EE 507 Random Processes
Homework 01

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Contents

Problem 1

All of you travelled to Pullman and WSU to begin your education here (maybe by airplane, maybe by car, maybe even by bus or on foot if you grew up in the area). Please consider your latitude (how far North or South you are) as a function of time on this journey. Before you started travelling, this time signal was subject to uncertainty. Please sketch two possible instances of this signal, explaining why each result may occur.

Solution

India and the West Coast of United States are almost at the opposite ends of the globe. Geographically, this leads to interesting possibilities as flights can be optimally made from both sides across the globe.

Flights from New Delhi, India to Pullman, WA, US can be broadly classified into two categories based on which ocean they fly over during their trip. Flights can come via the Atlantic Ocean or via the Pacific Ocean.

table 1 and table 2 show the latitude coordinates of the plane with respect to time and location, including layover times.

fig. 1 shows the two signal curves of latitudes vs time.

Table 1: Atlantic Ocean Route

City (Country)	Latitude	Journey Time (hrs)	Layover Time (hrs)
New Delhi (India)	28.6139° N	0	0
Doha (Qatar)	25.2854° N	4	3
Settle, WA (US)	47.6062° N	14	3
Pullman, WA (US)	46.7298° N	1	0

Table 2: Pacific Ocean Route

City (Country)	Latitude	Journey Time (hrs)	Layover Time (hrs)
New Delhi (India)	28.6139° N	0	0
Seoul (South Korea)	37.5665° N	7	14
Settle, WA (US)	47.6062° N	10	3
Pullman, WA (US)	46.7298° N	1	0

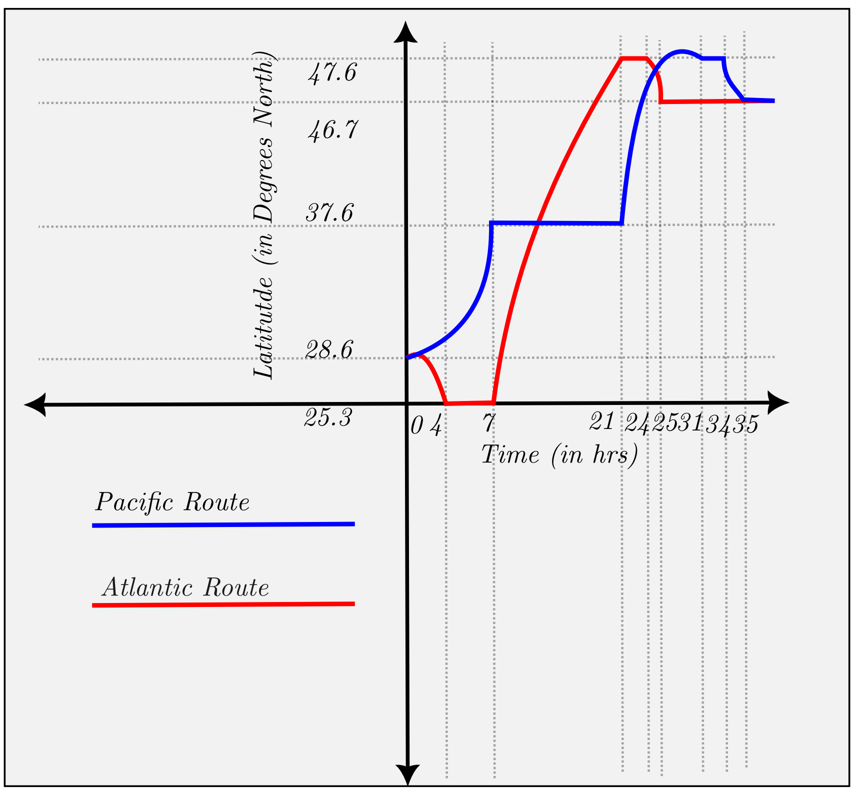

Figure 1: Latitude vs Time Plots for the two routes.

Problem 2

a. Please define the following terms:

   (i) Probability
   (ii) Outcome
   (iii) Event
   (iv) Sample Space
   (v) Axiom

b. Please describe, in words, the three axioms of probability.

c. It's important in defining an experiment to have the correct level of granularity (detail). Please give an example in which the granularity is too low, and one in which the granularity is too high.

Solution

a.
- **Outcome**
  A possible result of an experiment. Only one outcome is reached at the end of an experiment, i.e. Outcomes are always disjoint and never overlap.
- **Sample Space**
  The set of all possible outcomes of an experiment is called the sample space.
- **Event**
  For an experiment, an event is uniquely defined via the set of permissible outcomes as part of itself. In set theory, an event E is the subset of the power set $\wp$ of the sample space Ω of an experiment.
- **Probability**
  The probability of an event is defined as the ratio of the odds of that event occurring to the odds of all possible events occurring at the end of an experiment.
- **Axiom**
  A basic rule which cannot be proven but only assumed. Axioms when combined with other axioms, together can be used as a standalone set of rules to prove all subsequent results forming the base of a mathematical area.

b. The three axioms of probability are:

Axiom 1. *For an event A as part of an experiment, the probability $P(A) \geq 0$*

Probability of an event can never be less than zero. At the least the event can be an impossible event, whose probability would then be zero.

Axiom 2. *For the sample space Ω of an experiment, the probability $P(\Omega) = 1$.*

The set of all possible outcomes of an experiment is called the sample space. Since the sample space encompasses every possibility of an experiment, the probability of getting an outcome which is part of the sample space is 1, i.e. it is a certain event.

Axiom 3. *If events A and B are disjoint, i.e. $A \cap B = \phi$, then $P(A+B) = P(A) + P(B)$.*

If two events do not have any overlapping outcomes, then they are mutually exclusive events and thus their union is simply the sum of their individual possible outcomes.

c. Let's say the experiment is to determine whether a particular candidate is able to pass the course EE 507. Let's label this experiment E and its outcomes as 'Pass' and 'Fail'. We wish to determine P('Pass'):

- **Too Granular**: All of the candidate's grades for the previous six years of their university education. The candidate's relationship with their classmates and the instructor. The total time the candidate allotted to study for their real-time exams, the number of pages the candidate practised before the examination day. The breakfast the candidate had on the examination day and the duration of sound sleep the night before.
- **Good Granularity**: The candidate has taken a similar course in their previous degree and performed well. The candidate likes studying hard maths and spends a decent amount of their time in solving the given assignments.
- **Low Granularity**: The candidate comes from country XYZ, the candidate topped their English exams in Class 5. The candidate likes the number 507.

Problem 3

Consider any uncertain experiment, and let A, B, and C be three events defined for the experiment. Please prove the following equalities or inequalities. In doing so, you may use the three axioms, any equalities/inequalities developed in class, standard set theory and algebra concepts, and earlier parts of the problem.

a. $P(A\bar{B}) = P(A) - P(AB)$

b. If A is contained in B, then $P(A) \leq P(B)$.

c. $P(A + B) + P(\bar{A}) + P(\bar{B}) - P(\bar{A} + \bar{B}) = 1$

d. $P(AB) + P(\bar{A}C) + P(\bar{B}\bar{C}) \leq 1$

   Please also give an example showing that the bound can be achieved.

Solution

a. To Prove: $P(A\bar{B}) = P(A) - P(AB)$

   Proof.

   From the Law of Total Probability, we know that

   $$P(A) = P(A\bar{B}) + P(AB) \tag{1}$$

   Rearranging eq. (1), we get:

   $$P(A\bar{B}) = P(A) - P(AB) \tag{2}$$

   Hence Proved. 😄

   □

b. To Prove: If A is contained in B, then $P(A) \leq P(B)$

   Proof.

   To Prove: if $A \subseteq B$, then $P(A) \leq P(B)$

   From the Law of Total Probability, we know that

   $$P(B) = P(B\bar{A}) + P(BA) \tag{3}$$

   $$A \subseteq B \implies A \cap B = A \implies P(BA) = P(A)$$

   $$\text{or,} \quad P(B) = P(B\bar{A}) + P(A)$$

   But, by Axiom 1 of Probability

   $$P(B\bar{A}) \geq 0$$

   $$\implies P(A) \leq P(B)$$

   Hence Proved. 😄

   □

c. To Prove: $P(A+B)+P(\bar{A})+P(\bar{B})-P(\bar{A}+\bar{B})=1$

Proof.

$$
\begin{aligned}
LHS &= \{P(A)+P(B)-P(AB)\}+P(\bar{A})+P(\bar{B}) && \text{(Expanding the } LHS\text{)}\\
&\quad -\{P(\bar{A})+P(\bar{B})-P(\bar{A}\bar{B})\}\\
&= P(A)+P(B)-P(AB)+P(\bar{A}\bar{B})\\
&= P(A)-P(AB)+P(\bar{A}\bar{B})+P(B) && \text{(Rearranging the equation)}\\
&= P(A\bar{B})+P(\bar{A}\bar{B})+P(B) && \text{(Using eq. (1))}\\
&= P(\bar{B})+P(B) && \text{(Using eq. (3))}\\
&= 1 && \text{Hence Proved 😄}
\end{aligned}
$$

□

d. To Prove: $P(AB)+P(\bar{A}C)+P(\bar{B}\bar{C})\leq 1$
Also give an example for the equality case.

Proof.

$$
\begin{aligned}
AB\cap\bar{A}C\cap\bar{B}\bar{C} &= \phi && \text{(Pairwise Disjoint Sets)}\\
\implies P(AB)+P(\bar{A}C)+P(\bar{B}\bar{C}) &= P(AB+\bar{A}C+\bar{B}\bar{C}) && \text{(Axiom 3)}\\
&\leq 1 && \text{(Axiom 1)}\\
& && \text{Hence Proved 😄}
\end{aligned}
$$

□

For the special case of equality

$$P(AB)+P(\bar{A}C)+P(\bar{B}\bar{C})=1 \tag{4}$$

We may equate the constituent events of eq. (4) to trivial identities in order to potentially reverse engineer the relationships between the three events A, B and C:

Let

$$AB+\bar{A}C+\bar{B}\bar{C}=A+\bar{A} \tag{5}$$

Making comparisons, we get:

$$
\begin{aligned}
AB &= A\\
\implies A &\subseteq B \qquad (6)\\
\bar{A}C &= \bar{A}\\
\implies \bar{A} &\subseteq C \qquad (7)\\
\bar{B}\bar{C} &= 0\\
\implies \bar{B}\cap\bar{C} &= \phi\\
\implies B\cup C &= 1 \qquad (8)
\end{aligned}
$$

eq. (7) gives two cases: Either A and C are disjoint events which exhaustively represent the sample space (i.e. $A \cup C = \phi$ and $A + C = \Omega$) or C encompasses the whole sample space and A is merely a subset of it (i.e. $C = \Omega$ and $A \subseteq C$).

One of the solutions satisfying eq. (6), eq. (7) and eq. (8) is $A = B$ and $A \cap C = \phi$. fig. 2 represents this instance of the relationship between A, B and C.

Another solution satisfying eq. (6), eq. (7) and eq. (8) is $A \subseteq B \subseteq C$, represented by fig. 3.

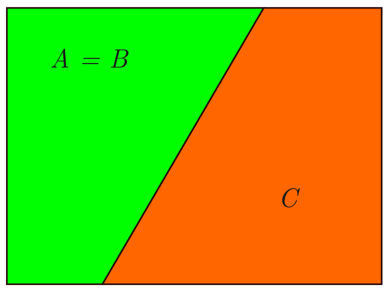

Figure 2: Instance 1 which satisfies the equality criterion.

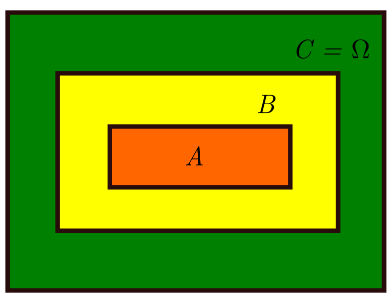

Figure 3: Instance 2 which satisfies the equality criterion.

Let
$$AB + \bar{A}C + \bar{B}\bar{C} = C + \bar{C} \tag{9}$$

Making comparisons, we get:

$$\begin{aligned} \bar{A}C &= C \\ \implies A &\subseteq \bar{C} \qquad (10) \\ \bar{B}\bar{C} &= \bar{C} \\ \implies B &\subseteq C \qquad (11) \\ AB &= 0 \\ \implies A \cap B &= \phi \qquad (12) \end{aligned}$$

A solution satisfying eq. (10), eq. (11) and eq. (12) is $A \subseteq B \subseteq C$, represented by fig. 4.

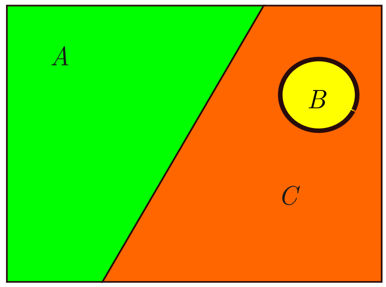

Figure 4: Instance 3 which satisfies the equality criterion.

Problem 4

Consider an experiment that has four outcomes, A, B, C, and $\square$.

a. How many events can be defined for this experiment? What are they?

b. What is the sample space for the experiment?

c. For this part, assume that $P(\{A, B, \square\}) = 0.8$, $P(\{A, C\}) = 0.4$ and $P(\{A, \square\}) = 0.4$.
Please find the probabilities of all events defined for this experiment.

d. For this part, assume that the probabilities $P(\{A, B, \square\})$ and $P(\{A, C\})$ are known, and that we can measure the probability of exactly one more event. Please find all other events such that the knowledge of the three events' probabilities allows us to compute the probabilities of all events.

Solution

Let the event be called E_4.
Given $\Omega = \{A, B, C, \square\}$

a. # Events $= 2^4 = 16$. They are the elements of the powerset $\wp$ made from Ω

$$\begin{aligned}\wp = \{&\{\phi\}, \{A\} \cdots \{\square\},\\ &\{A, B\}, \cdots \{C, \square\},\\ &\{A, B, C\}, \cdots \{B, C, \square\},\\ &\{A, B, C, \square\}\}\end{aligned}$$

Refer to table 3 for the full list.

b. Sample Space $\Omega = \{A, B, C, \square\}$

c. Since A, B, C, and $\square$ are all outcomes, they are disjoint events. Axiom 3 may be used to compute the probabilities of their unions.

$$\begin{aligned}P(\{A, B, \square\}) &= 0.8\\ \implies P(C) &= 0.2 && \text{(Axiom 2)}\\ P(\{A, C\}) &= 0.4\\ \implies P(A) &= 0.2\\ P(\{A, \square\}) &= 0.4\\ \implies P(\square) &= 0.2\\ \implies P(B) &= 0.4\end{aligned}$$

Refer to table 3 for the event-wise probabilities.

Table 3: Probabilities of all events of Experiment E_4. The 'Sufficient?' column is meant for part d. of the problem where knowledge of the event's probability value when combined with two other given probability values is sufficient for determining the probabilities of all possible events of E_4.

S.No.	*Event*	*Probability*	Sufficient?
1.	ϕ	0	No
2.	A	0.2	No
3.	B	0.4	Yes
4.	C	0.2	No
5.	$\square$	0.2	Yes
6.	$\{A, B\}$	0.6	Yes
7.	$\{A, C\}$	0.4	No
8.	$\{A, \square\}$	0.4	Yes
9.	$\{B, C\}$	0.6	Yes
10.	$\{B, \square\}$	0.6	No
11.	$\{C, \square\}$	0.4	Yes
12.	$\{A, B, C\}$	0.8	Yes
13.	$\{A, B, \square\}$	0.8	No
14.	$\{A, C, \square\}$	0.6	Yes
15.	$\{B, C, \square\}$	0.8	No
16.	$\{A, B, C, \square\}$	1.0	No

d. On similar lines as the previous part c. of this problem, since we are able to find out the values of $P(A)$ and $P(C)$ from the given values, we are only interested in knowing the values of either $P(B)$ or $P(\square)$. Only events which are supersets of unions of both B and $\square$ outcomes are unhelpful in determining their individual values. Refer to the 'Sufficient?' column in table 3.

Problem 5

You toss three fair coins. What is the probability that at least two show heads? Also, what is the probability that at least two show heads, given that the number of heads showing is even?

Solution

Let E be the experiment of tossing three fair coins. The sample space for E is
$\Omega = \{$TTT, TTH, THT, THH, HTT, HTH, HHT, HHH$\}$.
Let Event A: $\{$At least two heads$\}$
or $A = \{THH, HTH, HHT, HHH\}$

$\implies P(A) = \frac{4}{8}$

Let Event B: $\{$Even # of heads$\}$
or $B = \{TTT, THH, HTH, HHT\}$

$\implies P(B) = \frac{4}{8}$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$
$A \cap B = \{THH, HTH, HHT\}$

$P(A|B) = \frac{\frac{3}{8}}{\frac{4}{8}} = \frac{3}{4}$

Problem 6

A simple dartboard is shown below:

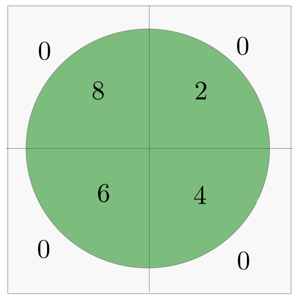

Figure 5: A dart board with given scores for hitting different regions.

Your favourite EE 507 instructor throws a dart at the board. He hits the board with probability 0.7 and misses the board with probability 0.3. Given that he hit the board, he is equally likely to hit any point on it. He receives a score of 0 if he misses the board, and receives the score shown if he hits the board.

a. What are the outcomes of this experiment? How many are there?

b. What are the events in this experiment? How many are there?

c. What is the probability that your instructor's score is more than 3?

d. Let's say we repeated the experiment twice, independently. What is the probability that the score is greater than 3 on the first try, and less than or equal to 3 on the second try?
Also what is the probability that the two scores are different?

Solution

Let the experiment of the instructor throwing the dart on the board be called E.

a. The outcomes can be listed as $\Omega = \{A_0, A_2, A_4, A_6, A_8\}$ where A_i refers to scoring i points with the dart. A_0 represents the dart missing the board and scoring zero points.
In total there are 5 outcomes for this experiment.

b. An event is a subset of the power set of the set of outcomes (Ω) of the experiment.

The power set of Ω is $\wp$:

$$\wp = \{\{\phi\}, \{A_0\}, \{A_2\}, \cdots \{A_8\},$$
$$\{A_0, A_2\}, \{A_0, A_4\} \cdots \{A_6, A_8\},$$
$$\cdots \{A_0, A_2, A_4, A_6, A_8\}\}$$

Say, a subset $\{A_0, A_4\}$ of $\wp$ represents the event {The dart misses the board OR The dart hits the board in the region with 4 points}.
There are $2^5 = 32$ events for this experiment.

c. $P(\text{Score} > 3) = P(\{A_4, A_6, A_8\})$.
Outcomes are always disjoint events, so we can use Axiom 3.
$P(\{A_4, A_6, A_8\}) = P(A_4) + P(A_6) + P(A_8) = 3 * 0.7/4 = 0.525$.

d. Repeated trials are independent trials. We can simply multiply the individual probabilities to obtain the final probability.
$P(\text{Score} > 3 \text{ on the first trial AND Score} < 3 \text{ on the second trials}) = P(\text{Score} > 3) * P(\text{Score} \leq 3) = 0.525(1 - 0.525) = 0.249375$

$P(\text{Both Scores Different}) = 1 - P(\text{Both Scores Same})$
$= 1 - P(\{\{A_0A_0\}, \{A_2A_2\}, \{A_4A_4\}, \{A_6A_6\}, \{A_8A_8\}\})$
$= 1 - [0.3^2 + 4 * \{\frac{0.7}{4}\}^2] = 0.7875$

Problem 7

Let's say an experiment has three outcomes.

a. How many events can be defined for this experiment?

Now we say we repeat the experiment twice (once more), independently.

b. How many outcomes does the combined experiment have?

c. How may events can be defined for the combined experiment?

d. How many events for the combined experiment can not be derived from the cartesian products for the individual experiments? Please give an example.

Solution

Let the experiment be called E_1 which has the three outcomes be A, B and C i.e. $\Omega = \{\text{A, B, C}\}$

a. # Events $= 2^n$ where n is the cardinality of the sample space Ω.
Here $n = 3$ and therefore # Events $= 2^3 = 8$.

Let the combined experiment be called E_2. The outcomes of this combined experiment will be represented by the new sample space $\Omega_2 = \{$AA, AB, AC, BA, BB, BC, CA, CB, CC$\}$.

b. # Outcomes = Cardinality ($n_2 = n^2$) of the sample space Ω_2.
Here $n_2 = 3^2 = 9$.

c. # Events $= 2^{n_2}$
Here $n_2 = 9$ and therefore # Events $= 2^9 = 512$.

d. # Events of the combined experiment which can be defined from the cartesian products of the individual experiments $= 2^n * 2^n = 2^{2n} = 2^6$.
Total # Events of the combined experiment $= 2^{n^2} = 2^9$.
Therefore, # Events which cannot be defined from the cartesian products of the individual experiments $= 2^{n^2} - 2^{2n} = 2^9 - 2^6 = 512 - 64 = 448$.

Examples for an event of E_2 which can be formed from the cartesian products of the events of E_1:

$$\{A\} \times \{A\} = \{AA\}$$
$$\{A, B, C\} \times \{A\} = \{AA, BA, CA\}$$
$$\{A, C\} \times \{B\} = \{AB, CB\}$$

Examples for events of E_2 which can NOT be formed from the cartesian products of the events of E_1:

$$\{AB, BC, CA\}$$
$$\{AA, CB\}$$
$$\{BC, CB\}$$