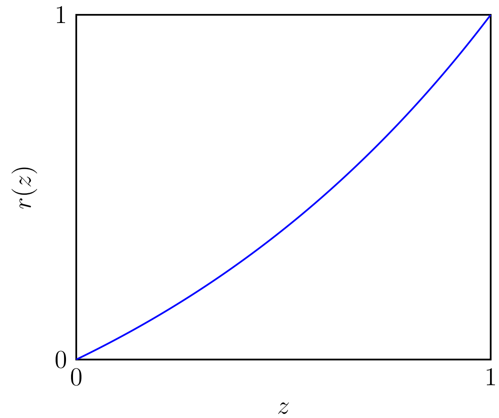
Integrating Newton's Aerodynamic Integral

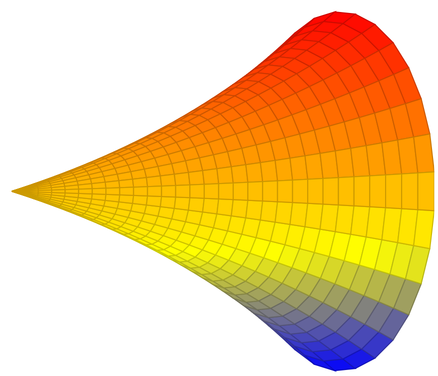
This set of tasks is an extension of the integral optimization problem of Adam Levy in his text "The Basics of Practical Optimization," Section 2.2.1.

Newton found that the aerodynamic drag on a surface of revolution (described by the continuous radial function $r(z)$) traveling in the z-direction is approximated by the (functional) integral

$$D(r(z)) = \int_0^1 \frac{r(z)r'(z)^3}{1+r'(z)^2}dz, \quad r(0)=0,\ r(1)=1. \tag{5}$$

The value of this definite integral depends on the specification of a function $r(z)$ over the domain $[0,1]$. The concept is illustrated below.





For example, consider the right circular cone described by $r(z) = z$. We have

$$D(r(z)) = \int_0^1 \frac{(z)(1)^3}{1+(1)^2}dz = \frac{1}{2}\int_0^1 z\,dz = \frac{1}{4}.$$

As a second example, consider the surface described by $r(z) = \sqrt{z}$ (note the compliance to the boundary conditions). We have

$$D(r(z)) = \int_0^1 \frac{\sqrt{z}\left(\frac{1}{2}z^{-1/2}\right)^3}{1+\left(\frac{1}{2}z^{-1/2}\right)^2}dz = \int_0^1 \frac{\frac{1}{8}z^{-1}}{1+\frac{1}{4}z^{-1}}dz = \frac{1}{8}\int_0^1 \frac{dz}{z+\frac{1}{4}} = \frac{\ln 5}{8} \approx 0.2012.$$

This second example exhibits less drag than the right circular cone. The goal is to find the function $r^*(z)$ that minimizes drag.

Instead of employing functional analysis methods, we will find an approximate solution by specifying a finite set of coordinates $\{z_k, r_k\}_{k=1}^m$ that lie on the curve (not including the boundary points $(z_0, r_0) = (0,0)$ and $(z_{m+1}, r_{m+1}) = (1,1)$) and using a piecewise linear assumption. In interval k, the slope is $s_k = \frac{r_{k+1}-r_k}{z_{k+1}-z_k}$ and we have

$$D \approx \sum_{k=0}^{m} \left(\frac{s_k^2}{1+s_k^2}\right)\left(\frac{r_{k+1}^2 - r_k^2}{2}\right). \tag{6}$$

If $m = 0$ we have only the boundary points (the right circular cone) and we recover $D = 1/4$. By including more and more intervals (larger m) we can more and more closely approximate the desired surface of revolution $r^*(z)$. The (equal-interval) optimization problem is now $\min D(r_1, r_2, \ldots, r_m)$ with $z_{k+1} - z_k = 1/(m+1)$. We could also allow variable width intervals and solve $\min D(r_1, r_2, \ldots, r_m, z_1, z_2, \ldots, z_m)$.

We could also compute the gradient of $D(r_1, r_2, \ldots, r_m)$; however, this would be cumbersome as each variable occurs in two terms of the summation. Instead, let's employ an approximation to the gradient. Each partial derivative can be approximated as

$$\frac{\partial D}{\partial r_k} \approx \frac{D(r_1, \ldots, r_k + \delta, \ldots, r_m) - D(r_1, \ldots, r_k, \ldots, r_m)}{\delta}. \tag{7}$$

where δ is some small change in the k^{th} coordinate. The natural question arises as to how to choose a good value of δ. We should choose as value as small as possible while not sacrificing acceptable precision. For example, if we choose $\delta = \varepsilon$, the machine precision, then we will retain at best one digit of precision. However, if we choose $\delta = 1$ we may have a very poor derivative estimate. A good compromise is to require the same accuracy in the both numerator and denominator of the approximation. To achieve this goal, we choose $\delta = \sqrt{\varepsilon}$.

We also must consider the inherent constraints in the problem. For example, there is the physical restriction that $r_k \geq 0$. We also have a monotonicity condition (which I believe is an assumption of Newton's derivation) $r_{k+1} \geq r_k$. All of the constraints are simply expressed as $r_{k+1} \geq r_k$ for $k = 0, 1, 2, \ldots, m$. If each constraint is not active at the optimal solution, then we can employ a continuous but nonsmooth penalty correction. That is, we recast as an unconstrained problem with modifed objective

$$D = \sum_{k=0}^{m} \left[\left(\frac{s_k^2}{1 + s_k^2} \right) \left(\frac{r_{k+1}^2 - r_k^2}{2} \right) + \exp\left(\max\left\{0, r_{k+1} - r_k\right\}\right) - 1 \right].$$

The extra terms evaluate to zero if the monotonicity condition is satisfied, but represents a large positive penalty for violating monotonicity. This new function is *not smooth* everywhere and this method should be used with caution and skepticism until we explore constrained optimization further.

Complete the following tasks.

1. Construct a function that returns the objective value and approximate gradient vector for the aerodynamic drag problem. The function should use the size of the input vector r to determine the uniformly-spaced z-values.

2. Solve the drag problem for the case $m = 4$ using the methods of gradient descent, conjugate gradient and BFGS quasi-Newton.

3. Choosing the most efficient method, solve the problem for cases of larger and larger m. In this way, approximate the minimum drag D^*.