

2.

$$X, Y \sim N(\mu_x=0, \mu_y=0, \sigma_x^2=1, \sigma_y^2=4, \rho=0.5)$$

2.1

$$f_{X,Y}(u,y) = \frac{1}{\sqrt{1-\rho^2} \sigma_x \sqrt{2\pi} \sigma_y \sqrt{2\pi}} e^{-\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{u-\mu_x}{\sigma_x} \right)^2 + \left(\frac{y-\mu_y}{\sigma_y} \right)^2 - 2 \frac{(u-\mu_x)(y-\mu_y)}{\sigma_x \sigma_y} \right\}}$$

↓
Only component (say $g(u,y)$) dependent on u and y .

Taking out the component $g(u,y)$ and putting the given values of $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2$ and ρ_{xy} :

$$g(u,y) = \left(\frac{u}{1} \right)^2 + \left(\frac{y}{2} \right)^2 - \frac{2 \left(\frac{u}{1} \right) \left(\frac{y}{2} \right)}{(1)(2)}$$

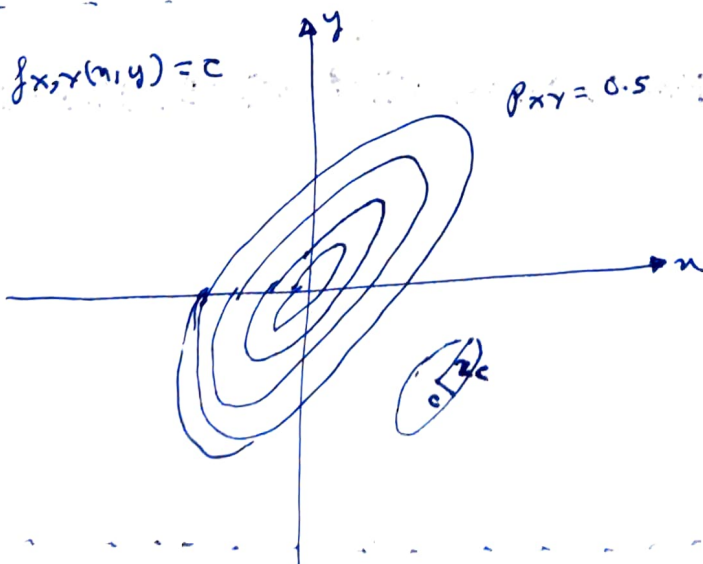
$$\Rightarrow f_{X,Y}(u,y) \xrightarrow[\text{is constant}]{\text{i.e.}} c \quad \text{for} \quad g(u,y) \xrightarrow[\text{is constant}]{\text{i.e.}} c$$

\Rightarrow Contour of $f_{X,Y}(u,y)$ is

$$g(u,y) = \left(\frac{u}{1} \right)^2 + \left(\frac{y}{2} \right)^2 - \frac{2}{2} u y = c$$

which is the equation for an ellipse, tilted and with positive slope.

2(a)

Ans

2(b)

$$f_X(n) = \mathcal{N}(n, 0, 1^2) = \frac{1}{1 \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \left(\frac{n}{1} \right)^2 \right\}} \quad \forall n$$

$$f_Y(y) = \mathcal{N}(y, 0, 2^2) = \frac{1}{2 \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \left(\frac{y}{2} \right)^2 \right\}} \quad \forall y$$

Ans

2(c)

$$f_{Y|X}(y|X=n) = \frac{f_{X,Y}(X=n, Y=y)}{f_X(X=n)} \quad \forall n, y$$

$$\Rightarrow f_{Y|X}(y|X=n) = \frac{1}{\sqrt{(1-\frac{1}{2})} \cdot 1 \cdot \sqrt{2\pi} \cdot 2 \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} \cdot \left\{ \frac{2}{2} \left(\frac{n}{1} \right)^2 + \left(\frac{y}{2} \right)^2 - 2 \times \frac{1}{2} \times \frac{n}{1} \times \frac{y}{2} \right\}}$$

 ~~$\forall n, y$~~

$$= \frac{1}{1 \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \left(\frac{n}{1} \right)^2 \right\}} \quad \forall (n, y)$$

$$\Rightarrow f_{Y|X}(y|X=n) = \frac{1}{\sqrt{3} \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \cdot \left\{ \left(\frac{n}{\sqrt{3}} \right)^2 + \left(\frac{y}{\sqrt{3}} \right)^2 - 2 \cdot 1 \cdot \left(\frac{n}{\sqrt{3}} \right) \left(\frac{y}{\sqrt{3}} \right) \right\}}$$

 $\forall (n, y)$

2(c)

$$\Rightarrow f_{Y|X}(y|X=n) = \sqrt{2\pi} \mathcal{N}(\mu_X=0, \mu_Y=0, \sigma_X^2=3, \sigma_Y^2=3, \rho_{XY}=1) \quad \forall (n, y)$$

$$\Rightarrow f_{Y|X}(y|X=n) = \mathcal{N}(\mu_Y=n, \sigma_Y^2=3) \quad \forall y$$

$$\Rightarrow f_{Y|X}(y|X=n) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left\{ \left(\frac{y-n}{\sqrt{3}} \right)^2 \right\}} \quad \forall y$$

2(c)

$$\Rightarrow f_{Y|X}(y|X=n) = \mathcal{N}(y=n, \mu_Y=n, \sigma_Y^2=3) \quad \forall y$$

Ans

2(d) Find n s.t. $E[Y|X=n] = -2$.

From 2(c), we know that $Y|X=n$ is fully correlated to $X=n$. ($P_{Y|X}, X=1$).

$\therefore Y$ and X ~~share~~ ^{have} the same support/domain of $n \in \mathbb{R}$ and $y \in \mathbb{R}$.

$$\therefore E_Y[Y|X=n] = -2 \equiv E_X[X=n] = -2$$

$$\text{But } E_X[X=n] = n$$

$$\underline{E_X[X=n]} \Rightarrow \boxed{n = -2} \text{ Ans}$$

2(d) Find n s.t. $E[Y|X=n] = -2$

From 2(c), we know that $Y|X=n$ is a gaussian with mean n .

$$\text{So } E[Y|X=n] = -2 \Rightarrow \boxed{n = -2} \text{ Ans}$$

$$2(e) \quad Z = X + Y - 1$$

Z is also a gaussian, we need to only compute μ_Z and σ_Z^2

to find $f_Z(z)$:

$$E(Z) = E(X) + E(Y) - E(1)$$

$$\text{or } \mu_Z = \mu_X + \mu_Y - 1$$

$$\text{or } \mu_Z = 0 + 0 - 1$$

$$\text{or } \boxed{\mu_Z = -1}$$

$$\sigma_Z^2 = E[\{X + Y - 1 - E[X + Y - 1]\}^2]$$

$$\text{or } \sigma_Z^2 = E[\{(X - \mu_X) + (Y - \mu_Y) + (-1 - E(-1))\}^2]$$

$$\text{or } \sigma_Z^2 = E[\{(X - \mu_X) + (Y - \mu_Y)\}^2]$$

$$\text{or } \sigma_Z^2 = E[(X - \mu_X)^2] + E[(Y - \mu_Y)^2] + 2E[(X - \mu_X)(Y - \mu_Y)]$$

$$\text{or } \sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y$$

$$\text{or } \sigma_Z^2 = 1^2 + 2^2 + 2 \times 0.5 \times 1 \times 2$$

$$\text{or } \boxed{\sigma_Z^2 = 7}$$

$$\therefore \boxed{Z \sim \mathcal{N}(z, \mu_Z = -1, \sigma_Z^2 = 7)} \quad \text{Ans}$$

2(f)

$$Z = 2X + 3Y$$

$$W = X - Y$$

2.5

$$E(Z) = 2\mu_X + 3\mu_Y$$

$$E(W) = \mu_X - \mu_Y$$

2(f)(i)
or $\boxed{\mu_Z = 0}$ Ans

2(f)(ii)
or $\boxed{\mu_W = 0}$ Ans

$$\sigma_Z^2 = E[\{(2X+3Y) - (2\mu_X+3\mu_Y)\}^2] \quad \sigma_W^2 = E[\{(X-Y) - (\mu_X-\mu_Y)\}^2]$$

$$\text{or } \sigma_Z^2 = E[\{2(X-\mu_X)\}^2 + \{3(Y-\mu_Y)\}^2 + 2 \cdot 2 \cdot 3 \cdot (X-\mu_X)(Y-\mu_Y)] \quad \text{or } \sigma_W^2 = E[\{X-\mu_X\}^2 + \{Y-\mu_Y\}^2 - 2 \cdot 1 \cdot 1 \cdot (X-\mu_X)(Y-\mu_Y)]$$

$$\text{or } \sigma_Z^2 = 4\sigma_X^2 + 9\sigma_Y^2 + 12\rho_{XY}\sigma_X\sigma_Y \quad \text{or } \sigma_W^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y$$

$$\text{or } \sigma_Z^2 = 4 \cdot 1^2 + 9 \cdot 2^2 + 12 \cdot (0.5) \cdot 1 \cdot 2 \quad \text{or } \sigma_W^2 = 1^2 + 2^2 - 2 \cdot (0.5) \cdot 1 \cdot 2$$

$$\text{or } \sigma_Z^2 = 4 + 36 + 12$$

$$\text{or } \sigma_W^2 = 1 + 4 - 2$$

2(f)(ii)
or $\boxed{\sigma_Z^2 = 52}$ Ans

2(f)(iv)
 $\boxed{\sigma_W^2 = 3}$ Ans

$$\text{2(f)} \quad \text{Cov}(Z, W) = E[(Z - \mu_Z)(W - \mu_W)]$$

$$\text{or } \text{Cov}(Z, W) = E[(2X+3Y-2\mu_X-3\mu_Y)(X-Y-\mu_X-(-\mu_Y))]$$

$$\text{or } \text{Cov}(Z, W) = E[\{2(X-\mu_X)+3(Y-\mu_Y)\}\{1(X-\mu_X)-1(Y-\mu_Y)\}]$$

$$\text{or } \text{Cov}(Z, W) = E[2(X-\mu_X)^2 - 3(Y-\mu_Y)^2 + 1(X-\mu_X)(Y-\mu_Y)]$$

$$\text{or } \text{Cov}(Z, W) = 2\sigma_X^2 - 3\sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y$$

$$\text{or } \text{Cov}(Z, W) = 2 \cdot 1^2 - 3 \cdot 2^2 + (0.5) \cdot 1 \cdot 2$$

$$\text{or } \text{Cov}(Z, W) = 2 - 12 + 1$$

2(f) $\boxed{\text{Cor}(Z, W) = -9}$ Ans

$$\rho_{Z,W} = \frac{\text{Cor}(Z, W)}{\sigma_Z \sigma_W} = \frac{-9}{\sqrt{52} \cdot \sqrt{3}} \approx -0.7206$$

2(f) $\boxed{f_{Z,W}(z, w) = N(\mu_Z=0, \mu_W=0, \sigma_Z^2=52, \sigma_W^2=3, \rho_{Z,W}=-0.7206)}$ Ans

2(g) $R = aX + bY \Rightarrow \sigma_R^2 = a^2 \sigma_X^2 + b^2 \sigma_Y^2$
 $\sigma_R^2 = a^2 \cdot 1^2 + b^2 \cdot 2^2$
 $\sigma_R^2 = a^2 + 4b^2$

$\text{Cor}(R, Y) = 0$ is a sufficient condition for R and Y to be independent, as both are gaussian.

$$\text{Cor}(R, Y) = E[(aX + bY - a\mu_X - b\mu_Y)(Y - \mu_Y)]$$

$$\text{or } \text{Cor}(R, Y) = E[a(X - \mu_X)(Y - \mu_Y) + b(Y - \mu_Y)^2]$$

$$\text{or } \text{Cor}(R, Y) = a \rho_{X,Y} \sigma_X \sigma_Y + b \sigma_Y^2$$

$$\text{or } \text{Cor}(R, Y) = a(0.5) \cdot 1 \cdot 2 + b \cdot 2^2$$

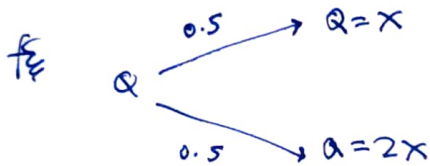
$$\text{or } \text{Cor}(R, Y) = a + 4b$$

2(g) For R, Y to be independent distributions, $a + 4b = 0$
 we can use any set of real numbers to do so,
 say $a = -4, b = 1$ Ans

$$2(b) \quad Q = aX$$

given $p(a=2) = 0.5$ $f(Q|a=2) = \cancel{2\sigma_x^2} \cancel{2\mu_x} f_X(X=2x)$

$$p(a=1) = 0.5 \quad f(Q|a=1) = f_X(X=x)$$



$$f_Q(q) = f_{Q|a}(Q|a=1) \cdot P(a=1) + f_{Q|a}(Q|a=2) \cdot P(a=2) \quad (\text{LTP})$$

$$\text{or } f_Q(q) = 0.5 \mathcal{N}(q, \mu_Q = \mu_x, \sigma_Q^2 = \sigma_x^2) + 0.5 \mathcal{N}(q, \mu_Q = 2\mu_x, \sigma_Q^2 = 2^2 \sigma_x^2)$$

2(h)

$$\text{or } f_Q(q) = 0.5 \mathcal{N}(q, \mu_Q = 0, \sigma_Q^2 = 1) + 0.5 \mathcal{N}(q, \mu_Q = 0, \sigma_Q^2 = 4) \quad \underline{\underline{\text{Ans}}}$$

—————X—————X—————X—————X—————

6.

(0,1)

(0,0)

(1,0)

A(1,1)

 $\alpha \in [0,1]$
 $\beta \geq \alpha$
 $\alpha \in [0,1]$
 $\beta \in [0,\alpha]$
 $\alpha \geq 1$ AND $\beta \geq 1$
 $\alpha \geq 1$
 $\beta \in [0,1]$

A

 $\alpha < 0$ or $\beta < 0$ X, Y are uniformly distributed in ΔAOB .

$$f_{X,Y}(x,y) = \frac{1}{\text{Area of } \Delta AOB} \quad (x,y) \in \Delta AOB$$

$$f_{X,Y}(x,y) = \frac{1}{\frac{1}{2} \times 1 \times 1} \quad \begin{matrix} x \in [0,1] \\ y \in [0,1] \end{matrix}$$

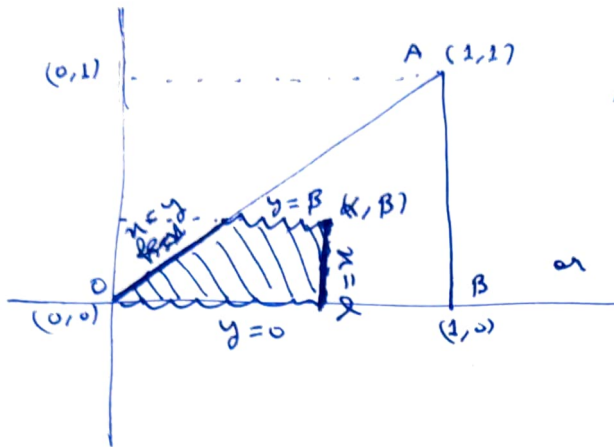
$$f_{X,Y}(x,y) = \begin{cases} 2 & \alpha \in [0,1] \\ & \beta \in [0,\alpha] \\ 0 & \text{else} \end{cases}$$

6(b) (Calculations on next pages.)

$$F_{X,Y}(x=\alpha, y=\beta) = \begin{cases} 0 & \alpha < 0 \text{ or } \beta < 0 \quad \text{A} \\ 2\alpha\beta - \beta^2 & \alpha \in [0,1], \beta \in [0,\alpha] \quad \text{B} \\ \alpha^2 & \alpha \in [0,1], \beta \geq \alpha \quad \text{C} \\ 2\beta - \beta^2 & \alpha \geq 1, \beta \in [0,\alpha] \quad \text{D} \\ 1 & \alpha \geq 1, \beta \geq 1 \quad \text{E} \end{cases}$$

For (α, β) in (B):

i.e. $\alpha \in (0, 1), \beta \in [0, \alpha]$



~~$F_{X,Y}(n,y)$~~

$$F_{X,Y}(n=\alpha, y=\beta) = \iint_{\text{shaded region}} 2 \, dn \, dy \quad (\alpha, \beta) \in \text{(B)}$$

$$\text{or } F_{X,Y}(n=\alpha, y=\beta) = \int_{y=0}^{y=\beta} \int_{n=y}^{n=\alpha} 2 \, dn \, dy \quad (\alpha, \beta) \in \text{(B)}$$

$$\text{or } F_{X,Y}(\alpha, \beta) = \int_{y=0}^{y=\beta} 2n \Big|_y^{\alpha} dy \quad (\alpha, \beta) \in \text{(B)}$$

$$\text{or } F_{X,Y}(\alpha, \beta) = \int_{y=0}^{y=\beta} 2(\alpha - y) dy \quad (\alpha, \beta) \in \text{(B)}$$

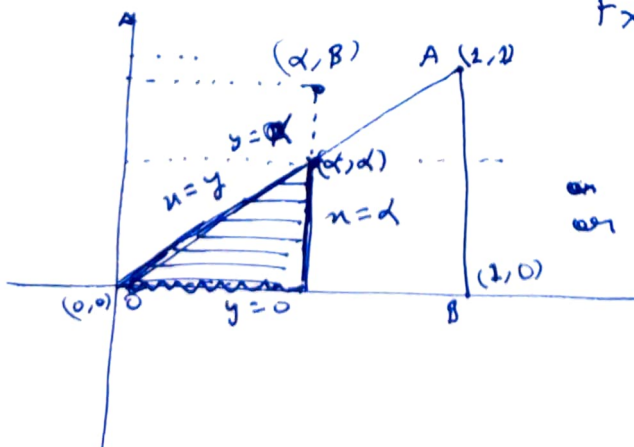
$$\text{or } F_{X,Y}(\alpha, \beta) = 2\alpha y - y^2 \Big|_0^{\beta} \quad (\alpha, \beta) \in \text{(B)}$$

$$\text{or } F_{X,Y}(\alpha, \beta) = 2\alpha\beta - \beta^2 \quad (\alpha, \beta) \in \text{(B)}$$

$$\text{or } \alpha \in (0, 1) \\ \beta \in [0, \alpha]$$

For (α, β) in (C):

i.e. $\alpha \in (0, 1), \beta \geq \alpha$



$$F_{X,Y}(n=\alpha, y=\beta) = \iint_{\text{shaded region}} 2 \, dn \, dy \quad (\alpha, \beta) \in \text{(C)}$$

$$\text{or } F_{X,Y}(n=\alpha, y=\beta) = \int_{y=0}^{y=\alpha} \int_{n=y}^{n=\alpha} 2 \, dn \, dy + \int_{y=\alpha}^{y=\beta} \int_{n=y}^{n=\alpha} 2 \, dn \, dy \quad (\alpha, \beta) \in \text{(C)}$$

or $F_{X,Y}(u=\alpha, y=\beta) = \int_{y=0}^{y=\alpha} 2(\alpha-y) dy \quad (\alpha, \beta) \in \textcircled{C}$

or $f_{X,Y}(x=\alpha, y=\beta) = 2\alpha^2 - \alpha^2 \quad (\alpha, \beta) \in \textcircled{C}$

or $F_{X,Y}(u=\alpha, v=\beta) = \alpha^2$ $(\alpha, \beta) \in \textcircled{C}$
 or $\alpha \in [0, 1], \beta \geq \alpha$

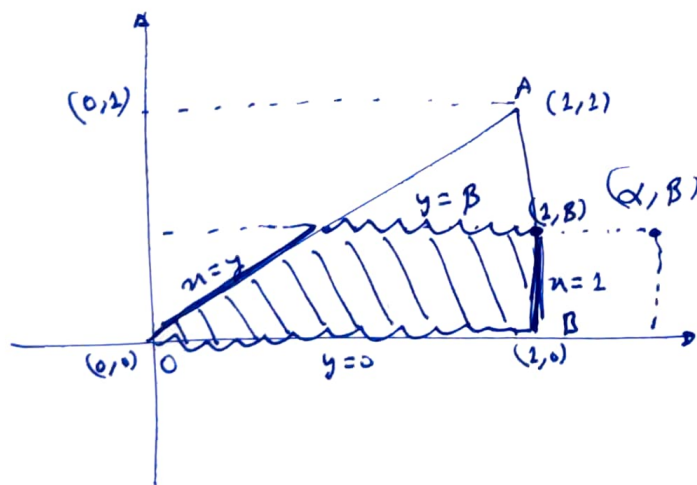
Area of the shaded Δ .

* $f_{X,Y}(u,v)$ in the region.

For $(\alpha, \beta) \in \text{in } \textcircled{D}$

i.e. $\alpha \geq 1$, $\beta \in [0, 1]$

$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2 \, dx \, dy \quad (\alpha, \beta) \in \textcircled{D}$$



(α, β) or $F_{xy}(\alpha, \beta) = \int_{y=0}^{\beta} \int_{x=0}^{\alpha} 2 \, dx \, dy; (\alpha, \beta) \in \textcircled{D}$

$$\text{or } F_{X,Y}(\alpha, \beta) = \int_{y=0}^{y=\beta} 2(1-y) dy \quad (\alpha, \beta) \in \textcircled{D}$$

$$\Rightarrow f_{X,Y}(\alpha, \beta) = 2\beta - \beta^2 \Big|_0^\beta \quad (\alpha, \beta) \in \textcircled{D}$$

$F_{X,Y}(\alpha, \beta) = 2\beta - \beta^2$
 $(\alpha, \beta) \in \textcircled{D}$
 $\alpha \geq 1, \beta \in (0, 1)$

$$f_X(n) = \int_{-\infty}^{\infty} f_{X,Y}(n,y) dy$$

$$\text{or } f_X(n) = \begin{cases} 0 & n < 0 \\ \int_{y=0}^{y=n} 2 dy & n \in [0,1] \\ 0 & n > 1 \end{cases}$$

6(c)

$$\text{or } f_X(n) = \begin{cases} 0 & n < 0 \\ 2n & n \in [0,1] \\ 0 & n > 1 \end{cases} \quad \underline{\underline{\text{Ans}}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(n,y) dn$$

$$\text{or } f_Y(y) = \begin{cases} 0 & y < 0 \\ \int_{n=y}^{n=1} 2 dn & y \in [0,1] \\ 0 & y > 1 \end{cases}$$

6(d)

$$\text{or } f_Y(y) = \begin{cases} 0 & y < 0 \\ 2(1-y) & y \in [0,1] \\ 0 & y > 1 \end{cases} \quad \underline{\underline{\text{Ans}}}$$

$$f_{X|Y}(u|Y=y) = \frac{f_{X,Y}(u, Y=y)}{f_Y(y)}$$

6(e)

or

$$f_{X|Y}(u|Y=y) = \begin{cases} \text{not defined} & y < 0 \\ \frac{1}{1-y} & y \in (0, 1) \\ \text{not defined} & y \geq 1 \end{cases}$$

Ans