Consider the general constrained problem (smooth functions)

$$\boxed{\begin{aligned} &\min \ f(x) \\ &\text{s.t.} \quad c_i(x) = 0 \quad i \in \mathcal{E} \\ &\qquad\quad c_i(x) \geq 0 \quad i \in \mathcal{I} \end{aligned}} \quad (*)$$

$\mathcal{E}, \mathcal{I}$ are index sets for identifying the equality and inequality constraint functions, respectively.

If $\mathcal{E} = \phi$ and $\mathcal{I} = \phi$ then we have an unconstrained problem for which we know

(a) x^* is a local minimizer $\Rightarrow$ $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ p.s.d.

(b) $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ p.d. $\Rightarrow$ x^* is a strict local minimizer.

We would like to have similar tests for constrained problems.

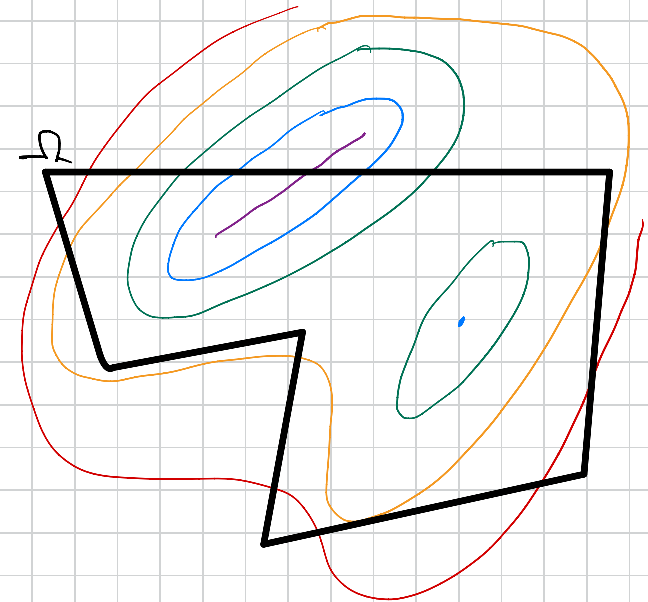
Some Definitions

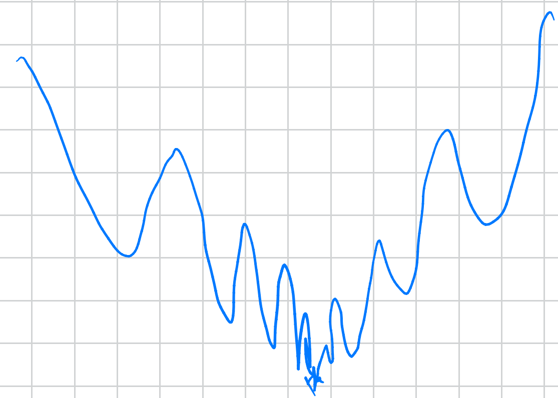
- x^* is a local minimizer of ⊛ if $f(x^*) \leq f(x)$ $\forall\, x \in \Omega \cap B(x^*, r)$ for some $r > 0$.

- x^* is a strict local minimizer of ⊛ if $f(x^*) < f(x)$ $\forall\, x \in \Omega \cap B(x^*, r)$ $x \neq x^*$, for some $r > 0$.

- x^* is an isolated local minimizer of ⊛ if $\exists$ neighborhood $\Omega \cap B(x^*, r)$, $r > 0$ for which x^* is the unique local minimizer.

- A constraint $c_i(x) = 0$ or $c_i(x) \leq 0$ is said to be active at x if $c_i(x) = 0$.





An Example

$$\boxed{\begin{aligned} \min \quad & f(x) = x_1 - x_2 \\ \text{s.t.} \quad & x_2 = x_1^3 \\ & x \in \mathbb{R}^2 \end{aligned}}$$

— or —

$$\boxed{\begin{aligned} \min \quad & f(x) = x_1 - x_2 \\ \text{s.t.} \quad & c(x) = x_1^3 - x_2 = 0 \\ & x \in \mathbb{R}^2 \end{aligned}}$$

For a feasible step s (small enough) at feasible point x, it must be true that $c(x+s) = 0$. So

$$c(x+s) = c(x) + s^T \nabla c(x) + \cdots$$

$$\Rightarrow \boxed{\nabla c(x)^T s = 0}$$

If x is a local minimizer then

$$f(x \pm s) \geq f(x). \quad \text{So}$$

$$f(x \pm s) = f(x) \pm s^T \nabla f(x) + \cdots$$

$$\Rightarrow \pm \nabla f(x)^T s \geq 0$$

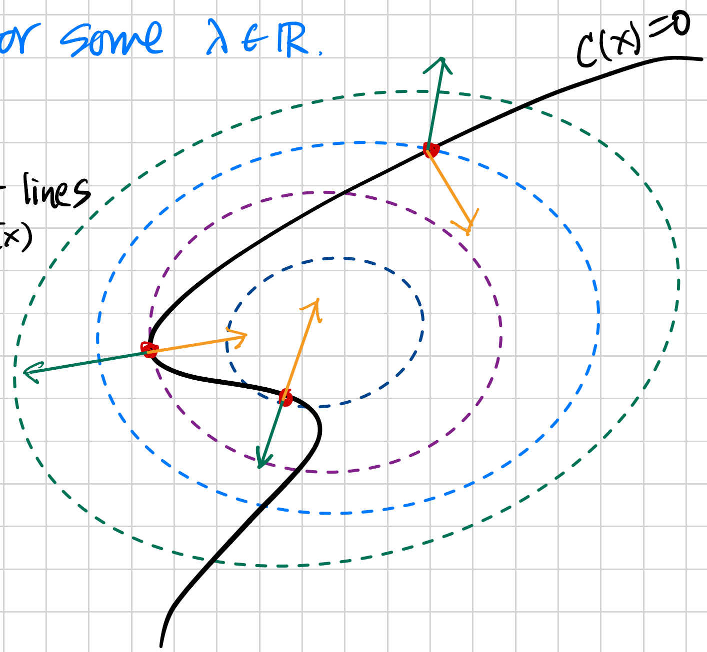
$$\Rightarrow \boxed{\nabla f(x)^T s = 0}$$

So, for a single constraint, the space of vectors perpendicular to $c(x) = 0$ is dimension 1. Thus, $\nabla f(x)$ and $\nabla c(x)$ are colinear.

If x is a local minimizer then

$$\lambda \nabla c(x) = \nabla f(x)$$

for some $\lambda \in \mathbb{R}$.



We define the Lagrangian as

$$L(x,\lambda) = f(x) - \lambda c(x).$$

Then we recover the first order necessary conditions as

$$\nabla L(x,\lambda) = 0.$$

In particular:

$$\nabla_x L = \nabla f(x) - \lambda \nabla c(x) = 0$$

$$\Rightarrow \boxed{\nabla f(x) = \lambda \nabla c(x)}$$

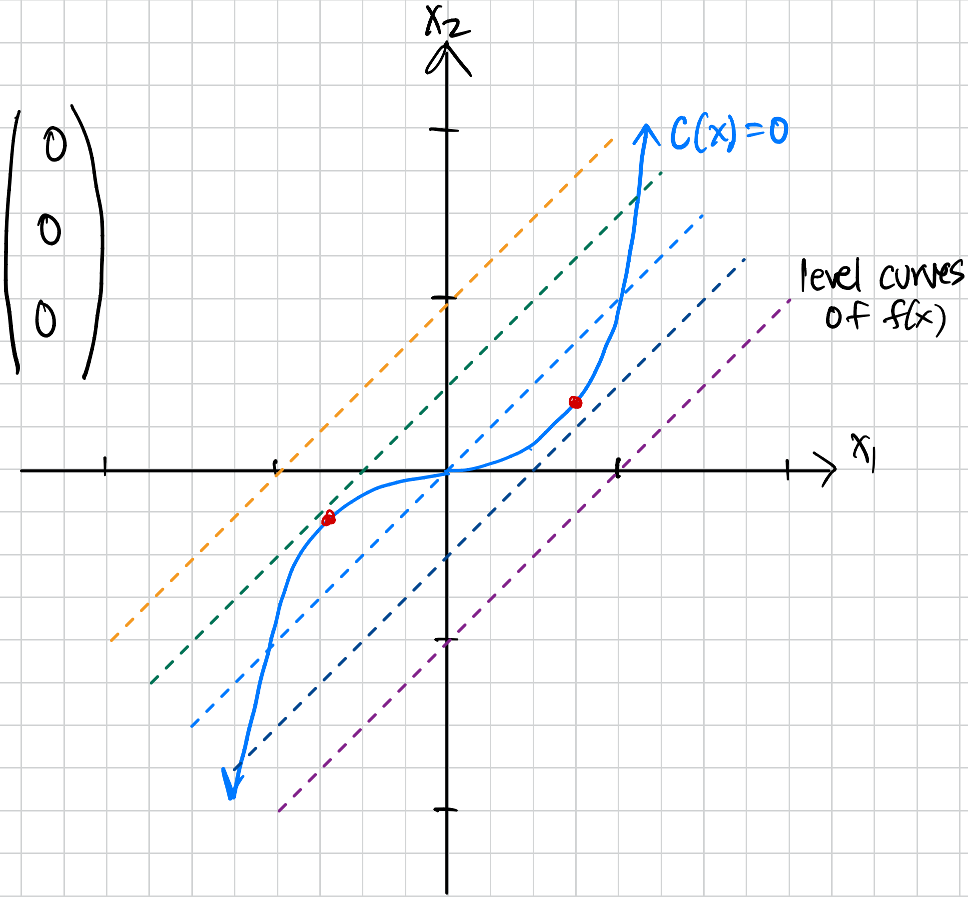
$$\nabla_\lambda L = -c(x) = 0$$

$$\Rightarrow \boxed{c(x) = 0}$$

For the current example

$$L(x,\lambda) = x_1 - x_2 - \lambda(x_1^3 - x_2)$$

$$\nabla L(x,\lambda) = \begin{pmatrix} 1 - 3\lambda x_1^2 \\ -1 + \lambda \\ -x_1^3 + x_2 \end{pmatrix} \overset{\text{set}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$



We find two solutions:

$$x_1 = \pm\sqrt{1/3}$$
$$x_2 = \pm\left(\sqrt{1/3}\right)^3$$
$$\lambda = 1$$