$K \rightarrow$ Chain 1 = $1 \rightarrow 2 \rightarrow 5 \rightarrow 6$

$K+1 \rightarrow$ Chain 2 = $3 \rightarrow 4 \rightarrow 7$

IF $\dfrac{\sum_{j=1}^{K} w_j}{\sum_{j=1}^{K} P_j} > \dfrac{\sum_{j=K+1}^{n} w_j}{\sum_{j=K+1}^{n} P_j}$ (K First)

IF $\dfrac{\sum_{j=1}^{K} w_j}{\sum_{j=1}^{K} P_j} < \dfrac{\sum_{j=K+1}^{n} w_j}{\sum_{j=K+1}^{n} P_j}$ (K+1 First)

1

Chain 1 = $\dfrac{6+18+8+17}{3+6+4+8}$

$= \dfrac{49}{21} = 2.3333\overline{3}$

Chain 2 : $\dfrac{12+8+18}{6+5+10} = \dfrac{38}{21} - 1.8095\overline{2}$

Since $\dfrac{\sum_{j=1}^{K} w_j}{\sum_{j=1}^{K} P_j} > \dfrac{\sum_{j=K+1}^{n} w_j}{\sum_{j=K+1}^{n} P_j}$, we will process chain 1 first before chain 2.

j	1	2	5	6	3	4	7
P_j	3	6	4	8	6	5	10
C_j	3	9	13	21	27	32	42

$\underbrace{1 \quad 2 \quad 5 \quad 6}_{\text{chain 1}} \quad \underbrace{3 \quad 4 \quad 7}_{\text{Chain 2}}$

(2) Since the chain can be broken,

$$\rho(1,2,\ldots,K) = \frac{\sum_{j=1}^{l^*} w_j}{\sum_{j=1}^{l^*} p_j} = \max_{1 \le l \le K}\left(\frac{\sum_{j=1}^{l} w_j}{\sum_{j=1}^{l} p_j}\right)$$

Chain 1 : $\rho(1,2,5,6) = \max\left\{\frac{6}{3}, \frac{18}{6}, \frac{8}{4}, \frac{17}{8}\right\}$

$\max\left\{\frac{6}{3}, \frac{6+18}{3+6}, \frac{6+18+8}{3+6+4}, \frac{6+18+8+17}{3+6+4+8}\right\}$

$\max\left\{2, 2.6666\bar{7}, 2.4615\bar{4}, 2.3333\bar{3}\right\} = 2.6666\bar{7}$ (Job 2)

Chain 2 : $\rho(3,4,7) = \max\left\{\frac{12}{6}, \frac{8}{5}, \frac{18}{10}\right\}$

$\max\left\{\frac{12}{6}, \frac{12+8}{6+5}, \frac{12+8+18}{6+5+10}\right\}$

$\max\left\{2, 1.81818, 1.8095\bar{2}\right\} = 2$ (Job 3)

$\rho(1,2,5,6) > \rho(3,4,7) = 2.6666\bar{7} > 2$

Scheduled $1 \to 2 \to ?$

$\rho(5,6) = \left\{\frac{8}{4}, \frac{8+17}{4+8}\right\} = \left\{2, 2.0833\bar{3}\right\} = 2.0833\bar{3}$ (Job 6)

$\rho(5,6) > \rho(3,4,7) = 2.0833\bar{3} > 2$

Scheduled $1 \to 2 \to 5 \to 6$

Since it's only chain 2 remaining, I used the ρ to schedule it. $\rho(3 > 4 > 7) = \{2, 1.818\bar{8}, 1.8095\bar{2}\}$

$1 \to 2 \to 5 \to 6 \to 3 \to 4 \to 7$

Jobs	1	2	3	4	5	6	7
P_j	3	6	8	5	4	8	10
r_j	2	0	0	2	10	10	10

Since $w_j = w$, $w = 1$

$$\frac{w}{p} = \left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{8}, \frac{1}{5}, \frac{1}{4}, \frac{1}{8}, \frac{1}{10}\right\}$$

$$= \left\{\underset{\text{Job 1}}{0.333\bar{3}}, \underset{\text{Job 2}}{0.1666\bar{7}}, \underset{\text{Job 3}}{0.125}, \underset{\text{Job 4}}{0.2}, \underset{\text{Job 5}}{0.25}, \underset{\text{Job 6}}{0.125}, \underset{\text{Job 7}}{0.1}\right\}$$

3

Arrange in increasing order

or

$$\left\{\underset{\text{Job 1}}{0.3333\bar{3}}, \underset{\text{Job 5}}{0.25}, \underset{\text{Job 4}}{0.2}, \underset{\text{Job 2}}{0.1666\bar{7}}, \underset{\text{Job 3,}}{0.125}, \underset{4}{0.125}, \underset{\text{Job 7}}{0.1}\right\}$$

For $1 \mid Prmp, r_j \mid \sum c_j$ (Shortest Remaining Processing Time)

$t = 0$, $J^c = \{2, 3\}$, Pick $J = \{2\}$

$P = 0 - 2$,

$t = 2$, $J^c = \{4, \cancel{1}\}$, Pick $J = \{1\}$

$P = 2 - 5$,

$t = 5$, $J^c = \{\cancel{2}, 3, 4\}$ Pick $J = \{2\}$

$P = 5 - 9$

$t = 9$, $J^c = \{3(8), 4(5)\}$, Pick $J = \{4\}$

$P = 9 - 10$

$t = 10$, $J^c = \{3(8), 4(4), 5(4), 6(8), 7(10)\}$, cont $J = \{4\}$

$P = 10 - 14$

$t = 14$, $J^c = \{3(8), 5(\cancel{4}), 6(8), 7(10)\}$, Pick $J = \{5\}$

$P = 14 - 18$

$t = 18$ $J^c = \{\cancel{3(8)}, 6(8), 7(10)\}$, Pick $J = \{3\}$

$P = 18 - 26$

Schedule : $5 \to 2 \to 4 \to 5 \to 3 \to 6 \to 7$

$t = 26$, $J^c = \{\cancel{6(8)}, 7(10)\}$, Pick $J = \{6\}$
$P = 26 - 34$
$t = 32$, $J^c = \{7(10)\}$ Pick $J = \{7\}$
$P = 34 - 44$

$= C_1 = 5$, $C_2 = 9$, $C_3 = 26$, $C_4 = 14$, $C_5 = 18$,
$C_6 = 34$, $C_7 = 44$

Since $w_j = 1$ for all jobs, the preemptive WSPT rule selects at any time the available job that maximises

$$\frac{w_j}{p_j^{\text{remaining}}} = \frac{1}{p_j^{\text{remaining}}}.$$

This equivalent to selecting the job with the smallest remaining processing time (SRPT), ∴ SRPT is optimal for $1 \mid prmp, r_j \mid \sum C_j$.

④ Scheduled: $3 \to 2 \to 1 \to 6 \to 4 \to 7 \to 5$

$\sum P_j = 4+8+12+7+6+9+9 = 55$

$J = \emptyset,\ J^c = \{1,2,3,4,5,6,7\},\ J' = \{1,2,3,4,5,6,7\}$

$h_1(55) = 3(55) = 165,\ h_2(55) = 77,\ h_3(55) = 166375,\ h_4(55) = 82.5,$

$h_5(55) = 57.41620,\ h_6(55) = 88,\ h_7 = 77$

$\min\{h_j(C_j)\} = h_5(55) = 57.41620$

$\therefore 55 - 6 = 49$

$h_1(49) = 3(49) = 147,\ h_2(49) = 77,\ h_3(49) = 49^3 = 117649,\ h_4(49) = 1.5(49) = 73.5$

$h_6 = 1.6(49) = 78.4,\ h_7(49) = 1.4(49) = 68.6$

$\min\{h_j(C_j)\} = h_7(49) = 68.6$

$\therefore 49 - 9 = 40$

$h_1(40) = 3(40) = 120,\ h_2(40) = 77,\ h_3(40) = 40^3 = 64000,\ h_4(40) = 1.5(40) = 60$

$h_6 = 1.6(40) = 64$

$\min\{h_j(C_j)\} = h_4(40) = 60$

$\therefore 40 - 7 = 33$

$h_1(33) = 3(33) = 99,\ h_2(33) = 77,\ h_3(33) = 33^3 = 35937,\ h_6(40) = 1.6(33) = 52.8$

$\min\{h_j(C_j)\} = h_6(33) = 52.8$

$\therefore 33 - 9 = 24$

$h_1(24) = 3(24) = 72,\ h_2(24) = 77,\ h_3(24) = 24^3 = 13824$

$\min\{h_j(C_j)\} = h_1(21) = 3(24) = 72$

$\therefore 24 - 4 = 20$

$h_2(20) = 77,\ h_3(20) = 20^3 = 8000$

$\min\{h_j(C_j)\} = h_2(20) = 77$

$\therefore 20 - 8 = 12$

$h_3(12) = 12^3 = 1728$

5 Chain 1 : 1 → 5 → 6 "3"

Chain 2 : 4 → 7

Chain 3 : 2 → 7

$\sum p_j = 4+8+12+7+6+9+9 = 55$

$J=\{\}$, $J^c=\{1,2,3,4,5,6,7\}$, $J'=\{3,6,7\}$

$h_3(55)=(55)^3=166375$, $h_6(55)=1.6(55)=88$, $h_7(55)=1.4(55)=77$

$\min\{h_j(\sum_{k\in J^c} P_k)\} = h_7(55)=77$, $\therefore 55-9=46$

$J=\{7\}$, $J^c=\{1,2,3,4,5,6\}$, $J'=\{2,3,4,6\}$

$h_2(46)=77$, $h_3(46)=(46)^3=97336$, $h_4(46)=1.5(46)=69$

$h_6(46)=1.6(46)=73.6$

$\min\{h_j(\sum_{k\in J^c} P_k)\} = h_4(46)=69$, $\therefore 46-7=39$

$J=\{4,7\}$, $J^c=\{1,2,3,5,6\}$, $J'=\{2,3,6\}$, check this side

$h_2(39)=77$, $h_3(39)=(39)^3=59319$, $h_6(39)=1.6(39)=62.4$

$\min\{h_j(\sum_{k\in J^c} P_k)\} = h_6(39)=62.4$, $\therefore 39-9=30$

$J=\{6,4,7\}$, $J^c=\{1,2,3,5\}$, $J'=\{1,2,3,5\}$

$h_1(30)=3(30)=90$, $h_2(30)=77$, $h_3(30)=(30)^3=27000$, $h_5(30)=50+\sqrt{30}=55.4772\bar{3}$

$\min\{h_j(\sum_{k\in J^c} P_k)\} = 55.4772\bar{3}$ $\therefore 30-6=24$

$J=\{5,6,4,7\}$, $J^c=\{1,2,3\}$, $J'=\{1,2,3\}$,

$h_1(24)=3(24)=72$, $h_2(24)=77$, $h_3(24)=(24)^3=13824$

$J=\{1,5,6,4,7\}$, $J^c=\{2,3\}$, $J'=\{2,3\}$ $\therefore 24-4=20$

$h_2(20)=77$, $h_3(20)=(20)^3=8000$

$\min\{h_j(\sum_{k\in J^c} P_k)\} = 77$, $\therefore 20-8=12$

$J=\{2,1,5,6,4,7\}$ $=$ $J^c=\{3\}$, $J'=\{3\}$

$h_3(12)=(12)^3=1728$

$\min\{h_j(\sum_{k\in J^c} P_k)\} = 1728$ $\therefore 12-12=0$

optimal Solution $= J=\{3,2,1,5,6,4,7\}$

Schedule : 3 → 2 → 1 → 5 → 6 → 4 → 7

(page 6)

$J = \{4,7\}$, $J^c = \{1,2,3,5,6\}$, $J' = \{2,3\}$

$h_2(39) = 77$, $h_3(39) = (39)^3 = 59319$

$\min\{h_j(\sum_{k\in J'} P_k)\} = h_2(39) = 77$, $39 - 8 = 31$

$J = \{2,4,7\}$, $J^c = \{1,3,5,6\}$, $J' = \{3,6\}$

$h_3(31) = (31)^3 = 29791$, $h_6(31) = 1.9(31) = 58.9$

$\min\{h_j(\sum_{k\in J'} P_k)\} = h_6(31) = 58.9$, $31 - 9 = 22$

$J = \{6,2,4,7\}$, $J^c = \{1,3,5\}$, $J' = \{1,3,5\}$

$h_1(22) = 3(22) = 66$, $h_3(22) = (22)^3 = 10648$, $h_5(22) = 50 + \sqrt{22}$
$= 54.69042$

$\min\{h_j(\sum_{k\in J'} P_k)\} = h_5(22) = 54.69042$

$\therefore 22 - 6 = 16$

$J = \{5,6,2,4,7\}$, $J^c = \{1,3\}$, $J' = \{1,3\}$

$h_1(16) = 3(16) = 48$, $h_3(16) = (16)^3 = 4096$

$\min\{h_j(\sum_{k\in J^c} P_k)\} = h_1(16) = 48$ $\therefore 16 - 4 = 12$

$J = \{1,5,6,2,4,7\}$, $J^c = \{3\}$, $J' = \{3\}$

$h_3(12) = (12)^3 = 1728$

$\min\{h_j(\sum_{k\in J^c} P_k)\} = h_3(12) = 1728$ $\therefore 12 - 12 = 0$

$J = \{3,1,5,6,2,4,7\}$

Schedule: $3 \to 1 \to 5 \to 6 \to 2 \to 4 \to 7$

6

Consider any time t and suppose two jobs k and l are available; $r_k \leq t$, $r_l \leq t$, Let x_k, x_l (processing times at t)

Assume $x_l < x_k$

S': process k first, then l

$C_k = t + x_k$

$C_l = t + x_k + x_l$

$C_k + C_l = (t + x_k) + (t + x_k + x_l) = 2t + 2x_k + x_l$

S'': process l first, then k

$C_l = t + x_l$

$C_k = t + x_l + x_k$

$C_l + C_k = (t + x_l) + (t + x_l + x_k) = 2t + 2x_l + x_k$

$(C_k + C_l) - (C_l + C_k)$

$= (2t + 2x_k + x_l) - (2t + 2x_l + x_k) = x_k - x_l$

Since $x_l < x_k$, $x_k - x_l > 0$

Hence, $C_l + C_k < C_k + C_l$.

Contradiction: If a schedule process k before l while $x_l < x_k$, we can interchange them and strictly decrease $\sum C_j$. Therefore, no optimal schedule can violate $x_k \leq x_l$ whenever k is processed before l. The machine must process the job with SPRT.

SRPT is optimal for $1 \mid prmp, r_j \mid \sum C_j$