EE 507, Homework 9
due December 4, 2015

Problem 1

a. What is a random process?

b. Why are random processes of interest to scientists and engineers?

c. Find a signal in the world around you that can be viewed as a random process. Actually measure a sample trajectory of this process.

d. When we defined random processes, we associated entire signals with outcomes of an experiment rather than using a different experiments to define the process values at particular times. What is the advantage of this approach?

Problem 2

An uncertain experiment has two equally likely outcomes, A and B. A random process $X(\omega, t)$, $t \in R$, is defined for this experiment as follows: $X(A, t) = cos(t)$ and $X(B, t) = sin(t)$.

a. Please find the first-order PDF for $X(t)$.

b. Please find the second-order PDF for $X(t)$.

c. Please find the nth-order PDF for $X(t)$. (Assume WLOG that the times in the joint PDF are increasing, i.e. $t_1 < t_2 < \ldots < t_n$).

d. Please find $E[X(t)]$ and $R_{XX}(t_1, t_2)$.

Problem 3

Consider a discrete-time random process $X[k]$, $k = 0, 1, 2, \ldots$, where each $X[k]$ is an independent random variable that equals 0 with probability 0.6 amd equals 1 with probability 0.4.

a. Please find the nth-order PMF for $X[k]$. (Assume without loss of generality that that the times in the joint PMF are increasing.)

b. Please find $E[X(t)]$, $R_{XX}(t,\tau)$, and $C_{XX}(t,\tau)$.

Now consider the process $Y[k] = \sum_{i=0}^{k-1} X[k]$, $k = 1, 2, 3, \ldots$.

c. Please find the first-order PMF for $Y[k]$.
d. Please find the nth-order PMF for $Y[k]$. (Assume without loss of generality that the times in the joint PMF are increasing.)
e. Please find $E[Y(t)]$, $R_{YY}(t,\tau)$, and $C_{YY}(t,\tau)$.
f. Please find $R_{XY}(t,\tau)$.

Problem 4

Let T be an exponential random variable with mean 1. We define a random process $X(t)$, $t \in R^+$, as follows: $X(t) = 1$ for $t \leq T$ and $X(t) = 0$ for $t \geq T$. Please find the second-order joint PDF of $X(t)$.

EE 507, Homework 10

due Dec. 7, 2015

Problem 1

a. Please find a sequence that converges in a mean square sense, but not with probability 1 (almost everywhere).
b. Please find a sequence that converges with probability 1 (almost everywhere), but does not converge in a mean square sense.

Problem 2

Consider the following two sequences: X_i, $i = 1, 2, 3, \ldots$, is a sequence of independent identically distributed random variables that are each uniform on $[0, 1]$. Meanwhile, Y_i, $i = 1, 2, 3, \ldots$, is defined as follows: $Y_i = max(X_1, X_2, \ldots, X_i)$.

a. Does X_i converge 1) in distribution?, 2) in probability?, 3) almost everywhere?, 4) in a mean square sense?, 5) everywhere?
b. Does Y_i converge 1) in distribution, 2) in probability?, 3) almost everywhere?, 4) in a mean square sense?, and 5) everywhere?

Problem 3

Consider a sequence of random variables $X_1, X_2, \ldots$ where each X_i is an independent random variable that is uniform on $[0.5, 1.5]$.

a. Please approximate the PDF of $Y = X_1 + \ldots + X_{1000}$.
b. Please approximate the PDF of $Z = X_1 X_2 \ldots X_{1000}$.

EE 507, Fall 2015, Final Exam.
Due December 11 at 5 PM in Sloan 347. This is a hard headline.

This exam has seven problems which each have multiple parts, and a further bonus problem. There are thirty parts in total (plus 2 bonus parts), which are each worth 5 points for a total of 150 points (plus 10 bonus points). Please explain your answer to each part carefully—you will only receive full credit if you explain your methods fully. You may use any reference materials (books, notes, homework and solutions, computer, etc) that you wish on the exam. However, you may NOT communicate with anyone else about the exam problems: the work on the exam must be entirely your own. *At the end of the exam, please write and sign a statement that the work on the exam is entirely your own.*

Problem 1

a. An experiment has three events A, B, and C defined for it, which satisfy $P(A) \geq 0.6$, $P(B) \geq 0.8$, and $P(C) \geq 0.9$. Further, you know that the events A and B are independent. Could the events A, B, and C be mutually disjoint, i.e. ABC is the null set? If yes, please prove. If no, please find the minimum probability of $P(ABC)$.
b. Two events A and B satisfy: 1) $P(A \,|\, B) \geq P(A)$ and $P(B) = 0.5$. Is it always true that $P(\overline{A} \,|\, \overline{B}) \geq P(\overline{A})$? Please prove or give a counterexample.
c. Please explain what a random variable is, and why the concept is important.
d. A probabilistic experiment has a finite number (say N) outcomes. A random variable X is defined from the experiment. If the CDF of X is known, is it always possible to determine the probability of every event in the experiment?

Problem 2

Two random variables X and Y are jointly Gaussian, with $m_x = 1$, $m_y = 2$, $\sigma_x^2 = 1$, $\sigma_y^2 = 4$, and $r = \frac{2}{3}$.

a. Please find: 1) the marginal pdfs of X and Y, 2) the conditional pdf of Y given $X = x$, and 3) the mean and variance of Y given $X = x$.
b. Please find the pdf of $Z = X + 2Y + 1$.
c. Consider two random variables A and B with the following properties: 1) A has a Gaussian distribution; and 2) the conditional distribution for B given $A = a$ is Gaussian for every a. Is it necessarily true that B is Gaussian? Please prove or give a counterexample.

Problem 3

A random variable X is uniformly distributed on $[0, 1]$. Unfortunately, the random variable X cannot be measured directly. Instead, a noisy measurement Y is taken of the random variable. Specifically, given that $X = x$, the measurement Y is a uniformly distributed on $[0, x^2]$. Please answer the following questions:

a. Please find the joint probability density function of X and Y.

b. Please find the marginal probability density function for Y.
c. Please find the conditional probability density function for X, given $Y = y$.
d. Please find the MMSE estimate for X given $Y = y$.
e. Please find the error in the MMSE estimate for X, given $Y = y$.
f. Please find the LMMSE estimate for X in terms of Y.

Problem 4

The random variable X is geometric with parameter 0.7 (that is, $P(X = k) = 0.3(0.7)^k$ for $k = 0, 1, 2, \ldots$). Please answer the following questions:

a. Please find the CDF, pdf, mean, and variance of $Y = 2(X - 3)$.
b. The random variable Z is defined as $Z = A(X-3)$, where the random variable A is independent of X and either equals 1 (with probability 0.4) or 2 (with probability 0.6). Please find the pdf of Z.

Problem 5

An experiment has three outcomes A, B, and C, which have probabilities 0.5, 0.3, and 0.2, respectively. A random process $X(t)$, $t \in R^+$, is defined from the experiment as follows: $X(A,t) = \sqrt{(}t)$, $X(B,t) = t$, $X(C,t) = t^2$. Also, a random variable Y is defined as follows: $Y(A) = 2$, $Y(B) = 2$, $Y(C) = 3$.

a. Please find the first-order pdf and pmf of $X(t)$.
b. Please find the second-order pmf of $X(t)$.
c. Please find the joint pmf of $X(2)$ and Y.
d. Please find the mean and autocorrelation of $X(t)$.
e. Is $X(t)$ a wide-sense stationary process?
f. Are samples of the random process $X(t)$ at two different times always, sometimes, or never independent?

Problem 6

A random process $X[k]$, $k \in Z+$, comprises a sequence of independent, identically distributed random variables. Specifically, each $X[k]$ has pdf $f_{X[k]}(x) = 2x$ for $0 \leq x \leq 1$ (and $f_{X[k]}(x) = 0$ otherwise). A second random process $Y[k]$ is defined as $Y[k] = min(X[0].X[1], \ldots, X[k])$.

a. What is the nth-order pdf of $X[k]$?
b. Please find the first-order CDF and pdf of $Y[k]$.
c. Please show that the sequence $Y[k]$ converges in probability to 0. Does it converge almost everywhere?
d. Please find the second-order pdf of $Y[k]$. (Hint: use a conditional-probability argument).

Problem 7

A wide sense stationary random process $X(t)$ has zero mean and autocorrelation $R_X X(t) = 2e^{-|t|}cos(2t)$.

a. Please find the variance of $X(1)$.
b. Please find the correlation coefficient between $X(2)$ and $X(4)$. Is it possible for $X(2)$ and $X(4)$ to be independent?
c. For what times t is $X(t)$ uncorrelated with $X(1)$?
d. Please find the power spectrum of $X(t)$.
e. The random process $Y(t)$ satisfies the differential equation $\frac{dY}{dt} + Y = X$. What is the power spectrum of $Y(t)$?

Problem 8: BONUS

Two random variables X and Y are uniformly distributed in the region shown. A sensor takes a measurement of Y. Specifically, the measurement is $Z = g(Y)$, where $g()$ is a function with the following properties: $g(0) = 0$, $g(1) = 1$, and $g()$ is monotonically strictly increasing (see below).

a. Please find the MMSE estimate for X given $Z = z$. (Your answer will be in terms of $g()$.)
b. Find the expected error in the MMSE estimate, and show that this error is the same no matter what $g()$ is. Why does this make sense?