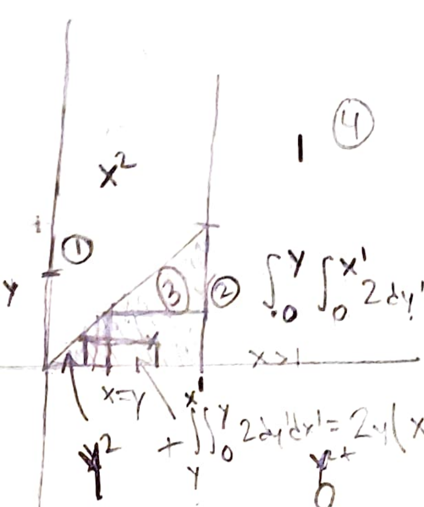
$$f_Y(y) = \int_y^1 2\,dx = \begin{cases} 2(1-y), & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{X|Y=y}(x|Y=y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \begin{cases} \frac{1}{1-y} & 0 < x \le 1,\ 0 < y \le x \\ 0 & x < 0 \text{ or } x > 1 \\ \text{not defined} & \text{o.w.} \end{cases}$$

$$F_{XY}(x,y) = \int_0^x\int_0^y f_{XY}(x',y')\,dy'\,dx' = \begin{cases} 0 & x<0 \text{ or } y<0 & ⓪ \\ x^2 & 0 \le x \le 1,\ y \ge x & ① \\ 2y-y^2 & x>1,\ 0 \le y \le 1 & ② \\ 2xy-y^2 & 0 \le x \le 1,\ 0 \le y \le 1 & ③ \\ 1 & x>1,\ y>1 & ④ \end{cases}$$

① $y \ge x,\ 0 \le x \le 1$

$$\int_{x'=0}^{x}\int_{y'=0}^{x'} 2\,dy'\,dx' = \int_{x'=0}^{x} 2x'\,dx' = x^2$$

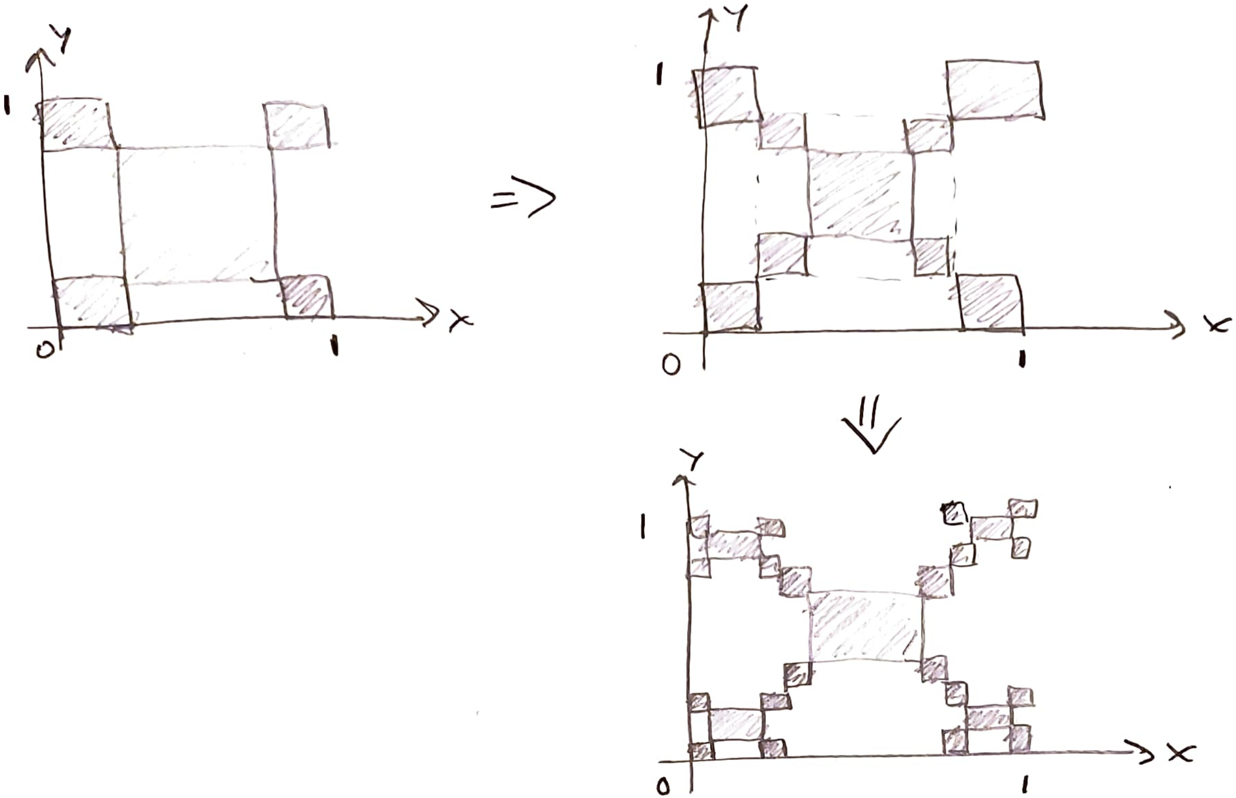


② $\int_0^y\int_0^{x'} 2\,dy'\,dx' + \int_y^1\int_0^y 2\,dy'\,dx' = \underbrace{\int_0^y 2x'\,dx'}_{y^2} + \underbrace{\int_y^1 2y\,dx'}_{2y(1-y)} = y^2 + 2y - 2y^2 = 2y - y^2$

③ $y^2 + \int_y^x\int_0^y 2\,dy'\,dx' = 2y(x-y) = y^2 + 2xy - 2y^2$

$= 2xy - y^2$

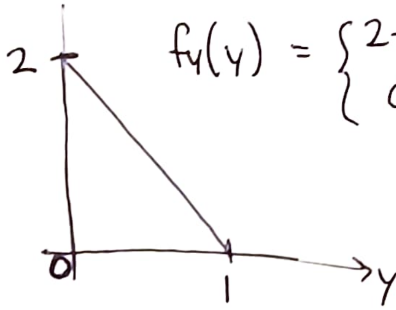
7.



...

infinitely many fractals!

Yes. The following distributions are examples of two RVs each uniform on $[0,1]$, uncorrelated, but not independent.



$$f_Y(y) = \begin{cases} 2-2y, & 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

a) Maximum probability occurs at $\alpha = 0$.

$f_Y(0) = 2$. choose $\alpha = 0$.

b) MMSE: $E[Y] = \int_0^1 y \cdot (2-2y)\,dy = 2 \cdot \left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1 = 2(1/2 - 1/3) = 1/3.$

Choose $\alpha = 1/3$

c) MMAE: $\arg\max_\alpha E[|\alpha - Y|] = \arg\max_\alpha \int_0^1 |\alpha - y| f_Y(y)\,dy$ $(\alpha \ge 0)$, $\alpha \le 1$

$\cancel{= \int_0^1 \alpha f_Y(y)dy + \int_0^1 -y f_Y(y) dy}$ $= \int_0^\alpha (\alpha - y) f_Y(y)dy + \int_\alpha^1 (y-\alpha) f_Y(y) dy$

$$= \int_0^\alpha \alpha f_Y(y)\,dy - \int_0^\alpha y f_Y(y) + \int_\alpha^1 y f_Y(y)\,dy - \int_\alpha^1 \alpha f_Y(y)\,dy$$

$$= \alpha(2\alpha - \alpha^2) + \frac{\alpha^2}{3}(2\alpha - 3) + \frac{\alpha^2(2\alpha-3)}{3} + \frac{1}{3} - \alpha(\alpha^2 - 2\alpha + 1)$$

$$= 2\alpha^2 - \alpha^3 + \frac{2}{3}\alpha^3 - \alpha^2 + \frac{2}{3}\alpha^3 - \alpha^2 + \frac{1}{3} - \alpha^3 + 2\alpha^2 - \alpha$$

$$= \alpha^3\left(-1 + \frac{2}{3} + \frac{2}{3} - 1\right) + \alpha^2(2 - 1 - 1 + 2) + \alpha(-1) + 1/3 \;. = g(\alpha)$$

we want to min. this quantity $g(\alpha)$: (by choosing α).

$$g(\alpha) = \alpha^3(-2/3) + 2\alpha^2 - \alpha + 1/3$$

$$g'(\alpha) = -2\alpha^2 + 4\alpha - 1 \quad = 0$$

$\Rightarrow$ choose $\alpha = \dfrac{-\sqrt{2} + 2}{2} = \underline{\underline{0.2929}}$

9a) MMSE $= E[Y|X] = x^2$. Overall Expected error $= E[(\hat{y}-Y)^2] = E[var(Y|X=x)]$.

$\Rightarrow var(Y|X) = E[Y^2|X] - E^2[Y|X] = \int_0^\infty y^2 \cdot \frac{1}{x^2} e^{-y/x^2} dy - (x^2)^2$

$= 2x^4 - x^4 = x^4$.

$E_x[x^4] = \int_0^1 x^4 \cdot 1\, dx = 1/5$.

expected error $=$ $E[(Y|X - \mu_{Y|X})^2] = \sigma^2_{Y|X} = x^4$.

9a) $\boxed{\hat{Y}_{MMSE|X} = x^2. \quad \text{Expected error} = x^4. \quad \text{Overall expected error} = \frac{1}{5}.}$

9b) $\hat{Y^2}_{MMSE|X} = E[Y^2|X] = 2x^4$. (shown in 9a). This is not $\hat{Y}^2_{MMSE|X}$ because $E[Y^2|X] \neq E^2[Y|X]$ since squaring is not a linear operation.

9c) $\hat{X}_{MMSE|Y} = E[X|Y]$. Need to find $f_{X|Y}$:

$$f_{X|Y} = \frac{f_{Y|X} \cdot f_X}{f_Y} = \frac{f_{Y|X} \cdot f_X}{\int_0^1 f_{Y|X} \cdot f_X\, dx} = \frac{\frac{1}{x^2} e^{-y/x^2}}{\int_0^1 \frac{1}{x^2} e^{-y/x^2} dx}$$

Let $u = \int_0^1 \frac{1}{x^2} e^{-y/x^2} dx$.

$du = \frac{1}{x^2} e^{-y/x^2} dx$.

$$\Rightarrow E[X|Y] = \int_0^1 x\, f_{X|Y}(x,y)\, dx = \int_0^1 \frac{\frac{1}{x} e^{-y/x^2}}{\int_0^1 \frac{1}{x^2} e^{-y/x^2} dx} dx$$

$$= \int \frac{1}{u} \cdot \frac{x \cdot \cancel{\frac{1}{x^2} e^{-y/x^2}}}{\cancel{\frac{1}{x^2} e^{-y/x^2}}} du = \int \frac{x}{u} du$$

??? How does one compute this integral?

9d) LMMSE : $\hat{Y}_{LMMSE|X} = aX + b$ where $a = \frac{cov(X,Y)}{Var(X)}$, $b =$

$var(x) = 1/12.$

$b = E[Y] - \frac{cov(X,Y)}{Var(x)} E[X]$

$E[Y] = \underset{X}{E}[\underset{Y}{E}[Y|X]] = \underset{x}{E}[x^2] = \int_0^1 x^2 dx = \frac{1}{3}.$

$E[X] = 1/2.$

$\int e^{-ax} dx = \frac{-1}{a} e^{-ax}$

$cov(x,y) = E[(X - E[x])(Y - E[Y])] = \underbrace{E[XY]} - E[x]E[Y].$

$E[xy] = \int_0^1 \int_x^\infty xy\, f_{XY}(x,y)\, dy\, dx = \int_0^1 \int_x^\infty xy \cdot \frac{1}{x^2} e^{-Y/x^2} \cdot 1\, dy\, dx$

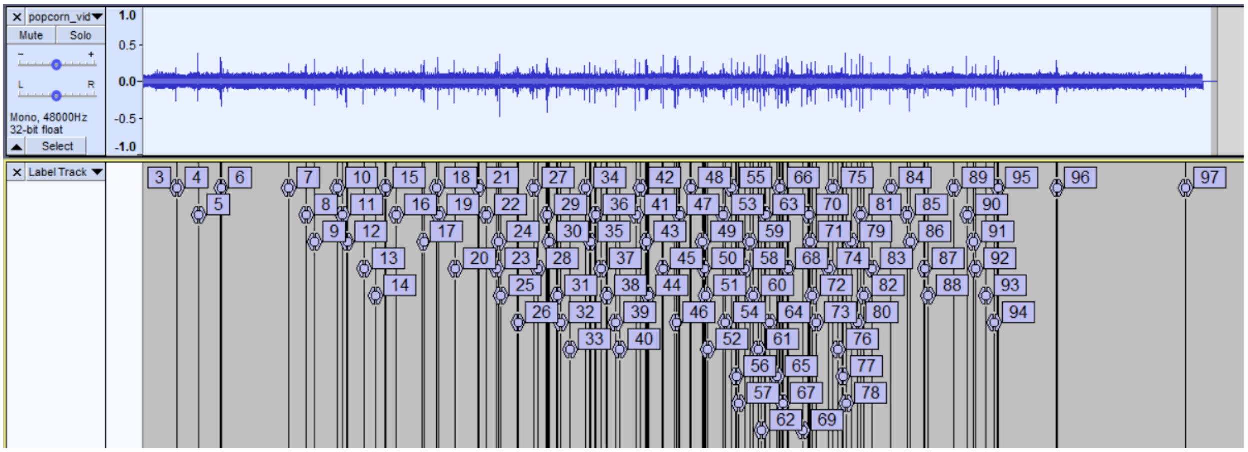
$= \int_0^1 x^3 dx = \frac{1}{4}.$

$1/4 - 1/6 = \frac{3}{12} - \frac{2}{12} = \frac{1}{12}$

$\Rightarrow a = \frac{cov(x,y)}{var(x)} = \frac{\frac{1}{4} - \frac{1}{3} \cdot \frac{1}{2}}{1/12} = 3 - 2 = 1.$ $a = 1.$

$b = E[Y] - 1 \cdot E[X] = \frac{1}{3} - 1/2 = 1/6.$

$\Rightarrow \boxed{\hat{Y}_{LMMSE|X} = 1 \cdot x + 1/6}$



I popped some popcorn and recorded the audio. The experiment of me popping popcorn produced a time signal. Running the experiment again would result in a different time signal, thus we can model the popcorn popping as a random process. In addition, I would expect this process to be ergodic, given constant humidity, temperature, and moisture content, as the popcorn popper should heat up about the same way every time.

10) a) A random process is a deterministic mapping from outcomes to time signals.

b) Random processes are observed in natural and engineered systems:

- phase of a carrier frequency
- load demand in a power system
- bit stream ~~in~~ a communication system.
- stock market prices
- brownian motion (dust in air)

c) Popcorn popping! I'll measure the time signal and plot the ~~[illegible]~~ sequence.

d) Time is continuous, so we can't define an uncountably infinite number of diffrent experiments. Also, we can find the average value of a signal. Much easier for analysis.

11a)

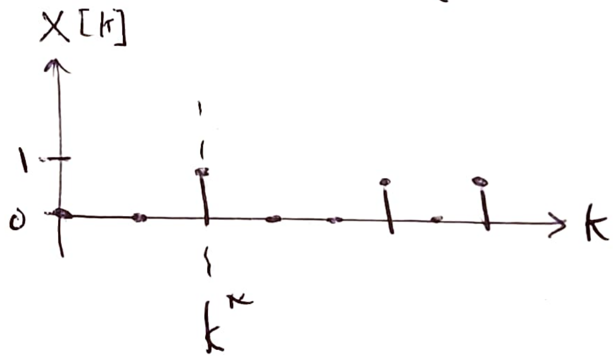
$X(\omega,t)=\cos t$ $\qquad$ $X(\omega,t)=\sin t$

Find first order PDF: $f_{X(t)} = \frac{1}{2}\delta(x-\sin t) + \frac{1}{2}\delta(x-\cos t)$

12a) $X[k]$, $k=0,1,2,\ldots$ each $X[k]$ is random variable $= \begin{cases} X[k] & P(X[k]) \\ 0 & 6/10 \\ 1 & 4/10 \end{cases}$

1^{st} order PMF $= \begin{cases} 0 & \text{w/ p. } 6/10 \\ 1 & \text{w/ p. } 4/10 \end{cases}$

$P_{X(k)} = 0.6\,\delta(x) + 0.4\,\delta(x-1)$



12. $X[k]$, $k = 0, 1, 2, \ldots$, each $X[k] = \begin{cases} 1 & \text{w/ p. } 4/10 \\ 0 & \text{w/ p. } 6/10 \end{cases}$

$$Y[k] = \sum_{i=0}^{k-1} X[i]$$

ex) $Y[3] = X[0] + X[1] + X[2]$ could be $0 + 1 + 1$ or $1 + 0 + 1$ or ...

$$Y[3] = \begin{cases} 0 & \text{w/ pr. } (0.6)^3 \\ 1 & \text{w/ pr. } (0.4)(0.6)^2 \cdot \binom{3}{1} \\ 2 & \text{w/ pr. } (0.4)^2(0.6)\binom{3}{2} \\ 3 & \text{w/ pr. } (0.4)^3 \end{cases}$$

$$Y[10] = \begin{cases} 0 & \text{w/ pr } (0.6)^{10} \\ 1 & \text{w/ pr. } (0.4)(0.6)^9\binom{10}{1} \\ \vdots & \\ 7 & \text{w/ pr. } (0.4)^7(0.6)^3\binom{10}{7} \\ \vdots & \\ 10 & \text{w/ pr. } (0.4)^{10} \end{cases}$$

$$\Rightarrow Y[k] = \begin{cases} 0 & \text{w/ pr. } (0.6)^k \\ n & \text{w/ pr. } (0.4)^n(0.6)^{k-n}\binom{k}{n} \\ k & \text{w/ pr. } (0.4)^k \end{cases}$$