Quadratic Programming

The general QP is

$$\min \; q(x) = \frac{1}{2} x^T G x + x^T c$$
$$\text{s.t.} \quad A_e x = b_e$$
$$A x \geq b$$
$$x \in \mathbb{R}^n$$

almost a linear program except for the quadratic term in the objective

or equivalently:

$$\boxed{\begin{aligned} \min \; & q(x) = \frac{1}{2} x^T G x + x^T c \\ \text{s.t.} \; & a_i^T x = b_i \quad i \in \mathcal{E} \\ & a_i^T x \geq b_i \quad i \in \mathcal{I} \\ & x \in \mathbb{R}^n \end{aligned}}$$

We will consider convex problems ($G \geq 0$) which serve as subproblems for later methods.

Equality Constrained QP

$$\boxed{\begin{aligned} \min\ & q(x) = \tfrac{1}{2}x^T G x + x^T c \\ \text{s.t.}\ & Ax = b \\ & x \in \mathbb{R}^n \end{aligned}}$$

The KKT conditions:

$$Gx + c - A^T\lambda = 0$$

$$Ax = b$$

which can be written in matrix form:

$$\begin{bmatrix} G & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

(First order necessary conditions)

Our assumptions:

(1) $G \geq 0$

(2) A is full rank

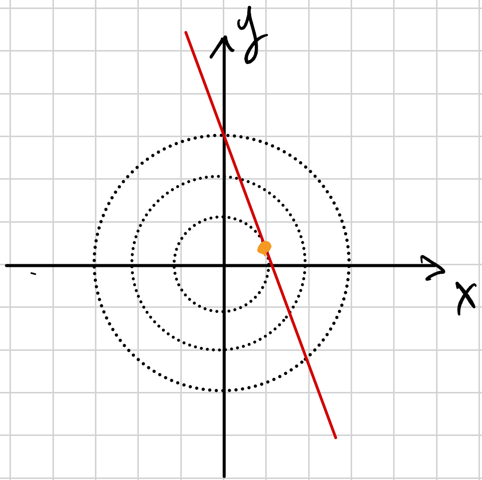
(3) $Z^T G Z > 0$

Example

$$\min \quad x^2+y^2$$
$$\text{s.t.} \quad 3x+y=3$$

$$f(x) = x^2+y^2 = \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix} \overset{G}{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x & y \end{bmatrix} \overset{c}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$3x+y=3 \Rightarrow \overset{A}{\begin{bmatrix} 3 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \overset{b}{3}$$



$$\text{KKT}: \begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & -1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$x_1^* = 0.9 \quad x_2^* = 0.3 \quad \lambda^* = 0.6$$

A computational Variant

Suppose x not necessarily optimal.
Find a step p so that $x^* = x + p$.

$$\begin{cases} G(x+p) + c - A^T\lambda = 0 \\ A(x+p) = b \end{cases}$$

$$\begin{bmatrix} G & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} p^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -Gx - c \\ b - Ax \end{bmatrix}$$

or

$$\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} -p^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} Gx + c \\ Ax - b \end{bmatrix}$$

Example: $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$Gx + c = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $Ax - b = 0$

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \Rightarrow p^* = \begin{pmatrix} -0.1 \\ +0.3 \end{pmatrix}$$

$$x^* = x + p^* = \begin{pmatrix} 0.9 \\ 0.3 \end{pmatrix} \quad ✓$$

Example: $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$Gx + c = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\quad Ax - b = 1$

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \Leftrightarrow p^* = \begin{pmatrix} -0.1 \\ -0.7 \end{pmatrix}$$

$$x^* = x + p^* = \begin{pmatrix} 0.9 \\ 0.3 \end{pmatrix} \quad ✓$$

An important condition

Let Z be the matrix whose columns form a basis for Null A. $(AZ=0)$.

Lemma. Let A have full row rank and $Z^TGZ > 0$. Then

$$K = \begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix} \text{ is invertible.}$$

Proof: Suppose w, v satisfying

$$\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If we show that $w=0, v=0$ is the unique solution then K is invertible.

First notice that $Aw=0$. So,

$$0 = [w^T \; v^T]\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix}$$

$$= [w^T \; v^T]\begin{bmatrix} Gw + A^Tv \\ Aw \end{bmatrix}$$

$$= w^TGw + w^TA^Tv + v^TAw$$

$$= w^TGw$$

Now $w \in \text{Null } A$ so $\exists\, u \ni w = Zu$, so,

$$0 = w^TGw = (Zu)^TG(Zu) = u^TZ^TGZu$$

Because $Z^TGZ > 0$, $u=0$ and also $w = Zu = 0$.

Finally $Gw + A^Tv = 0 \Rightarrow v = 0$. $\square$

Theorem. Let A have full row rank and assume $Z^TGZ > 0$. Then x^* satisfying

$$\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} -x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} c \\ -b \end{bmatrix}$$

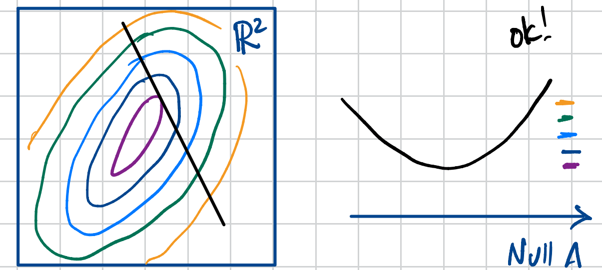
is the unique global minimizer of the ECQP.

Proof: Let $x \neq x^*$ be a feasible point and $p = x^* - x$. Using the facts that $Ap = 0$ and $p^TGx^* = p^T(-c + A^T\lambda^*) = -p^Tc$, we have

$$\begin{aligned} q(x) &= \tfrac{1}{2}(x^*-p)^TG(x^*-p) + c^T(x^*-p) \\ &= q(x^*) + \tfrac{1}{2}p^TGp - p^TGx^* - c^Tp \\ &= q(x^*) + \tfrac{1}{2}p^TGp \\ &> q(x^*). \end{aligned}$$

Thus x^* is the unique global minimizer. $\blacksquare$

Geometric Interpretation



Solving the KKT system

let $x^* = Y x_y + Z x_z$ (c)

Then $A(Z x_z) = (AZ) x_z = 0$

$A(Y x_y) = b - AZ x_z = b$

$$\min_x \; q(x) = \frac{1}{2} x^T G x + x^T c$$
$$\text{s.t.} \quad Ax = b$$

$$\min_{x_z} \; q(x) = \frac{1}{2} x_z^T Z^T G Z x_z + x_z^T \left(Z^T G Y x_y + Z^T c \right)$$

$$\Longrightarrow \boxed{\begin{aligned} & Z^T G Z x_z^* = -\left(Z^T G Y x_y^* + Z^T c \right) \quad \text{(a)} \\ & \& \; A Y x_y^* = b \quad \text{(b)} \end{aligned}}$$

Use (b) to solve for x_y^*.

Then use (a) to solve for x_z^*.

Then use (c) to find x^*.