Mathematics Gamification in Mobile App Software for Personalized Learning at Scale

Chee Wei Tan[+], Lin Ling*[+], Pei-Duo Yu[+], Ching Nam Hang[+], and Man Fai Wong[+]
[+]Nautilus Software Technologies Limited, *Princeton University, [+]City University of Hong Kong

2020 IEEE Integrated STEM Education Conference (ISEC) | 978-1-7281-7520-1/20/$31.00 ©2020 IEEE | DOI: 10.1109/ISEC49744.2020.9397846

Abstract – **The essence of science and engineering lies in the abstract thinking and logical reasoning skills. Advanced mathematical topics such as probability theory and modular arithmetic can be introduced to students at middle or pre-college schools to cultivate their capacity for logical thinking and problem-solving skills, as well as gaining mathematical competency required in fields of science and engineering. In this paper, we introduce the idea of mathematics gamification and its application to a mobile app educational software development. We define mathematics gamification as the process of embedding mathematical concepts and their logical manipulations in a puzzle game-like setting aided by computing technologies. This is a form of personalized learning technologies that facilitate learning with peers in a social environment. In particular, we first present PolyMath, a meticulously designed mobile app software with many varieties of bite-sized games. The key idea in mathematics gamification is to motivate the users to level up from easy to more challenging gameplay. Then we describe several mathematics gamification instances in PolyMath, and report its use in the annual Julia Robinson Mathematics Festivals in Hong Kong. The efficacy of mobile app software in a math circle environment opens up new pedagogical ways to teach and learn advanced mathematics.**

Index Terms – Educational Mobile App, Mathematics Gamification, Personalized Learning

INTRODUCTION

Mathematics is widely recognized as the bedrock of science, technology, and engineering. A central tenet in learning mathematics is to first observe a pattern and then logically study the observed pattern and how it relates to problem-solving. Marvin Minsky, in his 1970 Turing Award Lecture [1], asserted that "The computer scientist thus has a responsibility to education…how to help the children to debug their own problem-solving processes." Minsky pointed out that cultivating the capacity for logical thinking and problem-solving skills of students, while they are young, to learn foundational subjects such as mathematics is of the essence. The emphasis is on the tools and motivations for students to acquire problem-solving skills in lifelong learning of mathematics.

With the rapid development of information technology, software-aided learning has become more accessible than it once was. However, there is still a huge gap between traditional in-class learning and self-learning through online learning platforms or mobile devices. One of the significant differences lies in the communication and interaction with peers and the instructor. In traditional in-class learning, students may have a better learning outcome than in a self-learning mode due to the direct interaction with the instructor and other students. Indeed, peer engagement and collaborative learning in a social setting enhance the logical thinking and reasoning skills of learners. On the other hand, self-learning through online learning in personal mobile devices also has advantages such as accessibility and flexibility at the pace of the learner. But more importantly, it provides the learner with experiential learning – making mistakes without anxiety, akin to game-playing in mobile devices. It is worth noting that the primary purpose of educational software and technologies is not to replace the in-class learning that occurs between instructors and students but to enhance it [2]. The nexus of curriculum and games in software to strengthen the rigor of learning can be beneficial.

Game playing associated with mathematics learning is essentially the manipulative of mathematical objects or structures in a logical manner such to acquire useful mathematical insights that otherwise are not obvious or taught in traditional classrooms [3-5]. Also, the engaging game-like nature can potentially motivate students and serve as instructional tools for regular practice to gain proficiency in mathematics and numeracy. We define mathematics gamification as the process of embedding mathematical concepts and their logical manipulations in a puzzle game-like setting aided by computing technologies. Indeed, the idea is to view "toy examples" in mathematics as literally "toy" that can be manipulated or played with in software.

There are several ways to implement ideas of mathematics gamification in software that can be delivered to the students. We describe in this paper a mobile app software PolyMath™ [6] that allows students to learn advanced mathematics in a systematic manner and at their own pace. Furthermore, the software design provides a way to *teach for mastery*, where students are allowed to progressively move on to more challenging tasks only after they have fully grasped the topic on hand, akin to level-up in games. This requires software technologies that can track the individual student's learning pace, potentially allowing teachers to conduct mini game tournaments in the classroom to complement with traditional classroom teaching.

Another unique advantage of mathematics gamification in software is its ability to offer students instant feedback as they play, which is a crucial part of "debugging" their thinking process. In traditional classroom learning, students' answers often are graded and then returned to them weeks later. On the contrary, PolyMath™ enables students to get instant visual feedback as they play, so that they can reconsider the situation and correct their moves, which is an example of "thinking process debugging" on the go and in real-time in many varieties of bite-sized games.

MOBILE APP LEARNING SOFTWARE: POLYMATH APP

The games and puzzles in PolyMath™ are related to different fields in mathematics, and students are allowed to progressively move on to more challenging topics only after they have fully grasped the topic on hand. In the PolyMath™ app, mathematical problem sets are carefully crafted in a way such that they are initially easy, and then they become progressively challenging. In this way, students could gain confidence at the beginning and also develop the right intuition to tackle the more challenging mathematical aspects in the latter part of each problem set. From a pedagogical viewpoint, the harder problem parts, even when unsolved, would pique their curiosity. Hence, the intention is for the students to walk away with some sense of accomplishment and having a new understanding of a mathematical problem, rather than giving superficial answers to the problem sets with poor understanding. As illustrative examples, Figure I shows the opening interface that reports progress and navigates the user to some of the problem sets whose mathematical nature has been gamified.
In the following, we introduce three problem sets in the PolyMath app to give readers a taste of mathematics gamification.

I. Fish-Flavored Lollipops

The Fish-Flavored Lollipops game is a two-player game that is a variation of the ancient game "Nim", which has deep mathematical root and has been well studied [7]. Here is the problem:
There are 12 lollipops on the table, and the last lollipop is fish-flavored which tastes so disgusting that no one wants to eat it. Two players take turns to pick up lollipops from the table one by one, and whichever player that takes the last lollipop loses the game. The players can pick no more than 5 lollipops each turn, and they can not "skip" their turns (i.e., they have to pick at least one lollipops each turn).
The question is, is there a winning strategy for each player?

This problem can be solved by a method called ''backward induction.'' In the following, we briefly introduce how backward induction can be applied to solve this problem.

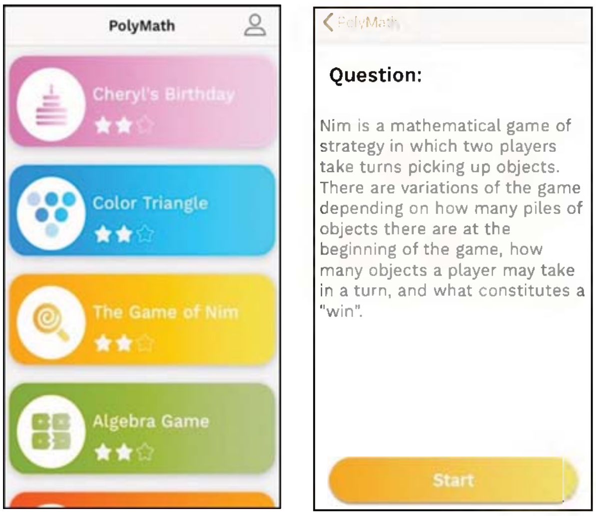

FIGURE I
SCREENSHOT OF POLYMATH APP AND THE NIM GAME

Let C denote the state such that the current player has a winning strategy and N denote the state such that the next player has a winning strategy. Obviously, it is an N state when there is only one lollipop, i.e., the next player will win the game since the first player cannot skip this turn and forced to take the last lollipop. Hence, the goal of both players is to change the game state into the N state during their turn to make sure they will win the game. Let us consider the state with two lollipops. Since the current player will take only one lollipop to avoid taking the last one and the next player is forced to take the last one, it is a C state. We can deduce that when there are three, four, five and six lollipops, the next player is always forced to take the last one. When there are seven lollipops, then whatever the first player takes, the game will change into a C state, i.e., the state with seven lollipops is an S state. Based on the above observation, we are able to assign {C, N} to each state starting from one lollipop. For example, we have {1: N, 2: C, 3: C, 4: C, 5: C, 6: C, 7: N, …}. Note that there are five consecutive C states due to the maximum number of lollipops being taken in each state is five. Hence, we can derive a winning strategy for each player.

We can see that, this puzzle can be challenging even for adults, especially when real physical manipulatives are not available to simulate the game. With the PolyMath™ app, we gamify the puzzle, so that students can play with the lollipops on their mobile devices to first gain intuitive ideas of developing a winning strategy, and then attempt to work out the math behind the game. By playing this puzzle, students can get familiar with basic algebra operations like division and modulo. More importantly, even younger students who have not learned these operations in class can gain an intuitive feeling of them by experientially manipulating the lollipops and observing the game outcome. Students can thus learn logical thinking and solving the problem using backward induction. Future extensions of

PolyMath™ app will allow learners to even craft a virtual AI player that can be shared with other human peers.

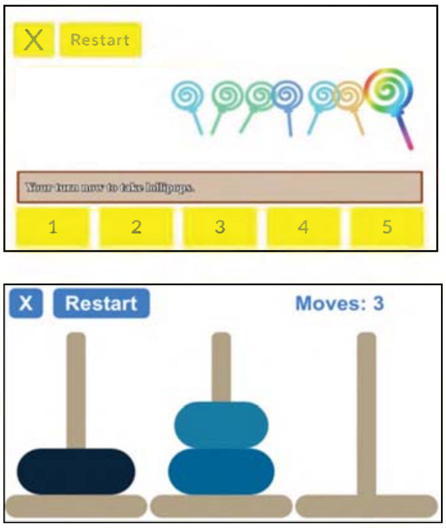

FIGURE II
THE GAMEPLAY SCREENSHOT OF THE FISH-FLAVORED LOLLIPOPS AND THE TOWER OF HANOI

II. Tower of Hanoi

Tower of Hanoi is a notable problem often introduced in different courses from elementary school to university. The problem Tower of Hanoi contains various concepts in mathematics and computer science such as recursion, recurrence equations and dynamic programming. The game is as follows:

In the Tower of Hanoi, there are three rods and some disks with different sizes, which can slide onto any rod as shown in the lower part of Figure II. The objective of the puzzle is to move the entire stack to another rod, and the process must obey the following three rules:

1. *Only one disk can be moved at a time.*
2. *Each move consists of taking the top disk from one of the rods and placing it on the top of another rod.*
3. *No larger disk cab be placed on top of a smaller disk.*

Users start playing from an easy level of a single disc and then progressing through levels with an increase in numbers of discs. After answering the easy parts, i.e., the problem with a single and two disks, users will be asked to answer a harder problem as shown in Figure III. More advanced levels then prompt the learners to solve recurrence equation incrementally as well as the "backward induction" strategy that the learners have encountered in the previous Fish-flavored Lollipop game. Hence, the games are typically designed in a way to reinforce the intended learning rubrics for a learner.

III. Number Game

We next turn to the Number game, which is a puzzle designed for students to explore randomness, probabilities as well as the concept of infinity. Here is the problem:

- I write two different numbers from 1 to 13, both are unknown to you, to two cards and put them upside-down on the table. You can choose one of the cards and flip it over to see the number on it. After seeing the number, you will have to make a guess on which card has a larger number on it. Is there a strategy that gives you more than 50% chance of winning?
- The same procedure is repeated, but this time the numbers I write down can be any integer $(\cdots, -2, -1, 0, 1, 2, \cdots)$. Again, is there a strategy that gives more than 50% chance of winning?

The first puzzle is pretty straightforward, and most students can answer it correctly. Even for students who haven't learned about probabilities, they can still answer it simply by making an intuitive guess.

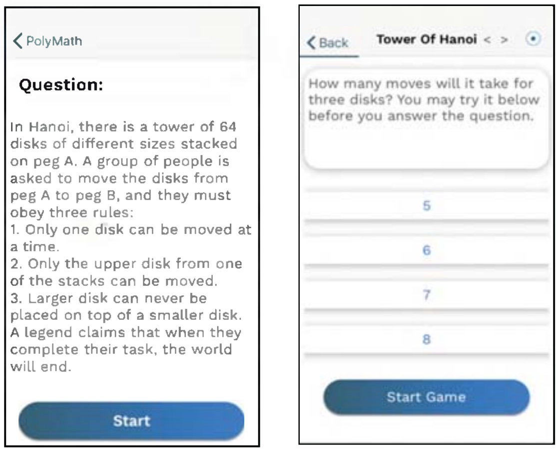

FIGURE III
TOWER OF HANOI IN POLYMATH APP

Let us analyze the first part of the problem, which is used as a foundation for our analysis of the second part, which is a much harder puzzle. Again, let us formulate the problem more formally. Let the two random numbers be x_1 and x_2, respectively, and $x_1 \neq x_2$. Since we are allowed to see one of the numbers, we can assume that x_1 is known, and our goal is to guess which one of x_1, x_2 is larger. In the first part of the puzzle, one important information is that $1 \leq x_1, x_2 \leq 13$. Using this information, we can express the probability that $x_1 > x_2$ as $p(x_1 > x_2) = \frac{x_1 - 1}{13}$, and similarly $p(x_1 < x_2) = \frac{13 - x_1}{13}$. Now we have our strategy:

1) If $p(x_1 > x_2) > p(x_1 < x_2)$ or $x_1 > 7$, then we guess x_1 is larger.
2) If $p(x_1 > x_2) < p(x_1 < x_2)$, or $x_1 < 7$, then we guess x_2 is larger.
3) If $p(x_1 > x_2) = p(x_1 < x_2)$, or $x_1 = 7$, then we just guess randomly.

Note that in both cases 1 and 2, we have 100% chance of winning, and even in the case 3 we still have 50% winning chance, so overall our winning chance using this strategy is over 50%.

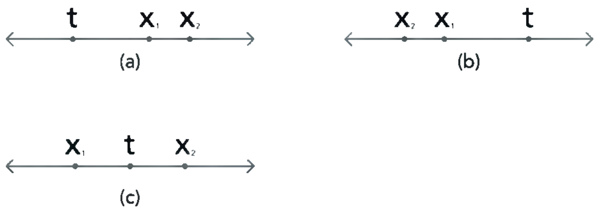

FIGURE IV

THREE CASES OF THE RELATIVE POSITION OF TWO NUMBERS AND T. IN CASE (A) AND (B), OUR STRATEGY IS EQUIVALENT TO RANDOM GUESS, WHICH GIVES 50% WINNING CHANCE. IN CASE (C), OUR STRATEGY GIVES 100% WINNING CHANCE.

Now let us move on to the second part of the puzzle, which is actually a variation of the famous "two envelopes problem" [8]. We will soon realize the difficulty here: the two numbers are not picked from a finite range anymore, so we cannot directly apply the same method we used for the first part, but we will see that the same philosophy of making the overall chance of winning over different cases larger than 50% can still be applied. Notice that, the key of the difficulty is that we cannot find a "middle point" for comparison as we did for the first part (the number 7), so we can try to choose a comparison point randomly. Before seeing x_1, we choose a random real number t, and use it as the number 7 in the first part.

At this point, the PolyMath™ app will provide a variety of strategies for the learner, namely:

1) If $x_1 > t$, then we guess x_1 is larger.
2) If $x_1 < t$, then we guess x_2 is larger.
3) If $x_1 = t$, then we guess randomly.

If the chosen t happens to be at one side of both x_1 and x_2 (i.e. $t \leq \min(x_1, x_2)$ or $t \geq \max(x_1, x_2)$), our strategy stated above is equivalent to random guess, which gives 50% chance of winning; if the t chosen happens to fall between x_1 and x_2, our chance of winning would be 100%. This is illustrated in Figure IV. Let p be the probability that t is in between x_1 and x_2, then we can express out overall winning chance as $p_w = 0.5 + p$. Now, we see that the generation of t is crucial in our strategy, since it has to be generated under a probability distribution such that the probability of t being in any range is greater than zero, otherwise p could become 0, making $p_w = 0.5$. Gaussian distribution, for example, satisfies this property and thus can be used to generate our t.

We can see that the second part of the puzzle requires the students to appreciate the concept of infinity and probability distribution, which can be quite challenging for younger students. However, despite that the math behind the puzzle can be deep for the students, the PolyMath™ app gamifies the famous "two envelopes problem" with a step-by-step demonstration so that learners can develop a mathematical intuition and hopefully are less likely to be anxious when they actually learn the knowledge at school. In the PolyMath™ app as shown in Figure V, we gamify the guessing process so that users can test their guessing strategy and see the average (expected) outcome after several guesses. Furthermore, users can compare their average outcome to the optimal strategy to evaluate their own strategy. This Number Game is associated with a deep mathematical problem with roots in information theory and probability theory. We observed that students who though may not have any notion of probability theory can still logically deduce nontrivial insights to problems that look deceivingly simple and straightforward.

SOFTWARE AIDED LEARNING IN MATH CIRCLE

The PolyMath™ app is used extensively at The Julia Robinson Mathematics Festival in Hong Kong (https://www.algebragame.app/jrmf/), which is held annually at the Singapore International School since 2017 in partnership with the U.S. Julia Robinson Mathematics Festivals (www.jrmf.org). The goal of the festivals is to encourage students to focus on collaborative problem-solving, as opposed to the competitive nature commonly

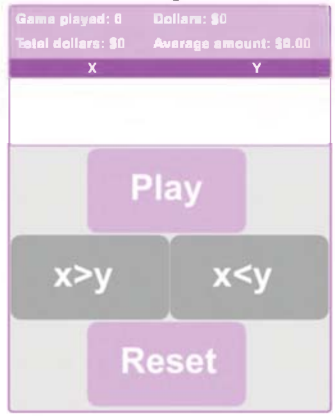

FIGURE V

SCREENSHOT OF THE NUMBER GAME

founded in mathematics examinations and contests. In this way, students can enjoy the richness and beauty of mathematics without any anxiety. At the festivals, there are typically twenty tables, each with its own unique fun and challenging mathematical theme, and each table was staffed by a facilitator. The facilitators are volunteers from academia or practicing mathematicians in the industry. Student participants can roam around from tables to tables and choose to work at a table that they find interests in. The spirit of the festivals is that there is no pressure for the student to finish any task on hand. The PolyMath™ app is used as the starting point for each problem set, providing students with experiential learning first to develop the intuition to the mathematical problems through observation and experimentation. For example, in Figure VI, student participants are learning the Chinese Remainder Theorem through by playing the game. After playing a few levels of the game, students can have some ideas about how the theorem works and apply it in the game. Another example, in Figure VII, is that students are playing with the digitized

Hanoi Tower and then figure out if there is any pattern to solve the puzzle. Even after the festival has concluded, the mobile app software can allow students (and also their parents and teachers) to relive the experience of the festival, and thus further encouraging collaborative problem-solving among peers.

FIGURE VI

STUDENT PARTICIPANTS DISCUSSED MATH PROBLEM THROUGH POLYMATH AT THE JRMF IN HONG KONG

Student participants at the festivals provide feedback that in turn generates new ideas to refine the process of mathematics gamification, allowing more logical aspects of mathematical problem sets to be digitalized and transformed into new mesmerizing game levels that can be shared with future learners. This would enhance the nature of the mathematical problem sets, when presented to the students, by appealing for its fun elements (after all the nature of a festival!) rather than its more abstract elements.

FIGURE VII

STUDENT PARTICIPANTS DISCUSSED THE PROBLEM OF HANOI TOWER THROUGH POLYMATH AT THE JRMF IN HONG KONG

Certainly, to train well for abstract thinking as required in advanced mathematics often starts with curious observations and simple experimentations. The PolyMath™ app serves as an experimental platform for students to more efficiently develop abstract thinking skills and intuition of problem-solving.

CONCLUSION

Our personalized learning technologies are based on mathematics gamification, which is defined as the process of embedding mathematical concepts and manipulations within puzzle-like instantiations. The PolyMath™ app software can be readily delivered to a large number of learners who learn at their own pace or in a social environment (e.g., Mathematics Festivals), encouraging peer learning and experiential learning. The benefit of a nexus of mathematics gamification with learning mathematics is clear: Through experimentation and curious observations, students acquire useful insights into the mathematical subject on hand that otherwise will not be obvious or found in traditional classrooms. The engaging game-like nature in mathematics gamification motivates students to peer-help one another as well as allowing classroom teachers to use them as instructional tools for students to gain proficiency in mathematics at their own pace.

REFERENCES

[1] Minsky, M. 1970. “Form and content in computer science.” Journal of the ACM, vol. 17, no. 2, pp. 197--215, 1970.

[2] Judge, S., Floyd, K. and Jeffs, T. 2015. “Using Mobile Media Devices and Apps to Promote Young Children’s Learning, Educating the Young Child.” Vol 10, (pp. 117-131), Springer.

[3] Tan, C. W. and Yu, P. and Ling, L. 2019. “Teaching Computational Thinking Using Mathematics Gamification in Computer Science Game Tournaments.” *Proceedings* of the Computational Thinking Education, pp 167-181, Springer, 2019.

[4] Thiele R. 1994. “Mathematical Games.” Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences, Vol.2, pp 1555−1567.

[5] Chang, R. C. and Yang, C. Y. 2016. “Developing a mobile app for game-based learning in middle school mathematics course.” Proceedings of 2016 International Conference on Applied System Innovation (ICASI).

[6] PolyMath App, The Algebra Game Project, 2019. [Online]. Available App Store: https://apps.apple.com/us/app/polymath-app/id1440456624, Google Play: https://play.google.com/store/apps/details?id=com.algebragamification.polymathapp

[7] Bouton, C. L. 1901. “Nim, a Game with a Complete Mathematical Theory,” Annals of Mathematics, Princeton 3, 35-39, 1901-1902.

[8] Sobel, H. 1994. “Two envelopes.” Theory and Decision, Vol 45, pp 69-96.