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EE 507 MIDTERM 2

17 NOVEMBER 2022

1.1

1. ~~MGF~~ $f_X(x) = 0.5e^{-|x|} \quad \forall x \in \mathbb{R}.$

$$\Rightarrow f_X(x) = \begin{cases} 0.5e^{x} & x<0 \\ 0.5e^{-x} & x \geq 0 \end{cases}$$

$$MGF_X(s) = E[e^{sX}]$$

$$\Rightarrow MGF_X(s) = \int_{x=-\infty}^{x=0} e^{sx}\, 0.5 e^{x}\, dx + \int_{x=0}^{x=\infty} e^{sx}\, 0.5 e^{-x}\, dx$$

$$\Rightarrow MGF_X(s) = 0.5 \int_{x=-\infty}^{x=0} e^{(s+1)x}\, dx + 0.5 \int_{x=0}^{x=\infty} e^{(s-1)x}\, dx$$

if $s > -1$ if $s < 1$

$$\Rightarrow MGF_X(s) = \frac{0.5}{s+1} - \frac{0.5}{s-1} \qquad s \in (-1, 1)$$

$$\Rightarrow MGF_X(s) = 0.5 \times \frac{2}{1-s^2} \qquad s \in (-1,1)$$

1 (a)

$$\Rightarrow \boxed{MGF_X(s) = \frac{1}{1-s^2} \qquad s \in (-1,1).}$$ Ans.

$$E[X^2] = \frac{d^2}{ds^2} MGF_X(s)\Big|_{s\to 0} = \frac{d}{ds}\left(\frac{d}{ds}(1-s^2)^{-1}\right)\Big|_{s=0}$$

$$\Rightarrow E[X^2] = \frac{d}{ds}\left(-(1-s^2)^{-2} \cdot -2s\right)\Big|_{s=0}$$

$$\Rightarrow E[X^2] = \frac{d}{ds}\left(2s(1-s^2)^{-2}\right)\Big|_{s=0}$$

$$\Rightarrow E[X^2] = 2(1-s^2)^{-2} + 2s(1-s^2)^{-3} \cdot -2 \cdot -2s\Big|_{s=0}$$

1(b)

$$\Rightarrow \boxed{E[X^2] = 2} \text{ Ans}$$

$\boxed{2.}$

X_i	Y_i	$Z_i = X_i + Y_i$	$P(Z_i$
0	0	0	
0	1	1	
0	2	2	
1	0	1	
1	1	2	
2	0	2	

Z_i	Possible (x_i, y_i)	$p(z_i) = \sum p(x_i, y_i) = \sum p(x_i) \cdot p(y_i)$
0	(0,0) ~~(0,2), (1,1), (2,0)~~	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 \cdot \frac{1}{3}$
1	(0,1), (1,0)	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 \cdot \frac{1}{3} + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^1 \frac{1}{3}$
2	(0,2), (1,1), (2,0)	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 \cdot \frac{1}{3} + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^1 \frac{1}{3} + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 \cdot \frac{1}{3}$

z_i	$p(z_i) = \sum p(x_i) p(y_i)$
0	$\frac{1}{9} \cdot 1 = \frac{1}{9} \cdot \frac{9}{9}$
1	$\frac{1}{9} \cdot \frac{5}{3} = \frac{1}{9} \cdot \frac{15}{9}$
2	$\frac{1}{9} \cdot \frac{19}{9}$

$$p(Z \le 2) = P(X + Y \le 2) = \frac{1}{9}\left\{\frac{9}{9} + \frac{15}{9} + \frac{19}{9}\right\} = \frac{43}{81}$$

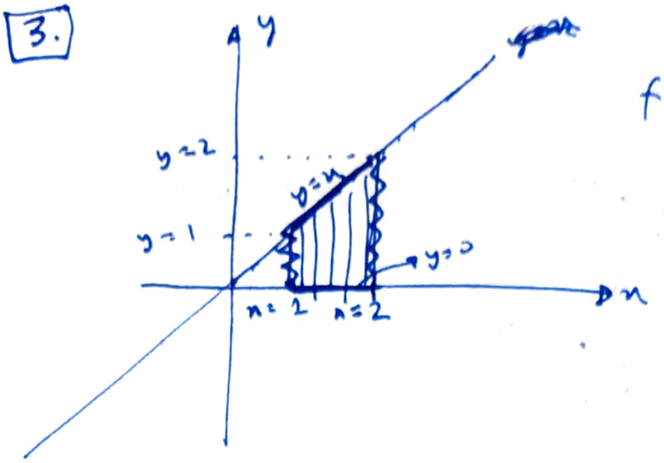
$\boxed{2.}$

$$\boxed{\begin{aligned} P(X+Y \le 2) &= \frac{43}{81} \\ P(X+Y \le 2) &\approx 0.5309 \end{aligned}}$$

Ans

——×——×——×——

3.



$f_{X,Y}(x,y)$ is uniform in the shaded region

$$f_{X,Y}(x,y) = \frac{1}{\text{Area of shaded region}} \quad \forall\ x \in [1,2],\ y \in [0,x]$$

$$\Rightarrow f_{X,Y}(x,y) = \frac{1}{\frac{(2+1)}{2} \cdot 1}$$

$$\Rightarrow \boxed{f_{X,Y}(x,y) = \begin{cases} \frac{2}{3} & x \in [1,2],\ y \in [0,x] \\ 0 & \text{else} \end{cases}}$$

PTO

PTO!

$$f_Y(y) = \int_{x=y}^{x=2} f_{X,Y}(x,y)\,dx + \int_{x=1}^{x=2} f_{X,Y}(x,y)\,dx$$

$$\Rightarrow f_Y(y) = \int_{x=y}^{x=2} \frac{2}{3}\,dx + \int_{x=1}^{x=2} \frac{2}{3}\,dx$$

$$\Rightarrow f_Y(y) = \frac{6-2y}{3} \quad \forall\ y \in [0,2]$$

$$\Rightarrow f_Y(y) = 2 - \frac{2y}{3} \quad \forall\ y \in [0,2]$$

$$f_{X|Y=y}(X, Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$\Rightarrow f_{X|Y=y}(X, Y=y) = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$$

$$\int_{y=0}^{y=2} 2 - \frac{2y}{3}\,dy$$

$$4 - \frac{2}{3}\cdot\frac{1}{2}\cdot 4$$

$\frac{2}{3}(2-y)$

$\frac{2}{3}\left(2y - \frac{y^2}{2}\right)$ $\quad 2 - \frac{2}{6}\frac{y^2}{} \quad 2 - \frac{4}{6} =$

$\frac{2}{3}(4-2)$ $\quad \frac{2-y}{3}$

$\frac{4}{3}$ $\quad \left.\frac{2}{3}y - \frac{y^2}{6}\right|_0^2$

$\frac{4}{3} - \frac{4}{6} = \frac{2}{3}$

$\frac{2}{3}(2-y)$

$\left.\frac{2}{3}\left(2y - \frac{y^2}{2}\right)\right|_{y=1}^{y=2}$

$\boxed{3.2}$

$$f_Y(y) = \begin{cases} \displaystyle\int_{x=1}^{x=2} f_{X,Y}(x,y)\,dx & y \in (0,1) \\ \displaystyle\int_{x=y}^{x=2} f_{X,Y}(x,y)\,dx & y \in [1,2) \end{cases}$$

$$\text{or } f_Y(y) = \begin{cases} \displaystyle\int_{x=1}^{x=2} \frac{2}{3}\,dx & y \in (0,1) \\ \displaystyle\int_{x=y}^{x=2} \frac{2}{3}\,dx & y \in [1,2] \end{cases}$$

3 (a)(i)

$$\text{or } \boxed{f_Y(y) = \begin{cases} \frac{2}{3} & y \in (0,1) \\ \cancel{\frac{2}{3} y} \quad \frac{2}{3}(2-y) & y \in [1,2) \end{cases}}$$

Ans

~~$f_{X,Y}(x,y$~~

$$f(x \mid Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$\Rightarrow f_{X|Y}(X|Y=y) = \begin{cases} \dfrac{\frac{2}{3}}{\frac{2}{3}} & y \in (0,1) \\ \dfrac{\frac{2}{3}}{\frac{2}{3}(2-y)} & y \in (1,2] \end{cases}$$

$\boxed{3(a)^{(ii)}}$

$$\Rightarrow \boxed{f_{X|Y}(X|Y=y) = \begin{cases} 1 & y \in [0,1] \\ \dfrac{1}{2-y} & y \in (1,2) \end{cases}}$$

Ans

$\boxed{3(a)^{(iii)}}$

$\boxed{\because f_{X|Y} \text{ depends on value of } y \Rightarrow X \text{ and } Y \text{ are NOT } \cancel{\text{rate}} \text{ independent.}}$

Ans

$$4 \quad \hat{X}_{X|Y_{MMSE}} = \mu_{X|Y} \quad \cancel{=\{}$$

$\Rightarrow$ ~~$\hat{X}_{X|Y_{MM}}$~~

$$\text{But} \quad \mu_{X|Y=y} = \int_{x=1}^{x=2} x \cdot f_{X|X}(X|Y=y)\, dx$$

$$\therefore \mu_{X|Y=y} = \begin{cases} \displaystyle\int_{x=1}^{x=2} x \cdot 1 \cdot dx & y \in [0,1] \\ \displaystyle\int_{x=y}^{x=2} \frac{1}{2-y} \cdot x \cdot dx & y \in (1,2] \end{cases}$$

$$\Rightarrow \quad \mu_{X|Y=y} = \begin{cases} \left.\dfrac{x^2}{2}\right|_1^2 & y \in (0,1) \\ \dfrac{1}{2-y} \cdot \left.\dfrac{x^2}{2}\right|_y^2 & y \in [1,2) \end{cases}$$

$$\Rightarrow \quad \mu_{X|Y=y} = \begin{cases} \dfrac{3}{2} & y \in (0,1) \\ \cancel{\dfrac{3}{2}} \cdot \dfrac{1}{2-y} \cdot \dfrac{4-y^2}{2} & y \in [1,2) \end{cases}$$

3(b)

$$\boxed{\Rightarrow \quad \cancel{MM:} \; \hat{X}_{X|Y=y \atop MMSE} = \begin{cases} \dfrac{3}{2} & y \in [0,1] \\ \cancel{\dfrac{3}{2} \cdot \dfrac{1}{2-y}} \; \dfrac{2+y}{2} & y \in [1,2] \end{cases}}$$

Ans

4. $f_X(x) = \mathcal{N}(x, \mu_X = 0, \sigma_X^2 = 4)$

or $f_X(x) = \dfrac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{2}\right)^2\right\}}$

$f_Y(y) = \mathcal{N}(y, \mu_Y = 0, \sigma_Y^2 = 9)$

or $f_Y(y) = \dfrac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y}{3}\right)^2\right\}}$

$f_{X,Y}(x,y) = \mathcal{N}(x, y, \mu_X = 0, \mu_Y = 0, \sigma_X^2 = 4, \sigma_Y^2 = 9, \rho = -\frac{1}{2})$

or $f_{X,Y}(x,y) = \dfrac{1}{2\sqrt{2\pi} \cdot 3\sqrt{2\pi}\sqrt{\left(1 - \left(-\frac{1}{2}\right)^2\right)}} e^{-\frac{1}{2\left(1-\left(-\frac{1}{2}\right)^2\right)}\left\{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 - 2\cdot\left(-\frac{1}{2}\right)\cdot\left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}$

4(a)

or $\boxed{f_{X,Y}(x,y) = \dfrac{1}{6\sqrt{3}\pi} e^{-\frac{1}{3/2}\left\{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}}$

4(a)

Again. (for clarity)

$\boxed{f_{X,Y}(x,y) = \dfrac{1}{6\sqrt{3}\pi} e^{-\frac{1}{2}\cdot\frac{4}{3}\left\{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}}$ Ans

Ans

4b)

$Z = X + Y$

$E(Z) = \mu_X + \mu_Y$

or $\mu_Z = 0 + 0$

$\boxed{\mu_Z = 0}$

$W = bX$

$E[W] = E[bX]$

or $\mu_W = b\,E[X]$

or $\mu_W = b\,\mu_X$

or $\boxed{\mu_W = 0}$

$\sigma_Z^2 = \sigma_X^2 + \sigma_X^2 + 2\rho\sigma_X\sigma_Y$ $\qquad$ $\sigma_W^2 = b^2\sigma_X^2$

or $\sigma_Z^2 = 4 + 9 + 2 \times -\frac{1}{2} \times 2 \times 3$ $\qquad$ or $\boxed{\sigma_W^2 = 4b^2}$

or $\sigma_Z^2 = 13 - 6$

or $\boxed{\sigma_Z^2 = 7}$

$Cov(Z, W) = Cov(X+Y, bX)$

or $Cov(Z,W) = E[\{X+Y - \cancel{\mu}E[X+Y]\}\{bX - E[bX]\}]$

or $Cov(Z,W) = E[\{X+Y\}\{bX\}]$

or $Cov(Z,W) = E[bX^2 + bXY]$

or $Cov(Z,W) = b\mu_{X^2} + b\underline{Cov(XY)}$

here, $Cov(X,Y) = E[XY]$ as $\mu_X\mu_Y = 0$.

or $Cov(Z,W) = b\{\sigma_X^2 + \mu_X^2\} + b\{\rho_{XY}\sigma_X\sigma_X\}$

or $Cov(Z,W) = b\left[4 + 0^2 + -\frac{1}{2} \cdot 2 \cdot 3\right]$

or $Cov(Z,W) = b[1]$

$\boxed{4(b)ii)}$

or $\boxed{Cov(Z,W) = b}$ $\Rightarrow$ $\boxed{Cov(Z,W) > 0 \ \forall \ b > 0}$ Ans

$$\rho_{Z,W} = \frac{Cov(Z,W)}{\sigma_Z \cdot \sigma_W}$$

or $$\rho_{Z,W} = \frac{b}{\sqrt{7} \cdot 2b}$$

or $$\boxed{\rho_{Z,W} = \frac{1}{2\sqrt{7}}}$$

4(b)(i)

$$\therefore \boxed{f_{Z,W}(z,w) \sim \mathcal{N}\left(z, w, \mu_Z = 0, \mu_W = 0, \sigma_Z^2 = 7, \sigma_W^2 = 4b^2, \rho_{ZW} = \frac{1}{2\sqrt{7}}\right)}$$ Ans

4(c) $$\hat{Y}_{Y|X=x}^{MMSE} = E[Y \mid X = x]$$

Let's try to find $f_{Y|X}(Y|X=x)$ first!

$$f_{Y|X}(Y|X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

or $$f_{Y|X}(Y|X=x) = \frac{\dfrac{1}{2\sqrt{2\pi} \cdot 3\sqrt{2\pi}\, \frac{\sqrt{3}}{2}} \cdot e^{-\frac{1}{2 \cdot \frac{3}{4}}\left\{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}}{\dfrac{1}{2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left\{\left(\frac{x}{2}\right)^2\right\}}}$$

or $\cancel{f_{Y|X}(Y|X=x) = \frac{1}{3\sqrt{2\pi}\cdot\frac{3}{4}}\, e^{-\frac{1}{2}\{}}$

or $f_{Y|X}(Y|X=x) = \dfrac{1}{\frac{3\sqrt{3}}{2}\cdot\sqrt{2\pi}}\, e^{-\frac{1}{2}\left\{\frac{1}{3}\left(\frac{x}{2}\right)^2 + \frac{4}{3}\left(\frac{y}{3}\right)^2 + \frac{4}{3}\left(\frac{x}{2}\right)\left(\frac{y}{3}\right)\right\}}$

or $f_{Y|X}(Y|X=x) = \dfrac{1}{\frac{3\sqrt{3}}{2}\cdot\sqrt{2\pi}}\, e^{-\frac{1}{2}\left\{\left(\frac{x}{2\sqrt{3}}\right)^2 + \left(\frac{y}{\frac{3\sqrt{3}}{2}}\right)^2 + 2\cdot 1\left(\frac{x}{2\sqrt{3}}\right)\left(\frac{y}{\frac{3\sqrt{3}}{2}}\right)\right\}}$

or $f_{Y|X}(Y|X=x) = \dfrac{1}{\left(\frac{3\sqrt{3}}{2}\right)\sqrt{2\pi}}\, e^{-\frac{1}{2}\left\{\left(\frac{y}{\frac{3\sqrt{3}}{2}} + \frac{x}{2\sqrt{3}}\right)^2\right\}}$

or $f_{Y|X}(Y|X=x) = \dfrac{1}{\left(\frac{3\sqrt{3}}{2}\right)\sqrt{2\pi}}\, e^{-\frac{1}{2}\left\{\left(\frac{y + \frac{3}{4}x}{\left(\frac{3\sqrt{3}}{2}\right)}\right)^2\right\}}$

or $\cancel{f_{Y|X}(Y|X=x) =}$

or $Y|X=x \sim \mathcal{N}\left(y,\ \mu_Y = -\frac{3}{4}x,\ \sigma_Y^2 = \frac{27}{4}\right)$

$\boxed{4(c)}$

$\Rightarrow \boxed{\hat{Y}_{\substack{Y|X=x \\ MMSE}} = \mu_{Y|X=x} = -\frac{3}{4}x}$ Ans

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5. $X \sim \text{unif}[0,1)$

$Y | X=x \sim \text{unif}[0,x]$

$Z | Y=y, X=x \sim \text{unif}[0,y)$

$$E[X^2Y^2Z^2] = \iiint x^2y^2z^2 \cdot f_{X,Y,Z}(x,y,z)\, dz\, dy\, dx$$

or $\cancel{E[X^2Y^2Z^2] = \iiint}$

But $f_{X,Y,Z}(x,y,z) = \cancel{f_{Z}} \cancel{f_{Z}}$

$$= f_Z(z | X=x, Y=y) \cdot f_{X,Y}(X=x, Y=y)$$

or $f_{X,Y,Z}(x,y,z) = f_Z(z|X=x,Y=y) \cdot f_{Y|X}(y|X=x) \cdot f_X(x)$

or $f_{X,Y,Z}(x,y,z) = \dfrac{1}{y} \cdot \dfrac{1}{x} \cdot \dfrac{1}{1} \quad \forall \; z \in [0,y), \; y \in [0,x], \; x \in [0,1)$

$$\therefore E[X^2Y^2Z^2] = \int_{x=0}^{x=1}\int_{y=0}^{y=x}\int_{z=0}^{z=y} x^2 \cdot y^2 \cdot z^2 \cdot \frac{1}{xy}\, dz\, dy\, dx$$

or $$E[X^2Y^2Z^2] = \int_{x=0}^{x=1}\int_{y=0}^{y=x}\int_{z=0}^{z=y} x y z^2\, dz\, dy\, dx$$

or $$E[X^2Y^2Z^2] = \int_{x=0}^{x=1} x \int_{y=0}^{y=x} y \cdot \left.\frac{z^3}{3}\right|_{z=0}^{z=y} dy\, dx$$

or $E[X^2Y^2Z^2] = \int_{x=0}^{x=1} x \int_{y=0}^{y=x} \frac{y^4}{3}\, dy\, dx$

or $E[X^2Y^2Z^2] = \int_{x=0}^{x=1} x \cdot \frac{y^5}{15} \Big|_{y=0}^{y=x} dx$

or $E[X^2Y^2Z^2] = \int_{x=0}^{x=1} \cancel{x \cdot x} \frac{x^6}{15}\, dx$

or $E[X^2Y^2Z^2] = \frac{x^7}{105} \Big|_{x=0}^{x=1}$

or $\boxed{E[X^2Y^2Z^2] = \frac{1}{105}}$ Ans

$\int\int\int xyz^2$

$xy\frac{y^3}{3}\Big|$

$x\frac{y^4}{3}$

$\cancel{\frac{xy^5}{3 \cdot 5}}$ $x\frac{y^4}{3}$

$x\frac{y^5}{3 \cdot 5}\Big|$

$x \cdot \frac{x^5}{3 \cdot 5}$ $\frac{x^6}{3 \cdot 5}\Big|$ $\frac{x^7}{3 \cdot 5 \cdot 7}\Big|_0^1$

6. Random Processes are mappings from outcomes of a probabilistic experiment to signals.

Ω: A, B, C

$A \rightarrow X(A,t) = \cos(t)$

$C \rightarrow X(C,t) = \begin{cases} 7 & \lfloor t \rfloor \text{ is even} \\ 0 & \lfloor t \rfloor \text{ is odd} \end{cases}$

$B \rightarrow X(B,t) = 2e^{-2t}$

where $\lfloor t \rfloor$ refers to the greatest integer lesser than or equal to t.

Virtually any physical process or simulated phenomenon can be represented as a random process.

Eg. Bus Voltages in a power grid are time-varying signals, and there are multiple buses (outcomes) in a power grid or we can measure a single bus voltage in multiple ways. (sample functions).

Ω: $i=1 \rightarrow V_1(t)$, $\vdots \rightarrow V_i(t)$, $i=14 \rightarrow V_{14}(t)$

$i \in 1$ to 14 $\quad\quad \Omega \equiv$ IEEE 14 Bus System.

$i \in N$

Study of Random Processes is important as it can be used to analyze, estimate, predict, forecast signals which can be very beneficial to us.

Eg. If tomorrow's load demand for a regional substation could be predicted accurately, we can optimize the scheduling of power generation, spot market rates, etc., which can save a grid operator millions of Dollars.

Ans

7. $X(t) = \frac{1}{t+1+C}$ $\quad t \in R^+$

$C \sim exp(1)$

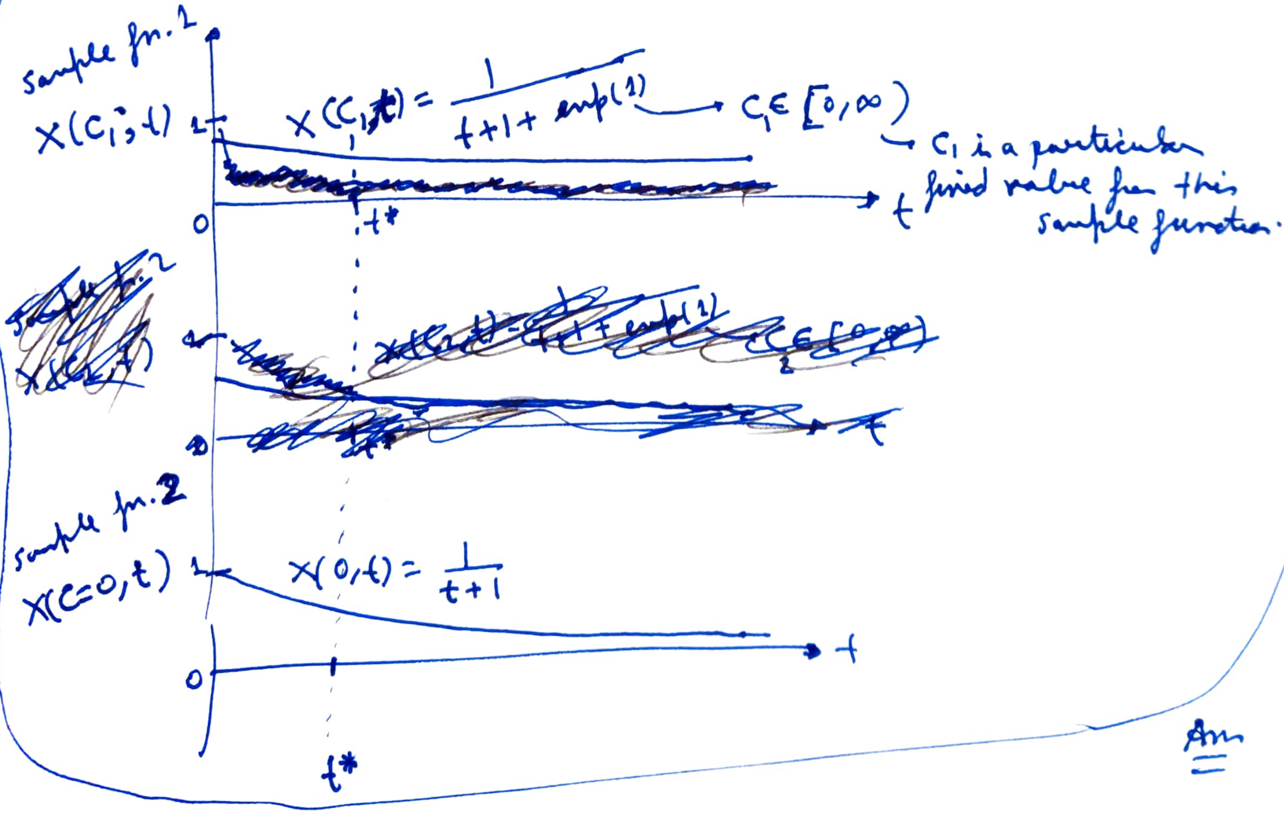
~~$f_C(c) =$~~ or $f_C(c) = e^{-c}$ $\quad c \ge 0$

7(a) $X(t)$ is a Random Process as

$X(t) = X(C,t)$ is a set of time varying signals which are all mapped to the outcome C of a ~~prob~~ probabilistic experiment, where

$C \sim exp(1)$.

So $X(C,t)$ can ~~be~~ have a sample function $X(C=2,t)$ or it can be sampled ~~at~~ time t^* to get a ~~unique~~ set of ~~$X(C,t^*)$~~ $X(C, t=t^*)$ $\forall$ $C \sim exp(1)$.



Ans

7.2

$X(t) = \frac{1}{t+1+C}$ $t \in \mathbb{R}^+$ $C \sim \exp(1)$

$C \in [0, \infty)$

$\Rightarrow$ ~~$X(t) \in \frac{1}{t+1}$ $\left[0, \frac{1}{t+1}\right)$~~

$\Rightarrow X(t) \in \left(0, \frac{1}{t+1}\right]$

Approach: Find CDF then pdf:

$$F_{X(t)}(x) = P(X(t) \leq x) = P\left(\frac{1}{t+1+C} \leq x\right)$$

$$\Rightarrow F_{X(t)}(x) = P\left(C+t+1 \geq \frac{1}{x}\right)$$

$$\Rightarrow F_{X(t)}(x) = P\left(C \geq \frac{1}{x} - (t+1)\right)$$

$$\Rightarrow F_{X(t)}(x) = 1 - P\left(C \leq \frac{1}{x} - (t+1)\right)$$

$$\Rightarrow F_{X(t)}(x) = \begin{cases} 0 & x \leq 0 \\ 1 - \left(1 - e^{-\left(\frac{1}{x} - (t+1)\right)}\right) & x \in \left(0, \frac{1}{t+1}\right] \\ 1 & x \geq \frac{1}{t+1} \end{cases}$$

$1 - e^{(t+1)} \cdot e^{-\frac{1}{x}}$

7(b)

$$\Rightarrow \boxed{f_{X(t)}(x) = \frac{d}{dx} F_{X(t)}(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{x^2} \cdot e^{-\frac{1}{x}} \cdot e^{(t+1)} & x \in \left(0, \frac{1}{t+1}\right] \\ 0 & x \geq \frac{1}{t+1} \end{cases}}$$ Ans

—X—X—X—X—