TODAY'S LECTURE (9/27)

1. Standard R.V.s
2. Functions of R.V.s

EXAM 1: week of Oct. 11th (week after next)

Example

$X \sim N(3, \sigma^2 = 4)$ $\quad \sigma = 2$

Find: $E(X) = 3$

$P(2 \le X \le 5) \Leftarrow$ in terms of the Standard Gaussian

$$P(2 \le X \le 5) = F_X(5) - F_X(2)$$

$F_X \to G\left(\frac{x-m}{\sigma}\right)$

$$= G\left(\frac{5-3}{2}\right) - G\left(\frac{2-3}{2}\right)$$

$$= \underbrace{G(1) - G(-0.5)}_{\text{look up on a table}}$$

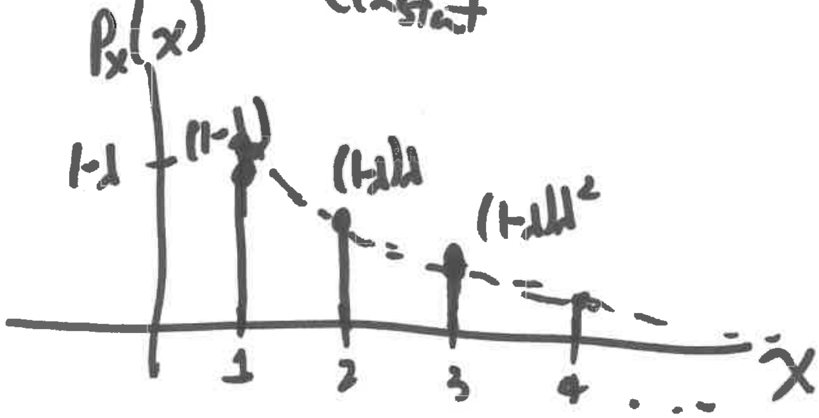
$G(-0.5) = 1 - G(0.5)$

Geometric R.V. X

PMF: $P_X(x) = \underline{(1-\lambda)} \cdot \lambda^{x-1}$, $x = 1, 2, 3, \ldots$

normalization constant

$\lambda \in (0,1)$



$E(X) = ?$ $\quad E(X) = \sum_x x\, P_X(x)$

$$E(X) = \sum_{x=1}^{\infty} x \cdot (1-\lambda) \cdot \lambda^{x-1} = (1-\lambda) \sum_{x=1}^{\infty} x \cdot \lambda^{x-1}$$

$\frac{1}{(1-\lambda)^2}$

$$\boxed{E(X) = \frac{1}{1-\lambda}}$$

Finding $A = \sum_{x=1}^{\infty} x \cdot \lambda^{x-1}$

difference

$$A = (1) \cdot \lambda^0 + (2) \cdot \lambda^1 + 3 \cdot \lambda^2 + 4 \cdot \lambda^3 + \ldots$$

$$\lambda A = \quad (1) \cdot \lambda^1 + 2 \cdot \lambda^2 + 3\lambda^3 + \ldots$$

$$(1-\lambda) A = \underbrace{\lambda^0 + \lambda^1 + \lambda^2 + \lambda^3 + \ldots}$$

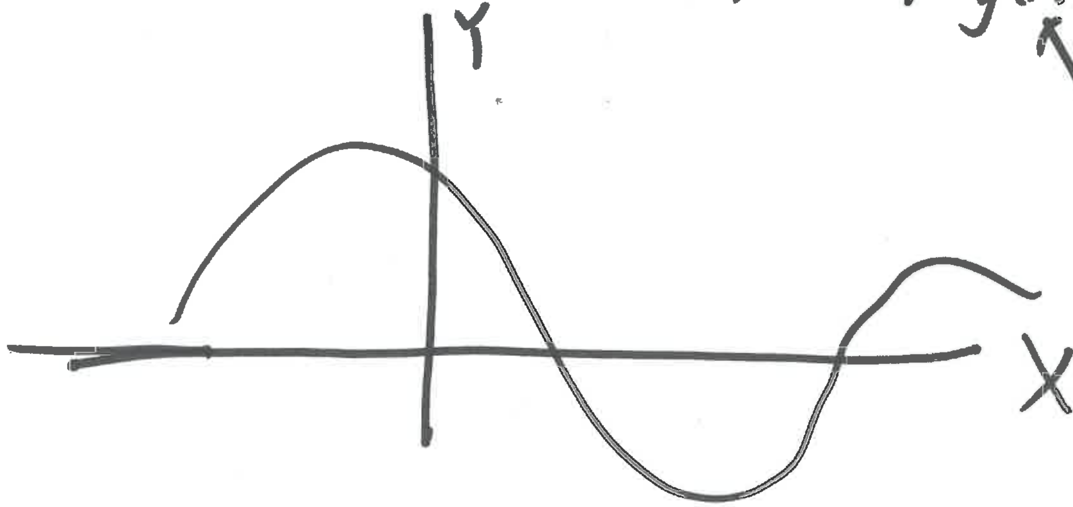
$$(1-\lambda) A = \frac{1}{1-\lambda} \Rightarrow A = \frac{1}{(1-\lambda)^2}$$

The next few lectures will be concerned with multiple random variables.

⇓

Want to understand what happen when you process random quantities, and understand interdependencies.

First (special) case: let's consider a function of a random variable, i.e. $Y = g(X)$ ← R.V.

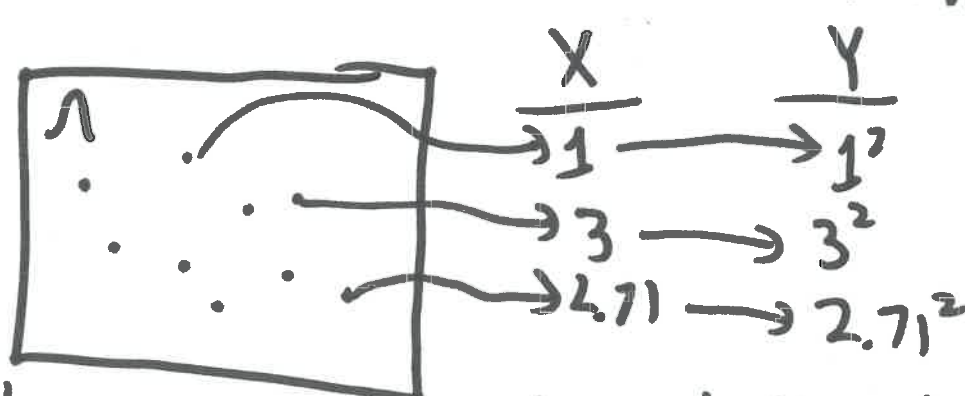


$Y = X^2$

$Y = \begin{cases} 1, & X > 0 \\ 0, & \text{otherwise} \end{cases}$

First insight: Y is also a random variable!

Can find CDFs, PDFs, mean, etc for Y.



$Y = X^2$

There's a mapping from outcomes to Y values! Y is also an R.V.

Since X decides $Y = g(X)$, should be able to infer CDF and PDF of Y from those of X.

$\Rightarrow$ General approach (works for continuous, discrete, or mixed R.V.s)

$\Downarrow$

1) Find the CDF of Y from information about X. $\Leftarrow$ from first principles
2) Find pdf by taking deriv.

Example: $X \sim \text{unif}[-1, 1)$

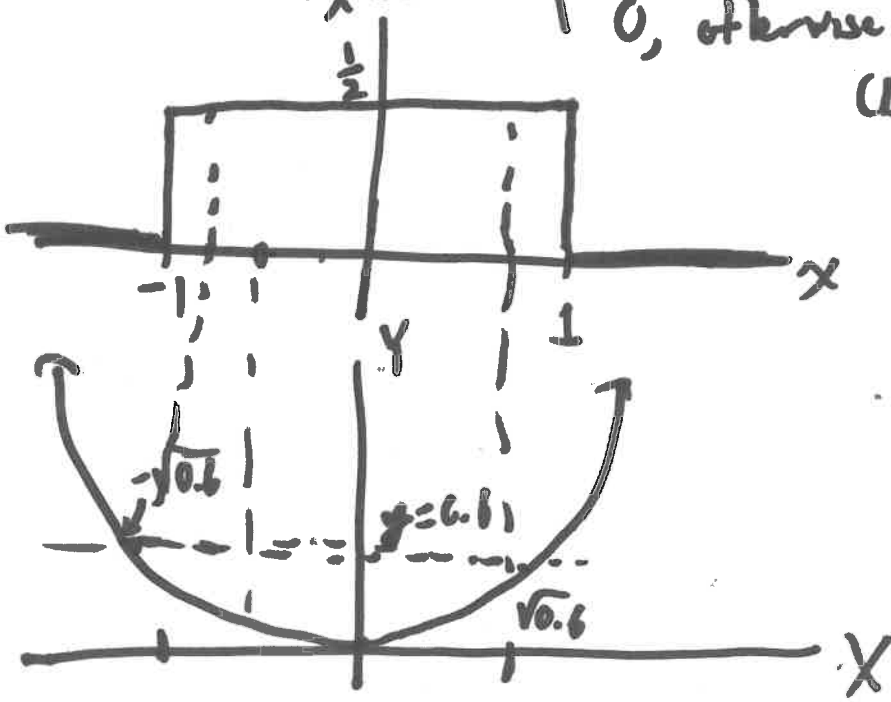
$Y = X^2$

$\Rightarrow$ Please find the pdf of Y

$Y = X^2$

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

expect Y in the range $(0,1)$



CDF: $F_Y(y) = P(Y \le y)$

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ ??, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 0, & y<0 \\ 0, & y>1 \\ e^{-\sqrt{y}} \cdot \frac{1}{2\sqrt{y}}, & 0 \le y < 1 \\ \frac{1}{e}\delta(y-1), & 1^- < y < 1^+ \end{cases}$$

$$f_Y(y) = e^{-\sqrt{y}} \cdot \frac{1}{2\sqrt{y}} + \frac{1}{e}\delta(y-1), \quad 0 \le y \le 1^+$$

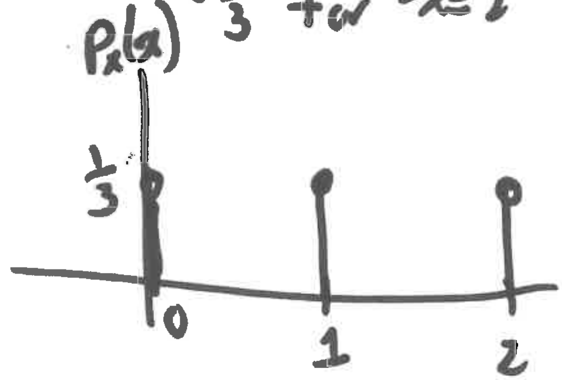
$$\boxed{f_Y(y) = e^{-\sqrt{y}} \cdot \frac{1}{2\sqrt{y}} \cdot \mathbb{1}(y) \cdot \mathbb{1}(1-y) + \frac{1}{e}\delta(y-1)}$$

$\Rightarrow$ Discrete-valued case

$\Downarrow$ often easier to work with probabilities directly

$$X \quad P_X(x) = \begin{cases} \frac{1}{3}, & \text{for } x=0 \\ \frac{1}{3} & \text{for } x=1 \\ \frac{1}{3} & \text{for } x=2 \end{cases} \qquad Y = (X-1)^2$$

Find $P_Y(y)$



X	Y
0	1
1	0
2	1

$Y = 0$ or 1

$P(Y=0) = P(X=1) = \frac{1}{3}$

$P(Y=1) = P(X=0) + P(X=2) = \frac{2}{3}$

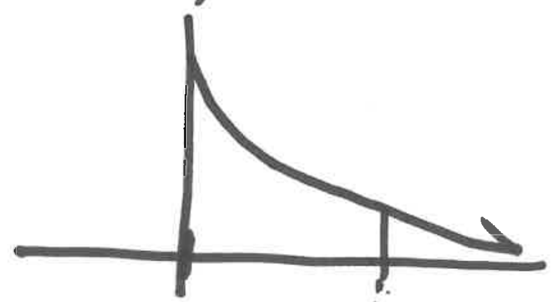
$$\boxed{P_Y(y) = \begin{cases} \frac{1}{3}, & y=0 \\ \frac{2}{3}, & y=1 \end{cases}}$$

Example 2

$X \sim \exp(1)$, $Y = \begin{cases} X^2 & \text{for } X \le 1 \\ 1, & \text{for } X > 1 \end{cases}$

Find PDF of Y

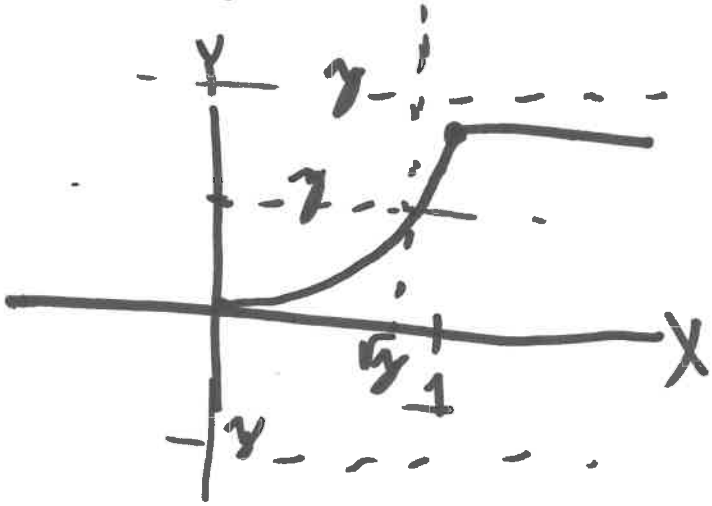
$f_X(x) = e^{-x}$, $x \ge 0$



Notice the Y should fall in $(0,1)$.

$\Rightarrow$ CDF of Y

$F_Y(y) = P(Y \le y) = \begin{cases} 0, & y < 0 \\ 1, & y \ge 1 \\ ??, & 0 \le y < 1 \end{cases}$



For $0 \le y < 1$

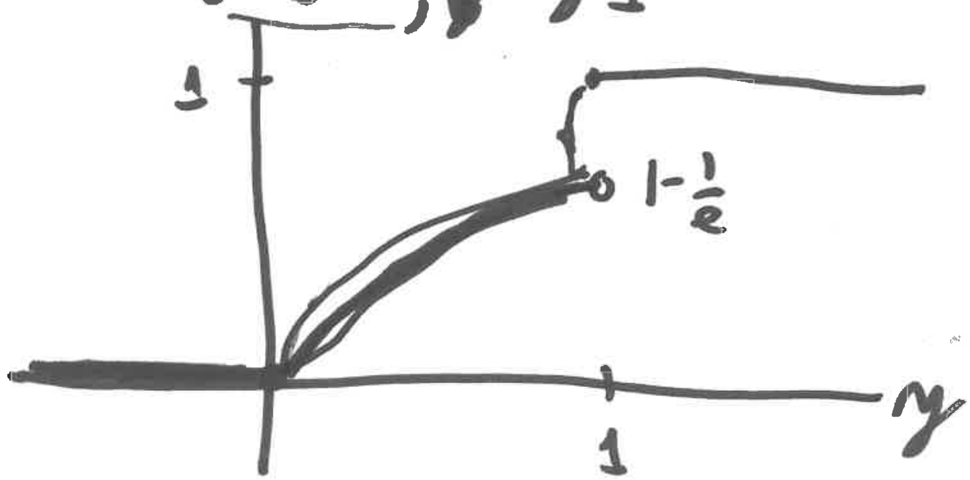
$P(Y \le y) = P(X^2 \le y)$

$= P(0 \le X \le \sqrt{y})$

$= \int_0^{\sqrt{y}} e^{-x}\, dx$

$= \left. \dfrac{e^{-x}}{-1} \right]_0^{\sqrt{y}} = 1 - e^{-\sqrt{y}}$

$F_Y(y) = \begin{cases} 0, & y < 0 \\ 1, & y \ge 1 \\ 1 - e^{-\sqrt{y}}, & 0 \le y < 1 \end{cases}$



Finding $F_Y(y) = P(Y \le y)$ for $0 \le y < 1$.

Example: $F_Y(0.6) = P(Y \le 0.6) = P(X^2 \le 0.6)$

$$= P(-\sqrt{0.6} \le X \le \sqrt{0.6})$$

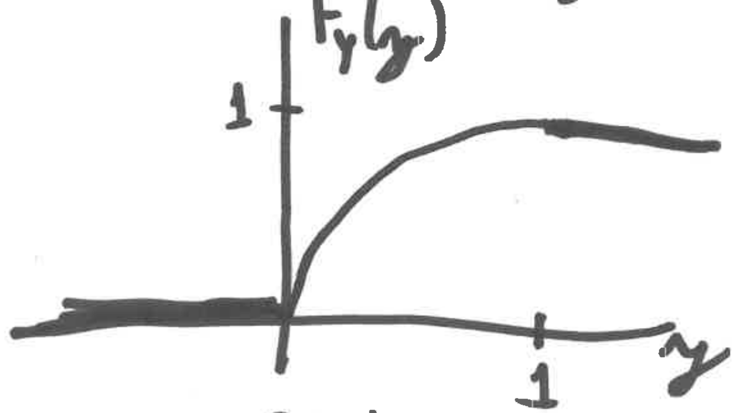
$$= \int_{-\sqrt{0.6}}^{\sqrt{0.6}} f_X(x)\,dx = \int_{-\sqrt{0.6}}^{\sqrt{0.6}} \left(\frac{1}{2}\right) dx$$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

For $0 \le y \le 1$, $F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx = \int_{-\sqrt{y}}^{\sqrt{y}} \left(\frac{1}{2}\right) dx$

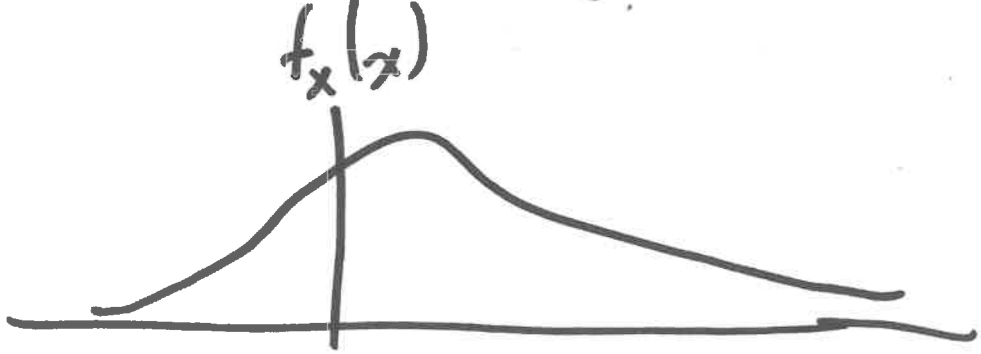
$$F_Y(y) = \frac{1}{2}x \Big]_{-\sqrt{y}}^{\sqrt{y}} = \frac{1}{2}\sqrt{y} - \left(\frac{1}{2}(-\sqrt{y})\right) = \sqrt{y}$$

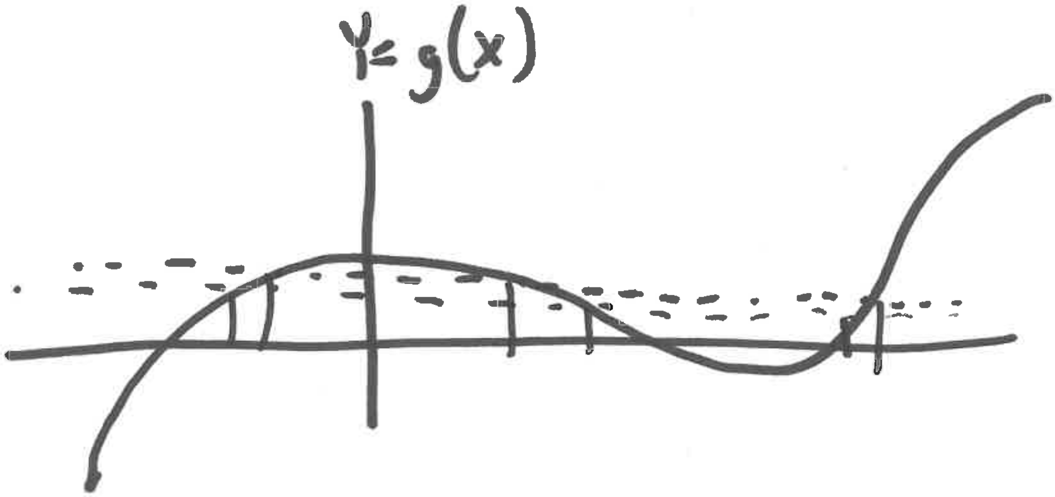
$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \sqrt{y}, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$



$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Is there a cute formula for the pdf of Y, which avoids finding the CDF?





$f_Y(y) = ?$

TODAY'S LECTURE

Functions of R.V.s

9/29

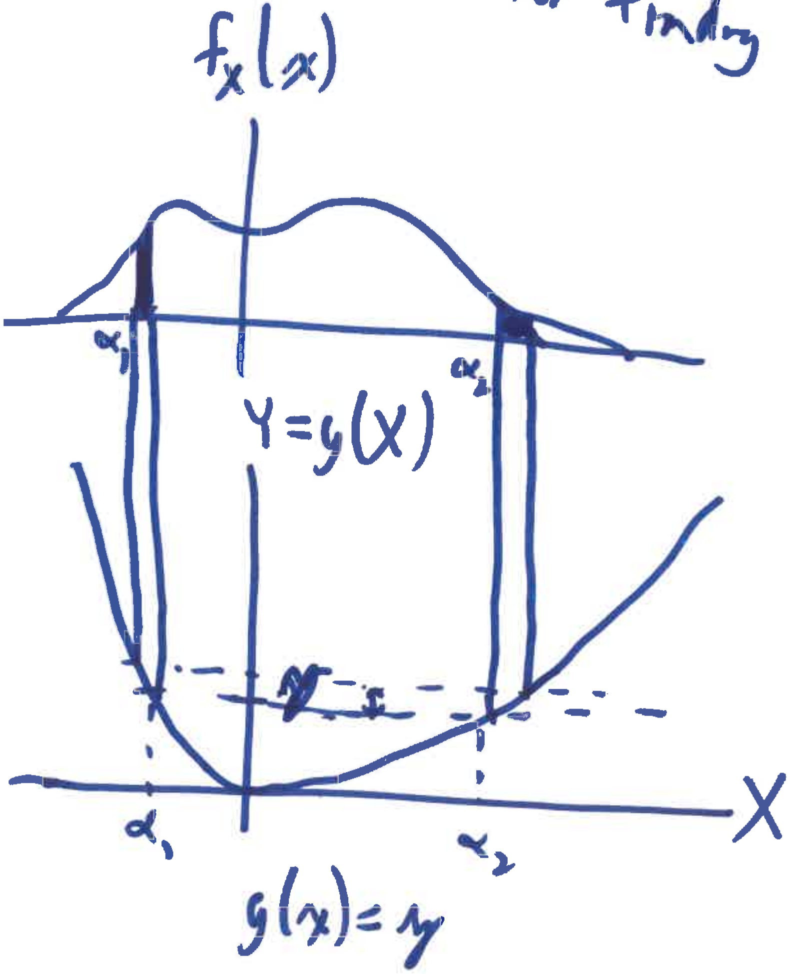
Midterm - week after next

Thursday, Oct. 13 - 6:00 PM - 9:00 PM

Finding PDFs of $Y = g(X)$ ← R.V.

General Approach: find CDF of Y, take deriv.

Cute formula for finding PDF of Y (works in only some cases)



$$f_Y(y) = \frac{P(y \le Y \le y + \delta)}{\delta}$$

1) Find solutions to $g(x) = y \Rightarrow \alpha_1, \alpha_2, \ldots, \alpha_n$

$$f_Y(y) = \frac{f_X(\alpha_1) \cdot (\text{width\_1}) + f_X(\alpha_2) \cdot (\text{width\_2}) + \ldots}{\delta}$$

$$f_Y(y) = \frac{f_X(\alpha_1)\cdot(\text{width\_1}) + f_X(\alpha_2)\cdot(\text{width\_2}) + \cdots}{\delta}$$

$\Rightarrow$ Note that: $\left.\frac{dg}{dx}\right|_{x=\alpha_1} = \frac{\delta}{\text{width\_1}} \Rightarrow \text{width\_1} = \frac{\delta}{\left|\left.\frac{dg}{dx}\right|_{x=\alpha_1}\right|}$

$$\cancel{f_Y(y) = f_X(\alpha_1)} \; \cancel{\frac{\delta}{\left|\left.\frac{dg}{dx}\right|_{x=\alpha_1}\right|}}$$

$$f_Y(y) = \frac{f_X(\alpha_1)\cdot\dfrac{\delta}{\left|\frac{dg(\alpha_1)}{dx}\right|} + f_X(\alpha_2)\cdot\dfrac{\delta}{\left|\frac{dg(\alpha_2)}{dx}\right|} + \cdots}{\delta}$$

$$\boxed{f_Y(y) = \sum_{i=1}^{n} \frac{f_X(\alpha_i)}{\left|\frac{dg(\alpha_i)}{dx}\right|}}$$

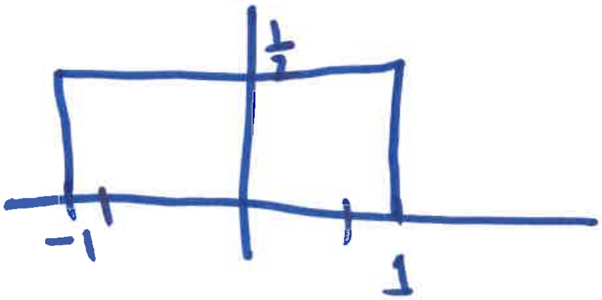
__Steps:__

1. Find solutions to $g(x) = y$
   $\Rightarrow$ call these $\alpha_1, \ldots, \alpha_n$
2. Plug in!

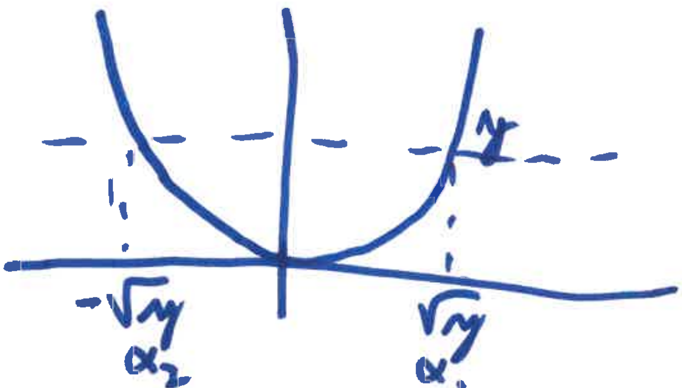
EXAMPLE

$X \sim \text{unif}[-1,1)$

$f_X(x)$



$Y = X^2$



$y = g(x) = x^2$

$\frac{dg}{dx} = 2x$

$\frac{dg(x_1)}{dx} = 2\sqrt{y}$

$\frac{dg(x_2)}{dx} = 2(-\sqrt{y})$

$f_Y(y) = ?$

Step 0) Notice that Y is in the range $(0,1]$

For $0 \le y \le 1$, let's find $f_Y(y)$ using the formula

Step 1: solve

$x^2 = y \Rightarrow x = \pm\sqrt{y}$

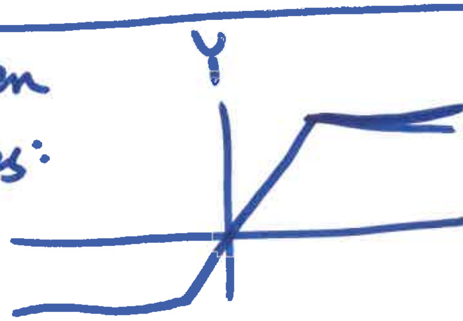
Step 2: plug in

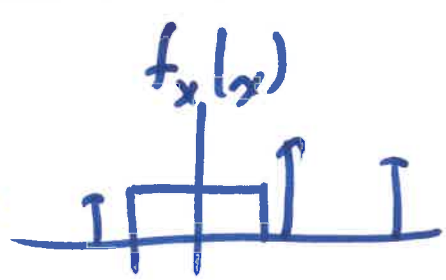
$$f_Y(y) = \frac{1/2}{|2\sqrt{y}|} + \frac{1/2}{|2(-\sqrt{y})|}$$

$$\boxed{f_Y(y) = \frac{1}{2\sqrt{y}} \quad 0 \le y \le 1}$$

(0 otherwise)

Problem Cases:





Let's find expectations of functions of R.V.s.

$Y = g(X)$

$E[Y] \Leftarrow$ this is well-defined, since Y is a R.V. and has an average.

$E[Y] = \int y \underbrace{f_Y(y)}\, dy$

⇒ We could therefore find $E(Y)$ by finding the PDF of Y, and then plugging in.

⇓

lot of work, want to avoid this.

⇒ Luckily, there's a direct approach!

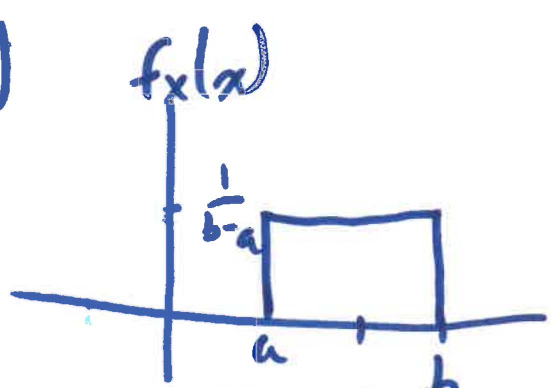
⇒ We can find

$E[g(X)]$ as $E(g(X)) = \underbrace{\int_{-\infty}^{\infty} g(x) f_X(x)\, dx}_{\text{can argue this by thinking about lots of trials..}}$

Example

$X \sim \text{unif}(a,b)$

$E(X) = \frac{a+b}{2}$



$E(X^2)$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$$

$$E(X^2) = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x^2\,dx$$

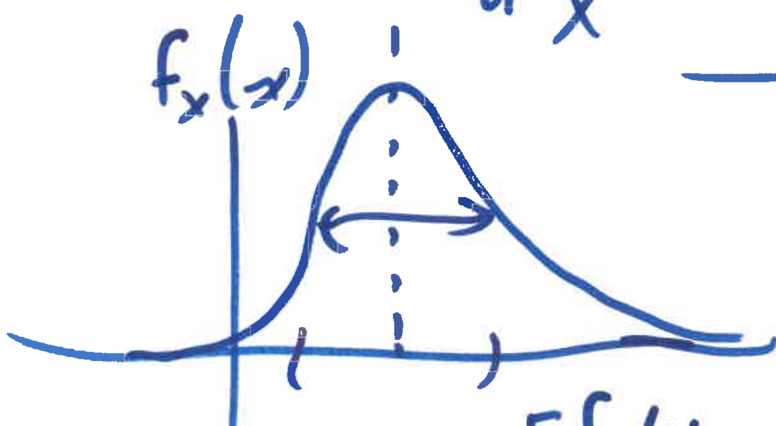
$$E(X^2) = \frac{1}{b-a}\cdot\frac{x^3}{3}\Big]_a^b = \frac{1}{b-a}\cdot\frac{1}{3}\cdot(b^3-a^3)$$

$$E(X^2) = \frac{1}{3}\cdot\frac{1}{\cancel{b-a}}\cancel{(b-a)}(b^2+ab+a^2)$$

$$\boxed{E(X^2) = \frac{1}{3}(b^2+ab+a^2)}$$

$\Rightarrow$ The example brings to mind that expectations of certain functions of random variables are important.

$E(X^2)$, $\underbrace{E(X^n)}_{\text{moments of } X}$ $\Leftarrow$ because arise in the physical world, Taylor approx, outliers,



Also care about the spread of an R.V. from the mean

$E[(X-E(X))^2] \Leftrightarrow$ variance

st. dev. $= \sqrt{E((X-E(X))^2}$

$E[(X-E(X))^n]$ $\Rightarrow$ central moments

The variance is a metric (statistic) that comes up a lot, so let's think further about it.

$$\boxed{var(X) = E[(X - E(X))^2]}$$

$$= E[X^2 - 2X\,E(X) + (E(X))^2]$$

Expectation is a linear operator!

$$var(X) = E(X^2) - E[2X\,E(X)] + E[(E(X))^2]$$

$$E(X^2) - E[2X\,E(X)] + E[(E(X))^2]$$

E(number) = number

$$var(X) = E(X^2) - 2E(X)\,E(X) + (E(X))^2$$

$$\boxed{var(X) = E(X^2) - (E(X))^2}$$

$$X \sim unif[a,b] \Rightarrow var(X) = ?$$

$$var(X) = E(X^2) - (E(X))^2$$ ⇐ use this formula, since we've found the terms.

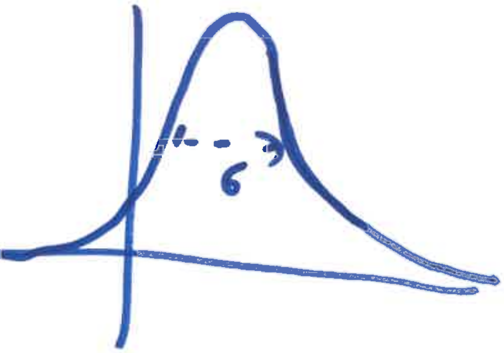
$$var(X) = \frac{1}{3}(b^2 + ab + a^2) - \left(\frac{a+b}{2}\right)^2$$

$$\boxed{var(X) = \frac{(b-a)^2}{12}}$$

Variance of a Gaussian R.V.

$X \sim N(m, \sigma^2)$

$Var(X) \overset{??}{=} \sigma^2$



$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$

$E(X) = m$

$$E\left((X-m)^2\right) = \int_{-\infty}^{\infty} (x-m)^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx$$

$$= \underbrace{\int_{-\infty}^{\infty} y^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \, dy}_{\text{can do this by parts, and probably should}}$$

$\Rightarrow$ But there's a weird trick

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \, dy = 1$$

$$\frac{d}{d\sigma}\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} \, dy\right) = \frac{d}{d\sigma}\left(\sigma\sqrt{2\pi}\right)$$

$$\int_{-\infty}^{\infty} \frac{d}{d\sigma}\left(e^{-\frac{y^2}{2\sigma^2}}\right) dy = \sqrt{2\pi}$$

10/4

TODAY'S LECTURE

1. More on statistics
2. Two random variables

Exam 1: next Thursday (10/13) in the evening (6PM-9PM), cancel class on that date.

HW Help on Friday (~~4~~ 9:30-11 AM)
Review session (Monday of next week, 10/10): 9:30 AM PT

} zoom

HW3: due 10/11.

Focus of the last class:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$\Downarrow$

Defined several statistics - moments, variance

$$\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = ?$$

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = 1 \Rightarrow \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = \sigma\sqrt{2\pi}$$

$$\frac{d}{d\sigma}\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\,dx\right) = \frac{d}{d\sigma}\left(\sigma\sqrt{2\pi}\right)$$

$$\int_{-\infty}^{\infty} \frac{d}{d\sigma}\left(e^{-\frac{x^2}{2\sigma^2}}\right) dx = \sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} \cdot \left(\frac{\cancel{2}x^2\sigma^{-3}}{\cancel{2}}\right) dx = \sqrt{2\pi}$$

$$\sigma^{-3}\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}}\,dx = \sqrt{2\pi}$$

gaussian R.V.

$\Downarrow$

$\mathrm{Var}(X) = \sigma^2$

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}}\,dx = \sigma^3\sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\,dx = \sigma^{3}\cancel{\sqrt{2\pi}} \cdot \frac{1}{\cancel{\sigma}\cancel{\sqrt{2\pi}}}$$

$$\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\,dx = \sigma^2$$

One more (weird) statistic

X is some R.V.

$$M_X(s) = E\left(e^{sX}\right) = \int_{-\infty}^{\infty} e^{sx} \cdot f_X(x)\,dx$$

Looks like Laplace transform!

$\Uparrow$

moment generating function

parameter (complex number)

Why do we care about the moment-generating function?

It's an equivalent representation for the pdf, tells us anything we want to know about the pdf.

$\Rightarrow$ Specifically, it makes computing moments of a random variable easier!

$$E(X^n) = \int_{-\infty}^{\infty} x^n \cdot f_X(x)\,dx$$

$$M_X(s) = E(e^{sX}) = E\left(1 + sX + \frac{(sX)^2}{2!} + \frac{(sX)^3}{3!} + \dots\right)$$

$$M_X(s) = 1 + s\,E(X) + \frac{s^2}{2!}E(X^2) + \frac{s^3}{3!}E(X^3) + \dots$$

accurate for s near zero...

To find $\boxed{E(X) = \frac{d}{ds} M_X(s)\Big|_{s=0}}$

$$\left(0 + E(X) + \cancel{s\,E(X^2)} + \cancel{\frac{s^2}{2!}E(X^3)} + \dots\right)\Big|_{s=0}$$

$= E(X)$ ✓

$$\boxed{\begin{aligned} E(X^2) &= \frac{d^2 M_X(s)}{ds^2}\Big|_{s=0} \\ E(X^n) &= \frac{d^n M_X(s)}{ds^n}\Big|_{s=0} \end{aligned}}$$

General strategy: find M.G.F., compute moments from there.

Let's use the M.G.F. to find the mean and variance for an exponential R.V.

$$X \sim \exp(\lambda) : f_X(x) = \lambda e^{-\lambda x}, \; x \geq 0$$

$$M_X(s) = E(e^{sX}) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$$

$$M_X(s) = \int_{0}^{\infty} e^{sx} \lambda e^{-\lambda x}\,dx$$

$$M_X(s) = \lambda \int_{0}^{\infty} e^{(s-\lambda)x}\,dx$$

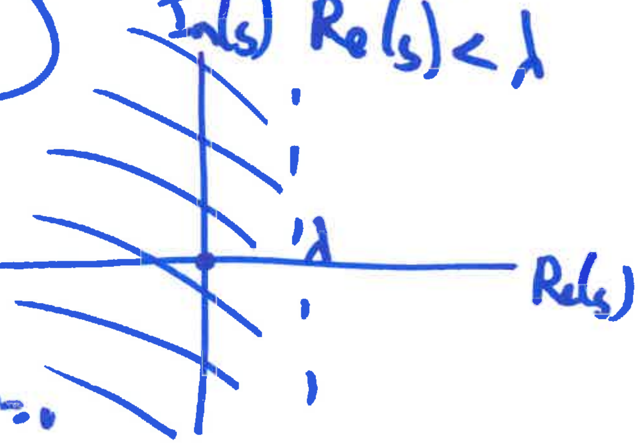
$$M_X(s) = \lambda \left. \frac{e^{(s-\lambda)x}}{(s-\lambda)} \right]_0^{\infty}$$

$$M_X(s) = \lambda\left(0 - \frac{1}{(s-\lambda)}\right)$$

$$M_X(s) = \frac{\lambda}{\lambda - s}$$

$Re(s) - \lambda < 0$

$Re(s) < \lambda$



$$E(X) = \frac{d}{ds} M_X(s)\Big|_{s=0} = \frac{d}{ds}\left(\frac{\lambda}{\lambda - s}\right)\Big|_{s=0}$$

$$= \frac{d}{ds}\left(\lambda(\lambda-s)^{-1}\right)\Big|_{s=0}$$

$$E(X) = +\lambda(\lambda-s)^{-2} \cdot (+1)\Big|_{s=0}$$

$$E(X) = \frac{\lambda}{(\lambda-s)^2}\Big|_{s=0} = \frac{1}{\lambda}$$

$$E(X^2) = \frac{d^2}{ds^2} M_X(s)\Big|_{s=0} = \frac{d}{ds}\left(\frac{\lambda}{(\lambda-s)^2}\right)\Big|_{s=0} = \frac{d}{ds}\left(\lambda(\lambda-s)^{-2}\right)\Big|_{s=0}$$

$$E(X^2) = \lambda\left(-2(\lambda-s)^{-3}\right)(-1)\Big|_{s=0} = \frac{2\lambda}{(\lambda-s)^3}\Big|_{s=0} = \frac{2}{\lambda^2}$$

$$\boxed{E(X^2) = \frac{2}{\lambda^2}}$$

$$\mathrm{var}(X) = E(X^2) - (E(X))^2 = \left(\frac{2}{\lambda^2}\right) - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$\boxed{\mathrm{var}(X) = \frac{1}{\lambda^2}}$$

M.G.F. also helps with the Gaussian case.

$\Rightarrow$ Also can think about the moment-generating function for $s = jw$.

For us, $M_X(jw) = E\left(e^{jwX}\right)$ is the equivalent of a Fourier transform.

Characteristic Function

$\Rightarrow$ The Fourier-transform form turns out to be ~~easier~~ helpful for finding pdfs of some functions of RVs.

$\Downarrow$ will skip the details...

How about statistics for discrete-valued R.V.s?

X is a discrete R.V.

$$E[g(X)] = \sum_{\substack{x \text{ s.t.} \\ P_X(x)>0}} g(x) P_X(x)$$

Example: $X = \begin{cases} 1, & \text{w.p. } 1/4 \\ 0, & \text{w.p. } 1/2 \\ -1, & \text{w.p. } 1/4 \end{cases}$ $\qquad E(X) = 0$

What is $E(X^2)$?

$$E(X^2) = (1)^2 \cdot \left(\tfrac{1}{4}\right) + (0)^2 \cdot \tfrac{1}{2} + (-1)^2 \cdot \tfrac{1}{4}$$

$$E(X^2) = \tfrac{1}{2}$$

$\Rightarrow$ Moments, variance: defined in the same way

$$E(X^n), \quad E\left((X - E(X))^2\right)$$

$\Rightarrow$ Moment generating function:

$$E(z^X) = M_X(z) \Leftarrow \text{find moments}$$

(complex argument)

How do we analyze pairs of random variables?

⇒ Overall idea: for any uncertain experiment, may want to consider multiple random quantities

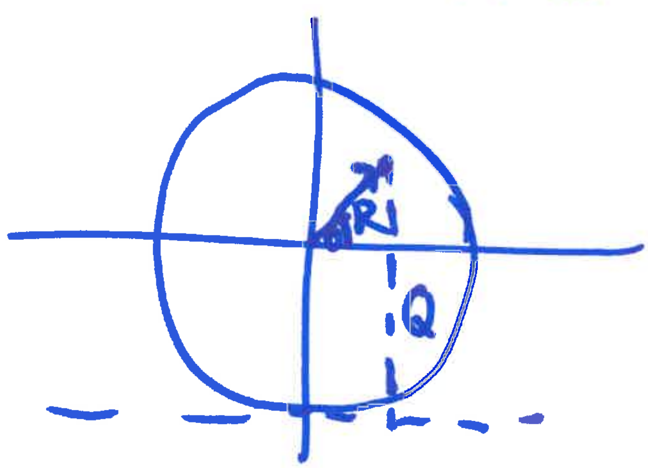
Examples

Throw two dice

X is the sum of the numbers showing

Y is the product.

⇒ Throw a dart at a dartboard



R: distance from the center

Q: distance from bottom

θ: angle from horizontal

TODAY'S LECTURE

1. Brief Aside
2. Pairs of RVs

10/6

Conditional Expectation: what does this mean?

$$\underline{\underline{E[g(X)|A]}} = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\,dx$$

Law of total expectations

Know $E[g(X)|A)$ and $E[g(X)|\bar{A})$

$$\checkmark \quad E[g(X)] \stackrel{?}{=} E[g(X)|A]P(A) + E[g(X)|\bar{A}]P(\bar{A})$$

Proof:

$$\checkmark \quad E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

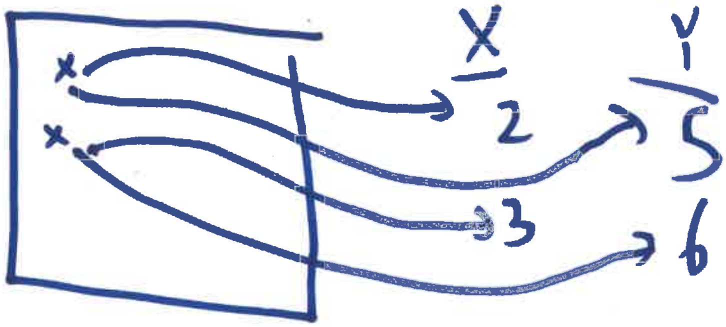
$$= \int_{-\infty}^{\infty} g(x)\left(f_{X|A}(x)P(A) + f_{X|\bar{A}}(x)P(\bar{A})\right)dx$$

$$= P(A)\int_{-\infty}^{\infty} g(x) f_{X|A}(x)\,dx + P(\bar{A})\int_{-\infty}^{\infty} g(x) f_{X|\bar{A}}(x)\,dx$$

$$= P(A)\,E[g(X)|A] + P(\bar{A})\,E[g(X)|\bar{A}]$$

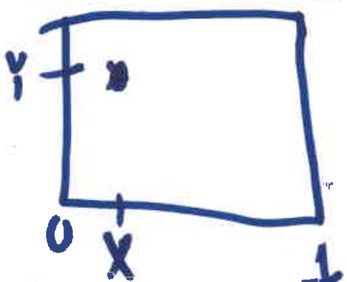
Pairs of Random Variables

⇒ Need to study co-dependence of different random quantities



⇒ Individual pdfs of the R.V.s are not enough!

Experiment 1



Choose a point at random in the unit square (uniformly)

X = horizontal coordinate

Y = vertical coordinate

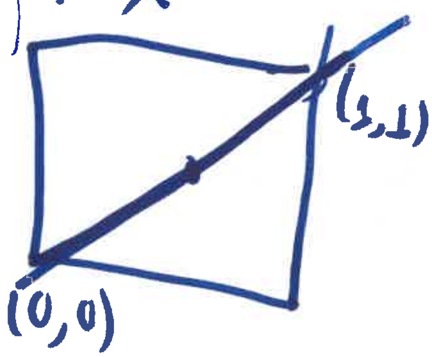
$X \sim unif(0,1)$

$Y \sim unif(0,1)$

Experiment 2

$X \sim unif(0,1)$

$Y = X$



$X \sim unif(0,1)$

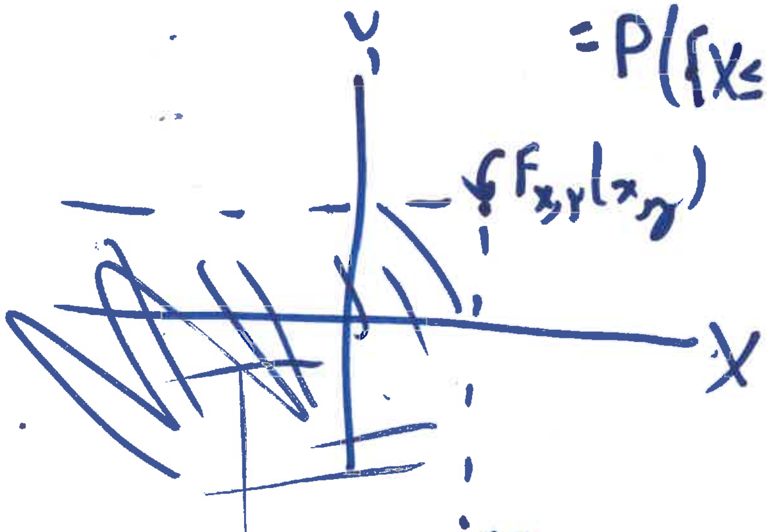
$Y \sim unif(0,1)$

Individual pdfs don't give insight into joint behavior!

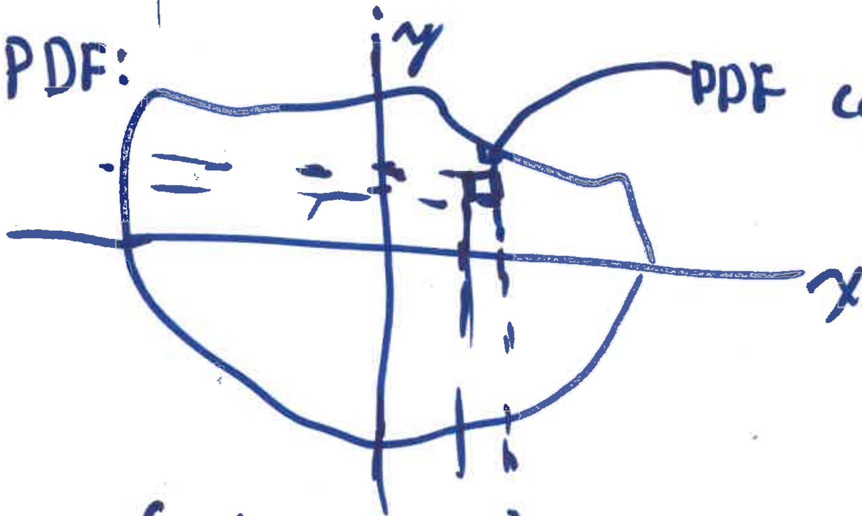
Let's define the notion of a joint CDF/PDF

CDF: $F_{X,Y}(x,y) = P(X \le x, Y \le y)$

$= P(\{X \le x\} \cap \{Y \le y\})$



PDF:



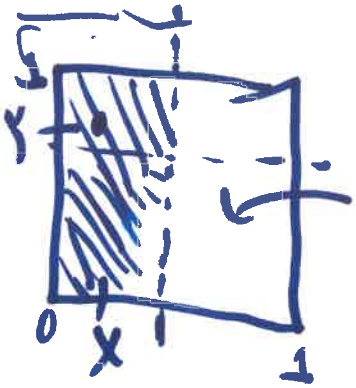
PDF captures probabilities of lying in differential elements (small squares), scaled by area

$$f_{X,Y}(x,y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x} F_{X,Y}(x,y)$$

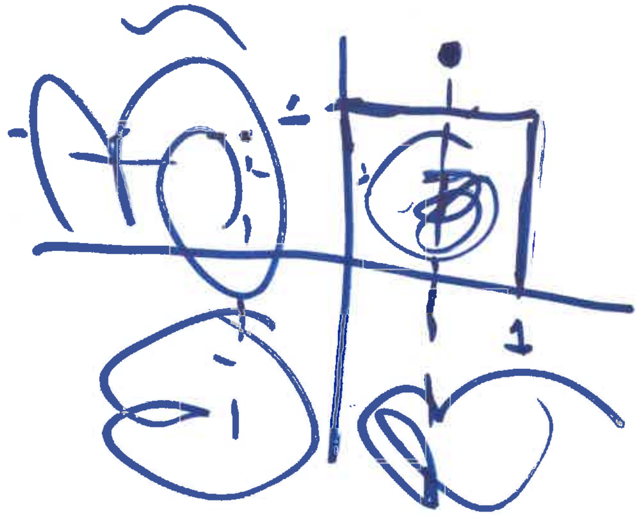
$$\boxed{f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\, \partial y} F_{X,Y}(x,y)}$$

$$\lim_{\delta \to 0} \frac{P(x \le X \le x+\delta,\ y \le Y \le y+\delta)}{\delta^2} \leftarrow P(\cancel{x \le X \le x+\delta})$$

✓



every point is equally likely



Can we find the CDF?

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

For $x < 0$, $F_{X,Y}(x,y) = 0$

For $y < 0$, $F_{X,Y}(x,y) = 0$

$F_{X,Y}(0.4, 0.7)$

For $0 \le x \le 1$, $0 \le y \le 1$, $F_{X,Y}(x,y) = \dfrac{\text{area of region (left/below)}}{\text{total area}}$

$$F_{X,Y}(x,y) = xy$$

$0 \le x \le 1$, $y > 1$, $F_{X,Y}(x,y) = \dfrac{\text{area of region}}{\text{total area}} = \dfrac{x \cdot 1}{1} = x$

$$F_{X,Y}(x,y) = x$$

$0 \le y \le 1$, $x > 1$, $F_{X,Y}(x,y) = y$

$y > 1$, $x > 1$, $F_{X,Y}(x,y) = 1$

$$\boxed{F_{X,Y}(x,y) = \begin{cases} xy, & 0 \le x \le 1,\ 0 \le y \le 1 \\ x, & 0 \le x \le 1,\ y > 1 \\ y, & 0 \le y \le 1,\ x > 1 \\ 1, & y > 1,\ x > 1 \\ 0, & \text{otherwise} \end{cases}}$$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial y\, \partial x} F_{X,Y}(x,y)$$

For $0 \le x < 1$, $0 \le y \le 1$, $\frac{\partial^2}{\partial y \cdot \partial x} \underbrace{\left(F_{X,Y}(x,y) \right)}_{xy} = \frac{\partial}{\partial y} \frac{\partial}{\partial x} (xy)$

$$= \frac{\partial}{\partial y} (y \cdot 1) = 1$$

$$\boxed{\begin{array}{l} f_{X,Y}(x,y) = 1, \quad 0 \le x \le 1, \ 0 \le y \le 1 \\ f_{X,Y}(x,y) = 0, \text{ otherwise} \end{array}}$$

$\Leftarrow$ We could have deduced this automatically

Properties of the CDF

$$F_{X,Y}(\infty, \infty) = 1$$

$$F_{X,Y}(\infty, y) = P(X \le \infty, Y \le y) = P(Y \le y) = F_Y(y)$$

$$F_{X,Y}(\infty, y) = F_Y(y)$$

$$F_{X,Y}(x, \infty) = F_X(x)$$

$$F_{X,Y}(-\infty, y) = 0 \qquad F_{X,Y}(x, -\infty) = 0$$

right continuous, top continuous

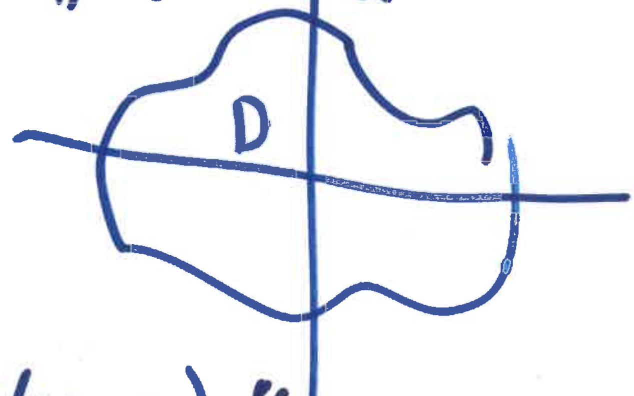
Properties of the pdf

Finding the CDF from the pdf:

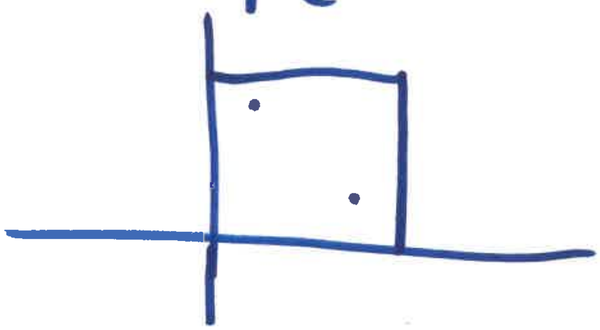
$$F_{X,Y}(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(x,y)\,dy\,dx$$

$$\boxed{F_{X,Y}(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(\alpha,\beta)\,d\beta\,d\alpha}$$

$$P(\{X,Y\}\in D)=\iint_D f_{X,Y}(\alpha,\beta)\,d\beta\,d\alpha$$



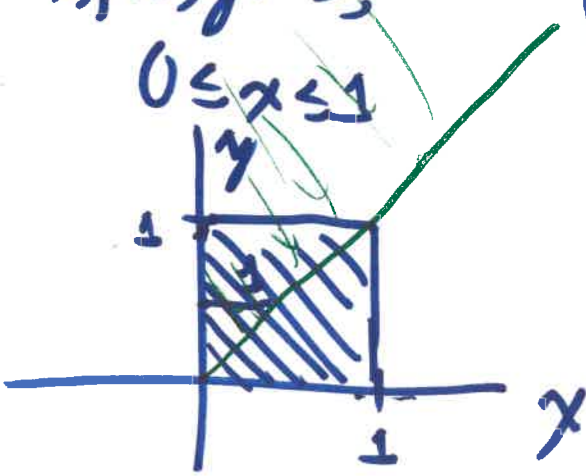
Example



$$P(X\le Y)=\iint_{\text{green region}} f_{X,Y}(x,y)\,dy\,dx$$

$f_{X,Y}(x,y)=1,$

$0\le x\le 1$



$$\boxed{P(X\le Y)=\int_0^1\int_0^y 1\,dx\,dy}$$

or $\int_0^1\int_x^1 1\,dy\,dx$

$$\int_0^1 x\Big]_0^y dy=\int_0^1 y\,dy=\frac{y^2}{2}\Big]_0^1=\frac{1}{2}$$

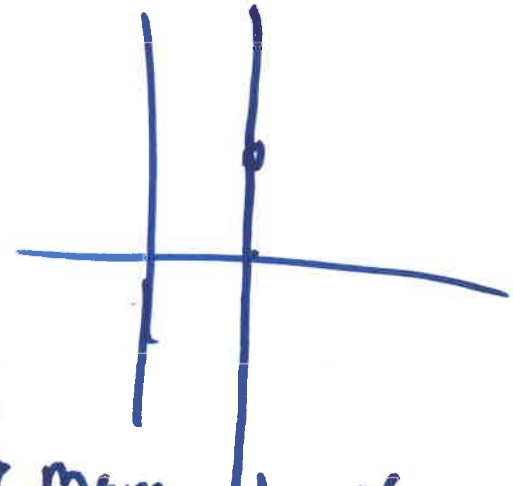
How can I find the pdf of X from the joint pdf?

$$f_{X,Y}(x,y)$$

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy}$$

$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}$$

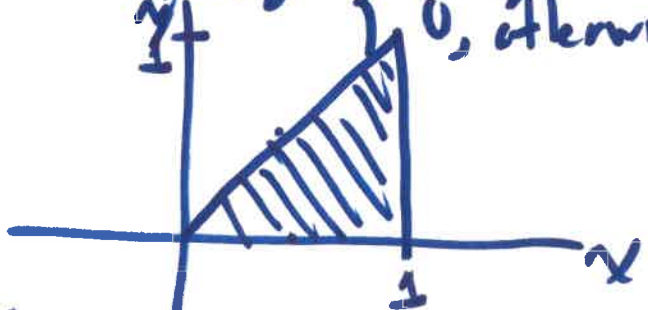
} marginal pdfs



Example

8 xy

$$f_{X,Y}(x,y) = \begin{cases} Cxy & \text{over the region below:} \\ 0, & \text{otherwise} \end{cases}$$



1) Find C

2) Find $f_X(x)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = 1$$

$$\Rightarrow \int_0^1 \int_0^x Cxy\,dy\,dx = 1$$

$$\Rightarrow C\int_0^1 x \int_0^x y\,dy\,dx = 1$$

$$\Rightarrow C\int_0^1 x \cdot \frac{y^2}{2}\Big]_0^x dx = 1$$

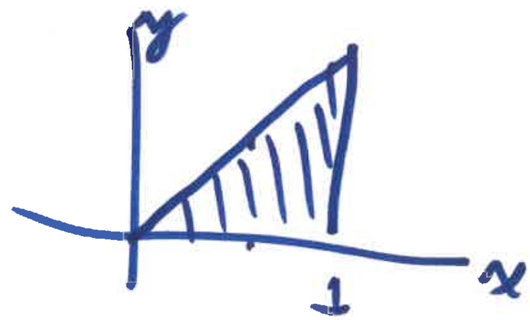
$$\Rightarrow C\int_0^1 x \cdot \frac{x^2}{2}\,dx = 1$$

$$C\int_0^1 \frac{x^3}{2}\,dx = 1$$

$$C \cdot \frac{x^4}{8}\Big]_0^1 = 1$$

$$C \cdot \frac{1}{8} = 1 \Rightarrow C = 8$$

$$f_{X,Y}(x,y) = \begin{cases} 8xy & \text{for } 0 \le x \le 1,\ 0 \le y \le x \\ 0, & \text{otherwise} \end{cases}$$



$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$f_X(x) = 0$ if $x < 0$ or $x > 1$

For $0 \le x \le 1$, $f_X(x) = \int_0^x 8xy\,dy$

$$f_X(x) = 8x \int_0^x y\,dy$$

$$= 8x \cdot \frac{x^2}{2} = 4x^3,\ 0 \le x \le 1$$

$$\boxed{f_X(x) = \begin{cases} 4x^3, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}}$$