EE 507 Assignment 03

OLUWAFEMI J. AJEIGBE

Tuesday 11th October, 2022

Problem 1

A random variable X has a probability density function $f_X(x) = Cx^{-3}, x \geq 1$.

(a) Please find the constant C.

(b) Find the mean and variance of X.

(c) Find the CDF of X.

(d) Please find the PDF of X given that X $\geq$ 2. Also, please find the mean value of X given that that X $\geq$ 2.

Solution

(a.)

To Find C, we can use one of the properties defined for pdfs:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(u)\mathrm{d}u \qquad (1) \\ &= \int_{1}^{\infty} Cu^{-3}\mathrm{d}u \\ &= C\left[\frac{-1}{2u^2}\right]_1^{\infty} \\ &= C\left[\frac{1}{2}\right] \\ 2 &= C \qquad (2) \end{aligned}$$

(b.)

For **mean** we have

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x \qquad (3) \\ &= \int_{1}^{\infty} x.2x^{-3}\mathrm{d}x \\ &= \int_{1}^{\infty} 2x^{-2}\mathrm{d}x \\ &= \left[-\frac{2}{x}\right]_1^{\infty} \\ &= 2 \end{aligned}$$

For **variance** we first find $E(X^2)$ which is given by:

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\mathrm{d}x \qquad (4)\\
&= \int_{1}^{\infty} x^2.2x^{-3}\mathrm{d}x\\
&= \int_{1}^{\infty} 2x^{-1}\mathrm{d}x\\
&= \Bigg[2log(x)\Bigg]_1^{\infty}\\
&= \infty \qquad (5)
\end{aligned}
$$

Thus we have,

$$
\begin{aligned}
Var(X) &= E[X^2] - [E(X)]^2 \qquad (6)\\
&= \infty - 4\\
&= \infty
\end{aligned}
$$

(c.)

To find the CDF of X, we use $F_X(x) = \int_{-\infty}^{x} f_X(u)\mathrm{d}u$, so for $x < 1$, we obtain $F_X(x) = 0$. For $x \geq 1$, we have:

$$
\begin{aligned}
F_X(x) &= \int_{1}^{x} 2u^{-3}\mathrm{d}u\\
&= \Bigg[-\frac{1}{u^2}\Bigg]_1^{x}\\
&= \frac{x^2-1}{x^2}.
\end{aligned}
$$

Thus,

$$
F_X(x) = \begin{cases} \frac{x^2-1}{x^2}, & \text{if } x \geq 1\\ 0, & \text{otherwise} \end{cases}
$$

(d.)

By law of conditioning, we have

$$f_{X|X\geq 2}(x) = \frac{P(X \geq 2|X = x).f_X(x)}{P(X \geq 2)} \tag{7}$$

such that:

$$\begin{aligned} P(X \geq 2) &= \int_2^{\infty} f_X(u)\mathrm{d}u \qquad (8) \\ &= \int_2^{\infty} 2u^{-3}\mathrm{d}u \\ &= \left[-\frac{1}{u^2}\right]_2^{\infty} \\ &= \frac{1}{4}, \qquad (9) \end{aligned}$$

$$f_X(x) = 2x^{-3} \tag{10}$$

and

$$P(X \geq 2|X = x) = \begin{cases} 0, & x < 2 \\ 1, & x \geq 2 \end{cases} \tag{11}$$

We substitute equations (9), (10), and (11) into (7), we get:

$$f_{X|X\geq 2}(x) = \frac{\Big(1\Big)\Big(2x^{-3}\Big)}{\frac{1}{4}}$$

$$= 8x^{-3}, \quad for \;\; x \geq 2$$

For **mean** we have

$$\begin{aligned}E[X|X \geq 2] &= \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x \\&= \int_{2}^{\infty} x.8x^{-3}\mathrm{d}x \\&= \int_{2}^{\infty} 8x^{-2}\mathrm{d}x \\&= \left[-\frac{8}{x}\right]_{2}^{\infty} \\&= 4\end{aligned} \tag{12}$$

Problem 2

(a) A Bernoulli random variable $X \sim B(p)$ is a discrete random variable, which equals 1 with probability p and equals 0 with probability $1-p$. Please find the pmf, CDF, and first five moments of a Bernoulli random variable X. (Please leave your answer in terms of p).

(b) A bi-directional Bernoulli random variable $X \sim BB(p)$ is a discrete random variable, which equals 1 with probability p and equals -1 with probability $1-p$. Please find the pmf, CDF, and first five moments of a bi-directional Bernoulli random variable.

Solution

(a.)

For pmf:

$$P_X(x) = \begin{cases} 1-p, & x=0 \\ p, & x=1 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

For CDF:

$$F_X(x) = \begin{cases} 1, & x \geq 0 \\ 1-p, & 0 \leq x < 1 \\ 0, & x < 0 \end{cases} \tag{2}$$

For the Bernoulli distribution, the range of X is $R_X = \{0,1\}$, and $P_X(1) = p$ and $P_X(0) = 1-p$. Thus the moments are expressed using:

$$\begin{aligned} E[X^n] &= \sum_x x^n.p_X(x) \\ E[X^n] &= 0.p_X(0) + 1.p_X(1) \\ &= 0.(1-p) + 1.p \\ &= p \end{aligned}$$

(b.)

For pmf:

$$p_X(x) = \begin{cases} 1-p, & x=-1 \\ p, & x=1 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

For CDF:

$$F_X(x) = \begin{cases} 1, & x \geq 1 \\ 1-p, & -1 \leq x < 1 \\ 0, & x < -1 \end{cases} \tag{4}$$

For the Bernoulli distribution, the range of X is $R_X = \{-1, 1\}$, and $P_X(1) = p$ and $P_X(-1) = 1-p$. Thus:

$$\begin{aligned} E[X^n] &= \sum_x x^n . p_X(x) \\ &= -1.p_X(-1) - 1.p_X(1) \\ &= -1.(1-p) - 1.p \\ &= 2p - 1, \quad for \quad x = -1 \end{aligned}$$

$$\begin{aligned} E[X^n] &= \sum_x x^n . p_X(x) \\ &= 1.p_X(-1) + 1.p_X(1) \\ &= 1.(1-p) + 1.p \\ &= 1, \quad for \quad x = 1 \end{aligned}$$

Problem 3

Consider a Gaussian random variable $Y \sim N(m = 1, \sigma^2 = 4)$

(a) Please find the mean, standard deviation, and second moment of Y.

(b) In terms of the standard Gaussian distribution, what is the probability that $2 \leq Y \leq 3$.

(c) Please design a zero-mean Gaussian random variable Z such that $P(Z \geq 1) = 0.3$

Solution

a.

$$Y \sim N(m, \sigma^2)$$

where m = mean, and σ = variance

The Gaussian Random Variable has a pdf of:

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} exp^{-\frac{(y-m)^2}{2\sigma^2}} \tag{1}$$

The mean of a Gaussian random variable by the analysis in our notes is given as:

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) \mathrm{d}x \tag{2}$$

$$E[Y] = \int_{-\infty}^{\infty} y \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-m)^2}{2\sigma^2}} \mathrm{d}y \tag{3}$$

$$E[Y] = m \qquad \text{where m = mean} \tag{4}$$

Therefore **mean = 1**

Also, standard deviation is given as:

$$\text{SD} = \sqrt{\text{variance}}$$

But variance is given as:

$$\begin{aligned}
\text{Var[Y]} &= \int_{-\infty}^{\infty} (y-m)^2 f_Y(y)\text{d}y &(5)\\
&= \int_{-\infty}^{\infty} (y-m)^2 \frac{1}{\sigma\sqrt{2\pi}} exp^{-\frac{(y-m)^2}{2\sigma^2}} \text{d}y &(6)\\
&= \sigma^2 &(7)\\
&= 4 &(8)
\end{aligned}$$

By this, the standard deviation = **2**

For the second moment, we have:

$$\begin{aligned}
\text{Var[Y]} &= E[Y^2] - [E(Y)]^2\\
4 &= E[Y^2] - [E(Y)]^2\\
E[Y^2] &= 4 + [E(Y)]^2\\
E[Y^2] &= 4 + [1]^2\\
E[Y^2] &= 5
\end{aligned}$$

b.

$P(2 \le Y \le 3) = F_Y(3) - F_Y(2)$

$$G\left(\frac{y-m}{\sigma}\right) = G\left(\frac{3-1}{2}\right) - G\left(\frac{2-1}{2}\right)$$

$$= G(1) - G(0.5)$$

$$= 0.8413 - 0.6915$$

$$= \mathbf{0.1499}$$

c.

$Y \sim N(0, \sigma)$

$$\begin{aligned} G(Z \leq 1) &= 1 - G(Z \geq 1) \\ G(Z \leq 1) &= 1 - 0.3 \\ &= 0.7 \end{aligned}$$

$$\begin{aligned} G\left(\frac{y-m}{\sigma}\right) &= 0.7 \\ G\left(\frac{1-0}{\sigma}\right) &= 0.7 \\ G^{-1}(0.7) &= \frac{1}{\sigma} \\ 0.5244 &= \frac{1}{\sigma} \\ \sigma = 1.907 \end{aligned}$$

Problem 4

A probabilistic experiment has three outcomes, A, B, and C which have probabilities of 0.2, 0.3, and 0.5 respectively. A random variable X is defined as follows: if the experiment has outcome A, then X is exponential with parameter $\lambda = 2$. If the experiment has outcome B, then X is exponential with parameter $\lambda = 1$. If the experiment has outcome C, then $X = 0$. Please find $P(A|X = x)$, $P(B|X = x)$, and $P(C|X = x)$ as a function of x

Solution

We have the following:

P(A) = 0.2, P(B) = 0.3, P(C) = 0.5,

$f_{X|A}(x) = 2e^{-2x}$, $f_{X|B}(x) = e^{-x}$, $f_{X|C}(x) = \delta(x)$

and
$f_X(x) = f_{X|A}(x)P(A) + f_{X|B}(x)P(B) + f_{X|C}(x)P(C)$

Thus:

$$\begin{aligned} P(A|X = x) &= \frac{f_{X|A}(x)P(A)}{f_X(x)} \\ &= \frac{0.4e^{-2x}}{0.4e^{-2x} + 0.3e^{-x} + 0.5\delta(x)} \end{aligned}$$

$$P(A|X = x) = \begin{cases} \frac{0.4e^{-2x}}{0.4e^{-2x}+0.3e^{-x}+0.5\delta(x)}, & x \geq 0^- \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

$$\begin{aligned} P(B|X = x) &= \frac{f_{X|B}(x)P(B)}{f_X(x)} \\ &= \frac{0.3e^{-x}}{0.4e^{-2x} + 0.3e^{-x} + 0.5\delta(x)} \end{aligned}$$

$$P(B|X = x) = \begin{cases} \frac{0.3e^{-x}}{0.4e^{-2x}+0.3e^{-x}+0.5\delta(x)}, & x \geq 0^- \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

$$
\begin{aligned}
P(C|X=x) &= \frac{f_{X|C}(x)P(C)}{f_X(x)} \\
&= \frac{0.5\delta(x)}{0.4e^{-2x} + 0.3e^{-x} + 0.5\delta(x)}
\end{aligned}
$$

$$
P(C|X=x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases} \tag{3}
$$

Problem 5

A random variable X is uniformly distributed on the interval [0,2]. Given $X = x$, the event A occurs with probability $1 - \frac{x}{2}$, and the event B occurs independently with probability $\frac{x}{2}$. Please answer the following questions:

(a) Please find P(A) and P(B).

(b) Find the PDF of X given A.

(c) Find P(A|B). Are A and B independent?

Solution

5a.

A continuous random variable X is said to have a uniform distribution over the interval [a,b], shown as X$\sim Uniform(a,b)$, if its PDF is given by:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & x < a \; or \; x > b \end{cases} \tag{1}$$

As such:

$$f_X(x) = \frac{1}{2}, \quad for \; 0 < x < 2$$

Thus by law of total probability:

$$\begin{aligned} P(A) &= \int_0^2 f_X(x) P(A|X = x) \mathrm{d}x \\ &= \frac{1}{2} \int_0^2 \left(1 - \frac{x}{2}\right) \mathrm{d}x \\ &= \frac{1}{2} \int_0^2 \left(1 - \frac{x}{2}\right) \mathrm{d}x \\ &= \frac{1}{4} \int_0^2 (2 - x) \mathrm{d}x \\ &= \frac{1}{4} \left[2x - \frac{x^2}{2}\right]_0^2 \\ &= \frac{1}{4}[2] \Rightarrow \frac{1}{2} \end{aligned}$$

$$
\begin{aligned}
P(B) &= \int_0^2 f_X(x) P(B|X = x) \mathrm{d}x \\
&= \frac{1}{2} \int_0^2 \left(\frac{x}{2}\right) \mathrm{d}x \\
&= \frac{1}{8} \left[x^2 \right]_0^2 \\
&= \frac{1}{8}[4] \Rightarrow \frac{1}{2}
\end{aligned}
$$

5b

To find the PDF of X given A, by Bayes rule, we have:

$$
P(A|X = x) = \frac{f_{X|A}(x) P(A)}{f_X(x)} \tag{2}
$$

By reverse conditioning, we get:

$$
\begin{aligned}
f_{X|A}(x) &= \frac{P(A|X = x) f_X(x)}{P(A)} \\
&= \frac{(1 - \frac{x}{2}) 0.5}{0.5} \\
&= 1 - \frac{x}{2}
\end{aligned}
$$

5c

Since $P(A|X = x)$ and $P(B|X = x)$ are independent, we have:

$$
\begin{aligned}
P(AB) &= \int_0^2 f_X(x) P(AB|X=x)\,\mathrm{d}x \\
&= \int_0^2 f_X(x) P(A|X=x) P(B|X=x)\,\mathrm{d}x \\
&= \int_0^2 \frac{1}{2}\left(1-\frac{x}{2}\right)\frac{x}{2}\,\mathrm{d}x \\
&= \frac{1}{4}\int_0^2 \left(x-\frac{x^2}{2}\right)\,\mathrm{d}x \\
&= \frac{1}{24}\left[3x^2-x^3\right]_0^2 \\
&= \frac{1}{24}[4] \Rightarrow \frac{1}{6}
\end{aligned}
$$

By conditional probability law, we have:

$$
\begin{aligned}
P(A|B) &= \frac{P(AB)}{P(B)} \\
&= \frac{\frac{1}{6}}{\frac{1}{2}} \Rightarrow \frac{1}{3}
\end{aligned}
$$

For independence, P(A|B) = P(A), therefore A and B are **not independent**.

Problem 6

Consider a geometric random variable Q with parameter p=0.7.

(a) Please find the pmf and mean of Q, given that $Q \leq 4$.

(b) Let $R = Q^2$. Please find the pmf of R.

Solution

a.

We know that the pmf of a random variable Q is equal to 1. That is:

$$\sum_{-\infty}^{\infty} p_Q(q) = 1 \tag{1}$$

Therefore we get:

$$\alpha \sum_{1}^{4} p_Q(q) \Rightarrow 1 \tag{2}$$

where

$$\alpha = \frac{1}{\sum_1^4 p_Q(q)}$$

But

$$P_Q(q) = (1-p)(p)^{q-1} \quad for \quad k = 1,2,3... \tag{3}$$

To find the pmf of Q, given that $Q \leq 4$. By equation 2 and 3, we get:

$$P_{Q|Q\leq 4}(q) = \frac{1}{0.7599}(0.3)(0.7)^{q-1}$$

$$P_{Q|Q\leq 4}(q) = \begin{cases} \frac{(0.3)(0.7)^{q-1}}{0.7599}, & q = 1,2,3,4 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

To find the mean of

$$
\begin{aligned}
E_{Q|Q\geq 4(q)} &= \sum_{q=1}^{4} q.p_{Q|Q\leq 4}(q) \\
&= \sum_{q=1}^{4} (q)(0.39479)(0.7)^{q-1} \\
&= 2.069489
\end{aligned}
$$

b.

To find the pmf of , we have:

$$
p_R(r) = \begin{cases} (0.3)(0.7)^{\sqrt{r}-1}, & r = 1, 4, 9, 16 \\ 0, & \text{otherwise} \end{cases} \tag{5}
$$

Problem 7

Consider throwing a dart at a circular dartboard with unit radius; assume that every point on the dartboard is equally likely. Let X be the distance of the dart from the center of the dartboard. Let $Z = X^a$ where $a > 0$. Please find the pdf of Z (leaving your answer in terms of a). Is Z a uniform random variable for some parameter value a?

Solution

$$P(X \leq x) = \begin{cases} 0, & x < 0 \\ \frac{\pi . x^2}{\pi . 1^2}, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

The pdf of Z is given as

$$f_Z(z) = \frac{d}{dz}\Big(F_Z(z)\Big) \tag{1}$$

Given $Z = X^a$

We have $F_Z(z) =$

$$P(Z \leq z) = P(X^a \leq z) = P(X \leq z^{\frac{1}{a}}) = \begin{cases} 0, & z^{\frac{1}{a}} < 0 \\ z^{\frac{2}{a}}, & 0 \leq z^{\frac{1}{a}} \leq 1 \\ 1, & z^{\frac{1}{a}} > 1 \end{cases}$$

By equation 1, we get:

$$f_Z(z) = \begin{cases} \frac{2}{a} z^{(\frac{2}{a} - 1)}, & 0 \leq z \leq 1 \\ 0, & z^{\frac{1}{a}} < 0, \ z^{\frac{1}{a}} > 1 \end{cases}$$

Z is a uniform random variable when a = 2.

Problem 8

The random variable X is uniform on [0,4]. A random variable Y is defined as follows: $Y = 0$ if $X < 1$, and $Y = X - 1$ if $X \geq 1$. Please find the pdf of Y.

Solution

X $\sim$ unif (0,4)

$$Y = \begin{cases} 0, & X < 1 \\ X - 1, & X \geq 1 \end{cases}$$

Note that $R_Y = [0,3]$. Therefore,

$$F_Y(y) = 0, \quad for \;\; y < 0$$
$$F_Y(y) = 1, \quad for \;\; y > 3$$

Also for $0 < y < 3$ we have:

$$= \int_0^{y+1} \frac{1}{4} \mathrm{d}x$$
$$= \frac{y+1}{4}$$

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{y+1}{4}, & 0 < y < 3 \\ 1, & y > 3 \end{cases}$$

Since we have:

$$f_Y(y) = \frac{d}{\mathrm{d}y}\Big(F_Y(y)\Big) \tag{1}$$

$$f_Y(y) = \frac{1}{4}\delta(y) + \frac{1}{4}, \quad for \;\; 0 \leq y \leq 3$$